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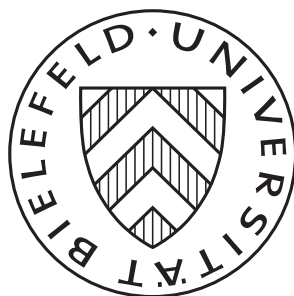


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Strong Solutions to Submodular Mean Field Games with Common Noise and Related McKean-Vlasov FBSDES

Jodi Dianetti



STRONG SOLUTIONS TO SUBMODULAR MEAN FIELD GAMES WITH COMMON NOISE AND RELATED MCKEAN-VLASOV FBSDES

JODI DIANETTI

ABSTRACT. This paper studies multidimensional mean field games with common noise and the related system of McKean-Vlasov forward-backward stochastic differential equations deriving from the stochastic maximum principle. We first propose some structural conditions which are related to the submodularity of the underlying mean field game and are a sort of opposite version of the well known Lasry-Lions monotonicity. By reformulating the representative player minimization problem via the stochastic maximum principle, the submodularity conditions allow to prove comparison principles for the forward-backward system, which correspond to the monotonicity of the best reply map. Building on this property, existence of strong solutions is shown via Tarski's fixed point theorem, both for the mean field game and for the related McKean-Vlasov forward-backward system. In both cases, the set of solutions enjoys a lattice structure, with minimal and maximal solutions which can be constructed by iterating the best reply map or via the *fictitious play* algorithm.

Keywords: Mean field games with common noise; FBSDE; Mean field FBSDE with conditional law; stochastic maximum principle; submodular cost function; Tarski's fixed point theorem; fictitious play.

AMS subject classification: 93E20, 91A15, 60H30, 60H10.

1. INTRODUCTION

Mean field games (MFGs, in short) have been proposed independently by [31] and [34], and arise as limit models for non-cooperative symmetric N -player games with interaction of mean field type as the number of players N tends to infinity. When the choice of the players is influenced by noises which are correlated, a common noise appears in the limiting MFG. While an extensive literature was developed around this topic (see, e.g., the two-volume book [15] and the references therein), a general understanding of the nature of the solutions to MFGs with common noise remains a central open problem in MFG theory. In particular, the existence of *strong solutions* (i.e., solutions which are adapted to the common noise) is known mainly under conditions which imply their uniqueness (see [2, 3, 16, 22, 32, 47]).

Crucially, despite mathematically very desirable, uniqueness in game theory represents an exceptional situation and multiple equilibria may arise. Hence, the main motivation of this paper is to (partially) overcome this limitation. This is done by proposing structural conditions and developing a theory which allows to study the existence and the approximation of strong solutions to MFGs with common noise, under lack of uniqueness.

The results. Consider independent Brownian motions W and B , which are independent from an initial condition ξ . For any B -adapted stochastic flow of probability measures μ , a representative player minimizes, by choosing an admissible control α , the cost functional

$$(1.1) \quad J(\alpha, \mu) = \mathbb{E} \left[\int_0^T h(t, X_t, \mu_t, \alpha_t) dt + g(X_T, \mu_T) \right],$$

subject to $dX_t = b(t, X_t, \mu_t, \alpha_t) dt + \sigma(t, X_t) dW_t + \sigma^\circ(t, X_t) dB_t, \quad X_0 = \xi.$

A strong MFG equilibrium (or solution) is a process μ which coincides with the flow of conditional probabilities $(\mathbb{P}_{X_t^\mu|B})_t$ with respect to the common noise B of a trajectory X^μ , with X^μ being optimal when optimizing against μ itself. By enforcing some additional conditions, one can reformulate the representative player minimization problem (parametrized by μ) via the stochastic maximum principle (SMP, in short). Doing so, any optimal trajectory X^μ for μ corresponds to the forward component X of a solution (X, Y, Z, Z°) of the fully coupled forward-backward stochastic differential equation (FBSDE, in short)

$$(1.2) \quad \begin{aligned} dX_t &= b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + \sigma(t, X_t)dW_t + \sigma^\circ(t, X_t)dB_t, & X_0 &= \xi, \\ dY_t &= -D_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + Z_t dW_t + Z_t^\circ dB_t, & Y_T &= D_x g(X_T, \mu_T), \end{aligned}$$

where $\hat{\alpha}(t, x, \nu, y) := \arg \min_{a \in A} H(t, x, \nu, y, a)$ is the (unique, under suitable assumptions) minimizer of the Hamiltonian $H(t, x, \nu, y, a) := b(t, x, \nu, a)y + h(t, x, \nu, a)$. In this spirit, one wishes to relate any strong MFG equilibrium μ to a solution (X, Y, Z, Z°) to the conditional McKean-Vlasov (MKV, in short) FBSDE

$$\begin{aligned} dX_t &= b(t, X_t, \mathbb{P}_{X_t|B}, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t|B}, Y_t))dt + \sigma(t, X_t)dW_t + \sigma^\circ(t, X_t)dB_t, & X_0 &= \xi, \\ dY_t &= -D_x H(t, X_t, \mathbb{P}_{X_t|B}, Y_t, \hat{\alpha}(t, X_t, \mathbb{P}_{X_t|B}, Y_t))dt + Z_t dW_t + Z_t^\circ dB_t, & Y_T &= D_x g(X_T, \mathbb{P}_{X_T|B}). \end{aligned}$$

This paper addresses the problems above by assuming some structural conditions which are related to the submodularity of the underlying mean field game. It is crucial to underline that (see [23] or Remark 2.10 below) the submodularity property represents in MFGs a sort of antithetic version of the well known Lasry-Lions monotonicity condition, which is typically related to the uniqueness of the equilibria (see [15]). Indeed, the submodularity conditions appears (implicitly) in a number of linear-quadratic models (see, e.g., [6]) and in many works discussing multiple equilibria in MFGs (see, e.g., [5, 18, 22]), although this property is not exploited therein.

One of the starting points of this work was to investigate whether the submodularity of the game could be related to a comparison principle, in the parameter μ , of the related family of multidimensional FBSDEs (1.2). When the state process is 1-dimensional, this idea can be easily motivated in the following example. Take $h(t, x, \nu, a) = f(t, x, \nu) + l(t, x, a)$ such that, for $\phi \in \{f(t, \cdot, \cdot), g\}$, ϕ is C^1 in x and $\phi_x(x, \bar{\nu}) \leq \phi_x(x, \nu)$, for $\nu \leq \bar{\nu}$ (in a suitable sense). If b and $\hat{\alpha}$ do not depend on the measure ν , then it follows that

$$H_x(t, x, \bar{\nu}, y, \hat{\alpha}(t, x, y)) \leq H_x(t, x, \nu, y, \hat{\alpha}(t, x, y)) \quad \text{and} \quad g_x(x, \bar{\nu}) \leq g_x(x, \nu), \quad \text{for } \nu \leq \bar{\nu}.$$

The latter inequality suggests to investigate a suitable comparison principle for the FBSDE (1.2) which allows to conclude that $X^\mu \leq X^{\bar{\mu}}$ when $\mu \leq \bar{\mu}$. Such a comparison principle is clearly related to the monotonicity of the *best reply map* $\mu \mapsto (\mathbb{P}_{X_t^\mu|B})_t$.

In the sequel, two classes of structural conditions are proposed, for either separable or nonseparable Hamiltonians. For separable Hamiltonians, the monotonicity of the best reply map is proved, from which a new comparison principle for multidimensional FBSDEs follows. When the Hamiltonians are nonseparable, we instead employ the comparison principle in [19] in order to show that the best reply map is increasing.

Building on the monotonicity of the best reply map, the main results of this paper are the following:

- (1) Existence of strong solutions to the MFG problem and to the related MKV FBSDE is shown via Tarski's fixed point theorem (see [46]). Moreover, both sets of solutions enjoy a lattice structure, and there exist minimal and maximal solutions.
- (2) The algorithm which consists in iterating the best reply map (equivalently, in solving iteratively the FBSDE (1.2)) converges either to the minimal or to the maximal solution, when suitably initialized.

- (3) The *fictional play algorithm* (which consists in solving a sequence of control problems, at each iteration solving the representative player minimization problem parametrized by an arithmetic average of the optimal distributions computed in the previous steps) converges either to the minimal or to the maximal solution, when suitably initialized.

The approach presented in this paper focuses on the reformulation of the representative player minimization problem via the SMP, and one might investigate similar results in terms of the –so called– master equation or of the MFG system (see, e.g., [15]). These questions are left for future research.

Related literature. We now discuss the relations between the results described above and the ones known in the literature.

Strong solutions to MFGs with common noise. A probabilistic formulation of MFGs with common noise was presented in [16], together with several notions of equilibrium. There, the definition of a weak solution was introduced, where (in line with the theory of stochastic differential equations) the equilibrium measure flow is not necessarily adapted to the filtration generated by the common noise, but only conditionally independent to it. In [16], the existence of weak solutions is established in a very general setting, while the existence of a strong solution is deduced (in a similar spirit of the Yamada-Watanabe theory) under the Lasry-Lions monotonicity condition, which automatically implies the uniqueness of the equilibrium.

Along with [16], several methods and structural conditions have been proposed in order to study the existence of (strong) solutions. The MKV FBSDE deriving from the SMP was first used in [13] in order to solve linear-convex MFGs with no common noise. Later, strong solutions to MFGs with common noise have been studied via the MKV FBSDE in [2], for a model with terminal cost satisfying convexity and weak monotonicity properties. A similar approach was used in [3, 32] for more general models, while linear-quadratic models were solved in [22, 47]. Furthermore, one can study MFGs by using MKV FBSDEs in which the backward component represents the minimal cost of the optimization problem at equilibrium. This method was first used in [17] for MFGs with no common noise, and later employed in [8] in order to show the existence of a weak solution to MFGs with absorbing boundary conditions and common noise. MFGs with common noise are also studied via partial differential equations (PDEs, in short) (see, e.g., the recent [11]) and through the master equation (see, e.g., [7, 9, 15, 27, 39]). In particular, the recent [39] provides a well posedness theory for the master equation related MFGs with common noise by introducing some *anti-monotonicity conditions*. These conditions seem to be related to the submodularity conditions proposed in this paper, though none of the two sets of assumptions includes the other. Existence of strong solutions is also discussed in some examples in [23, 24] for submodular MFGs.

In relation to the previous works, this paper presents a theory establishing the existence of strong solutions without relying on uniqueness. Moreover, coefficients which may be discontinuous in the measure as well as degenerate settings are allowed: In particular, first order MFGs with (see, e.g., [11]) and without common noise can be treated.

Approximation of strong solutions. The problem of providing algorithms able to approximate MFG solutions was first addressed in [10], where the authors studied, through PDE methods, the convergence of the fictional play in potential MFGs with no common noise. This algorithm was further studied for MFGs of stopping and with absorbing boundary conditions in [25] and, with the help of machine learning techniques, in [26, 44, 53]. Another learning procedure able to approximate MFG equilibria is the one studied, for submodular MFGs, in [23, 24]. This algorithm consists in iterating the best reply map, and it seems to be quite promising when combined with reinforcement learning methods, as shown in the recent [28, 35].

In this work, both the learning procedure which consists in iterating the best reply map and the fictitious play are further developed for MFGs with common noise and for conditional MKV FBSDEs, and their convergence is studied by using stability results for BSDEs. It is worth underlining that, in contrast with the papers mentioned above (but [35]), these convergence results hold under lack of uniqueness of the equilibria.

FBSDEs, MKV FBSDEs and comparison principles for FBSDEs. Solvability of coupled FBSDEs was first discussed in [4] and soon has become a relevant topic in (applied) probability theory (see [21, 30, 36, 42] and the monograph [38], among many others). An important use of FBSDEs in stochastic control theory comes from the SMP (see [43] or the monograph [15]). This approach allows to characterize (under suitable linear-convex assumptions) the optimal control in terms of the minimizer of the Hamiltonian of the control problem and of the solution of the system. Building on the SMP, MKV FBSDEs were introduced in [12, 13, 14] in order to study control problems involving also the distribution of the controlled dynamics.

While comparison principles are classical for stochastic differential equations (see, e.g., [45]) and for backward stochastic differential equations (see, e.g., [55] and the references therein), they are known to fail for FBSDEs (see, e.g., [20] or Example 6.2 at p. 22 in [38]). When the backward component is one dimensional, two types of comparison principles are known. One is the comparison principle proved in [20, 37], which allows to compare the *decoupling fields* of the systems (a function which allows to express the backward component in terms of the forward one). Such results typically hinge on comparison principles for a suitable parabolic partial differential equation, which is solved by the decoupling field. The second type of comparison principles allows to compare both the forward and the backward component (see [52]). In multidimensional cases, comparison principles are even more difficult to obtain, and strong monotonicity assumptions are required. To the best of the author's knowledge, such results were obtained, in multidimensional settings, only in [19, 52] for the forward and the backward components.

In this paper, a comparison principle is obtained for multidimensional parameter-dependent-FBSDEs corresponding (via the SMP) to minimization problems which are submodular in the parameter (for further details, see Corollary 2.12 and Remark 2.13 below). Moreover, existence of minimal and maximal strong solutions to the conditional MKV FBSDE is established by lattice theoretical methods. This approach does not require the time horizon to be small and the Lipschitz continuity in the measure flow (see [15]), or to enforce the –so called– monotonicity conditions (see [2, 3, 32]).

Submodular games and MFGs. In game theory, the submodularity property was first introduced by Topkis [48] for static N -player games and describes games in which players minimize cost functions which are submodular in the vector of strategies chosen by all the players. Such a class of games plays an important role in economical applications (see the monograph [50] and the references in it). In MFGs, the submodularity has already been used in [1] for a class of stationary discrete time games, in [51] for a class of finite state MFGs with exit, in [16] for optimal timing MFGs and in [23] for MFGs involving one-dimensional Itô-diffusions. As shown in the recent [24], the submodularity property allows to treat qualitatively different formulations of MFGs (such as MFGs with singular controls, optimal stopping, reflecting boundary conditions and finite-state problems).

In terms of methodology, the closest contribution to the present work is [23] and a detailed comparison is, at this point, necessary. First of all, this paper provides a theory for MFGs with common noise and interprets the submodularity property from the FBSDE point of view, while [23] does not. In particular, extra technical care is needed in order to deal with stochastic flows of measures and the approximation result is proved using stability of backward stochastic differential equations. Furthermore, this paper also establishes the convergence of

the fictitious play, which was not studied in [23]. Secondly, while the state space in [23] must be one-dimensional, here multidimensional settings are considered. This important issue is overcome by formulating the fixed point problem on a lattice of state processes (see Remark 2.23 below). Moreover, in the present paper, structural conditions are presented for which the drift of the state process can depend on the measure and in which the Hamiltonian is nonseparable, while the setting in [23] excludes this possibility (apart very specific examples). Finally, differently to [23], this paper also allows to treat problems where the volatility of the state process is state dependent.

Outline of the paper. The rest of the paper is organized as follows. Section 2 presents the FBSDE-approach to submodular MFGs, by introducing the submodularity conditions and by stating both some intermediate results and the main results. The intermediate results are proved in Section 3, while the main results are proved in Section 4. Examples are discussed in Section 5, while an auxiliary lemma is proved in Appendix A.

General notation. Set $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$. For $d \in \mathbb{N}_0$ and $x, y \in \mathbb{R}^d$, denote by xy the scalar product in \mathbb{R}^d , as well as by $|\cdot|$ the Euclidean norm in \mathbb{R}^d . For $x \in \mathbb{R}^d$, x^\top indicates the transpose of x . For $x, y \in \mathbb{R}^d$ and $c \in \mathbb{R}$, set $x \leq y$ if $x^i \leq y^i$ for each $i = 1, \dots, d$, as well as $x \leq c$ if $x^i \leq c$ for each $i = 1, \dots, d$. Moreover, define $x \wedge y := (x^1 \wedge y^1, \dots, x^d \wedge y^d)$ and $x \vee y := (x^1 \vee y^1, \dots, x^d \vee y^d)$, where $x^i \wedge y^i := \min\{x^i, y^i\}$ and $x^i \vee y^i := \max\{x^i, y^i\}$ for each $i = 1, \dots, d$. For a separable metric space (E, ℓ) , $\mathcal{P}(E)$ denotes the set of probability measures on the Borel sets of E . For $p > 0$, $\mathcal{P}_p(E)$ is the set of $\nu \in \mathcal{P}(E)$ such that $\int_E \ell^p(y, y_0) d\nu(y) < \infty$ for some (and thus for all) $y_0 \in E$. The p -Wassertein distance on $\mathcal{P}_p(E)$ is defined as

$$(1.3) \quad \ell_{E,p}(\bar{\nu}, \nu) := \inf \left\{ \left(\int_{E \times E} \ell^p(x, y) dN(x, y) \right)^{1/p} \mid N \in \Pi(\bar{\nu}, \nu) \right\}, \quad \text{for } \bar{\nu}, \nu \in \mathcal{P}_p(E),$$

where $\Pi(\bar{\nu}, \nu)$ denotes the set of probability measures on $E \times E$ with marginals $\bar{\nu}$ and ν . For $d \in \mathbb{N}_0$, $p > 0$ and $\mu \in \mathcal{P}(\mathbb{R}^d)$, set $\|\mu\|_p := \int_{\mathbb{R}^d} |y|^p d\mu(y)$. Unless otherwise stated, C indicates a generic positive constant, which may change from line to line.

2. PROBLEM FORMULATION AND MAIN RESULTS

2.1. MFG problem and related MKV FBSDE. Take $d, d_1, d_2, k \in \mathbb{N}_0$, a d_1 -dimensional Brownian motion $W = (W^1, \dots, W^{d_1})$ and a d_2 -dimensional Brownian motion $B = (B^1, \dots, B^{d_2})$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume the processes W and B to be independent, and independent from a d -dimensional square integrable random variable $\xi = (\xi^1, \dots, \xi^d)$. Denote by $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ (resp. $\mathbb{F}^B := (\mathcal{F}_t^B)_{t \in [0, T]}$) the right continuous extension of the filtration generated by (W, B, ξ) (resp. (B, ξ)), augmented by \mathbb{P} -null sets.

Consider a set $A \subset \mathbb{R}^k$, $A = A^1 \times \dots \times A^k$ for closed intervals $A^i \subset \mathbb{R}$, $i = 1, \dots, k$. The set of admissible (open loop) controls \mathcal{A} is defined as

$$\mathcal{A} := \{\alpha : \Omega \times [0, T] \rightarrow A \mid \alpha = (\alpha^1, \dots, \alpha^k) \text{ is square integrable } \mathbb{F}\text{-progressively measurable}\}.$$

On the set $\mathcal{P}_2(\mathbb{R}^d)$, consider the 2-Wasserstein distance $\ell_{\mathbb{R}^d, 2}$. The coefficients of the problem are given through (deterministic) measurable functions

$$\begin{aligned} (b, h) &: [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \rightarrow \mathbb{R}^d \times \mathbb{R}, \quad b := (b^1, \dots, b^d), \\ (\sigma, \sigma^\circ) &: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d_1} \times \mathbb{R}^{d \times d_2}, \quad \sigma := (\sigma_j^i)_{j=1, \dots, d_1}^{i=1, \dots, d}, \quad \sigma^\circ := (\sigma_j^{i, \circ})_{j=1, \dots, d_2}^{i=1, \dots, d}, \\ g &: \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}, \end{aligned}$$

which will be subject to the assumptions below (see Assumptions 2.2, 2.4, 2.7, 2.8 and 2.14).

Introduce the finite measure $\pi := \delta_0 + dt + \delta_T$ on the Borel σ -algebra of the interval $[0, T]$, where δ_0 and δ_T denote the Dirac measures at time 0 and T , respectively. The set of possible (stochastic) distributions of players is denoted by

$$\mathcal{M}_B^2 := \left\{ \mu : \Omega \times [0, T] \rightarrow \mathcal{P}_2(\mathbb{R}^d) \mid \mu \text{ is } \mathbb{F}^B\text{-progr. meas. and } \mathbb{E} \left[\int_0^T \|\mu_t\|_2^2 d\pi(t) \right] < \infty \right\},$$

while any controlled state processes will be an element of the set

$$\mathbb{H}^2 := \left\{ M : \Omega \times [0, T] \rightarrow \mathbb{R}^d \mid M \text{ is } \mathbb{F}\text{-progr. meas. and } \mathbb{E} \left[\int_0^T |M_t|^2 d\pi(t) \right] < \infty \right\}.$$

The sets \mathcal{M}_B^2 and \mathbb{H}^2 are actually sets of equivalence classes of processes; that is, elements of \mathcal{M}_B^2 (and of \mathbb{H}^2) are identified if they coincide $\mathbb{P} \otimes \pi$ -a.e. in $\Omega \times [0, T]$. Also, for technical reasons, the processes in \mathcal{M}_B^2 and in \mathbb{H}^2 are not assumed to be continuous. For an \mathbb{F} -progressively measurable process M (taking values in any Polish space), similarly to Remark 1 in [47] one can show that the conditional law $\mathbb{P}[M_t \in \cdot \mid \mathcal{F}_t^B]$ of M_t given the filtration \mathcal{F}_t^B is such that $\mathbb{P}[M_t \in \cdot \mid \mathcal{F}_t^B] = \mathbb{P}[M_t \in \cdot \mid \mathcal{F}_T^B]$ \mathbb{P} -a.s., for any $t \in [0, T]$. Therefore, in the sequel, $\mathbb{P}[M_t \in \cdot \mid \mathcal{F}_t^B]$ will be simply denoted by $\mathbb{P}_{M_t|B}$.

2.1.1. *The MFG problem.* Consider a mean field game (MFG, in short) in which, for any $\mu \in \mathcal{M}_B^2$, the representative player minimizes the cost functional

$$J(\alpha, \mu) := \mathbb{E} \left[\int_0^T h(t, X_t, \mu_t, \alpha_t) dt + g(X_T, \mu_T) \right], \quad \alpha \in \mathcal{A},$$

where the process $X = (X^1, \dots, X^d)$ denotes the solution to the controlled stochastic differential equation (SDE $_{(\alpha, \mu)}$, in short)

$$(2.1) \quad dX_t = b(t, X_t, \mu_t, \alpha_t) dt + \sigma(t, X_t) dW_t + \sigma^\circ(t, X_t) dB_t, \quad t \in [0, T], \quad X_0 = \xi.$$

For the moment, just assume that, for any $\alpha \in \mathcal{A}$ and $\mu \in \mathcal{M}_B^2$, the SDE $_{(\alpha, \mu)}$ admits a unique strong solution and the integrals in the cost functional $J(\alpha, \mu)$ are well defined, possibly equal to $+\infty$. For $\mu \in \mathcal{M}_B^2$, a control $\alpha^\mu \in \mathcal{A}$ is said to be optimal for μ if $J(\alpha^\mu, \mu) \leq J(\alpha, \mu)$ for any other $\alpha \in \mathcal{A}$. When such a control exists, one refers to the solution X^μ to the SDE $_{(\alpha^\mu, \mu)}$ (2.1) as to optimal trajectory, and to the couple (X^μ, α^μ) as to optimal pair.

The focus of this paper is on the following notion of equilibrium.

Definition 2.1. *A (strong) MFG equilibrium is a process $\mu \in \mathcal{M}_B^2$ such that*

$$\mu_t = \mathbb{P}_{X_t^\mu|B}, \quad \text{for any } t \in [0, T],$$

for some optimal trajectory X^μ .

Since $\mathbb{P}_{X_t^\mu|B} = \mathbb{P}[X_t^\mu \in \cdot \mid \mathcal{F}_t^B]$, any equilibrium μ as in Definition 2.1 is actually strong (see [16]), in the sense that it is adapted to the filtration generated by the common noise B .

2.1.2. *The related FBSDEs.* We next introduce the forward-backward stochastic differential equation (FBSDE, in short) related to the MFG problem with common noise. Define the Hamiltonians of the minimization problems by

$$H(t, x, \mu, y, a) = b(t, x, \mu, a)y + h(t, x, \mu, a), \quad (t, x, \mu, y, a) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times A,$$

and assume for the moment that the function $a \mapsto H(t, x, \mu, y, a)$ admits a unique minimizer $\hat{\alpha}(t, x, \mu, y)$ (see Assumption 2.2 and Remark 2.3 below). Namely, set

$$\hat{\alpha}(t, x, \mu, y) := \arg \min_{a \in A} H(t, x, \mu, y, a) \in A, \quad (t, x, \mu, y) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d.$$

For any $\mu \in \mathcal{M}_B^2$, following the stochastic maximum principle approach, one can characterize the optimal pair for μ in terms of the solution (X, Y, Z, Z°) to the FBSDE $_\mu$

$$(2.2) \quad \begin{aligned} dX_t &= b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + \sigma(t, X_t)dW_t + \sigma^\circ(t, X_t)dB_t, \\ dY_t &= -D_x H(t, X_t, \mu_t, Y_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + Z_t dW_t + Z_t^\circ dB_t, \end{aligned}$$

with boundary conditions $X_0 = \xi$ and $Y_T = D_x g(X_T, \mu_T)$. Recall that such a solution is defined as a \mathbb{F} -progressively measurable process $(X, Y, Z, Z^\circ) : \Omega \times [0, T] \rightarrow \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d_1} \times \mathbb{R}^{d \times d_2}$ such that

$$(2.3) \quad \mathbb{E} \left[\sup_{t \in [0, T]} (|X_t|^2 + |Y_t|^2) + \int_0^T (|Z_t|^2 + |Z_t^\circ|^2) dt \right] < \infty,$$

and such that system (2.2) and the boundary conditions hold. Also, for future reference, it is convenient to introduce the notation

$$(2.4) \quad \begin{aligned} \hat{b}(t, x, \mu, y) &:= b(t, x, \mu, \hat{\alpha}(t, x, \mu, y)), \\ \hat{h}(t, x, \mu, y) &:= D_x H(t, x, \mu, y, \hat{\alpha}(t, x, \mu, y)), \\ \hat{g}(x, \mu) &:= D_x g(x, \mu), \end{aligned}$$

which allows to rewrite the system (2.2) in the more compact form

$$\begin{aligned} dX_t &= \hat{b}(t, X_t, \mu_t, Y_t)dt + \sigma(t, X_t)dW_t + \sigma^\circ(t, X_t)dB_t, \quad X_0 = \xi, \\ dY_t &= -\hat{h}(t, X_t, \mu_t, Y_t)dt + Z_t dW_t + Z_t^\circ dB_t, \quad Y_T = \hat{g}(X_T, \mu_T). \end{aligned}$$

In this spirit, one wishes to relate any (strong) equilibrium μ of the MFG problem with a (strong) solution (X, Y, Z, Z°) of the following the McKean-Vlasov FBSDE (MKV FBSDE, in short)

$$(2.5) \quad \begin{aligned} dX_t &= \hat{b}(t, X_t, \mathbb{P}_{X_t|B}, Y_t)dt + \sigma(t, X_t)dW_t + \sigma^\circ(t, X_t)dB_t, \quad X_0 = \xi, \\ dY_t &= -\hat{h}(t, X_t, \mathbb{P}_{X_t|B}, Y_t)dt + Z_t dW_t + Z_t^\circ dB_t, \quad Y_T = \hat{g}(X_T, \mathbb{P}_{X_T|B}). \end{aligned}$$

A solution of the MKV FBSDE is a solution (X, Y, Z, Z°) of the FBSDE $_\mu$ (2.2) with $\mu = (\mathbb{P}_{X_t|B})_{t \in [0, T]}$.

2.2. Optimal controls, FBSDEs and best reply maps. In order to reformulate the representative player control problem in terms of the the FBSDE $_\mu$, it is natural to work under some regularity conditions.

Assumption 2.2. For constants $K, \kappa, \lambda > 0$ and fixed $(\nu_0, a_0) \in \mathcal{P}_2(\mathbb{R}^d) \times A$, assume that:

- (1) $b(t, x, \mu, a) = b_1(t, x, \mu) + b_2(t)a$ with b_1 continuous, $h(t, x, \mu, \cdot)$ is convex in a , h is lower semi-continuous in (x, μ, a) and g is lower semi-continuous in (x, μ) .
- (2) The functions $b_1, \sigma, \sigma^\circ$ are C^1 w.r.t. x , and

$$\begin{aligned} |b_2(t)| + |b_1(t, 0, \nu_0)| + |\sigma(t, 0)| + |\sigma^\circ(t, 0)| &\leq K, \\ |D_x b_1(t, x, \mu)| + |D_x \sigma(t, x)| + |D_x \sigma^\circ(t, x)| &\leq K, \end{aligned}$$

for any $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

- (3) The function $h(t, \cdot, \mu, \cdot)$ is C^1 in (x, a) , the function $g(\cdot, \mu)$ is C^1 in x and

$$\begin{aligned} \kappa(|a|^2 - 1) &\leq h(t, x, \mu, a) \leq K(1 + |x|^2 + \|\mu\|_2^2 + |a|^2), \\ -\kappa &\leq g(x, \mu) \leq K(1 + |x|^2 + \|\mu\|_2^2 + |a|^2), \\ |D_x h(t, x, \mu, a)| + |D_a h(t, x, \mu, a)| + |D_x g(x, \mu)| &\leq K(1 + |x| + \|\mu\|_1 + |a|), \end{aligned}$$

for any $(t, x, \mu, a) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A$. Moreover, the functions $D_x h(t, \cdot, \mu, \cdot)$, $D_a h(t, \cdot, \mu, \cdot)$ and $D_x g(\cdot, \mu)$ are Lipschitz continuous in (x, a) , uniformly in (t, μ) .

(4) The function h is λ -convex in a ; i.e.,

$$h(t, x, \mu, \bar{a}) - h(t, x, \mu, a) - D_a h(t, x, \mu, a)(\bar{a} - a) \geq \lambda |\bar{a} - a|^2,$$

for any $\bar{a}, a \in A$ and $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

In particular, these conditions ensure some regularity of the feedback $\hat{\alpha}$.

Remark 2.3. According to Lemma 3.3 at p. 137 Vol. I in [15], $\hat{\alpha}(t, x, \mu, y)$ is a singleton, and the function $(t, x, \mu, y) \mapsto \hat{\alpha}(t, x, \mu, y)$ is measurable, locally bounded and such that

$$\begin{aligned} |\hat{\alpha}(t, \bar{x}, \mu, \bar{y}) - \hat{\alpha}(t, x, \mu, y)| &\leq L(|\bar{x} - x| + |\bar{y} - y|), \\ |\hat{\alpha}(t, x, \mu, y)| &\leq L(1 + |x| + \|\mu\|_1 + |y|), \end{aligned}$$

for a constant $L > 0$, for any $\bar{x}, x, \bar{y}, y \in \mathbb{R}^d$ and $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$.

For $\mu \in \mathcal{M}_B^2$, since the process μ is adapted the noise B , system FBSDE $_{\mu}$ (2.2) can be seen as an FBSDE with deterministic coefficients. Then, referring to Definition 1.8 at p. 16 in Vol. II of [15], we enforce the following requirement, which is satisfied in linear-convex models or under some nondegeneracy of the diffusion terms (see the discussion in Subsection 5.1).

Assumption 2.4. For any $\mu \in \mathcal{M}_B^2$, the strong uniqueness holds for the FBSDE $_{\mu}$ (2.2).

The previous condition, together with Remark 2.3, ensures the existence of optimal controls in strong formulation; that is, controls which are adapted to the noises and to the initial condition, and that can be realized on the underlying (fixed) filtered probability space. This result is stated in the following lemma. Its proof, which exploits the –so called– *compactification method* and the stochastic maximum principle, is difficult to refer and it is provided in Appendix A.

Lemma 2.5 (Existence and characterization of optimal controls). *Under Assumptions 2.2, 2.4, for any $\mu \in \mathcal{M}_B^2$, there exists a unique optimal control α^{μ} . Moreover, such a control is determined by $\alpha^{\mu} = (\hat{\alpha}(t, X_t, \mu_t, Y_t))_{t \in [0, T]}$, where (X, Y, Z, Z°) is the unique solution to the FBSDE $_{\mu}$ (2.2).*

Thanks to the previous lemma, one can define the *best reply maps* Γ , \mathcal{R} and R as

$$(2.6) \quad \begin{aligned} \Gamma(\mu) &:= X^{\mu}, \quad \mathcal{R}(\mu) := (\mathbb{P}_{X_t^{\mu}|B})_{t \in [0, T]}, \quad \mu \in \mathcal{M}_B^2, \\ R(M) &:= \Gamma((\mathbb{P}_{M_t|B})_{t \in [0, T]}), \quad M \in \mathbb{H}^2, \end{aligned}$$

where X^{μ} denotes the optimal trajectory for μ . In this framework, MFG equilibria coincide with fixed points of the best reply map \mathcal{R} ; namely,

$$\mu \in \mathcal{M}_B^2 \text{ is a MFG equilibrium if and only if } \mu = \mathcal{R}(\mu).$$

Remark 2.6 (Characterization of MFG equilibria). *Define the projection map $p : \mathbb{H}^2 \rightarrow \mathcal{M}_B^2$, $p(M) := (\mathbb{P}_{M_t|B})_{t \in [0, T]}$. and observe that, in this notation, we have*

$$R(M) = \Gamma \circ p(M), \text{ for any } M \in \mathbb{H}^2, \quad \text{and} \quad \mathcal{R}(\mu) = p \circ \Gamma(\mu), \text{ for any } \mu \in \mathcal{M}_B^2.$$

It follows that, if $\mu \in \mathcal{M}_B^2$ is a fixed point of \mathcal{R} , then $\Gamma(\mu)$ is a fixed point of R . Analogously, if $M \in \mathbb{H}^2$ is a fixed point of R , then the flow $p(M)$ is a fixed point of the map \mathcal{R} . In other words, the function p is a bijection (with inverse Γ) between the sets

$$(2.7) \quad \{\text{Fixed points of } R\} \xrightarrow{p} \{\text{Fixed points of } \mathcal{R}\} = \{\text{MFG equilibria}\}.$$

Moreover, Lemma 2.5 has a relevant immediate consequence: *The process $\mu \in \mathcal{M}_B^2$ is a MFG equilibrium if and only if $\mu = (\mathbb{P}_{X_t|B})_{t \in [0, T]}$ with (X, Y, Z, Z°) solution to the MKV FBSDE (2.5). Also, the process (X, Y, Z, Z°) is a solution to the MKV FBSDE (2.5) if and only if $R(X) = X$.*

2.3. Submodularity and monotonicity of the best reply maps. On the set $\mathcal{P}_2(\mathbb{R}^d)$, introduce the *first order stochastic dominance* \leq^{st} , which is defined, for $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$, as

$$\mu \leq^{\text{st}} \bar{\mu} \text{ if and only if } \int_{\mathbb{R}^d} \varphi(x) d\mu(x) \leq \int_{\mathbb{R}^d} \varphi(x) d\bar{\mu}(x) \text{ for any } \varphi \in \Phi.$$

Here Φ is the set of measurable functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ which are nondecreasing (i.e., $\varphi(x) \leq \varphi(\bar{x})$ if $x \leq \bar{x}$). The relation \leq^{st} is actually an order relation on the set $\mathcal{P}_2(\mathbb{R}^d)$, and $(\mathcal{P}_2(\mathbb{R}^d), \leq^{\text{st}})$ is a partially ordered set. We also observe that, for two square integrable random variables $\zeta, \bar{\zeta}$, we have

$$(2.8) \quad \mathbb{P}_{\zeta|B} \leq^{\text{st}} \mathbb{P}_{\bar{\zeta}|B} \quad \mathbb{P}\text{-a.s.} \quad \text{if } \zeta \leq \bar{\zeta} \quad \mathbb{P}\text{-a.s.}$$

The following two classes of assumptions are the key structural conditions which will allow to employ a lattice theoretical approach.

Assumption 2.7 (Separable Hamiltonian and mean field independent dynamics).

- (1) (*Separability*) $k = d$ and, for $i = 1, \dots, d$, one has
 - (a) $b^i(t, x, \mu, a) = b_1^i(t, x^i) + b_2^i(t) a^i$ for functions $b_1^i : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $b_2^i : [0, T] \rightarrow \mathbb{R}$;
 - (b) $\sigma^i = (\sigma_j^i)_{j=1, \dots, d_1}$ and $\sigma^{i, \circ} = (\sigma_j^{i, \circ})_{j=1, \dots, d_2}$ only depend on x^i ;
 - (c) $h(t, x, \mu, a) = f(t, x, \mu) + \sum_{i=1}^d l^i(t, x^i, a^i)$ for functions $l^i : [0, T] \times \mathbb{R} \times A^i \rightarrow \mathbb{R}$, $f : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$.
- (2) (*Submodularity conditions*) For dt a.a. $t \in [0, T]$, for $\phi \in \{f(t, \cdot, \cdot), g\}$, we have
 - (a) $\phi(x \wedge \bar{x}, \mu) + \phi(x \vee \bar{x}, \mu) \leq \phi(x, \mu) + \phi(\bar{x}, \mu)$ for all $x, \bar{x} \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$;
 - (b) $\phi(\bar{x}, \bar{\mu}) - \phi(x, \bar{\mu}) \leq \phi(\bar{x}, \mu) - \phi(x, \mu)$ for all $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ with $x \leq \bar{x}$ and $\mu \leq^{\text{st}} \bar{\mu}$.

Assumption 2.8 (Nonseparable Hamiltonian, affine drift and costs concave in the state).

- (1) *The coefficients of the SDE have the following structure*
 - (a) $b(t, x, \mu, a) = b_0(t, \mu) + \bar{b}_1(t)x + b_2(t)a$ for functions

$$b_0 : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}^d,$$

$$\bar{b}_1 : [0, T] \rightarrow \mathbb{R}^{d \times d}, \quad \bar{b}_1(t) = ((\bar{b}_1(t))_j^i)_{j=1, \dots, d}^{i=1, \dots, d},$$

$$b_2 : [0, T] \rightarrow \mathbb{R}^{d \times k}, \quad b_2(t) = ((b_2(t))_\ell^i)_{\ell=1, \dots, k}^{i=1, \dots, d};$$
 - (b) For dt a.a. $t \in [0, T]$, one has $b_0(t, \mu) \leq b_0(t, \bar{\mu})$ for any $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mu \leq^{\text{st}} \bar{\mu}$ and $(\bar{b}_1(t))_j^i, (b_2(t))_\ell^i \geq 0$ for any $i, j = 1, \dots, d, \ell = 1, \dots, k$;
 - (c) $\sigma^i = (\sigma_j^i)_{j=1, \dots, d_1}$ and $\sigma^{i, \circ} = (\sigma_j^{i, \circ})_{j=1, \dots, d_2}$ only depend on x^i .
- (2) (*Growth conditions*) $|D_x h(t, x, \mu, a)| + |D_x g(x, \mu)| \leq K(1 + \|\mu\|_1)$, for any $(t, x, \mu, a) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A$.
- (3) (*Submodularity conditions*) For dt a.a. $t \in [0, T]$, we have
 - (a) $h(t, x, \mu, a \wedge \bar{a}) + h(t, x, \mu, a \vee \bar{a}) \leq h(t, x, \mu, a) + h(t, x, \mu, \bar{a})$ for all $x \in \mathbb{R}^d$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $a, \bar{a} \in A$;
 - (b) $h(t, \bar{x}, \bar{\mu}, \bar{a}) - h(t, \bar{x}, \bar{\mu}, a) \leq h(t, x, \mu, \bar{a}) + h(t, x, \mu, a)$ for all $x, \bar{x} \in \mathbb{R}^d$ with $x \leq \bar{x}$, $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mu \leq^{\text{st}} \bar{\mu}$ and $a, \bar{a} \in A$ with $a \leq \bar{a}$;
 - (c) $D_x h(t, \bar{x}, \bar{\mu}, \bar{a}) \leq D_x h(t, x, \mu, a)$ and $D_x g(\bar{x}, \bar{\mu}) \leq D_x g(x, \mu)$ for all $x, \bar{x} \in \mathbb{R}^d$ with $x \leq \bar{x}$, $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mu \leq^{\text{st}} \bar{\mu}$ and $a, \bar{a} \in A$ with $a \leq \bar{a}$.

A function ϕ satisfying Condition 2a in Assumption 2.7 is said to be submodular in x , while a function ϕ satisfying Condition 2b in Assumption 2.7 is said to have decreasing differences in x and μ . Using this terminology, Condition 3a in Assumption 2.8 means that the function $h(t, x, \mu, \cdot)$ is submodular in a , while Condition 3b that the function $h(t, \cdot, \cdot, \cdot)$ have decreasing differences in a and (x, μ) .

The conditions above are particularly easy to check for *mean-field interactions of scalar type* and *mean-field interactions of order-1*. Examples are discussed in Subsection 5.2.

Remark 2.9. *Conditions 2.7 and 2.8 represent a natural counterpart, for stochastic differential MFGs, of the submodularity conditions introduced in [48] for static N -player games. For this reason, when either Condition 2.7 or 2.8 is satisfied, we say the MFG to be submodular. Similar (but less general) conditions have been used in [23] (see also [1, 16, 24]).*

Remark 2.10 (Submodularity and Lasry-Lions monotonicity condition). *A well known structural condition which ensures uniqueness of equilibria in MFGs is the –so called– Lasry-Lions monotonicity condition, which is typically required for separable Hamiltonians. In the notation of Condition 2 in Assumption 1, the Lasry-Lions monotonicity condition is satisfied if*

$$(2.9) \quad \int_{\mathbb{R}^d} (\phi(x, \bar{\mu}) - \phi(x, \mu)) d(\bar{\mu} - \mu)(x) \geq 0, \quad \text{for any } \mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d).$$

Instead, when Condition 2 in Assumption 1 is verified, then the map $\phi(\cdot, \bar{\mu}) - \phi(\cdot, \mu)$ is nonincreasing for $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$ with $\mu \leq^{\text{st}} \bar{\mu}$, and we obtain

$$\int_{\mathbb{R}^d} (\phi(x, \bar{\mu}) - \phi(x, \mu)) d(\bar{\mu} - \mu)(x) \leq 0, \quad \text{for any } \mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d) \text{ with } \mu \leq^{\text{st}} \bar{\mu}.$$

The latter represents, roughly speaking, an opposite version of condition (2.9).

Either of the previous sets of conditions implies the monotonicity of the best reply maps. This property is formalized in the following proposition, which is proved in Subsections 3.1 and 3.2.

Proposition 2.11 (Monotonicity of best reply maps). *Under Assumptions 2.2, 2.4 and either Assumption 2.7 or 2.8, the following monotonicity properties hold true:*

- (1) *The map Γ is increasing; i.e., $\Gamma(\mu) \leq \Gamma(\bar{\mu})$ for any $\mu, \bar{\mu} \in \mathcal{M}_B^2$ with $\mu \leq^{\text{st}} \bar{\mu}$;*
- (2) *The map \mathcal{R} is increasing; i.e., $\mathcal{R}(\mu) \leq^{\text{st}} \mathcal{R}(\bar{\mu})$ for any $\mu, \bar{\mu} \in \mathcal{M}_B^2$ with $\mu \leq^{\text{st}} \bar{\mu}$;*
- (3) *The map R is increasing; i.e., $R(M) \leq R(\bar{M})$ for any $M, \bar{M} \in \mathbb{H}^2$ with $M \leq \bar{M}$.*

We observe that, in light of Lemma 2.5, Proposition 2.11 can be restated in terms of a comparison principle for the forward component of the solution of the FBSDE (2.2).

Corollary 2.12 (Comparison principle). *Under Assumptions 2.2, 2.4 and either Assumption 2.7 or 2.8, let $\mu, \bar{\mu} \in \mathcal{M}_B^2$ and let $(X, Y, Z, Z^\circ), (\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ)$ be the solutions to the FBSDE (2.2) with $\mu, \bar{\mu}$, respectively. If $\mu \leq^{\text{st}} \bar{\mu}$, then $X \leq \bar{X}$.*

Remark 2.13. *While the previous result is known under Assumption 2.8, it is genuinely new under Assumption 2.7. Indeed, under Assumption 2.8, Corollary 2.12 follows from a comparison principle for FBSDEs (Theorem 2.2 in [19]), and such a result will actually be used in order to prove Proposition 2.11 (see Subsection 3.2). On the other hand, under Assumption 2.7, Corollary 2.12 covers relevant cases not covered by Theorem 2.2 in [19], as, for example, the cases in which h and g are convex in x (notice indeed that some monotonicity assumptions in [19] need to be inverted to be employed the present context).*

Comparison principles for multidimensional FBSDEs are difficult to obtain. Indeed, the corresponding decoupling field is the solution of a parabolic system, for which comparison principles fail in general (see, e.g., [20] or Example 6.2 at p. 22 in [38]). Corollary 2.12 offers instead a new natural point of view on this problem, and it connects a comparison principle to properties of the underlying optimization problem. Roughly speaking, for parameter-dependent-FBSDEs corresponding (via the stochastic maximum principle) to a minimization problem which is submodular in the parameter, a comparison principle for the forward component holds.

2.4. Lattice of processes and a priori estimates. Given processes $M, \bar{M} \in \mathbb{H}^2$, set

$$M \leq \bar{M} \text{ if and only if } M_t \leq \bar{M}_t \text{ } \mathbb{P} \otimes \pi\text{-a.e. in } \Omega \times [0, T],$$

and define a lattice structure on \mathbb{H}^2 by setting

$$(2.10) \quad (M \wedge \bar{M})_t := M_t \wedge \bar{M}_t \quad \text{and} \quad (M \vee \bar{M})_t := M_t \vee \bar{M}_t, \quad \mathbb{P} \otimes \pi\text{-a.e. in } \Omega \times [0, T].$$

The set (\mathbb{H}^2, \leq) is a partially ordered set, and the operations \wedge, \vee provide a lattice structure on \mathbb{H}^2 , which is compatible with the order relation \leq . Similarly, for processes $\mu, \bar{\mu} \in \mathcal{M}_B^2$, set

$$\mu \leq^{\text{st}} \bar{\mu} \text{ if and only if } \mu_t \leq^{\text{st}} \bar{\mu}_t \text{ } \mathbb{P} \otimes \pi\text{-a.e. in } \Omega \times [0, T].$$

The set $(\mathcal{M}_B^2, \leq^{\text{st}})$ is a partially ordered set, but it does not have a lattice structure when $d > 1$ (see Remark 2.23 below). Moreover, from (2.8), we have that the projection map $p : \mathbb{H}^2 \rightarrow \mathcal{M}_B^2$ (see Remark 2.6) is increasing; that is

$$(2.11) \quad p(M) \leq^{\text{st}} p(\bar{M}) \quad \text{if} \quad M \leq \bar{M}, \quad \text{for any } M, \bar{M} \in \mathbb{H}^2.$$

Taking ess inf and ess sup with respect to the measure $\mathbb{P} \otimes \pi$, define the processes

$$(2.12) \quad \underline{\Gamma} := \text{ess inf}\{R(M) | M \in \mathbb{H}^2\} \quad \text{and} \quad \bar{\Gamma} := \text{ess sup}\{R(M) | M \in \mathbb{H}^2\}.$$

In order to ensure the square integrability of the processes $\underline{\Gamma}$ and $\bar{\Gamma}$, we impose the following requirements.

Assumption 2.14. *The drift b_1 satisfies $b_1(t, x, \mu) \leq K(1 + |x|)$ for any $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, and either of the following conditions is satisfied:*

- (1) *A is compact;*
- (2) *For K as in Assumption 2.2, the functions h, g satisfy*

$$\begin{aligned} g(x, \mu) &\leq K(1 + |x|^2), \\ h(t, x, \mu, a) &\leq K(1 + |x|^2 + |a|^2), \end{aligned}$$

for any $(t, x, \mu, a) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A$.

From Assumption 2.14 and Proposition 2.11 we obtain the following result, which is proved in Subsection 3.3.

Lemma 2.15 (A priori estimates). *Under Assumptions 2.2, 2.4, 2.14 and either Assumption 2.7 or 2.8, we have*

$$(2.13) \quad \sup_{\mu \in \mathcal{M}_B^2} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^\mu|^2 \right] < \infty.$$

Moreover, $\underline{\Gamma}, \bar{\Gamma} \in \mathbb{H}^2$.

In particular, it is worth underlining that the fact that $\underline{\Gamma}, \bar{\Gamma} \in \mathbb{H}^2$ is a consequence of the monotonicity of the best reply map (see Proposition 2.11).

2.5. Main results: Existence and approximation of solutions. We are now ready to discuss the main results of this paper.

2.5.1. *Solutions to the MKV FBSDE.* Define inductively two sequences of processes as follows:

(1) Set $\underline{X}^0 := \underline{\Gamma}$ and, for $n \geq 1$, define $(\underline{X}^n, \underline{Y}^n, \underline{Z}^n, \underline{Z}^{\circ,n})$ as the solution to the FBSDE

$$\begin{aligned} d\underline{X}_t^n &= \hat{b}(t, \underline{X}_t^n, \mathbb{P}_{\underline{X}_t^{n-1}|B}, \underline{Y}_t^n)dt + \sigma(t, \underline{X}_t^n)dW_t + \sigma^\circ(t, \underline{X}_t^n)dB_t, \quad \underline{X}_0^n = \xi, \\ d\underline{Y}_t^n &= -\hat{h}(t, \underline{X}_t^n, \mathbb{P}_{\underline{X}_t^{n-1}|B}, \underline{Y}_t^n)dt + \underline{Z}_t^n dW_t + \underline{Z}_t^{\circ,n} dB_t, \quad \underline{Y}_T^n = \hat{g}(\underline{X}_T^n, \mathbb{P}_{\underline{X}_T^{n-1}|B}); \end{aligned}$$

(2) Set $\overline{X}^0 := \overline{\Gamma}$ and, for $n \geq 1$, define $(\overline{X}^n, \overline{Y}^n, \overline{Z}^n, \overline{Z}^{\circ,n})$ as the solution to the FBSDE

$$\begin{aligned} d\overline{X}_t^n &= \hat{b}(t, \overline{X}_t^n, \mathbb{P}_{\overline{X}_t^{n-1}|B}, \overline{Y}_t^n)dt + \sigma(t, \overline{X}_t^n)dW_t + \sigma^\circ(t, \overline{X}_t^n)dB_t, \quad \overline{X}_0^n = \xi, \\ d\overline{Y}_t^n &= -\hat{h}(t, \overline{X}_t^n, \mathbb{P}_{\overline{X}_t^{n-1}|B}, \overline{Y}_t^n)dt + \overline{Z}_t^n dW_t + \overline{Z}_t^{\circ,n} dB_t, \quad \overline{Y}_T^n = \hat{g}(\overline{X}_T^n, \mathbb{P}_{\overline{X}_T^{n-1}|B}). \end{aligned}$$

In particular, notice that, in light of Lemma 2.5 and of (2.6), the sequences $(\underline{X}^n)_n$ and $(\overline{X}^n)_n$ are obtained by iterating the best reply map R ; that is,

$$(2.14) \quad \underline{X}^0 = \underline{\Gamma}, \quad \underline{X}^{n+1} = R(\underline{X}^n), \quad n \in \mathbb{N} \quad \text{and} \quad \overline{X}^0 = \overline{\Gamma}, \quad \overline{X}^{n+1} = R(\overline{X}^n), \quad n \in \mathbb{N}.$$

The convergence of the sequences $(\underline{X}^n, \underline{Y}^n, \underline{Z}^n, \underline{Z}^{\circ,n})_n$, $(\overline{X}^n, \overline{Y}^n, \overline{Z}^n, \overline{Z}^{\circ,n})_n$ is subject to the following continuity condition.

Assumption 2.16. *The functions $h(t, \cdot, \cdot, \cdot)$, $D_x h(t, \cdot, \cdot, \cdot)$, g , and $D_x g$ are jointly continuous in (x, μ, a) on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A$, where on $\mathcal{P}_2(\mathbb{R}^d)$ we consider the $\ell_{\mathbb{R}^d, 2}$ Wasserstein distance (see (1.3)).*

Remark 2.17 (Continuity of $\hat{\alpha}$). *Following the discussion before equation (3.57) at p. 209 in Vol. II of [15], it is possible to show that, under Assumption 2.2 and 2.16, the feedback $\hat{\alpha}$ is continuous in (x, μ, y) , for any $t \in [0, T]$.*

Introduce an order relation on the set of the solutions of the MKV FBSDE. Given two solutions (X, Y, Z, Z°) and $(\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ)$, if $X \leq \bar{X}$, set $(X, Y, Z, Z^\circ) \leq (\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ)$. With such an order relation, the set of solutions of the MKV FBSDE is a partially ordered set.

The following theorem is the first main result of this paper and it is proved in Subsection 4.1.

Theorem 2.18. *Under Assumptions 2.2, 2.4, 2.14 and either Assumption 2.7 or 2.8, the following statements hold true:*

(1) *The set of solutions to the MKV FBSDE (2.5) is a nonempty complete lattice (compatible with the order relation \leq): in particular, there exist minimal and maximal solutions $(\underline{X}, \underline{Y}, \underline{Z}, \underline{Z}^\circ)$ and $(\overline{X}, \overline{Y}, \overline{Z}, \overline{Z}^\circ)$, respectively.*

(2) *If the additional Assumption 2.16 holds, then:*

(a) *The sequence $(\underline{X}^n)_n$ is monotone increasing, it converges to \underline{X} , $\pi \otimes \mathbb{P}$ -a.e., and*

$$\lim_n \mathbb{E} \left[\sup_{t \in [0, T]} (|\underline{X}_t^n - \underline{X}_t|^2 + |\underline{Y}_t^n - \underline{Y}_t|^2) + \int_0^T (|\underline{Z}_t^n - \underline{Z}_t|^2 + |\underline{Z}_t^{\circ,n} - \underline{Z}_t^\circ|^2) dt \right] = 0;$$

(b) *The sequence $(\overline{X}^n)_n$ is monotone decreasing, it converges to \overline{X} , $\pi \otimes \mathbb{P}$ -a.e., and*

$$\lim_n \mathbb{E} \left[\sup_{t \in [0, T]} (|\overline{X}_t^n - \overline{X}_t|^2 + |\overline{Y}_t^n - \overline{Y}_t|^2) + \int_0^T (|\overline{Z}_t^n - \overline{Z}_t|^2 + |\overline{Z}_t^{\circ,n} - \overline{Z}_t^\circ|^2) dt \right] = 0.$$

2.5.2. *MFG equilibria.* We next interpret Theorem 2.18 in terms of the MFG problem. Set

$$\mu^{\text{Min}} := (\mathbb{P}_{\underline{\Gamma}_t|B})_{t \in [0, T]} \quad \text{and} \quad \mu^{\text{Max}} := (\mathbb{P}_{\overline{\Gamma}_t|B})_{t \in [0, T]},$$

and define inductively two sequences of processes $(\underline{\mu}^n)_n, (\overline{\mu}^n)_n \subset \mathcal{M}_B^2$ as follows:

$$(2.15) \quad \underline{\mu}^0 := \mu^{\text{Min}}, \quad \underline{\mu}^{n+1} := \mathcal{R}(\underline{\mu}^n), \quad n \in \mathbb{N} \quad \text{and} \quad \overline{\mu}^0 := \mu^{\text{Max}}, \quad \overline{\mu}^{n+1} := \mathcal{R}(\overline{\mu}^n), \quad n \in \mathbb{N}.$$

Notice that $\underline{\mu}^n = (\mathbb{P}_{\underline{X}_t^n|B})_{t \in [0, T]}$ and $\bar{\mu}^n = (\mathbb{P}_{\bar{X}_t^n|B})_{t \in [0, T]}$. In order to discuss the convergence of the sequences $(\underline{\mu}^n)_n, (\bar{\mu}^n)_n$, introduce the set \mathcal{C}^d of continuous functions $\varphi : [0, T] \rightarrow \mathbb{R}^d$, equipped with the sup norm. On $\mathcal{P}_2(\mathcal{C}^d)$, consider the 2-Wasserstein distance $\ell_{\mathcal{C}^d, 2}$ (see (1.3)).

The following theorem is the second main result of this paper. Its proof is given in Subsection 4.2.

Theorem 2.19. *Under Assumptions 2.2, 2.4, 2.14 and either Assumption 2.7 or 2.8, the following statements hold true:*

(1) *The set of MFG equilibria is a nonempty complete lattice (compatible with the order relation \leq^{st}): in particular, there exist minimal and maximal solutions $\underline{\mu}$ and $\bar{\mu}$, respectively.*

(2) *If the additional Assumption 2.16 holds, then:*

(a) *The sequence $(\underline{\mu}^n)_n$ is such that $\underline{\mu}^n \leq^{\text{st}} \underline{\mu}^{n+1}$ for any $n \in \mathbb{N}$ and*

$$\lim_n \ell_{\mathcal{C}^d, 2}(\underline{\mu}^n, \underline{\mu}) = 0, \quad \mathbb{P}\text{-a.s.}, \quad \text{and} \quad \lim_n \mathbb{E}[\ell_{\mathcal{C}^d, 2}^2(\underline{\mu}^n, \underline{\mu})] = 0;$$

(b) *The sequence $(\bar{\mu}^n)_n$ is such that $\bar{\mu}^{n+1} \leq^{\text{st}} \bar{\mu}^n$ for any $n \in \mathbb{N}$ and*

$$\lim_n \ell_{\mathcal{C}^d, 2}(\bar{\mu}^n, \bar{\mu}) = 0, \quad \mathbb{P}\text{-a.s.}, \quad \text{and} \quad \lim_n \mathbb{E}[\ell_{\mathcal{C}^d, 2}^2(\bar{\mu}^n, \bar{\mu})] = 0.$$

2.5.3. *Convergence of the fictitious play.* We now study the convergence of the learning procedure introduced (in the context of MFGs) in [10]. Take $\hat{X}^1 \in \mathbb{H}^2$, $\hat{\mu}^1 = (\mathbb{P}_{\hat{X}_t^1|B})_{t \in [0, T]}$ and, for $n > 1$, define $(\hat{X}^n, \hat{Y}^n, \hat{Z}^n, \hat{Z}^{\circ, n})$ as the solution to the FBSDE

$$\begin{aligned} d\hat{X}_t^n &= \hat{b}(t, \hat{X}_t^n, \hat{\mu}_t^{n-1}, \hat{Y}_t^n)dt + \sigma(t, \hat{X}_t^n)dW_t + \sigma^\circ(t, \hat{X}_t^n)dB_t, \quad \hat{X}_0^n = \xi, \\ d\hat{Y}_t^n &= -\hat{h}(t, \hat{X}_t^n, \hat{\mu}_t^{n-1}, \hat{Y}_t^n)dt + \hat{Z}_t^n dW_t + \hat{Z}_t^{\circ, n} dB_t, \quad \hat{Y}_T^n = \hat{g}(\hat{X}_T^n, \hat{\mu}_T^{n-1}), \end{aligned}$$

where $\hat{\mu}^n$ is defined by

$$\hat{\mu}_t^n := \frac{1}{n} \sum_{k=1}^n \mathbb{P}_{\hat{X}_t^k|B}.$$

Notice that, in light of Lemma 2.5 and of (2.6), the sequences $(\hat{X}^n)_n$ and $(\hat{\mu}^n)_n$ can be written in terms of the best reply maps Γ and \mathcal{R} ; that is,

$$(2.16) \quad \hat{X}^{n+1} = \Gamma(\hat{\mu}^n), \quad n \in \mathbb{N}_0 \quad \text{and} \quad \hat{\mu}^{n+1} = \frac{1}{n+1}(\mathcal{R}(\hat{\mu}^n) + n\hat{\mu}^n), \quad n \in \mathbb{N}_0.$$

The sequence $(\hat{\mu}^n)_n$ is referred to as fictitious play.

With $(\underline{X}, \underline{Y}, \underline{Z}, \underline{Z}^\circ)$ and $\underline{\mu}$ as in Theorems 2.18 and 2.19, we have the following convergence result. Its proof follows arguments similar to those in the proof of Theorems 2.18 and 2.19 (see Section 4), and it is therefore only sketched in Subsection 4.3.

Theorem 2.20. *Under Assumptions 2.2, 2.4, 2.14, 2.16 and either Assumption 2.7 or 2.8, if $\hat{X}^1 = \underline{\Gamma}$, then the following statements hold true:*

(1) *The sequence $(\hat{X}^n)_n$ is monotone increasing, it converges to \underline{X} , $\pi \otimes \mathbb{P}$ -a.e., and*

$$\lim_n \mathbb{E} \left[\sup_{t \in [0, T]} (|\hat{X}_t^n - \underline{X}_t|^2 + |\hat{Y}_t^n - \underline{Y}_t|^2) + \int_0^T (|\hat{Z}_t^n - \underline{Z}_t|^2 + |\hat{Z}_t^{\circ, n} - \underline{Z}_t^\circ|^2) dt \right] = 0;$$

(2) *The sequence $(\hat{\mu}^n)_n$ is such that $\hat{\mu}^n \leq^{\text{st}} \hat{\mu}^{n+1}$ for any $n \in \mathbb{N}_0$ and*

$$\lim_n \ell_{\mathcal{C}^d, 2}(\hat{\mu}^n, \underline{\mu}) = 0, \quad \mathbb{P}\text{-a.s.}, \quad \text{and} \quad \lim_n \mathbb{E}[\ell_{\mathcal{C}^d, 2}^2(\hat{\mu}^n, \underline{\mu})] = 0.$$

Clearly, analogous convergence results to the maximal solutions $(\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ)$ and $\bar{\mu}$ hold if $\hat{X}^0 = \bar{\Gamma}$.

2.5.4. *Some remarks.* This subsection collects some observations on Theorems 2.18, 2.19 and 2.20.

Remark 2.21 (Discontinuity in the measure μ). *It is worth observing that Claim 1 in Theorem 2.18 establishes the existence of minimal and maximal strong solutions to the conditional MKV FBSDE (2.5) through a lattice theoretical fixed point theorem (see Subsection 4.1). This approach is very different from the ones already used in the literature. In particular, it does not require the time horizon to be small and the Lipschitz continuity in the measure flow (see [15]), or the use of the –so called– monotonicity conditions (see [2, 3, 32]).*

Remark 2.22 (Explicit initialization of the algorithms). *The initialization of the sequences in (2.14), (2.15) and in (2.16) can be made more explicit if the control set A is assumed to be compact.*

For example, by the monotonicity of the best reply maps, the conclusions of Claim 2a in Theorem 2.18 can be obtain for any sequence $(M^n)_n$ of type $M^0 \in \mathbb{H}^2$, $M^{n+1} := R(M^n)$ for $n \in \mathbb{N}$, whenever M^0 verifies $M^0 \leq \underline{\Gamma}$. In particular, when the assumptions of Theorem 2.18 are satisfied and the control set A is compact, one can choose M^0 as the solution of the SDE

$$dM_t^0 = (\tilde{b}_1(t, M_t^0) + \tilde{b}_2(t))dt + \sigma(t, M_t^0)dW_t + \sigma^\circ(t, M_t^0)dB_t, \quad t \in [0, T], \quad M_0^0 = \xi,$$

where

$$\tilde{b}_1(t, x) := \inf_{\mu \in \mathcal{M}_B^2} b_1(t, x, \mu_t) \quad \text{and} \quad \tilde{b}_2(t) := \inf_{a \in A} b_2(t)a.$$

By the comparison principle for SDEs (see, e.g., [45]), one obtains $M^0 \leq X^{\alpha, \mu}$, for any $\alpha \in \mathcal{A}$ and $\mu \in \mathcal{M}_B^2$, where $X^{\alpha, \mu}$ denotes the solution of (2.1). Hence, by the definition of $\underline{\Gamma}$ in (2.12), one has $M^0 \leq \underline{\Gamma}$, so that the convergence of M^n to \underline{X} holds.

Obviously, analogous considerations can be made also for the initialization of the sequences $(\bar{X}^n)_n$ and \hat{X}^n .

Remark 2.23. *In comparison with [23], the approach used in this paper allows to treat multidimensional settings. In particular, the fixed point problem in [23] was formulated on the lattice related to the partially ordered set $(\mathcal{P}(\mathbb{R}), \leq^{\text{st}})$. However, although the first order stochastic dominance induces a lattice structure on $\mathcal{P}(\mathbb{R})$, it does not induce a lattice order on $\mathcal{P}(\mathbb{R}^d)$ for $d > 1$ (cf. [33] and [40]). Thus, Tarski's fixed point theorem (see [46]) can not be employed on the set $(\mathcal{P}(\mathbb{R}^d), \leq^{\text{st}})$. This technical limitation is overcome in the present work by formulating the fixed point problem on the lattice of processes related to the partially ordered set (\mathbb{H}^2, \leq) . Indeed, while the set $\mathcal{P}(\mathbb{R}^d)$ has good compactness properties and is a natural choice in order to use topological fixed point theorems, the set \mathbb{H}^2 has a more natural lattice structure and represents a better choice when using Tarski's fixed point theorem.*

3. PROOFS OF PROPOSITION 2.11 AND OF LEMMA 2.15

3.1. Proof of Proposition 2.11 under Assumption 2.7. We recall that under Assumption 2.7 the drift b_1 does not depend on the measure argument ν . In this subsection, for $\alpha \in \mathcal{A}$, we denote by X^α the solution to the SDE $_\alpha$ (2.1).

3.1.1. *A preliminary lemma.* We first prove that the set of controlled trajectories is a lattice.

Lemma 3.1. *For $\alpha, \bar{\alpha} \in \mathcal{A}$, the controls $\alpha^\wedge, \alpha^\vee \in \mathcal{A}$ defined by*

$$\alpha_t^{i, \wedge} := \alpha_t^i \mathbf{1}_{\{X_t^{i, \alpha} < X_t^{i, \bar{\alpha}}\}} + \bar{\alpha}_t^i \mathbf{1}_{\{X_t^{i, \alpha} \geq X_t^{i, \bar{\alpha}}\}} \quad \text{and} \quad \alpha_t^{i, \vee} := \bar{\alpha}_t^i \mathbf{1}_{\{X_t^{i, \alpha} < X_t^{i, \bar{\alpha}}\}} + \alpha_t^i \mathbf{1}_{\{X_t^{i, \alpha} \geq X_t^{i, \bar{\alpha}}\}},$$

are such that $X^\alpha \wedge X^{\bar{\alpha}} = X^{\alpha^\wedge}$ and $X^\alpha \vee X^{\bar{\alpha}} = X^{\alpha^\vee}$.

Proof. Take $\alpha, \bar{\alpha} \in \mathcal{A}$ and set $X := X^\alpha, \bar{X} := X^{\bar{\alpha}}$. The controls $\alpha^\wedge, \alpha^\vee$ are clearly admissible. For any $i = 1, \dots, d$, by the Meyer-Itô formula for continuous semimartingales (see, e.g., Theorem 68 at p. 216 in [45]), one has

$$\begin{aligned} X_t^i \wedge \bar{X}_t^i &= X_t^i + 0 \wedge (\bar{X}_t^i - X_t^i) \\ &= X_t^i + \int_0^t \mathbb{1}_{\{\bar{X}_s^i - X_s^i \leq 0\}} d(\bar{X}^i - X^i)_s - \frac{1}{2} L_t^0(\bar{X}^i - X^i) \\ &= \xi^i + \int_0^t \left(\mathbb{1}_{\{X_s^i < \bar{X}_s^i\}} (b_1^i(s, X_s^i) + b_2^i(s) \alpha_s^i) + \mathbb{1}_{\{X_s^i \geq \bar{X}_s^i\}} (b_1^i(s, \bar{X}_s^i) + b_2^i(s) \bar{\alpha}_s^i) \right) ds \\ &\quad + \int_0^t \left(\mathbb{1}_{\{X_s^i < \bar{X}_s^i\}} \sigma^i(s, X_s^i) + \mathbb{1}_{\{X_s^i \geq \bar{X}_s^i\}} \sigma^i(s, \bar{X}_s^i) \right) dW_s \\ &\quad + \int_0^t \left(\mathbb{1}_{\{X_s^i < \bar{X}_s^i\}} \sigma^{i,\circ}(s, X_s^i) + \mathbb{1}_{\{X_s^i \geq \bar{X}_s^i\}} \sigma^{i,\circ}(s, \bar{X}_s^i) \right) dB_s \\ &\quad - \frac{1}{2} L_t^0(\bar{X}^i - X^i), \end{aligned}$$

where $L_t^0(\bar{X}^i - X^i)$ is the local time of the process $\bar{X}^i - X^i$ at 0 (see, e.g., Chapter IV in [45]). Denote by $[\bar{X}^i - X^i, \bar{X}^i - X^i]$ the quadratic variation of the process $\bar{X}^i - X^i$ (see, e.g., p. 66 in [45]). Using the characterization of local times (see, e.g., Corollary 3 at p. 230 in [45]), we obtain

$$\begin{aligned} L_t^0(\bar{X}^i - X^i) &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leq \bar{X}_s^i - X_s^i \leq \varepsilon\}} d[\bar{X}^i - X^i, \bar{X}^i - X^i]_s \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leq \bar{X}_s^i - X_s^i \leq \varepsilon\}} |\sigma^i(s, \bar{X}_s^i) - \sigma^i(s, X_s^i)|^2 ds \\ &\quad + \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leq \bar{X}_s^i - X_s^i \leq \varepsilon\}} |\sigma^{i,\circ}(s, \bar{X}_s^i) - \sigma^{i,\circ}(s, X_s^i)|^2 ds \\ &\leq 2K^2 \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leq \bar{X}_s^i - X_s^i \leq \varepsilon\}} |\bar{X}_s^i - X_s^i|^2 ds \\ &\leq 2K^2 \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{0 \leq \bar{X}_s^i - X_s^i \leq \varepsilon\}} \varepsilon^2 ds = 0. \end{aligned}$$

Combining together the latter two equalities and using the definition of α^\wedge , we conclude that

$$X_t^i \wedge \bar{X}_t^i = \xi^i + \int_0^t \left(b_1^i(s, X_s^i \wedge \bar{X}_s^i) + b_2^i(s) \alpha_s^{i,\wedge} \right) ds + \int_0^t \sigma^i(s, X_s^i \wedge \bar{X}_s^i) dW_s + \int_0^t \sigma^{i,\circ}(s, X_s^i \wedge \bar{X}_s^i) dB_s.$$

Since the index i is generic, this proves that $X \wedge \bar{X} = X^{\alpha^\wedge}$.

In the same way, the process $X \vee \bar{X} = \bar{X} + 0 \vee (X - \bar{X})$ solves the SDE controlled by α^\vee , completing the proof of the lemma. \square

3.1.2. Proof of Proposition 2.11. By monotonicity of the map p (see (2.11)), it is sufficient to prove Claim 1.

Take $\mu, \bar{\mu} \in \mathcal{M}_B^2$ with $\mu \leq^{st} \bar{\mu}$ and let $(\bar{X}, \bar{\alpha})$ and (X, α) be the optimal pairs related to $\bar{\mu}$ and μ , respectively.

By the admissibility of α^\vee and the optimality of $\bar{\alpha}$, thanks to Lemma 3.1 we can write

$$\begin{aligned} (3.1) \quad 0 \leq J(\alpha^\vee, \bar{\mu}) - J(\bar{\alpha}, \bar{\mu}) &= \mathbb{E} \left[\int_0^T \left(f(t, X_t \vee \bar{X}_t, \bar{\mu}_t) - f(t, \bar{X}_t, \bar{\mu}_t) \right) dt \right] \\ &\quad + \mathbb{E} \left[\int_0^T \sum_{i=1}^d \left(l^i(t, X_t^i \vee \bar{X}_t^i, \alpha_t^{i,\vee}) - l^i(t, \bar{X}_t^i, \bar{\alpha}_t^i) \right) dt \right] \\ &\quad + \mathbb{E} [g(X_T \vee \bar{X}_T, \bar{\mu}_T) - g(\bar{X}_T, \bar{\mu}_T)]. \end{aligned}$$

Next, from Condition 2 in Assumption 2.7 we have

$$\begin{aligned} \mathbb{E} \left[\int_0^T \left(f(t, X_t \vee \bar{X}_t, \bar{\mu}_t) - f(t, \bar{X}_t, \bar{\mu}_t) \right) dt \right] &\leq \mathbb{E} \left[\int_0^T \left(f(t, X_t, \bar{\mu}_t) - f(t, X_t \wedge \bar{X}_t, \bar{\mu}_t) \right) dt \right] \\ &\leq \mathbb{E} \left[\int_0^T \left(f(t, X_t, \mu_t) - f(t, X_t \wedge \bar{X}_t, \mu_t) \right) dt \right], \end{aligned}$$

as well as

$$\begin{aligned} \mathbb{E} [g(X_T \vee \bar{X}_T, \bar{\mu}_T) - g(\bar{X}_T, \bar{\mu}_T)] &\leq \mathbb{E} [g(X_T, \bar{\mu}_T) - g(X_T \wedge \bar{X}_T, \bar{\mu}_T)] \\ &\leq \mathbb{E} [g(X_T, \mu_T) - g(X_T \wedge \bar{X}_T, \mu_T)]. \end{aligned}$$

Moreover, by the definition of α^\vee and α^\wedge , we see that

$$\begin{aligned} \mathbb{E} \left[\int_0^T \left(l^i(t, X_t^i \vee \bar{X}_t^i, \alpha_t^{i,\vee}) - l^i(t, \bar{X}_t^i, \bar{\alpha}_t^i) \right) dt \right] \\ = \mathbb{E} \left[\int_0^T \mathbb{1}_{\{X_t^i \geq \bar{X}_t^i\}} \left(l^i(t, X_t^i, \alpha_t^i) - l^i(t, \bar{X}_t^i, \alpha_t^{i,\wedge}) \right) dt \right] \\ = \mathbb{E} \left[\int_0^T \left(l^i(t, X_t^i, \alpha_t^i) - l^i(t, X_t^i \wedge \bar{X}_t^i, \alpha_t^{i,\wedge}) \right) dt \right]. \end{aligned}$$

Finally, the latter three inequalities allow to estimate (3.1), thus obtaining

$$\begin{aligned} 0 \leq J(\alpha^\vee, \bar{\mu}) - J(\bar{\alpha}, \bar{\mu}) &\leq \mathbb{E} \left[\int_0^T \left(f(t, X_t, \mu_t) - f(t, X_t \wedge \bar{X}_t, \mu_t) \right) dt \right] \\ &+ \mathbb{E} \left[\int_0^T \sum_{i=1}^d \left(l^i(t, X_t^i, \alpha_t^i) - l^i(t, X_t^i \wedge \bar{X}_t^i, \alpha_t^{i,\wedge}) \right) dt \right] \\ &+ \mathbb{E} [g(X_T, \mu_T) - g(X_T \wedge \bar{X}_T, \mu_T)], \end{aligned}$$

which, thanks to Lemma 3.1, can be written as

$$(3.2) \quad 0 \leq J(\alpha^\vee, \bar{\mu}) - J(\bar{\alpha}, \bar{\mu}) \leq J(\alpha, \mu) - J(\alpha^\wedge, \mu).$$

Hence, by optimality of α , the control α^\wedge is also a minimizer for $J(\cdot, \mu)$, and, by uniqueness of optimal controls (see Lemma 2.5), we conclude that $\alpha = \alpha^\wedge$. Therefore, by Lemma 3.1 we obtain $X \wedge \bar{X} = X$; that is, $X \leq \bar{X}$, completing the proof.

3.2. Proof of Proposition 2.11 under Assumption 2.8. Again, by monotonicity of the map p (see (2.11)), it is sufficient to prove only Claim 1.

The proof is divided in two steps.

Step 1. In this step we prove the monotonicity of the feedback function $\hat{\alpha}$ providing the optimal controls (see Lemma 2.5). In particular, we show that, for any $(t, x, \mu, y), (t, \bar{x}, \bar{\mu}, \bar{y}) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$,

$$(3.3) \quad \text{if } x \leq \bar{x}, \bar{y} \leq y, \mu \leq^{\text{st}} \bar{\mu}, \text{ then } \hat{\alpha}(t, x, \mu, y) \leq \hat{\alpha}(t, \bar{x}, \bar{\mu}, \bar{y}).$$

Fix $(t, x, \mu, y), (t, \bar{x}, \bar{\mu}, \bar{y}) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ with $x \leq \bar{x}, \bar{y} \leq y, \mu \leq^{\text{st}} \bar{\mu}$. Set $\alpha := \hat{\alpha}(t, x, \mu, y)$ and $\bar{\alpha} := \hat{\alpha}(t, \bar{x}, \bar{\mu}, \bar{y})$. Since $\bar{\alpha}$ minimizes the function $a \mapsto yb_2(t)a + h(t, \bar{x}, \bar{\mu}, a)$, we have

$$0 \leq \bar{y}b_2(t)\alpha \vee \bar{\alpha} + h(t, \bar{x}, \bar{\mu}, \alpha \vee \bar{\alpha}) - \bar{y}b_2(t)\bar{\alpha} + h(t, \bar{x}, \bar{\mu}, \bar{\alpha}),$$

so that, by using Condition 3a in Assumption 2.8 and the linearity of the map $a \mapsto yb_2(t)a$, we obtain

$$\begin{aligned} 0 &\leq \bar{y}b_2(t)\alpha \vee \bar{\alpha} + h(t, \bar{x}, \bar{\mu}, \alpha \vee \bar{\alpha}) - \bar{y}b_2(t)\bar{\alpha} + h(t, \bar{x}, \bar{\mu}, \bar{\alpha}) \\ &\leq \bar{y}b_2(t)\alpha + h(t, \bar{x}, \bar{\mu}, \alpha) - \bar{y}b_2(t)\alpha \wedge \bar{\alpha} + h(t, \bar{x}, \bar{\mu}, \alpha \wedge \bar{\alpha}). \end{aligned}$$

From the latter inequality, we can use Condition 3b in Assumption 2.8, together with the fact that the map $y \mapsto yb_2(t)(\alpha - \alpha \wedge \bar{\alpha})$ is nonincreasing (by Condition 1b in Assumption 2.8), in order to obtain

$$\begin{aligned} 0 &\leq \bar{y}b_2(t)\alpha + h(t, \bar{x}, \bar{\mu}, \alpha) - \bar{y}b_2(t)\alpha \wedge \bar{\alpha} + h(t, \bar{x}, \bar{\mu}, \alpha \wedge \bar{\alpha}) \\ &\leq yb_2(t)\alpha + h(t, x, \mu, \alpha) - yb_2(t)\alpha \wedge \bar{\alpha} + h(t, x, \mu, \alpha \wedge \bar{\alpha}). \end{aligned}$$

This implies that $\alpha \wedge \bar{\alpha}$ minimizes the function $a \mapsto yb_2(t)a + h(t, x, \mu, a)$. Therefore, since the minimizer α is unique by Remark 2.3, we obtain that $\alpha \wedge \bar{\alpha} = \alpha$, so that (3.3) is proved.

Step 2. Take $\mu, \bar{\mu} \in \mathcal{M}_B^2$ with $\mu \leq^{\text{st}} \bar{\mu}$ and let $(\bar{X}, \bar{\alpha})$ and (X, α) be the optimal pairs related to $\bar{\mu}$ and μ , respectively. By Lemma 2.5, X and \bar{X} are given by the unique solutions (X, Y, Z, Z°) and $(\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ)$ to the FBSDEs depending on μ and $\bar{\mu}$, respectively.

We now aim at using the monotonicity of $\hat{\alpha}$ proved in Step 1 (see (3.3)) and Assumption 2.8 to employ a comparison theorem for FBSDEs (see Theorem 2.2 in [19]) in order to show that

$$(3.4) \quad X \leq \bar{X}.$$

First, observe that Assumptions $\mathcal{A}1 - \mathcal{A}4$ in [19] (with the extra growth in μ and $\bar{\mu}$) clearly follow from our Assumptions 2.2 and 2.8, together with the Lipschitz continuity and the growth of $\hat{\alpha}$ (see Remark 2.3). In particular, we underline that, (with minimal adjustment) the results in [19] cover also our case, in which the growth conditions in Assumptions $\mathcal{A}1 - \mathcal{A}4$ in [19] are replaced with

$$|\hat{b}(t, x, \nu, y)| \leq C(1 + |x| + \|\nu\|_1 + |y|), \quad |\hat{h}(t, x, \nu, y)| + |\hat{g}(x, \nu)| \leq C(1 + \|\nu\|_1 + |y|),$$

for any $(t, x, \nu, y) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$. We remain to check the conditions in [19] on the monotonicity of $(\hat{b}, \hat{h}, \hat{g})$. To do so, take $(t, x, \nu, y), (t, \bar{x}, \bar{\nu}, \bar{y}) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$ with $x \leq \bar{x}, \bar{y} \leq y, \nu \leq^{\text{st}} \bar{\nu}$. From the monotonicity $\hat{\alpha}$ (see (3.3)) and from Condition 1b in Assumption 2.8 we obtain:

$$(3.5) \quad \hat{b}(t, x, \nu, y) \leq \hat{b}(t, \bar{x}, \bar{\nu}, \bar{y}).$$

From Conditions 1b and 3c in Assumption 2.8, and from the monotonicity $\hat{\alpha}$ we have:

$$(3.6) \quad \hat{h}(t, \bar{x}, \bar{\nu}, \bar{y}) \leq \hat{h}(t, x, \nu, y) \quad \text{and} \quad \hat{g}(\bar{x}, \bar{\nu}) \leq \hat{g}(x, \nu).$$

The conditions (3.5) and (3.6) are a slightly different version of Assumptions $\mathcal{A}5 - \mathcal{A}7$ and of the conditions in the statement of Theorem 2.2 in [19]. In particular, the inequalities in (3.6) have opposite direction with respect to the conditions in [19], and the monotonicity in y in (3.5) is inverted as well. This implies that we can employ a slightly different version of Theorem 2.2 in [19], which gives $X \leq \bar{X}$ and $\bar{Y} \leq Y$. This completes the proof of the proposition.

3.3. Proof of Lemma 2.15. We begin by proving the uniform a priori estimate (2.13) on the moments of the optimally controlled state processes. For $\alpha \in \mathcal{A}$ and $\mu \in \mathcal{M}_B^2$, let $X^{\alpha, \mu}$ denotes the solution to the SDE $_{(\alpha, \mu)}$ (2.1). By boundedness of b_1 in μ , classical estimates give

$$(3.7) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{\alpha, \mu}|^2 \right] \leq C \left(1 + \mathbb{E}[|\xi|^2] + \mathbb{E} \left[\int_0^T |\alpha_t|^2 dt \right] \right).$$

If now A is compact, the estimate (2.13) is obvious. Suppose instead that Condition 2 in Assumption 2.14 holds. Denote by α^μ the optimal control for μ (by Lemma 2.5 such a control exists unique) and let $\bar{\alpha} \in \mathcal{A}$ be the control constantly equal to a_0 (see Assumption 2.2). By using Condition 3 in Assumption 2.2 and the optimality of α^μ , we obtain

$$\kappa \left(\mathbb{E} \left[\int_0^T |\alpha_t^\mu|^2 dt \right] - T - 1 \right) \leq J(\alpha^\mu, \mu) \leq J(\bar{\alpha}, \mu) \leq C \left(1 + \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{\bar{\alpha}, \mu}|^2 + \int_0^T |\bar{\alpha}_t|^2 dt \right] \right),$$

so that, by (3.7), we deduce that

$$\mathbb{E} \left[\int_0^T |\alpha_t^\mu|^2 dt \right] \leq C \left(1 + \mathbb{E} \left[\int_0^T |\bar{\alpha}_t|^2 dt \right] \right) = C(1 + a_0 T),$$

for a constant C which does not depend on μ . From the latter estimate, we can again employ (3.7) in order to conclude that

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^\mu|^2 \right] \leq C,$$

which (since the generic constant C does not depend on μ) implies the uniform estimate (2.13).

We next prove that $\bar{\Gamma} \in \mathbb{H}^2$. In doing so, the monotonicity of the best reply map R (see Proposition 2.11) will play an essential role. First, observe that there exists a sequence $(N^n)_n \subset \mathbb{H}^2$ such that

$$\bar{\Gamma} = \sup_n R(N^n), \quad \mathbb{P} \otimes \pi\text{-a.e. in } \Omega \times [0, T].$$

Recalling the notation introduced in (2.10), construct the sequences

$$N^{\wedge, n} := N^n \wedge \dots \wedge N^0 \quad \text{and} \quad N^{\vee, n} := N^n \vee \dots \vee N^0, \quad n \in \mathbb{N}.$$

Clearly, we have

$$\mathbb{E} \left[\int_0^T (|N_t^{\wedge, n}|^2 + |N_t^{\vee, n}|^2) d\pi(t) \right] \leq 2 \sum_{j=1}^n \mathbb{E} \left[\int_0^T |N_t^j|^2 d\pi(t) \right],$$

so that the processes $N^{\wedge, n}$ and $N^{\vee, n}$ are square integrable; that is,

$$(3.8) \quad N^{\wedge, n}, N^{\vee, n} \in \mathbb{H}^2.$$

Observe next that, since $N^{\wedge, n} \leq N^n \leq N^{\vee, n}$, by Proposition 2.11 we have $R(N^{\wedge, n}) \leq R(N^n) \leq R(N^{\vee, n})$, so that

$$(3.9) \quad R(N^{\wedge, n}) \wedge 0 \leq R(N^n) \wedge 0 \leq R(N^n) \vee 0 \leq R(N^{\vee, n}) \vee 0.$$

Also, since the sequences $(N^{\wedge, n})_n$ and $(N^{\vee, n})_n$ are monotone, again by Proposition 2.11 we have that $(R(N^{\wedge, n}))_n$ is decreasing and that $(R(N^{\vee, n}))_n$ is increasing. Therefore, the sequences

$$(3.10) \quad (|R(N^{\wedge, n}) \wedge 0|^2)_n \quad \text{and} \quad (|R(N^{\vee, n}) \vee 0|^2)_n \quad \text{are increasing.}$$

We can now estimate $|\bar{\Gamma}|^2$. Using (3.9) we find

$$\begin{aligned} |\bar{\Gamma}|^2 &= \left| \sup_n R(N^n) \right|^2 = \left| \sup_n (R(N^n) \wedge 0 + R(N^n) \vee 0) \right|^2 \\ &\leq \sup_n |R(N^n) \wedge 0|^2 + \sup_n |R(N^n) \vee 0|^2 \\ &\leq \sup_n |R(N^{\wedge, n}) \wedge 0|^2 + \sup_n |R(N^{\vee, n}) \vee 0|^2. \end{aligned}$$

Hence, by the monotonicity in (3.10), we obtain

$$|\bar{\Gamma}|^2 \leq \lim_n \left(|R(N^{\wedge, n}) \wedge 0|^2 + |R(N^{\vee, n}) \vee 0|^2 \right).$$

The latter allows to use Fatou's lemma in order to find

$$\begin{aligned} \mathbb{E} \left[\int_0^T |\bar{\Gamma}_t|^2 d\pi(t) \right] &\leq \mathbb{E} \left[\int_0^T \lim_n \left(|R(N^{\wedge, n})_t \wedge 0|^2 + |R(N^{\vee, n})_t \vee 0|^2 \right) d\pi(t) \right] \\ &\leq \liminf_n \mathbb{E} \left[\int_0^T \left(|R(N^{\wedge, n})_t \wedge 0|^2 + |R(N^{\vee, n})_t \vee 0|^2 \right) d\pi(t) \right] \\ &\leq \liminf_n \mathbb{E} \left[\int_0^T \left(|R(N^{\wedge, n})_t|^2 + |R(N^{\vee, n})_t|^2 \right) d\pi(t) \right] < \infty, \end{aligned}$$

where the last inequality follows from (2.13) and from (3.8). Hence we have $\bar{\Gamma} \in \mathbb{H}^2$.

In the same way, it is possible to show that $\underline{\Gamma} \in \mathbb{H}^2$, completing the proof of the lemma.

4. PROOF OF THEOREMS 2.18, 2.19 AND 2.20

4.1. **Proof of Theorem 2.18.** We prove the two claims separately.

4.1.1. *Proof of Claim 1: the fixed point argument.* Define the set

$$L := \{M \in \mathbb{H}^2 | \underline{\Gamma} \leq M \leq \bar{\Gamma}\}.$$

The partially ordered set (L, \leq) is a complete lattice (see (2.10)), and, by the definition of $\underline{\Gamma}$ and $\bar{\Gamma}$ (see (2.12)), we have $R(M) \in L$ for any $M \in \mathbb{H}^2$. Moreover, by Proposition 2.11, the best reply map R is increasing from L into itself. We can therefore employ Tarski's fixed point theorem (see [46]) in order to deduce that the set of fixed points of the map $R : L \rightarrow L$ is a nonempty complete lattice. In light of Remark 2.6, the set of fixed points of the map R coincides with the set of (forward components of the) solutions to the MKV FBSDE (2.5), completing the proof of Claim 1.

4.1.2. *Proof of Claim 2: convergence of the algorithm.* It is sufficient to prove Claim 2a, as the proof of Claim 2b follows by similar arguments. The proof is divided in four steps.

Step 1. We begin by observing that, by the definition of $\underline{\Gamma}$ (see (2.12)) and by the monotonicity of R (see Proposition 2.11), one has

$$\underline{X}^0 = \underline{\Gamma} \leq R(\underline{\Gamma}) = \underline{X}^1,$$

so that, iterating the map R , the monotonicity of R gives

$$(4.1) \quad \underline{X}^n \leq \underline{X}^{n+1}, \quad n \in \mathbb{N}.$$

Therefore, one can define the process

$$(4.2) \quad \underline{X} := \sup_n \underline{X}^n = \lim_n \underline{X}^n, \quad \mathbb{P} \otimes \pi\text{-a.e. in } \Omega \times [0, T].$$

Obviously, since $\underline{\Gamma} \leq \underline{X}^n \leq \bar{\Gamma}$ for every $n \in \mathbb{N}$, we have

$$(4.3) \quad |\underline{X}_t|^2, |\underline{X}_t^n|^2 \leq |\underline{\Gamma}_t|^2 + |\bar{\Gamma}_t|^2 \text{ for any } n \in \mathbb{N}, \quad \mathbb{P} \otimes \pi\text{-a.e. in } \Omega \times [0, T],$$

so that, by integrability of $\underline{\Gamma}$ and of $\bar{\Gamma}$ (see Lemma 2.15) and by the dominated convergence theorem one obtains

$$(4.4) \quad \lim_n \mathbb{E} \left[\int_0^T |\underline{X}_t^n - \underline{X}_t|^2 d\pi(t) \right] = 0.$$

Moreover, consider the stochastic flows $\underline{\mu}^n, \underline{\mu} \in \mathcal{M}_B^2$ given by

$$(4.5) \quad \underline{\mu}^n = (\mathbb{P}_{\underline{X}_t^n | B})_{t \in [0, T]}, \quad n \in \mathbb{N}, \quad \underline{\mu} := (\mathbb{P}_{\underline{X}_t | B})_{t \in [0, T]}.$$

Notice that the processes $\underline{\mu}^n$ are those defined in (2.15). From (4.1) and (4.2), we have that the sequence $|\underline{X}_t^n - \underline{X}_t|^2$ is decreasing and it converges to 0. Moreover, by (4.3) and by Lemma 2.15, we have $\mathbb{E}[|\underline{X}_t^0 - \underline{X}_t|^2] \leq 4\mathbb{E}[|\underline{\Gamma}_t|^2 + |\bar{\Gamma}_t|^2] < \infty$ for π -a.a. $t \in [0, T]$. Thus, by the monotone convergence theorem for the conditional expectation, together with elementary properties of the 2-Wasserstein distance $\ell_{\mathbb{R}^d, 2}$ on \mathbb{R}^d (see (1.3)), we obtain

$$\lim_n \ell_{\mathbb{R}^d, 2}^2(\underline{\mu}_t^n, \underline{\mu}_t) \leq \lim_n \mathbb{E}[|\underline{X}_t^n - \underline{X}_t|^2 | B] = 0, \quad \mathbb{P}\text{-a.s., for } \pi\text{-a.a. } t \in [0, T],$$

Furthermore, by (4.1) and the monotonicity of the projection map p (see (2.11)), the latter limit is monotonic w.r.t. \leq^{st} ; that is,

$$(4.6) \quad \underline{\mu}_t^n \leq^{\text{st}} \underline{\mu}_t^{n+1} \leq^{\text{st}} \underline{\mu}_t, \quad \text{and} \quad \ell_{\mathbb{R}^d, 2}(\underline{\mu}_t^n, \underline{\mu}_t) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \mathbb{P}\text{-a.s., for } \pi\text{-a.a. } t \in [0, T].$$

By the definition of $(\underline{X}^n, \underline{Y}^n, \underline{Z}^n, \underline{Z}^{\circ,n})$ and $\underline{\mu}^n$, the process $(\underline{X}^n, \underline{Y}^n, \underline{Z}^n, \underline{Z}^{\circ,n})$ solves the system

$$(4.7) \quad \begin{aligned} d\underline{X}_t^n &= \hat{b}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t^n)dt + \sigma(t, \underline{X}_t^n)dW_t + \sigma^\circ(t, \underline{X}_t^n)dB_t, \quad \underline{X}_0^n = \xi, \\ d\underline{Y}_t^n &= -\hat{h}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t^n)dt + \underline{Z}_t^n dW_t + \underline{Z}_t^{\circ,n} dB_t, \quad \underline{Y}_T^n = \hat{g}(\underline{X}_T^n, \underline{\mu}_T^{n-1}). \end{aligned}$$

for any $n \in \mathbb{N}$ with $n \geq 1$. The idea of the proof is now to take (monotonic) limits in the system above in order to approximate the minimal solution of the MKV FBSDE (2.5). This will be done in the subsequent two steps.

Step 2. In this step we will prove that the process $(\underline{Y}^n, \underline{Z}^n, \underline{Z}^{\circ,n})$ converges to the unique strong solution $(\underline{Y}, \underline{Z}, \underline{Z}^\circ)$ of the backward stochastic differential equation (BSDE, in short)

$$(4.8) \quad d\underline{Y}_t = -\hat{h}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t)dt + \underline{Z}_t dW_t + \underline{Z}_t^\circ dB_t, \quad \underline{Y}_T = \hat{g}(\underline{X}_T, \underline{\mu}_T).$$

Recall that a solution of such a BSDE is defined as a process $(\underline{Y}, \underline{Z}, \underline{Z}^\circ)$ such that

$$(4.9) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |\underline{Y}_t|^2 + \int_0^T (|\underline{Z}_t|^2 + |\underline{Z}_t^\circ|^2) dt \right] < \infty,$$

and such that the equation (4.8) is satisfied.

By (4.7), the processes $(\underline{Y}^n, \underline{Z}^n, \underline{Z}^{\circ,n})$ solve the BSDEs

$$d\underline{Y}_t^n = -\hat{h}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t^n)dt + \underline{Z}_t^n dW_t + \underline{Z}_t^{\circ,n} dB_t, \quad \underline{Y}_T^n = \hat{g}(\underline{X}_T^n, \underline{\mu}_T^{n-1}).$$

Therefore, by stability properties for BSDEs (see, e.g., Theorem 4.2.3 at p. 84 in [55]) we obtain a first estimate

$$(4.10) \quad \begin{aligned} &\mathbb{E} \left[\sup_{t \in [0, T]} |\underline{Y}_t^n - \underline{Y}_t|^2 + \int_0^T (|\underline{Z}_t^n - \underline{Z}_t|^2 + |\underline{Z}_t^{\circ,n} - \underline{Z}_t^\circ|^2) dt \right] \\ &\leq C \mathbb{E} \left[|\hat{g}(\underline{X}_T^n, \underline{\mu}_T^{n-1}) - \hat{g}(\underline{X}_T, \underline{\mu}_T)|^2 + \int_0^T |\hat{h}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t^n) - \hat{h}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t)|^2 dt \right], \end{aligned}$$

for a suitable constant C which does not depend on n . Hence, we proceed by estimating the right hand side in (4.10).

We begin by studying the convergence of the terms with \hat{g} . Write

$$(4.11) \quad \begin{aligned} \mathbb{E} \left[|\hat{g}(\underline{X}_T^n, \underline{\mu}_T^{n-1}) - \hat{g}(\underline{X}_T, \underline{\mu}_T)|^2 \right] &\leq 2\mathbb{E} \left[|D_x g(\underline{X}_T^n, \underline{\mu}_T^{n-1}) - D_x g(\underline{X}_T, \underline{\mu}_T^{n-1})|^2 \right] \\ &\quad + 2\mathbb{E} \left[|D_x g(\underline{X}_T, \underline{\mu}_T^{n-1}) - D_x g(\underline{X}_T, \underline{\mu}_T)|^2 \right]. \end{aligned}$$

From the monotonicity of the sequence $(\underline{\mu}^n)_n$ in (4.6) and the submodularity assumptions (in particular, either Condition 2 in Assumption 2.7 or Condition 3 in Assumption 2.8), we have

$$D_x g(\underline{X}_T, \underline{\mu}_T^n) - D_x g(\underline{X}_T, \underline{\mu}_T) \geq D_x g(\underline{X}_T, \underline{\mu}_T^{n+1}) - D_x g(\underline{X}_T, \underline{\mu}_T) \geq 0,$$

so that

$$|D_x g(\underline{X}_T, \underline{\mu}_T^n) - D_x g(\underline{X}_T, \underline{\mu}_T)|^2 \geq |D_x g(\underline{X}_T, \underline{\mu}_T^{n+1}) - D_x g(\underline{X}_T, \underline{\mu}_T)|^2.$$

Also, by the growth condition on $D_x g$ (from Condition 3 in Assumption 2.2), from (4.3) we obtain

$$\begin{aligned} \mathbb{E} [|D_x g(\underline{X}_T, \underline{\mu}_T^0) - D_x g(\underline{X}_T, \underline{\mu}_T)|^2] &\leq C \mathbb{E} [1 + |\underline{X}_T|^2 + \|\underline{\mu}_T^0\|_1^2 + \|\underline{\mu}_T\|_1^2] \\ &\leq C \mathbb{E} [1 + |\underline{\Gamma}_T|^2 + |\bar{\Gamma}_T|^2 + \|\mathbb{P}_{\underline{\Gamma}_T|B}\|_1^2 + \|\mathbb{P}_{\underline{X}_T|B}\|_1^2] \\ &\leq C \mathbb{E} [1 + |\underline{\Gamma}_T|^2 + |\bar{\Gamma}_T|^2] < \infty, \end{aligned}$$

where the latter estimates follows from Lemma 2.15. Therefore, by the limits in (4.6) and the continuity of $D_x g$ (from Assumption 2.16), we can use the monotone convergence theorem and obtain

$$(4.12) \quad \lim_n \mathbb{E} \left[|D_x g(\underline{X}_T, \underline{\mu}_T^n) - D_x g(\underline{X}_T, \underline{\mu}_T)|^2 \right] = 0.$$

Finally, by using the uniform Lipschitz continuity of $D_x g$ (from Condition 3 in Assumption 2.2) and the limits in (4.4), we obtain

$$(4.13) \quad \lim_n \mathbb{E} \left[|D_x g(\underline{X}_T^n, \underline{\mu}_T^{n-1}) - D_x g(\underline{X}_T, \underline{\mu}_T^{n-1})|^2 \right] \leq C \lim_n \mathbb{E} [|\underline{X}_T^n - \underline{X}_T|^2] = 0,$$

so that, combining (4.12) and (4.13) into (4.11), we conclude that

$$(4.14) \quad \lim_n \mathbb{E} \left[|\hat{g}(\underline{X}_T^n, \underline{\mu}_T^{n-1}) - \hat{g}(\underline{X}_T, \underline{\mu}_T)|^2 \right] = 0.$$

We next study the convergence of the terms with \hat{h} in (4.10). We write

$$(4.15) \quad \mathbb{E} \left[\int_0^T |\hat{h}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t) - \hat{h}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t)|^2 dt \right] \\ \leq 2\mathbb{E} \left[\int_0^T |\hat{h}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t) - \hat{h}(t, \underline{X}_t, \underline{\mu}_t^{n-1}, \underline{Y}_t)|^2 dt \right] \\ + 2\mathbb{E} \left[\int_0^T |\hat{h}(t, \underline{X}_t, \underline{\mu}_t^{n-1}, \underline{Y}_t) - \hat{h}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t)|^2 dt \right] \\ =: 2\mathbb{E} \left[\int_0^T (|\hat{H}_t^{1,n}|^2 + |\hat{H}_t^{2,n}|^2) dt \right],$$

and we proceed by estimating separately the two terms in the right hand side of (4.15).

We begin with the term involving $\hat{H}^{1,n}$. By the Lipschitz continuity of $D_x h$ in (x, a) (from Condition 3 in Assumption 2.2) we obtain

$$|\hat{H}_t^{1,n}|^2 \leq C \left(|\underline{X}_t^n - \underline{X}_t|^2 + |\hat{\alpha}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t) - \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t^{n-1}, \underline{Y}_t)|^2 \right. \\ \left. + |(D_x b_1(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}) - D_x b_1(t, \underline{X}_t, \underline{\mu}_t^{n-1})) \underline{Y}_t|^2 \right),$$

so that, using the Lipschitz continuity of $\hat{\alpha}$ in x (see Remark 2.3) we find

$$(4.16) \quad |\hat{H}_t^{1,n}|^2 \leq C \left(|\underline{X}_t^n - \underline{X}_t|^2 + |(D_x b_1(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}) - D_x b_1(t, \underline{X}_t, \underline{\mu}_t^{n-1})) \underline{Y}_t|^2 \right).$$

Moreover, by continuity of $D_x b_1$ in x (indeed, under either Assumption 2.7 or 2.8, $D_x b_1$ does not depend on the measure) and the limits in (4.2), we get

$$|(D_x b_1(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}) - D_x b_1(t, \underline{X}_t, \underline{\mu}_t^{n-1})) \underline{Y}_t|^2 \rightarrow 0, \text{ as } n \rightarrow \infty,$$

and, by boundedness of $D_x b_1$ we have

$$|(D_x b_1(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}) - D_x b_1(t, \underline{X}_t, \underline{\mu}_t^{n-1})) \underline{Y}_t|^2 \leq C |\underline{Y}_t|^2.$$

Since the left hand side in the latter inequality is integrable (by (4.9)), by the dominated convergence theorem, we obtain

$$\lim_n \mathbb{E} \left[\int_0^T |(D_x b_1(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}) - D_x b_1(t, \underline{X}_t, \underline{\mu}_t^{n-1})) \underline{Y}_t|^2 dt \right] = 0.$$

The latter equality, together with (4.16) and (4.4), allows to conclude that

$$(4.17) \quad \lim_n \mathbb{E} \left[\int_0^T |\hat{H}_t^{1,n}|^2 dt \right] = 0.$$

We continue by estimating the term in (4.15) involving $\hat{H}^{2,n}$. Observe that, under both Assumptions 2.7 or 2.8, we have $D_x b_1(t, \underline{X}_t, \underline{\mu}_t^{n-1}) = D_x b_1(t, \underline{X}_t, \underline{\mu}_t)$, so that

$$(4.18) \quad \hat{H}_t^{2,n} = D_x h(t, \underline{X}_t, \underline{\mu}_t^{n-1}, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t^{n-1}, \underline{Y}_t)) - D_x h(t, \underline{X}_t, \underline{\mu}_t, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t)).$$

Now, under Assumption 2.8, we have shown in the proof of Proposition 2.11 (see (3.3)) that $\hat{\alpha}$ is monotone in the measure ν . On the other hand, under Assumption 2.7, $\hat{\alpha}$ does not depend on the measure. In both cases, from the monotonicity of the sequence $(\underline{\mu}^n)_n$ in (4.6) and the submodularity assumptions (i.e., either Assumption 2.7 or 2.8), we have

$$\begin{aligned} & D_x h(t, \underline{X}_t, \underline{\mu}_t^n, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t^n, \underline{Y}_t)) - D_x h(t, \underline{X}_t, \underline{\mu}_t, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t)) \\ & \geq D_x h(t, \underline{X}_t, \underline{\mu}_t^{n+1}, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t^{n+1}, \underline{Y}_t)) - D_x h(t, \underline{X}_t, \underline{\mu}_t, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t)) \geq 0, \end{aligned}$$

so that

$$\begin{aligned} & |D_x h(t, \underline{X}_t, \underline{\mu}_t^n, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t^n, \underline{Y}_t)) - D_x h(t, \underline{X}_t, \underline{\mu}_t, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t))|^2 \\ & \geq |D_x h(t, \underline{X}_t, \underline{\mu}_t^{n+1}, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t^{n+1}, \underline{Y}_t)) - D_x h(t, \underline{X}_t, \underline{\mu}_t, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t))|^2. \end{aligned}$$

Also, by the growth conditions on $D_x h$ (from Condition 3 in Assumption 2.2) and on $\hat{\alpha}$ (from Remarks 2.3), from (4.3) we obtain

$$(4.19) \quad \begin{aligned} & \mathbb{E} \left[\int_0^T |D_x h(t, \underline{X}_t, \underline{\mu}_t^0, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t^0, \underline{Y}_t)) - D_x h(t, \underline{X}_t, \underline{\mu}_t, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t))|^2 dt \right] \\ & \leq C \mathbb{E} \left[\int_0^T \left(1 + |\underline{\Gamma}_T|^2 + |\bar{\Gamma}_T|^2 + |\underline{Y}_t|^2 \right) dt \right] < \infty, \end{aligned}$$

where the latter estimates follows from Lemma 2.15 and from (4.9). Moreover, in case of Assumption 2.7 we have that $\hat{\alpha}$ does not depend on the measure ν , while under Assumption 2.8, we have that the function $\hat{\alpha}$ is continuous in ν (see Remark 2.17). Hence, in both cases we have

$$(4.20) \quad \lim_n |D_x h(t, \underline{X}_t, \underline{\mu}_t^n, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t^n, \underline{Y}_t)) - D_x h(t, \underline{X}_t, \underline{\mu}_t, \hat{\alpha}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t))|^2 = 0.$$

Therefore, by using (4.18), from the limits in (4.20) and the estimates in (4.19), we can invoke the monotone convergence theorem in order to obtain

$$(4.21) \quad \lim_n \mathbb{E} \left[\int_0^T |\hat{H}_t^{2,n}|^2 dt \right] = 0.$$

Finally, combining together (4.17) and (4.21) into (4.15), we obtain

$$\lim_n \mathbb{E} \left[\int_0^T |\hat{h}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t) - \hat{h}(t, \underline{X}_t, \underline{\mu}_t, \underline{Y}_t)|^2 dt \right] = 0.$$

The latter can be plugged, together with (4.14), into the first estimate (4.10), so to conclude that

$$(4.22) \quad \lim_n \mathbb{E} \left[\sup_{t \in [0, T]} |\underline{Y}_t^n - \underline{Y}_t|^2 + \int_0^T (|\underline{Z}_t^n - \underline{Z}_t|^2 + |\underline{Z}_t^{\circ, n} - \underline{Z}_t^\circ|^2) dt \right] = 0,$$

which completes the proof of the convergence of $(\underline{Y}^n, \underline{Z}^n, \underline{Z}^{\circ, n})$.

Step 3. In this step we will prove that the process \underline{X} is the unique strong solution to the SDE (2.1) controlled by the feedback $\hat{\alpha}(t, \cdot, \underline{\mu}_t, \underline{Y}_t)$.

Indeed, by Lipschitz continuity of the data $b_1, \sigma, \sigma^\circ$ and of the feedback $\hat{\alpha}$ (see Remark 2.3), there exists a unique strong solution X to the SDE

$$(4.23) \quad dX_t = \hat{b}(t, X_t, \underline{\mu}_t, \underline{Y}_t) dt + \sigma(t, X_t) dW_t + \sigma^\circ(t, X_t) dB_t, \quad X_0 = \xi.$$

By (4.7), the process \underline{X}^n solves the SDE

$$d\underline{X}^n = \hat{b}(t, \underline{X}_t^n, \underline{\mu}_t^{n-1}, \underline{Y}_t^n)dt + \sigma(t, \underline{X}_t^n)dW_t + \sigma^\circ(t, \underline{X}_t^n)dB_t, \quad \underline{X}_0^n = \xi.$$

Therefore, by standard stability results for SDEs, we obtain

$$(4.24) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |\underline{X}_t^n - X_t|^2 \right] \leq C \left(\mathbb{E} \left[\int_0^T |b_1(t, X_t, \underline{\mu}_t^{n-1}) - b_1(t, X_t, \underline{\mu}_t)|^2 dt \right] + \mathbb{E} \left[\int_0^T |\hat{\alpha}(t, X_t, \underline{\mu}_t^{n-1}, \underline{Y}_t^n) - \alpha(t, X_t, \underline{\mu}_t, \underline{Y}_t)|^2 dt \right] \right).$$

We proceed by estimating separately the two terms in the right-hand side of (4.24).

Under Assumption 2.7, the drift b_1 does not depend on the measure ν , so that

$$b_1(t, X_t, \underline{\mu}_t^{n-1}) = b_1(t, X_t, \underline{\mu}_t).$$

On the other hand, under Assumption 2.8, one has

$$b_1(t, X_t, \underline{\mu}_t^{n-1}) - b_1(t, X_t, \underline{\mu}_t) = b_0(t, \underline{\mu}_t^{n-1}) - b_0(t, \underline{\mu}_t),$$

with b_0 continuous and bounded (from Assumption 2.14). Hence, thanks to the limits in (4.6), by the dominated convergence theorem and we obtain

$$(4.25) \quad \lim_n \mathbb{E} \left[\int_0^T |b_1(t, X_t, \underline{\mu}_t^{n-1}) - b_1(t, X_t, \underline{\mu}_t)|^2 dt \right] = 0.$$

We next estimate the term with $\hat{\alpha}$. By Lipschitz continuity of $\hat{\alpha}$ (see Remark 2.3) and by the convergence established in (4.22) we have

$$(4.26) \quad \lim_n \mathbb{E} \left[\int_0^T |\hat{\alpha}(t, X_t, \underline{\mu}_t^{n-1}, \underline{Y}_t^n) - \alpha(t, X_t, \underline{\mu}_t^{n-1}, \underline{Y}_t)|^2 dt \right] \leq C \lim_n \mathbb{E} \left[\int_0^T |\underline{Y}_t^n - \underline{Y}_t|^2 dt \right] = 0.$$

Moreover, under Assumption 2.7, $\hat{\alpha}$ does not depend on the measure ν , so that

$$\hat{\alpha}(t, X_t, \underline{\mu}_t^{n-1}, \underline{Y}_t) = \hat{\alpha}(t, X_t, \underline{\mu}_t, \underline{Y}_t).$$

On the other hand, under Assumption 2.8, we have shown in the proof of Proposition 2.11 (see (3.3)) that $\hat{\alpha}$ is monotone in the measure ν , therefore

$$\hat{\alpha}(t, X_t, \underline{\mu}_t, \underline{Y}_t) - \hat{\alpha}(t, X_t, \underline{\mu}_t^n, \underline{Y}_t) \geq \hat{\alpha}(t, X_t, \underline{\mu}_t, \underline{Y}_t) - \hat{\alpha}(t, X_t, \underline{\mu}_t^{n+1}, \underline{Y}_t) \geq 0,$$

and

$$|\hat{\alpha}(t, X_t, \underline{\mu}_t, \underline{Y}_t) - \hat{\alpha}(t, X_t, \underline{\mu}_t^n, \underline{Y}_t)|^2 \geq |\hat{\alpha}(t, X_t, \underline{\mu}_t, \underline{Y}_t) - \hat{\alpha}(t, X_t, \underline{\mu}_t^{n+1}, \underline{Y}_t)|^2 \geq 0.$$

Also, by the growth conditions of $\hat{\alpha}$ (see Remark 2.3), from (4.3) we can estimate

$$\begin{aligned} & \mathbb{E} \left[\int_0^T |\hat{\alpha}(t, X_t, \underline{\mu}_t, \underline{Y}_t) - \hat{\alpha}(t, X_t, \underline{\mu}_t^0, \underline{Y}_t)|^2 dt \right] \\ & \leq C \mathbb{E} \left[\int_0^T (1 + |X_t|^2 + |\underline{Y}_t|^2 + |\underline{\Gamma}_t|^2 + |\bar{\Gamma}_t|^2) dt \right] < \infty, \end{aligned}$$

where the latter integrability follows from Lemma 2.15, (4.9) and from the fact that X is the solution of the SDE (4.23). Therefore, by continuity of $\hat{\alpha}$ in the measure (see Remark 2.17), we can invoke the monotone convergence theorem in order to obtain

$$(4.27) \quad \lim_n \mathbb{E} \left[\int_0^T |\hat{\alpha}(t, X_t, \underline{\mu}_t^{n-1}, \underline{Y}_t) - \alpha(t, X_t, \underline{\mu}_t, \underline{Y}_t)|^2 dt \right] = 0.$$

Finally, combining (4.25), (4.26) and (4.27) into (4.24), we get

$$\lim_n \mathbb{E} \left[\sup_{t \in [0, T]} |\underline{X}_t^n - X_t|^2 \right] = 0.$$

By uniqueness of the limit and by (4.4), we conclude that $\underline{X} = X$, which in turn implies that \underline{X} is the unique strong solution to the SDE (4.23). Moreover, by the previous limits, we obtain a stronger convergence with respect to (4.4); that is,

$$(4.28) \quad \lim_n \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^n - X_t|^2 \right] = 0.$$

Step 4. Combining together (4.5) from Step 1, (4.8) from Step 2 and the fact that \underline{X} is the solution to (4.23) as seen in Step 3, we have that the process $(\underline{X}, \underline{Y}, \underline{Z}, \underline{Z}^\circ)$ is a solution to the MKV FBSDE (2.5). Moreover, from (4.22) and (4.28) we obtain

$$\mathbb{E} \left[\sup_{t \in [0, T]} (|\underline{X}_t^n - \underline{X}_t|^2 + |\underline{Y}_t^n - \underline{Y}_t|^2) + \int_0^T (|\underline{Z}_t^n - \underline{Z}_t|^2 + |\underline{Z}_t^{\circ, n} - \underline{Z}_t^\circ|^2) dt \right] = 0,$$

proving the claimed convergence.

We finally prove that $(\underline{X}, \underline{Y}, \underline{Z}, \underline{Z}^\circ)$ is the minimal solution to the MKV FBSDE. Let (X, Y, Z, Z°) be another solution to the MKV FBSDE, set $\mu := (\mathbb{P}_{X_t|B})_{t \in [0, T]}$. Since (X, Y, Z, Z°) is a solution, as in Remark 2.6 we have $X = R(X)$, so that, by the definition of $\underline{\Gamma}$ (see (2.12)), we have $\underline{X}^0 = \underline{\Gamma} \leq R(X) = X$. By monotonicity of R (see Proposition 2.11) we obtain $\underline{X}^1 = R(\underline{X}^0) \leq R(X) = X$. Therefore, by iterating the map R we deduce that $\underline{X}^n \leq X$ for any $n \in \mathbb{N}$, and, taking limits as in (4.2), we conclude that $\underline{X} \leq X$. Thus, $(\underline{X}, \underline{Y}, \underline{Z}, \underline{Z}^\circ)$ is the minimal solution to the MKV FBSDE (2.5), which completes the proof of the theorem.

4.2. Proof of Theorem 2.19. We first prove Claim 1. Thanks to Remark 2.6, from Claim 1 in Theorem 2.18 we deduce that the set of MFG equilibria \mathcal{M} is nonempty.

We now show the lattice property of \mathcal{M} . Let F be the set of solutions to the MKV FBSDE (2.5). By Claim 1 in Theorem 2.18, the set F is a complete lattice. This means that, given two solutions $\Sigma = (X, Y, Z, Z^\circ)$ and $\bar{\Sigma} = (\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ)$ of (2.5), there exist two (unique) solutions

$$\Sigma \wedge^F \bar{\Sigma} = (X^\wedge, Y^\wedge, Z^\wedge, Z^{\circ, \wedge}) \quad \text{and} \quad \Sigma \vee^F \bar{\Sigma} = (X^\vee, Y^\vee, Z^\vee, Z^{\circ, \vee})$$

of the MKV FBSDE (2.5) which verify the conditions:

- (1) $X^\wedge \leq X, \bar{X}$;
- (2) $\bar{X} \leq X^\wedge$ for any other solution $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{Z}^\circ)$ of (2.5) with $\tilde{X} \leq X, \bar{X}$;
- (3) $X, \bar{X} \leq X^\vee$;
- (4) $X^\vee \leq \hat{X}$ for any other solution $(\hat{X}, \hat{Y}, \hat{Z}, \hat{Z}^\circ)$ of (2.5) with $X, \bar{X} \leq \hat{X}$.

Next, recall that the projection map $p : F \rightarrow \mathcal{M}$ is a bijection with inverse Γ (see Remark 2.6). Given $\mu, \bar{\mu} \in \mathcal{M}$, we can define the flows $\mu \wedge^{\mathcal{M}} \bar{\mu} := p(\Gamma(\mu) \wedge^F \Gamma(\bar{\mu}))$ and $\mu \vee^{\mathcal{M}} \bar{\mu} := p(\Gamma(\mu) \vee^F \Gamma(\bar{\mu}))$. Since $\Gamma(\mu) \wedge^F \Gamma(\bar{\mu}) \leq \Gamma(\mu), \Gamma(\bar{\mu})$, by monotonicity of the projection map p (see (2.11)) we have $\mu \wedge^{\mathcal{M}} \bar{\mu} \leq^{\text{st}} \mu, \bar{\mu}$. Moreover, if $\nu \in \mathcal{M}$ with $\nu \leq^{\text{st}} \mu, \bar{\mu}$, then, by Proposition 2.11, we have $\Gamma(\nu) \leq \Gamma(\mu), \Gamma(\bar{\mu})$, so that $\Gamma(\nu) \leq \Gamma(\mu) \wedge^F \Gamma(\bar{\mu})$ and $\nu = p \circ \Gamma(\nu) \leq^{\text{st}} p(\Gamma(\mu) \wedge^F \Gamma(\bar{\mu})) = \mu \wedge^{\mathcal{M}} \bar{\mu}$. This proves that $\mu \wedge^{\mathcal{M}} \bar{\mu}$ is the biggest element ν of \mathcal{M} (w.r.t. the order relation \leq^{st}) such that $\nu \leq^{\text{st}} \mu, \bar{\mu}$. Analogously, one can see that the stochastic flow $\mu \vee^{\mathcal{M}} \bar{\mu} := p(\Gamma(\mu) \vee^F \Gamma(\bar{\mu}))$ is the smallest element ν of \mathcal{M} such that $\mu, \bar{\mu} \leq^{\text{st}} \nu$. Therefore, the set \mathcal{M} has a lattice structure compatible with the binary relation \leq^{st} . In particular, this argument shows that there exist minimal and maximal elements of \mathcal{M} , which are respectively given by

$$(\mathbb{P}_{\underline{X}_t|B})_{t \in [0, T]} \quad \text{and} \quad (\mathbb{P}_{\bar{X}_t|B})_{t \in [0, T]},$$

where \underline{X} and \bar{X} are the forward components of the minimal and the maximal solutions to the MKV FBSDE (2.5) (see Claim 1 in Theorem 2.18).

The proof of the Claim 2 in Theorem 2.19 hinges on the proof of Claim 2 in Theorem 2.18. It is sufficient to prove the Claim 2a, as the proof of Claim 2b follows by similar arguments.

Recalling (4.6), we directly have

$$\underline{\mu}^n \leq^{\text{st}} \underline{\mu}^{n+1}, \quad n \in \mathbb{N}.$$

Moreover, by elementary properties of the 2-Wasserstein distance, we have

$$(4.29) \quad \ell_{\mathcal{C}^d, 2}^2(\underline{\mu}^n, \underline{\mu}) \leq \mathbb{E} \left[\sup_{t \in [0, T]} |\underline{X}_t^n - \underline{X}_t|^2 \middle| B \right], \quad n \in \mathbb{N}.$$

By (4.1), (4.2), and (4.28), we know that

$$\lim_n \sup_{t \in [0, T]} |\underline{X}_t^n - \underline{X}_t|^2 = 0, \quad \mathbb{P}\text{-a.s.}$$

Moreover, by (4.1) we have

$$\sup_{t \in [0, T]} |\underline{X}_t^n - \underline{X}_t|^2 \leq \sup_{t \in [0, T]} |\underline{X}_t^1 - \underline{X}_t|^2, \quad \text{for } n \geq 1,$$

and, since \underline{X}^1 and \underline{X} are both solutions to SDEs, we also have

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\underline{X}_t^1 - \underline{X}_t|^2 \right] \leq 2 \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\underline{X}_t^1|^2 \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |\underline{X}_t|^2 \right] \right) < \infty.$$

Therefore, by the monotone convergence theorem for conditional expectations, we can take limits in (4.29) in order to deduce that

$$\lim_n \ell_{\mathcal{C}^d, 2}(\underline{\mu}^n, \underline{\mu}) \leq \lim_n \left(\mathbb{E} \left[\sup_{t \in [0, T]} |\underline{X}_t^n - \underline{X}_t|^2 \middle| B \right] \right)^{1/2} = 0, \quad \mathbb{P}\text{-a.s.}$$

Similarly, taking limits in expectation in (4.29), by the monotone convergence theorem we conclude that

$$\lim_n \mathbb{E}[\ell_{\mathcal{C}^d, 2}^2(\underline{\mu}^n, \underline{\mu})] \leq \lim_n \mathbb{E} \left[\sup_{t \in [0, T]} |\underline{X}_t^n - \underline{X}_t|^2 \right] = 0,$$

which completes the proof of the theorem.

4.3. Sketch of the proof of Theorem 2.20. In light of the analysis of Subsections 4.1 and 4.2, we limit our self to prove the monotonicity of the sequences $(\hat{X}^n)_n$ and $(\hat{\mu}^n)_n$, as well as their convergence to limit points $\hat{X} \in H^2$ and $\hat{\mu} \in \mathcal{M}_B^2$ such that $(\mathbb{P}_{\hat{X}_t|B})_{t \in [0, T]} = \hat{\mu}$.

We prove this monotonicity by induction. Clearly, since $\hat{X}^1 = \underline{\Gamma}$, by the definition of $\underline{\Gamma}$ (see (2.12)) we obtain $\hat{X}^1 \leq \hat{X}^2$. Therefore, we have $\hat{\mu}^1 = (\mathbb{P}_{\hat{X}_t^1|B})_{t \in [0, T]} \leq^{\text{st}} (\mathbb{P}_{\hat{X}_t^2|B})_{t \in [0, T]}$, so to obtain

$$\hat{\mu}^1 \leq^{\text{st}} \frac{1}{2} \left((\mathbb{P}_{\hat{X}_t^2|B})_{t \in [0, T]} + \hat{\mu}^1 \right) = \hat{\mu}^2.$$

Assume now that $\hat{X}^{n-1} \leq \hat{X}^n$ and $\hat{\mu}^{n-1} \leq^{\text{st}} \hat{\mu}^n$ for some $n \in \mathbb{N}_0$. By the definition of \hat{X}^{n+1} (see (2.16)) and the monotonicity of the best reply map Γ (see Lemma 2.11), we have

$$\hat{X}^n = \Gamma(\hat{\mu}^{n-1}) \leq \Gamma(\hat{\mu}^n) = \hat{X}^{n+1}.$$

Therefore, using again (2.16) and the monotonicity of the best reply map \mathcal{R} , we deduce that

$$\begin{aligned} \hat{\mu}^n &= \frac{1}{n+1} (n\hat{\mu}^n + \hat{\mu}^n) \\ &\leq^{\text{st}} \frac{1}{n+1} (\mathcal{R}(\hat{\mu}^{n-1}) + (n-1)\hat{\mu}^{n-1} + \hat{\mu}^n) \\ &= \frac{1}{n+1} (\mathcal{R}(\hat{\mu}^n) + (n-1)\hat{\mu}^n + \hat{\mu}^n) = \hat{\mu}^{n+1}. \end{aligned}$$

Thus, the sequences $(\hat{X}^n)_n$ and $(\hat{\mu}^n)_n$ are increasing.

By monotonicity, one can define the limit processes

$$\hat{X} := \sup_n \hat{X}^n = \lim_n \hat{X}^n \quad \text{and} \quad \hat{\mu} := \sup_n \hat{\mu}^n = \lim_n \hat{\mu}^n, \quad \mathbb{P} \otimes \pi\text{-a.e. in } \Omega \times [0, T].$$

In particular, we have $\mathbb{P}_{\hat{X}_t^n|B} \rightarrow \mathbb{P}_{\hat{X}_t|B}$ weakly as $n \rightarrow \infty$, so that

$$\hat{\mu}_t^n := \frac{1}{n} \sum_{k=1}^n \mathbb{P}_{\hat{X}_t^k|B} \rightarrow \mathbb{P}_{\hat{X}_t|B}, \quad \text{weakly as } n \rightarrow \infty.$$

Hence, by uniqueness of the limit, we have $(\mathbb{P}_{\hat{X}_t|B})_{t \in [0, T]} = \hat{\mu}$.

The rest of the proof (i.e., the characterization of \hat{X} and $\hat{\mu}$ as the minimal MFG solutions as well as the improved convergence in \mathbb{H}^2 and in the 2-Wasserstein distances) can now be recovered following the same lines as in the proof of Theorems 2.18 and 2.19 (see Subsections 4.1 and 4.2).

5. EXAMPLES

In this section, we provide some examples meeting the assumptions made in Section 2.

5.1. Sufficient conditions for strong uniqueness for FBSDEs. In the following, we provide two natural set-ups in which Assumption 2.4 is satisfied.

5.1.1. *Linear-convex models.* Let $b, \sigma, \sigma^\circ, h, g$ satisfy Assumptions 2.2 and 2.14. Assume moreover that:

$$(5.1) \quad \begin{aligned} b(t, x, \mu, a) &= b_0(t, \mu) + \bar{b}_1(t)x + b_2(t)a \\ \sigma^i(t, x^i) &= p^i(t) + q^i(t)x^i \\ \sigma^{i,\circ}(t, x^i) &= p^{i,\circ}(t) + q^{i,\circ}(t)x^i \\ h, g &\text{ are convex in } (x, a). \end{aligned}$$

Under the additional condition (5.1), the existence of an optimal control (for any $\mu \in \mathcal{M}_B^2$) can be shown as in Theorem 5.2 at p. 68 in [54], while uniqueness follows from the strict convexity of h in a (see Assumption 2.2). In this case, the necessary and sufficient conditions of the stochastic maximum principle are fulfilled (see, e.g., [15]), so that any solution of the FBSDE (2.2) provides an optimal control. Therefore, by uniqueness of the optimal control, the solution of FBSDE $_\mu$ (2.2) is unique (on any stochastic basis) and Assumption 2.4 is satisfied.

It is also worth underlining that, when Condition (5.1) holds, the continuity assumptions in the measure (see Assumption 2.2) are not needed for the existence of the optimal controls (see Lemma 2.5).

5.1.2. *Nondegenerate case.* A second relevant example in which Assumption 2.4 is satisfied is when A is compact and the effect of the noises is nondegenerate. For example, for $b, \sigma, \sigma^\circ, h, g$ which satisfy Assumption 2.2, we can enforce the additional requirements:

- (1) A is compact;
- (2) $d = d_1$ and the matrix $\sigma(t, x)$ is invertible with inverse $\sigma(t, x)^{-1}$ such that $\sigma(t, x)^{-1} \leq \bar{K}$ for any $(t, x) \in [0, T] \times \mathbb{R}^d$ and some $\bar{K} > 0$.

In this case, following the same rational as in the proof of Lemma 4.1 in [41], one can use a Girsanov transformation in order to prove that, for any $\mu \in \mathcal{M}_B^2$, the FBSDE $_\mu$ (2.2) admits a unique strong solution, so that Assumption 2.4 is fulfilled. Clearly, the same conclusion can be obtained by requiring the same nondegeneracy condition on the matrix $(\sigma, \sigma^\circ) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d_1} \times \mathbb{R}^{d \times d_2}$.

5.2. Checking the submodularity conditions. In the following, we illustrate how to check the submodularity conditions and we provide some examples.

For $m \in \mathbb{N}$, consider a measurable function $\gamma : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}$. The function γ is said to have *decreasing differences* in x and y if

$$\gamma(\bar{x}, \bar{y}) - \gamma(x, \bar{y}) \leq \gamma(\bar{x}, y) - \gamma(x, y), \text{ for any } x, \bar{x} \in \mathbb{R}^d, \bar{y}, y \in \mathbb{R}^m \text{ with } x \leq \bar{x} \text{ and } y \leq \bar{y}.$$

Observe that, if $\gamma \in C^2(\mathbb{R}^d \times \mathbb{R}^m)$, then γ has decreasing differences in x and y if and only if

$$(5.2) \quad \frac{\partial^2 \gamma}{\partial x^i \partial y^j}(x, y) \leq 0, \quad \text{for each } (x, y) \in \mathbb{R}^d \times \mathbb{R}^m, \quad i = 1, \dots, d, \quad j = 1, \dots, m.$$

Moreover, γ is said to be *submodular* in x if

$$\gamma(x \wedge \bar{x}, y) + \gamma(x \vee \bar{x}, y) \leq \gamma(x, y) + \gamma(\bar{x}, y), \text{ for any } x, \bar{x} \in \mathbb{R}^d, \quad y \in \mathbb{R}^m.$$

This condition is always satisfied in the case $d = 1$. If $d \geq 2$ and $\gamma(\cdot, y)$ is twice-differentiable in x , then γ is submodular in x if and only if, for each fixed y , for any $i = 1, \dots, d$ the function $\gamma(\cdot, y)$ has decreasing differences in x_i and $(x_j)_{j \neq i}$ (see Theorem 2.6.1 and Corollary 2.6.1 at p. 44 in [49]). Hence, in the case of twice-differentiable cost functions γ , the submodularity in x corresponds to having

$$(5.3) \quad \frac{\partial^2 \gamma}{\partial x^i \partial x^j}(x, y) \leq 0, \quad \text{for each } (x, y) \in \mathbb{R}^d \times \mathbb{R}^m, \quad i = 1, \dots, d \text{ and } j = 1, \dots, d \text{ with } j \neq i.$$

For $n, m \in \mathbb{N}_0$ and a measurable function $\varphi = (\varphi^1, \dots, \varphi^m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$, φ is said to be nondecreasing (resp. nonincreasing) if $\varphi(x) \leq \varphi(\bar{x})$ for every $x, \bar{x} \in \mathbb{R}^n$ with $x \leq \bar{x}$ (resp. $\bar{x} \leq x$). Define the set

$$(5.4) \quad \Phi_2^{n;m} := \{\text{measurable nondecreasing } \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ with } \varphi(x) \leq C(1 + |x|^2), \quad C > 0\},$$

and, for $\varphi \in \Phi_2^{d;m}$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, set

$$\langle \varphi, \mu \rangle := (\langle \varphi^1, \mu \rangle, \dots, \langle \varphi^m, \mu \rangle)^\top := (\int_{\mathbb{R}^d} \varphi^1(x) d\mu(x), \dots, \int_{\mathbb{R}^d} \varphi^m(x) d\mu(x))^\top.$$

It is straightforward to observe that, for two \mathbb{R}^d -valued square integrable r.v.'s ζ and $\bar{\zeta}$, if $\zeta \leq \bar{\zeta}$ \mathbb{P} -a.s., then $\langle \varphi, \mathbb{P}_\zeta \rangle \leq \langle \varphi, \mathbb{P}_{\bar{\zeta}} \rangle$ for any $\varphi \in \Phi_2^{d;m}$.

All the considerations above easily allows to construct functions $\phi : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ which are submodular in x and have decreasing differences in x and μ .

Example 5.1 (Mean-field interaction of scalar type). *Consider a mean-field interaction of scalar type; that is, $\phi(x, \mu) = \gamma(x, \langle \varphi, \mu \rangle)$ for given measurable maps $\gamma : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^m$, $m \in \mathbb{N}_0$. If $\varphi \in \Phi_2^{d;m}$ and the function γ has decreasing differences in x and y and it is submodular in x , then the function ϕ has decreasing differences in x and μ and it is submodular in x .*

Example 5.2 (Mean-field interactions of order-1). *Another example is provided by the interactions of order-1, i.e. when ϕ is of the form $\phi(x, \mu) = \int_{\mathbb{R}^d} \gamma(x, y) d\mu(y)$. It is easy to check that, if the function γ has decreasing differences in x and y and it is submodular in x , then the function ϕ has decreasing differences in x and μ and it is submodular in x .*

For the sake of illustration, we present two explicit examples which verify either Condition 2 in Assumption 2.7 or Condition 3 in Assumption 2.8.

Example 5.3. *For a function $\varphi \in \Phi_2^{d;m}$, $m \in \mathbb{N}_0$, Condition 2 in Assumption 2.7 holds in the following two cases:*

- (1) (Convex costs) $d = m$ and $\phi(x, \mu) = \gamma(x - \langle \varphi, \mu \rangle)$ or $\phi(x, \mu) = \int_{\mathbb{R}^d} \gamma(x - \varphi(y)) d\mu(y)$, for a convex function $\gamma \in C^2(\mathbb{R}^d)$, when either $d = 1$ or γ verifies the condition $\partial \gamma / \partial x^i \partial x^j = 0$ for $i \neq j$ (this holds, e.g., for $\gamma(x) = |x|^2$);

- (2) (Multiplicative costs) $\phi(x, \mu) = \gamma_1(x)\gamma_2(\langle \varphi, \mu \rangle)$, for a nonincreasing (resp. nondecreasing) function $\gamma_1 \in C^2(\mathbb{R}^d)$ and a nondecreasing (resp. nonincreasing) function $\gamma_2 \in C^1(\mathbb{R}^m)$, such that either $d = 1$ or $\gamma_2 \geq 0$ and γ_1 verifies (5.3).

Example 5.4. Consider, for $t \in [0, T]$ and $m \in \mathbb{N}_0$, functions $\varphi_t \in \Phi_2^{d;m}$ and $\psi_t \in \Phi_2^{d;k}$, as well as functions $\gamma_1 \in C^1([0, T] \times \mathbb{R}^d)$ and $\gamma_2 \in C([0, T] \times \mathbb{R}^m)$. Take costs h and g of the form

$$\begin{aligned} h(t, x, \mu, a) &= \gamma_1(t, x)\gamma_2(t, \langle \varphi_t, \mu \rangle) + l(a, \mu), \\ g(x, \mu) &= \gamma_1(T, x)\gamma_2(T, \langle \varphi_T, \mu \rangle). \end{aligned}$$

Assume that the function $D_x\gamma_1(t, x)\gamma_2(t, y)$ is nonincreasing in (x, y) for any $t \in [0, T]$. Then, Condition 3 in Assumption 2.8 is satisfied in the following two cases:

- (1) (Convex costs) $l(a, \mu) = \gamma_3(a - \langle \psi_t, \mu \rangle)$ or $l(a, \mu) = \int_{\mathbb{R}^d} \gamma_3(a - \psi_t(y))d\mu(y)$, for a convex function $\gamma_3 \in C^2(\mathbb{R}^k)$, when either $k = 1$ or γ_3 verifies $\partial\gamma_3/\partial a^i \partial a^j$ for $i \neq j$ (this holds, e.g., for $\gamma_3(a) = |a|^2$);
- (2) (Multiplicative costs) $l(a, \mu) = \gamma_3(a)\gamma_4(\langle \psi_t, \mu \rangle)$, for a nonincreasing (resp. nondecreasing) function $\gamma_3 \in C^2(\mathbb{R}^k)$ and a nondecreasing (resp. nonincreasing) function $\gamma_4 : \mathbb{R}^m \rightarrow \mathbb{R}$, such that $k = 1$ or $\gamma_4 \geq 0$ and γ_3 verifies (5.3) in a .

5.3. Linear-quadratic submodular MFGs. Building on the discussion of Subsections 5.1 and 5.2, we now provide some more explicit examples of linear-quadratic MFGs that can be treated with the approach presented in Section 2.

Take b, σ, σ° as in (5.1). Consider, for $t \in [0, T]$ and $m \in \mathbb{N}_0$, functions $\varphi_t \in \Phi_2^{d;d}$ and $\psi_t \in \Phi_2^{d;k}$, symmetric nonnegative semi-definite matrices P_t and R_t and a matrix Q_t .

Example 5.5. If b_0 does not depend on μ , take $k = d$ and

$$\begin{aligned} h(t, x, \mu, a) &= xP_t x + (x - \langle \varphi_t, \mu \rangle)Q_t(x - \langle \varphi_t, \mu \rangle) + aR_t a, \\ g(x, \mu) &= xP_T x + (x - \langle \varphi_T, \mu \rangle)Q_T(x - \langle \varphi_T, \mu \rangle). \end{aligned}$$

All the assumptions of Theorems 2.18 and Theorem 2.19 (in particular, Assumption 2.7) are fulfilled if

- (1) $(P_t)_j^i + (Q_t)_j^i \leq 0$, for $i, j = 1, \dots, d$ with $i \neq j$;
- (2) Q_t is symmetric nonnegative semi-definite and $(Q_t)_j^i \geq 0$, for $i, j = 1, \dots, d$;
- (3) either $|\varphi_t(x)| \leq K$ for any $(t, x) \in [0, T] \times \mathbb{R}^d$ or A is compact.

Example 5.6. If b_0 depends on μ , assume that b_0, \bar{b}_1 and b_2 satisfy Condition 1b in Assumption 2.8 and take

$$\begin{aligned} h(t, x, \mu, a) &= \langle \varphi_t, \mu \rangle Q_t x + aR_t a + (a - \langle \psi_t, \mu \rangle)P_t(a - \langle \psi_t, \mu \rangle) \\ g(x, \mu) &= \langle \varphi_T, \mu \rangle Q_T x. \end{aligned}$$

All the assumptions of Theorems 2.18 and Theorem 2.19 (in particular, Assumption 2.8) are fulfilled if

- (1) $(R_t)_j^i + (P_t)_j^i \leq 0$, for $i, j = 1, \dots, d$ with $i \neq j$;
- (2) $(P_t)_j^i \geq 0$, for $i, j = 1, \dots, d$;
- (3) $(Q_t)_j^i \leq 0$, for $i, j = 1, \dots, d$;
- (4) Either $|\varphi_t(x)| + |\psi_t(x)| \leq K$ for any $(t, x) \in [0, T] \times \mathbb{R}^d$ or A is compact.

In this case, we point out that the conditions $\kappa(|a|^2 - 1) \leq h(t, x, \mu, a)$ and $-\kappa \leq g(x, \mu)$ in Assumption 2.2 are not satisfied, but Lemma 2.5 can be recovered via the sufficient conditions of the stochastic maximum principle (see Subsection 5.1.1).

In both the previous examples, the additional continuity requirement of Assumption 2.16 is satisfied, e.g., if φ_t and ψ_t are bounded and continuous.

APPENDIX A. PROOF OF LEMMA 2.5

Recall that the process μ is fixed. The proof is divided in three steps.

Step 1. We introduce a weak formulation of the control problem and we use the results of [29] to have a first existence result for optimal controls.

A tuple $\rho = (\Omega^\rho, \mathcal{F}^\rho, \mathbb{F}^\rho, \mathbb{P}^\rho, \xi^\rho, W^\rho, B^\rho, \mu^\rho, \alpha^\rho)$ is said to be an admissible weak control if

- (1) $(\Omega^\rho, \mathcal{F}^\rho, \mathbb{F}^\rho, \mathbb{P}^\rho)$, $\mathbb{F}^\rho = (\mathcal{F}_t^\rho)_{t \in [0, T]}$, is a filtered probability space satisfying the usual conditions;
- (2) $\xi^\rho : \Omega^\rho \rightarrow \mathbb{R}^d$ is an \mathcal{F}_0^ρ -measurable square integrable random variable;
- (3) $(W^\rho, B^\rho) : \Omega^\rho \times [0, T] \rightarrow \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is an $(\Omega^\rho, \mathcal{F}^\rho, \mathbb{F}^\rho, \mathbb{P}^\rho)$ -Brownian motion;
- (4) $\mu^\rho : \Omega^\rho \times [0, T] \rightarrow \mathcal{P}_2(\mathbb{R}^d)$ is an \mathbb{F}^ρ -progressively measurable process;
- (5) $\mathbb{P}^\rho \circ (\xi^\rho, W^\rho, B^\rho, \mu^\rho)^{-1} = \mathbb{P} \circ (\xi, W, B, \mu)^{-1}$;
- (6) $\alpha^\rho : \Omega^\rho \times [0, T] \rightarrow A$ is square integrable \mathbb{F}^ρ -progressively measurable process.

Let \mathcal{A}^w denotes the set of admissible weak controls. Given $\rho \in \mathcal{A}^w$, we define the cost functional

$$J^w(\rho) := \mathbb{E}^\rho \left[\int_0^T h(t, X_t^\rho, \mu_t^\rho, \alpha_t^\rho) dt + g(X_T^\rho, \mu_T^\rho) \right],$$

where \mathbb{E}^ρ denotes the expectation under the probability measure \mathbb{P}^ρ and the process $X^\rho = (X^{1, \rho}, \dots, X^{d, \rho})$ denotes the solution to the controlled SDE

$$dX_t^\rho = (b_1(t, X_t^\rho, \mu_t^\rho) + b_2(t) \alpha_t^\rho) dt + \sigma(t, X_t^\rho) dW_t^\rho + \sigma^\circ(t, X_t^\rho) dB_t^\rho, \quad t \in [0, T], \quad X_0^\rho = \xi^\rho.$$

By Assumption 2.2, such a solution X^ρ exists unique on the stochastic basis $(\Omega^\rho, \mathcal{F}^\rho, \mathbb{F}^\rho, \mathbb{P}^\rho)$ and it is \mathbb{F}^ρ -adapted. A weak control $\bar{\rho} \in \mathcal{A}^w$ is said to be optimal if $J^w(\bar{\rho}) \leq J^w(\rho)$ for any $\rho \in \mathcal{A}^w$.

In light of Assumption 2.2, Corollary 4.8 in [29] (slightly adapted to our setting, in order to deal with the extra stochastic term μ) ensures the existence of a weak control

$$(A.1) \quad \bar{\rho} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\xi}, \bar{W}, \bar{B}, \bar{\mu}, \bar{\alpha}) \in \mathcal{A}^w,$$

which minimizes the cost functional J^w over \mathcal{A}^w . Clearly, for any $\alpha \in \mathcal{A}$ one have that $\rho^\alpha = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \xi, W, B, \mu, \alpha) \in \mathcal{A}^w$ and that $J(\alpha, \mu) = J^w(\rho^\alpha)$, so that

$$(A.2) \quad J^w(\bar{\rho}) = \inf_{\rho \in \mathcal{A}^w} J^w(\rho) \leq \inf_{\alpha \in \mathcal{A}} J(\alpha, \mu).$$

Step 2. We now characterize the control $\bar{\alpha}$ using the necessary conditions of the stochastic maximum principle. In order to do so, we underline that we will work on the stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ with noises $(\bar{\xi}, \bar{W}, \bar{B}, \bar{\mu})$. In particular, the filtration $\bar{\mathbb{F}}$ may be larger than the filtration generated by the noises $(\bar{\xi}, \bar{W}, \bar{B}, \bar{\mu})$.

To simplify the notation, set $\bar{X} := X^{\bar{\rho}}$. Define next the adjoint processes $(\bar{Y}, \bar{Z}, \bar{Z}^\circ, \bar{M})$ as the solution of the BSDE

$$d\bar{Y}_t = D_x H(t, \bar{X}_t, \bar{\mu}_t, \bar{Y}_t, \bar{\alpha}_t) dt + \bar{Z}_t d\bar{W}_t + \bar{Z}_t^\circ d\bar{B}_t + d\bar{M}_t, \quad \bar{Y}_T = D_x g(\bar{X}_T, \bar{\mu}_T).$$

Since the control $\bar{\rho}$ is optimal for J^w in the class of weak controls \mathcal{A}^w , clearly it is optimal also in the smaller class $\mathcal{A}_{\bar{\rho}}$ of controls $\rho = (\Omega^\rho, \mathcal{F}^\rho, \mathbb{F}^\rho, \mathbb{P}^\rho, \xi^\rho, W^\rho, B^\rho, \mu^\rho, \alpha^\rho) \in \mathcal{A}^w$ such that $(\Omega^\rho, \mathcal{F}^\rho, \mathbb{F}^\rho, \mathbb{P}^\rho, \xi^\rho, W^\rho, B^\rho, \mu^\rho) = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\xi}, \bar{W}, \bar{B}, \bar{\mu})$. In other words, the control $\bar{\alpha}$ is optimal on the stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}}, \bar{\xi}, \bar{W}, \bar{B}, \bar{\mu})$ determined by $\bar{\rho}$. Therefore, by the stochastic maximum principle (see Theorem 1.59 at p. 98 in the Vol. II of [15]) we have that

$$\bar{\alpha}_t \in \arg \min_{a \in A} H(t, \bar{X}_t, \bar{\mu}_t, \bar{Y}_t, a), \quad \bar{\mathbb{P}} \otimes dt\text{-a.e.},$$

so that, by Remark 2.3, we know that

$$\bar{\alpha}_t = \hat{\alpha}(t, \bar{X}_t, \bar{\mu}_t, \bar{Y}_t), \quad \bar{\mathbb{P}} \otimes dt\text{-a.e.}$$

This implies that, for coefficients $\hat{b}, \hat{h}, \hat{g}$ as in (2.4), the process $(\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ, \bar{M})$ is a solution of the FBSDE

$$(A.3) \quad \begin{aligned} d\bar{X}_t &= \hat{b}(t, \bar{X}_t, \bar{\mu}_t, \bar{Y}_t)dt + \sigma(t, \bar{X}_t)d\bar{W}_t + \sigma^\circ(t, \bar{X}_t)d\bar{B}_t, & \bar{X}_0 &= \bar{\xi}, \\ d\bar{Y}_t &= -\hat{h}(t, \bar{X}_t, \bar{\mu}_t, \bar{Y}_t)dt + \bar{Z}_t d\bar{W}_t + \bar{Z}_t^\circ d\bar{B}_t + d\bar{M}_t, & \bar{Y}_T &= \hat{g}(\bar{X}_T, \bar{\mu}_T). \end{aligned}$$

The system (A.3) is a FBSDE (in a random environment) with coefficients $(\hat{b}, \sigma, \sigma^\circ, \hat{h}, \hat{g})$ and noises $(\bar{\xi}, \bar{W}, \bar{B}, \bar{\mu})$ on the stochastic basis $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ (we refer to Chapter 1 in the Vol. II of [15] for further details on FBSDEs in a random environment).

Step 3. We finally identify the optimal control α^μ by reconstructing a copy of the solution of the FBSDE (A.3) on the original probability space.

From the definition of weak control, we have $\bar{\mathbb{P}} \circ (\bar{\xi}, \bar{W}, \bar{B}, \bar{\mu})^{-1} = \mathbb{P} \circ (\xi, W, B, \mu)^{-1}$, so that the process $\bar{\mu}$ is adapted to the filtration generated by \bar{B} . This allows to use Remark 1.16 at p. 15 in Vol. II in [15] to deduce that the martingale term \bar{M} is null, so that the process $(\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ)$ is a solution to the FBSDE

$$(A.4) \quad \begin{aligned} d\bar{X}_t &= \hat{b}(t, \bar{X}_t, \bar{\mu}_t, \bar{Y}_t)dt + \sigma(t, \bar{X}_t)d\bar{W}_t + \sigma^\circ(t, \bar{X}_t)d\bar{B}_t, & \bar{X}_0 &= \bar{\xi}, \\ d\bar{Y}_t &= -\hat{h}(t, \bar{X}_t, \bar{\mu}_t, \bar{Y}_t)dt + \bar{Z}_t d\bar{W}_t + \bar{Z}_t^\circ d\bar{B}_t, & \bar{Y}_T &= \hat{g}(\bar{X}_T, \bar{\mu}_T). \end{aligned}$$

Moreover, by Assumption 2.4, such a system satisfies the strong uniqueness property. Therefore, by Theorem 1.33 at p. 34 in Vol. II of [15], there exists a solution (X, Y, Z, Z°) of the FBSDE with coefficients $(\hat{b}, \sigma, \sigma^\circ, \hat{h}, \hat{g})$, and with noises (ξ, W, B, μ) on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Moreover, such a solution is such that

$$\mathbb{P} \circ (X, Y, Z, Z^\circ, \xi, W, B, \mu)^{-1} = \bar{\mathbb{P}} \circ (\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}^\circ, \bar{\xi}, \bar{W}, \bar{B}, \bar{\mu})^{-1}.$$

The latter, allows to conclude that

$$\mathbb{P} \circ (X, \mu, (\hat{\alpha}(t, X_t, \mu_t, Y_t))_{t \in [0, T]})^{-1} = \bar{\mathbb{P}} \circ (\bar{X}, \bar{\mu}, (\hat{\alpha}(t, \bar{X}_t, \bar{\mu}_t, \bar{Y}_t))_{t \in [0, T]})^{-1},$$

so that, setting $\alpha^\mu := (\hat{\alpha}(t, X_t, \mu_t, Y_t))_{t \in [0, T]}$, one has $J(\alpha^\mu, \mu) = J^w(\bar{\rho})$ and, by (A.2), we conclude that the control α^μ is optimal for the functional $J(\cdot, \mu)$ on \mathcal{A} .

Finally, if $\tilde{\alpha} \in \mathcal{A}$ is another optimal control for $J(\cdot, \mu)$, we can use the same arguments as in the beginning of this step in order to show that $\tilde{\alpha} = (\hat{\alpha}(t, \tilde{X}_t, \mu_t, \tilde{Y}_t))_{t \in [0, T]}$ with $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{Z}^\circ)$ is a solution of the FBSDE $_\mu$ (2.2). Again, by strong uniqueness (Assumption 2.4), we have $(X, Y, Z, Z^\circ) = (\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{Z}^\circ)$ and $\tilde{\alpha} = (\hat{\alpha}(t, \tilde{X}_t, \mu_t, \tilde{Y}_t))_{t \in [0, T]} = (\hat{\alpha}(t, X_t, \mu_t, Y_t))_{t \in [0, T]} = \alpha^\mu$, proving the uniqueness of the optimal control.

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REFERENCES

- [1] S. ADLAKHA, R. JOHARI, AND G. Y. WEINTRAUB, *Equilibria of dynamic games with many players: Existence, approximation, and market structure*, J. Econom. Theory, 156 (2015), pp. 269–316.
- [2] S. AHUJA, *Wellposedness of mean field games with common noise under a weak monotonicity condition*, SIAM J. Control Optim., 54 (2016), pp. 30–48.
- [3] S. AHUJA, W. REN, AND T.-W. YANG, *Forward–backward stochastic differential equations with monotone functionals and mean field games with common noise*, Stochastic Process. Appl., 129 (2019), pp. 3859–3892.
- [4] F. ANTONELLI, *Backward-forward stochastic differential equations*, Ann. Appl. Probab., (1993), pp. 777–793.
- [5] M. BARDI AND M. FISCHER, *On non-uniqueness and uniqueness of solutions in finite-horizon mean field games*, ESAIM Control Optim. Calc. Var., 25 (2019), p. 44.
- [6] A. BENSOUSSAN, K. SUNG, S. C. P. YAM, AND S.-P. YUNG, *Linear-quadratic mean field games*, J. Optim. Theory Appl., 169 (2016), pp. 496–529.

- [7] C. BERTUCCI, *Monotone solutions for mean field games master equations: continuous state space and common noise*, arXiv preprint arXiv:2107.09531, (2021).
- [8] M. BURZONI AND L. CAMPI, *Mean field games with absorption and common noise with a model of bank run*, arXiv preprint arXiv:2107.00603, (2021).
- [9] P. CARDALIAGUET, F. DELARUE, J.-M. LASRY, AND P.-L. LIONS, *The Master Equation and the Convergence Problem in Mean Field Games*, Annals of Mathematics Studies, Princeton University Press, 2019.
- [10] P. CARDALIAGUET AND S. HADIKHANLOO, *Learning in mean field games: the fictitious play*, ESAIM Control Optim. Calc. Var., 23 (2017), pp. 569–591.
- [11] P. CARDALIAGUET AND P. E. SOUGANIDIS, *On first order mean field game systems with a common noise*, Ann. Appl. Probab., 32 (2022), pp. 2289–2326.
- [12] R. CARMONA AND F. DELARUE, *Mean field forward-backward stochastic differential equations*, Electron. Commun. Probab., 18 (2013), pp. 1–15.
- [13] ———, *Probabilistic analysis of mean-field games*, SIAM J. Control Optim., 51 (2013), pp. 2705–2734.
- [14] ———, *Forward-backward stochastic differential equations and controlled McKean–Vlasov dynamics*, Ann. Probab., 43 (2015), pp. 2647–2700.
- [15] ———, *Probabilistic Theory of Mean Field Games with Applications I-II*, Springer, 2018.
- [16] R. CARMONA, F. DELARUE, AND D. LACKER, *Mean field games with common noise*, Ann. Probab., 44 (2016), pp. 3740–3803.
- [17] R. CARMONA AND D. LACKER, *A probabilistic weak formulation of mean field games and applications*, Ann. Appl. Probab., 25 (2015), pp. 1189–1231.
- [18] A. CECCHIN, P. D. PRA, M. FISCHER, AND G. PELINO, *On the convergence problem in mean field games: a two state model without uniqueness*, SIAM J. Control Optim., 57 (2019), pp. 2443–2466.
- [19] Y. CHEN AND P. LUO, *Existence of global solutions for multi-dimensional coupled FBSDEs with diagonally quadratic generators*, arXiv preprint arXiv:2111.07246, (2021).
- [20] J. CVITANIĆ AND J. MA, *Hedging options for a large investor and forward-backward SDE’s*, Ann. Appl. Probab., 6 (1996), pp. 370–398.
- [21] F. DELARUE, *On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case*, Stochastic Process. Appl., 99 (2002), pp. 209–286.
- [22] F. DELARUE AND R. F. TCHUENDOM, *Selection of equilibria in a linear quadratic mean-field game*, Stochastic Process. Appl., 130 (2020), pp. 1000–1040.
- [23] J. DIANETTI, G. FERRARI, M. FISCHER, AND M. NENDEL, *Submodular mean field games: Existence and approximation of solutions*, Ann. Appl. Probab., 31 (2021), pp. 2538–2566.
- [24] ———, *A unifying framework for submodular mean field games*, Math. Oper. Res., to appear, (2022).
- [25] R. DUMITRESCU, M. LEUTSCHER, AND P. TANKOV, *Linear programming fictitious play algorithm for mean field games with optimal stopping and absorption*, arXiv preprint arXiv:2202.11428, (2022).
- [26] R. ELIE, J. PÉROLAT, M. LAURIÈRE, M. GEIST, AND O. PIETQUIN, *Approximate fictitious play for mean field games*, arXiv preprint arXiv:1907.02633, (2019).
- [27] W. GANGBO, A. R. MÉSZÁROS, C. MOU, AND J. ZHANG, *Mean field games master equations with nonseparable Hamiltonians and displacement monotonicity*, Ann. Probab., 50 (2022), pp. 2178 – 2217.
- [28] X. GUO, A. HU, R. XU, AND J. ZHANG, *Learning mean-field games*, Advances in Neural Information Processing Systems, 32 (2019).
- [29] U. G. HAUSSMANN AND J. P. LEPELTIER, *On the existence of optimal controls*, SIAM J. Control Optim., 28 (1990), pp. 851–902.
- [30] Y. HU AND S. PENG, *Solution of forward-backward stochastic differential equations*, Probab. Theory Related Fields, 103 (1995), pp. 273–283.
- [31] M. HUANG, R. P. MALHAMÉ, AND P. E. CAINES, *Large population stochastic dynamic games: closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle*, Commun. Inf. Syst., 6 (2006), pp. 221–252.
- [32] Z. HUANG AND S. TANG, *Mean field games with common noises and conditional distribution dependent fbsdes*, arXiv preprint arXiv:2105.11984, (2021).
- [33] T. KAMAE, U. KRENGEL, AND G. L. O’BRIEN, *Stochastic inequalities on partially ordered spaces*, Ann. Probab., 5 (1977), pp. 899–912.
- [34] J.-M. LASRY AND P.-L. LIONS, *Mean field games*, Jpn. J. Math., 2 (2007), pp. 229–260.
- [35] K. LEE, D. RENGARAJAN, D. KALATHIL, AND S. SHAKKOTTAI, *Reinforcement learning for mean field games with strategic complementarities*, in International Conference on Artificial Intelligence and Statistics, PMLR, 2021, pp. 2458–2466.
- [36] J. MA, P. PROTTER, AND J. YONG, *Solving forward-backward stochastic differential equations explicitly—a four step scheme*, Probab. Theory Related Fields, 98 (1994), pp. 339–359.

- [37] J. MA, Z. WU, D. ZHANG, AND J. ZHANG, *On well-posedness of forward-backward SDEs—a unified approach*, Ann. Appl. Probab., 25 (2015), pp. 2168–2214.
- [38] J. MA AND J. YONG, *Forward-Backward Stochastic Differential Equations and their Applications*, Springer Science & Business Media, 1999.
- [39] C. MOU AND J. ZHANG, *Mean field game master equations with anti-monotonicity conditions*, arXiv preprint arXiv:2201.10762, (2022).
- [40] A. MÜLLER AND M. SCARSINI, *Stochastic order relations and lattices of probability measures*, SIAM J. Control Optim., 16 (2006), pp. 1024–1043.
- [41] K. NAM AND Y. XU, *Coupled FBSDEs with measurable coefficients and its application to parabolic PDEs*, J. Math. Anal. Appl., (2022), p. 126403.
- [42] E. PARDOUX AND S. TANG, *Forward-backward stochastic differential equations and quasilinear parabolic pdes*, Probab. Theory Related Fields, 114 (1999), pp. 123–150.
- [43] S. PENG, *A general stochastic maximum principle for optimal control problems*, SIAM J. Control Optim., 28 (1990), pp. 966–979.
- [44] S. PERRIN, J. PÉROLAT, M. LAURIÈRE, M. GEIST, R. ELIE, AND O. PIETQUIN, *Fictitious play for mean field games: Continuous time analysis and applications*, arXiv preprint arXiv:2007.03458, (2020).
- [45] P. E. PROTTER, *Stochastic Integration and Differential Equations*, Springer, 2nd ed., 2005.
- [46] A. TARSKI, *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math., 5 (1955), pp. 285–309.
- [47] R. F. TCHUENDOM, *Uniqueness for linear-quadratic mean field games with common noise*, Dyn. Games Appl., 8 (2018), pp. 199–210.
- [48] D. M. TOPKIS, *Equilibrium points in nonzero-sum n -person submodular games*, SIAM J. Control Optim., 17 (1979), pp. 773–787.
- [49] ———, *Supermodularity and Complementarity*, Princeton University Press, 2011.
- [50] X. VIVES, *Oligopoly Pricing: Old Ideas and New Tools*, MIT press, 1999.
- [51] P. WIĘCEK, *Total reward semi-Markov mean-field games with complementarity properties*, Dyn. Games Appl., 7 (2017), pp. 507–529.
- [52] Z. WU AND M. XU, *Comparison theorems for forward backward SDEs*, Statist. Probab. Lett., 79 (2009), pp. 426–435.
- [53] Q. XIE, Z. YANG, Z. WANG, AND A. MINCA, *Provable fictitious play for general mean-field games*, arXiv preprint arXiv:2010.04211, (2020).
- [54] J. YONG AND X. Y. ZHOU, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, vol. 43, Springer Science & Business Media, 1999.
- [55] J. ZHANG, *Backward Stochastic Differential Equations*, Springer, 2017.