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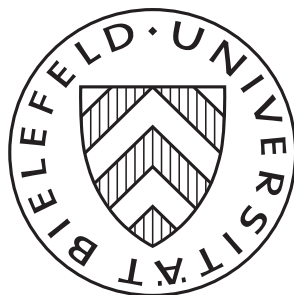


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The Role of Price Normalization in Imperfectly Competitive Economies

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Abstract

This note presents an analytical method to examine when, why, and how rules of price normalization have an impact on noncompetitive equilibria in Arrow-Debreu economies. For simple prototype economies it is shown that under smooth strict convexity of preferences and technologies regular monopolistic equilibria depend generically on the parameters of the normalization map. Conversely, for an example with consumer preferences from a parametrized class of quasi-concave/quasi-convex utility functions monopolistic equilibria are shown to be invariant under normalization for all parameters under convex technologies as well as under fixed costs or increasing returns.

The generic dependence of allocations on rules of price normalization implies alterations of characteristics of outcomes in a much wider class of applications than models of monopolistic competition only. Several examples of so-called Cournot-Walras oligopolies are discussed occurring in typical models with monopolistic competition, international trade, welfare economics, public economics, and macroeconomics revealing the generic impact of price normalization on noncompetitive outcomes in Arrow-Debreu economies.

Contents

1	Introduction	2
2	Normalization matters for allocations	3
2.1	A prototype model: The Robinson-Crusoe economy	3
2.2	Monopolistic Equilibria with Price-Wage Normalization	4
3	Nonregular monopolistic equilibria	8
4	Cournot-Walras Oligopolies	12
4.1	Cournot-Walras Duopolies	12
4.2	General Cournot-Walras Oligopolies	15
	Bibliography	19

1 Introduction

Economists working in the theory of general equilibrium in the late 1960ies and early seventies (among them Arrow & Hahn, 1971; Marschak & Selten, 1974, and many others) turned with great enthusiasm and effort to analyze extensions of the competitive paradigm to imperfectly competitive behavioral models of the firm within Arrow-Debreu economies. At the same time they realized - somewhat disturbed and unexpectedly - that equilibria with profit maximizing monopolistic firms are defined for and may depend in an essential way on the choice of an arbitrary numeraire commodity, or more generally, on the choice of a rule which converts relative prices into nominal or absolute prices. As it turns out, such a choice is necessary in order to define monopolistic equilibria in the first place.

This requirement is in contrast to the conditions needed for competitive equilibria of such economies whose supporting equilibrium prices are always determined only up to a positive scalar factor, referred to as relative prices. In other words, under monopolistic competition, prices, values of commodity bundles, incomes, profits, GDP, etc. are all well defined nominal entities because of a specifically chosen rule of price normalization, while for competitive equilibria they are not since the choice is not necessary. Moreover, this leaves the allocative generality of the static general equilibrium paradigm with the disturbing feature that an innocuously looking and often arbitrary choice in many cases induces effects on the real outcomes, the equilibrium allocations.

Following the publication of the article by Gabszewicz & Vial (1972) these facts have become known more widely with an increased awareness of their appearance in different areas of economic theory. Their consequences for allocative results, however, have been revealed and discussed mostly only case by case and not systematically in search for a theoretical explanation, except for a few investigations (among them Roberts & Sonnenschein, 1976, 1977; Böhm, 1994; Dierker & Grodal, 1999).

With the number of research attempts growing over the past decades introducing equilibria for Arrow-Debreu economies with strategic rules for agents or non-competitive structures,

- analyzing variants of Nash equilibria for markets rather than competitive outcomes,
- introducing taxes and tariffs in models of international trade or with regulation,
- designing discretionary regulatory fiscal policy in industrial organization,
- investigating rules for second best optimization,
- constructing trading mechanisms as alternatives to markets,

there is a need for further understanding why and under which conditions a normalization rule for the class of GE models makes a difference for the allocative outcome i.e. why *real* allocations of Arrow-Debreu economies and their properties are not invariant under perturbations of a scaling device which has no apparent economic grounding within the structure of the model. A dependence of equilibrium allocations on rules of price normalization implies, in principle, alterations of characteristics of outcomes in a much wider class of applications than models of monopolistic competition only, as in international trade, welfare economics, public finance, applied game theory, industrial organization, financial economics, or incomplete markets. Thus, dependence of outcomes requires a reexamination of properties such as existence or uniqueness of equilibria or their stability, welfare, incomes and their distribution, but also for the implementability or incentive compatibility of second best solutions, for pricing policies with public goods or externalities.

This note proposes an analytical method to examine why, when, and how rules of price normalization have an impact on noncompetitive equilibria in Arrow-Debreu economies. Section 2 presents the conditions for two fundamental propositions stating generic dependence of regular monopolistic equilibria on price normalization rules for prototype economies with strict convexity of consumer preferences and technologies. Nonregular equilibria occur when strategic behavior provides no advantage for the monopolistic producer compared to equilibria under competitive behavior, i.e. when the monopolistic equilibrium coincides with the competitive one. Section 3 presents an example with consumer preferences from a parametrized class of quasi-concave utility functions for which monopolistic equilibria are immune to normalization for all strictly convex cost functions, as well as in situations with fixed costs or increasing returns. Section 4 presents and discusses further applications of the methodology of Section 2.2 to examples of oligopolies occurring in typical models of international trade, welfare economics, public economics, and of macroeconomics.

2 Normalization matters for allocations

2.1 A prototype model: The Robinson-Crusoe economy

Consider an economy with

- one consumer supplying labor $\ell \geq 0$, consuming a consumption good $x \geq 0$ with differentiable utility function $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}$, which is strictly quasi-concave, strictly increasing in x , strictly decreasing in ℓ , $U_x(x, \ell) > 0$, $U_\ell(x, \ell) < 0$, so that inverse demand, i.e. an optimal consumption-labor plan $(x, \ell) \geq 0$ at commodity price p , wage rate w , and under the budget constraint $px \leq w\ell$, satisfies

$$(2.1) \quad \frac{p}{w} \stackrel{!}{=} -\frac{U_x(x, \ell)}{U_\ell(x, \ell)};$$

- one producer with differentiable and strictly monotonically increasing real cost function (inverse of production function) $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $x \mapsto c(x) = \ell$ satisfying the so-called convex Inada conditions

$$(2.2) \quad \begin{aligned} xc'(x) - c(x) &> 0 && \text{for all } x > 0 \\ c(0) = c'(0) &= 0 && \text{and } \lim_{x \rightarrow \infty} c'(x) = \infty. \end{aligned}$$

This implies that the set of feasible allocations $(x, \ell) = (x, c(x)) \geq 0$ is identifiable with $x \in X \equiv \mathbb{R}_+$. Let the unique Pareto optimal allocation be denoted

$$(2.3) \quad x^* := \arg \max_x U(x, c(x))$$

which coincides with the unique competitive equilibrium.

- The objective inverse demand function $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for all feasible allocations is defined as

$$(2.4) \quad \frac{p}{w} = D(x) := -\frac{U_x(x, c(x))}{U_\ell(x, c(x))}$$

which determines supporting relative prices p/w with income consistency (wages plus profit) for each feasible allocation $(x, c(x))$. Without loss of generality, let the utility function and the cost function be chosen so that $c'(x^*) = 1 = D(x^*)$,

- For any price-wage pair (p, w) , nominal profit $\Pi = px - wc(x) = w(x\frac{p}{w} - c(x))$ is attainable for the producer via x if $\Pi = w(xD(x) - c(x))$. This implies the attainable/feasible profit function $\Pi : \mathbb{R}_+^2 \times X \rightarrow \mathbb{R}$ under the demand restriction $p = wD(x)$ as

$$(2.5) \quad \begin{aligned} \Pi(p, w, x) &:= \{px - wc(x) \mid x \in X, p = wD(x)\} = \Pi(wD(x), w, x) \\ &= w\Pi(D(x), 1, x) \equiv w\pi(x) := w[xD(x) - c(x)]. \end{aligned}$$

Feasible profit Π is homogeneous of degree one in (p, w) for every feasible allocation $x \in X$, so that the profit function is unbounded if $\pi(x) := xD(x) - c(x) > 0$ for some $x \in X$. Therefore, if profit maximization of a price-wage setting producer is to have a solution in this general equilibrium environment some normalization of prices or wages is *necessary* to guarantee existence of an equilibrium.

2.2 Monopolistic Equilibria with Price-Wage Normalization

Let a unit of labor be chosen as the numeraire and the nominal wage set to $\bar{w} > 0$. Then, $\tilde{x} \in X$ is a monopolistic equilibrium if it solves

$$(2.6) \quad \begin{aligned} \tilde{x}(\bar{w}) &:= \arg \max_{x \in X} \Pi_{\bar{w}}(x) := \arg \max_{x \in X} \Pi(\bar{w}D(x), \bar{w}, x) \\ &= \arg \max_{x \in X} \bar{w}\Pi(D(x), 1, x) \\ &= \arg \max_{x \in X} \bar{w}[xD(x) - c(x)] = \arg \max_{x \in X} \pi(x) \end{aligned}$$

where $\pi(x) := xD(x) - c(x) = \Pi(D(x), 1, x)$ is real profit measured in units of labor.

Next, let a unit of output be chosen as the numeraire and the output price set to $\bar{p} > 0$. Then,

$$(2.7) \quad \begin{aligned} \tilde{x}(\bar{p}) &:= \arg \max_{x \in X} \Pi_{\bar{p}}(x) := \arg \max_{x \in X} \Pi(\bar{p}, \bar{p}/D(x), x) \\ &= \arg \max_{x \in X} \bar{p}\Pi(1, 1/D(x), x) \\ &= \arg \max_{x \in X} \bar{p} \frac{xD(x) - c(x)}{D(x)} = \arg \max_{x \in X} \frac{\pi(x)}{D(x)} \end{aligned}$$

implying a different allocation as monopolistic equilibrium which maximizes real profit measured in units of output.

As a third possibility, consider a linear mapping restricting prices and wages on a simplex with slope $-\alpha$ satisfying

$$(2.8) \quad \Delta(\alpha) := \{(p, w) \in \mathbb{R}_+^2 \mid p + \alpha w = 1 + \alpha, \alpha \geq 0\}.$$

Together with $p = wD(x)$, for every feasible allocation $x \in X$, this yields a pair of supporting nominal price-wage values

$$(2.9) \quad p = \frac{(1 + \alpha)D(x)}{\alpha + D(x)} \quad \text{and} \quad w = \frac{1 + \alpha}{\alpha + D(x)}$$

making an allocation $x \in X$ a utility maximizer with income consistency at the price-wage pair (p, w) . As a consequence, the associated profit is given by

$$(2.10) \quad \Pi(\alpha, x) := \Pi \left(\frac{(1 + \alpha)D(x)}{\alpha + D(x)}, \frac{1 + \alpha}{\alpha + D(x)}, x \right) = \frac{1 + \alpha}{\alpha + D(x)} \pi(x)$$

implying as monopolistic equilibrium

$$(2.11) \quad \begin{aligned} \tilde{x}(\alpha) &:= \arg \max_{x \in X} \Pi(\alpha, x) := \arg \max_{x \in X} \frac{1 + \alpha}{\alpha + D(x)} \pi(x) \\ &= \arg \max_{x \in X} \frac{\pi(x)}{\alpha + D(x)}. \end{aligned}$$

Thus, an interior monopolistic equilibrium $\tilde{x}(\alpha) > 0$ under normalization on the simplex differs generically with α since the maximizers change with the slope $-\alpha$ of the constraint set, the simplex. Equations (2.6), (2.7), and (2.10) reveal that there are two effects responsible for this result.

1. The linear relation $p = wD(x)$ combined with the linear homogeneity of Π in (p, w) implies that the profit map is multiplicatively separable in the wage level times real profits in terms of labor as numeraire (see (2.6)). Thus, Π is essentially a one-dimensional map of feasible allocations x multiplied by a wage level independent of the output price.
2. The normalization constraint (chosen here as a linear manifold) implies a constraint on the price-wage domain of the profit function Π making its positive upper contour sets *smaller and depending on* α when prices and wages are restricted to $(p, w) \in \Delta(\alpha) \subsetneq \mathbb{R}_+^2$. Together with the linear relation $p = wD(x)$ supportable nominal prices and wages are the unique solution of the two linear constraints for any feasible allocation $x \in X$. In other words, due to the multiplicative form of nominal profits the joint effect from the value of the normalization parameter and the dependence on the objective demand function appears as a *change of the objective function to be maximized* on the set of feasible allocations X .

The formulation includes the two boundary cases with choices of the corresponding numeraire since, by construction, $\tilde{x}(\bar{p}) = \tilde{x}(\alpha)$ for $\alpha = 0$, and that $\tilde{x}(\bar{w}) = \tilde{x}(\alpha)$ for $\lim \alpha = +\infty$. Formally, one obtains the following proposition.

Proposition 2.1. *Assume that the inverse demand function D and the real cost function c are twice continuously differentiable. Let $\tilde{x}(\alpha) > 0$ denote an interior monopolistic equilibrium of the economy for the linear normalization*

$$(2.12) \quad \Delta(\alpha) = \{(p, w) \geq 0 \mid p + \alpha w = 1 + \alpha, \alpha \geq 0\}$$

defined in (2.8), such that the equilibrium allocation is a local maximizer of nominal profit with $x^ \neq \tilde{x}(\alpha)$. Then, the equilibrium allocation and the associated profit are non-constant functions for any level of the parameter $\alpha \geq 0$.*

Proof. Let $\tilde{x}(\alpha) > 0$ denote the maximizer for (2.11) satisfying the first and second order conditions for a regular monopolistic equilibrium, i.e.

$$(2.13) \quad \pi'(\alpha + D) - \pi D' = 0 \quad \text{and} \quad \pi''(\alpha + D) - \pi D'' < 0.$$

Since $\alpha + D > 0$ and $\pi > 0$, $D' = 0 = \pi' = \tilde{x}D' + D - c'$ implies $D = c'$, i.e. $x^* = \tilde{x}(\alpha)$ is the competitive and not a monopolistic equilibrium. Therefore, under regularity, π' and D' must be nonzero and have the same sign.

Applying the Envelop Theorem to (2.10) resp. (2.11) yields

$$(2.14) \quad \frac{d}{d\alpha} \left(\frac{(1 + \alpha)\pi(\tilde{x}(\alpha))}{\alpha + D(\tilde{x}(\alpha))} \right) = \frac{\partial}{\partial \alpha} \left(\frac{(1 + \alpha)\pi(\tilde{x})}{\alpha + D(\tilde{x})} \right) = \frac{\pi(\tilde{x})(D(\tilde{x}) - 1)}{(\alpha + D(\tilde{x}))^2} \neq 0 \quad \text{since } D(\tilde{x}(\alpha)) \neq 1$$

$$(2.15) \quad \frac{d}{d\alpha} \left(\frac{\pi(\tilde{x}(\alpha))}{\alpha + D(\tilde{x}(\alpha))} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\pi(\tilde{x})}{\alpha + D(\tilde{x})} \right) = -\frac{\pi(\tilde{x}(\alpha))}{(\alpha + D(\tilde{x}(\alpha)))^2} < 0$$

confirming that the maximal profit level is non-constant for all $\alpha \geq 0$. Applying the Implicit Function Theorem to the first order necessary condition (2.13) one obtains

$$(2.16) \quad \frac{d\tilde{x}}{d\alpha}(\alpha) = -\frac{\pi'}{\pi''(\alpha + D) - \pi D''} \neq 0$$

because of regularity (2.13) together with $\pi'(\tilde{x}(\alpha)) \neq 0$. \square

Thus, in smooth economies, regular monopolistic allocations and payoffs differ under different normalizations. As a consequence, qualitative properties associated with the equilibrium such as efficiency, optimality, welfare, or stability in a dynamic setting may be sensitive to parametric changes of the normalization as well. This makes the significance of specific economic properties of a monopolistic equilibrium for a given α weak which, in the end, may require a justification for a particular normalization within a theory of allocations for non-competitive environments, since independence from normalization as in the case of competitive equilibria is not guaranteed. In contrast, however, notice that independence of allocations from normalization implies an indeterminacy of nominal equilibrium values, i.e. for any two $\alpha \neq \alpha'$ one has $\tilde{x}(\alpha) = \tilde{x}(\alpha')$ but $\Pi(\alpha, \tilde{x}(\alpha)) \neq \Pi(\alpha', \tilde{x}(\alpha'))$.

While the above analysis proposes a local differentiable answer to the sensitivity of *monopolistic allocations and payoffs* on parameters of linear normalization rules, there exists an equivalent and intuitive geometric analysis in *price-wage space* showing the general dependence of monopolistic equilibria on the choice of the price-wage normalization. This characterizes the equilibrium as a minimizer for a wider class of normalization maps on the upper contour sets of feasible profit.

Let $X^+ := \{x \in X \mid \pi(x) \geq 0\}$ and define the unit profit contour for supporting pairs $(p, w) \in \mathbb{R}_+^2$ as

$$(2.17) \quad \begin{aligned} \Pi(1) &:= \left\{ (p, w) \in \mathbb{R}_+^2 \mid \Pi(p, w, x) = \Pi \left(\frac{D(x)}{\pi(x)}, \frac{1}{\pi(x)}, x \right) \equiv 1, x \in X^+ \right\} \\ &= \left\{ (p, w) \in \mathbb{R}_+^2 \mid p = \frac{D(x)}{\pi(x)}, w = \frac{1}{\pi(x)}, x \in X^+ \right\}. \end{aligned}$$

The homogeneity of the profit function in (p, w) implies that price-wage pairs for positive profits form a cone

$$(2.18) \quad C^+ := \{(p, w) \in \mathbb{R}_+^2 \mid (p, w) = (\lambda p', \lambda w'), (p', w') \in \Pi(1), \lambda \geq 0\}$$

generated by the unit contour $\Pi(1)$, i.e. $(p, w) \in \Pi(1)$ implies that $\Pi(\lambda p, \lambda w, x) \gg 0$ for all $\lambda > 0$. Then, as shown above, a monopolistic equilibrium for the normalization $\Delta(\alpha)$ is given by

$$(2.19) \quad \tilde{x}(\alpha) := \arg \max \{ \Pi(p, w, x) \mid x \in X^+, (p, w) \in C^+ \cap \Delta(\alpha) \}.$$

The strict monotonicity of the profit function on C^+ implies that equilibria depend 'generically' on the parameter α since the intersection $C^+ \cap \Delta(\alpha)$ is 'thin' and non-constant. Therefore, in smooth economies, regular monopolistic allocations and payoffs differ under different linear normalizations.

The characterization of an equilibrium in (2.19) suggests that its geometric features hold for a wider class of normalization mappings. Let $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $(p, w) \mapsto F(p, w)$ denote an increasing, homogeneous, and convex function. Then, any positive contour

$$(2.20) \quad I_F(d) := \{ (p, w) \in \mathbb{R}_+^2 \mid F(p, w) = d > 0 \}$$

defines a set of positive normalized price-wage pairs and

$$(2.21) \quad \tilde{x}_F(d) := \arg \max \{ \Pi(p, w, x) \mid x \in X^+, (p, w) \in C^+ \cap I_F(d) \}$$

as an associated equilibrium. Notice that, in particular, all norms $\| \cdot \|$ are candidates for price normalization maps. As a consequence, monopolistic equilibria are minimizers of the normalization map over the upper contour set

$$(2.22) \quad \Pi^+(1) := \{ (p, w) \in \mathbb{R}_+^2 \mid \Pi(p, w, x) \geq 1, x \in X^+ \}$$

of the unit profit contour.

Proposition 2.2. *Let $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $(p, w) \mapsto F(p, w)$ denote a continuous homogeneous convex normalization map for twice differentiable inverse demand and cost functions.*

1. $\tilde{x} \in X^+$ is a monopolistic equilibrium for F if and only if

$$(2.23) \quad (\tilde{p}, \tilde{w}) = \arg \min \{ F(p, w) \mid (p, w) \in \Pi^+(1) \}.$$

2. If $\tilde{x} \in X^+$ with $\pi(\tilde{x}) > 0$, then \tilde{x} is a monopolistic equilibrium for the normalization map

$$(2.24) \quad F_{\tilde{x}}(p, w) := \max \{ p, D(\tilde{x})w \}.$$

Proof.

- 1.

" \Rightarrow " Let $(\tilde{p}, \tilde{w}, \tilde{x})$ denote an equilibrium for F and suppose that (\tilde{p}, \tilde{w}) is not an F -minimizer. There exists (p, w, x) with $F(p, w) < F(\tilde{p}, \tilde{w})$ and $\Pi(p, w, x) \geq \Pi(\tilde{p}, \tilde{w}, \tilde{x})$. There exists $\lambda > 1$ such that $F(\lambda p, \lambda w) = F(\tilde{p}, \tilde{w})$. But then $\Pi(\lambda p, \lambda w, x) > \Pi(p, w, x) \geq \Pi(\tilde{p}, \tilde{w}, \tilde{x})$, contradicting that \tilde{x} is an equilibrium.

" \Leftarrow " Let (\bar{p}, \bar{w}) be an F -minimizer, but $\bar{x} \in X^+$ is not a profit maximizer for $F(p, w) = F(\bar{p}, \bar{w}) = \bar{d}$. There exists (p, w, x) with $F(p, w) = \bar{d}$ and $0 < \bar{\lambda} < 1$ such that

$$(2.25) \quad \Pi(p, w, x) > \Pi(\bar{p}, \bar{w}, \bar{x}) = \Pi(\bar{\lambda} p, \bar{\lambda} w, x).$$

Then, $1 \geq \lambda > \bar{\lambda}$ implies

$$(2.26) \quad \Pi(\lambda p, \lambda w, x) > \Pi(\bar{p}, \bar{w}, \bar{x}) \quad \text{and} \quad F(\lambda p, \lambda w) < \bar{d} = F(\bar{p}, \bar{w})$$

contradicting that (\bar{p}, \bar{w}) is an F -minimizer for $(p, w) \in \Pi^+(1)$.

2. Let

$$(\tilde{p}, \tilde{w}) := \left(\frac{D(\tilde{x})}{\pi(\tilde{x})}, \frac{1}{\pi(\tilde{x})} \right).$$

By construction, $F_{\tilde{x}}(p, w) < F_{\tilde{x}}(\tilde{p}, \tilde{w})$ holds if and only if $(p, w) \ll (\tilde{p}, \tilde{w})$, which implies $\Pi(p, w, \tilde{x}) < \Pi(\tilde{p}, \tilde{w}, \tilde{x}) = 1$ for $(p, w) \not\geq (\tilde{p}, \tilde{w})$ by the strict monotonicity of Π in (p, w) . Thus, (\tilde{p}, \tilde{w}) is an $F_{\tilde{x}}$ minimizer on $\Pi^+(1)$ and \tilde{x} is a monopolistic equilibrium. \square

The proposition shows that, with smoothness, any allocation with positive profits can be supported as a monopolistic equilibrium for appropriate choices of a convex normalization function. This creates an arbitrary indeterminacy of monopolistic equilibria arising from an artificial device, the normalization function, which is necessary to define *nominal values, prices and profits*.

3 Nonregular monopolistic equilibria

Let consumer preferences be given by the monotonic quasi-concave utility function $U : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $(x, \ell) \mapsto U(x, \ell)$, such that

$$(3.1) \quad U(x, \ell) := \frac{s + x}{r + \ell}, \quad r > 0, \quad s \geq 0$$

which is neither strictly quasi-concave nor concave. It belongs to a class $U(x, \ell) = (s + u(x))/(r + v(\ell))$ representing separable preferences between consumption and labor (a case often studied in the early days of utility and demand theory by Samuelson, 1947; Leontief, 1947; Houthakker, 1960; Debreu, 1960; Böhm & Barten, 1982, and others). Since the indifference curves of (3.1) are straight lines U is also quasi-convex, i.e. all lower contour sets are convex as well. Competitive commodity demand and labor supply at price p , wages w , and profits $d \geq 0$ are the solution

$$(3.2) \quad (x, \ell) = \arg \max_{(x', \ell')} \{U(x', \ell') \mid px' \leq w\ell' + d\}$$

which satisfies the necessary condition

$$(3.3) \quad \frac{p}{w} \stackrel{!}{=} -\frac{U_x(x, \ell)}{U_\ell(x, \ell)} = \frac{r + \ell}{s + x}.$$

Together with a real cost function $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ one obtains feasible utility levels as

$$(3.4) \quad V(x) := U(x, c(x)) = \frac{s + x}{r + c(x)}$$

and an objective inverse demand function $D : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as

$$(3.5) \quad \frac{p}{w} = D(x) := \frac{r + c(x)}{s + x},$$

for every feasible allocation $(x, \ell) = (x, c(x))$, with $x \geq 0$. Notice that $D(x) = 1/V(x)$, for all $x \geq 0$. If the real cost function satisfies the convex Inada conditions (2.2), D is U-shaped and

strictly convex with a unique global minimum at some $x^* > 0$.

The economy has a unique interior Pareto optimal allocation $(x^*, \ell^*) = (x^*, c(x^*))$ given by

$$(3.6) \quad x^* = \arg \max V(x) = \arg \max \frac{s+x}{r+c(x)}$$

which is also the unique competitive equilibrium with relative prices $(p/w)^*$ satisfying

$$(3.7) \quad \left(\frac{p}{w}\right)^* = D(x^*) = \frac{r+c(x^*)}{s+x^*} = c'(x^*) > \frac{c(x^*)}{x^*}.$$

Furthermore, real profit in terms of labor $\pi(x) = (rx - sc(x))/(s+x)$ is a strictly concave function attaining a unique global maximum at the Pareto optimal allocation x^* since

$$(3.8) \quad \pi'(x) = \frac{rs + sc(x) - sc'(x)(s+x)}{(s+x)^2} = 0 \iff \frac{r+c(x^*)}{s+x^*} = c'(x^*).$$

In other words, the Pareto optimal allocation $x^* > 0$ is a global minimizer of the relative price in terms of labor *and* a global maximizer of real profit in terms of labor.

Monopolistic equilibria under decreasing returns

To investigate the role of price-wage normalization for monopolistic equilibria in this economy, let first labor serve as numeraire with a constant wage rate $\bar{w} > 0$. This implies the profit function

$$(3.9) \quad \Pi_{\bar{w}}(x) := \Pi(\bar{w}D(x), \bar{w}, x) = \bar{w}(xD(x) - c(x)) = \bar{w}\pi(x) := \bar{w} \frac{rx - sc(x)}{s+x}.$$

For $s = 0$, π takes on the constant value $\bar{w}\pi(x) = \bar{w}r > 0$ for all $x > 0$ since the real revenue function $xD(x) = r + c(x)$ is equal to the real cost function with an additive shift of r . Therefore, any feasible supportable allocation is a monopolistic equilibrium.

For $s > 0$, (3.8) implies that

$$(3.10) \quad \tilde{x}(\bar{w}) = \arg \max_{x \in X} \bar{w}\pi(x) = x^*.$$

Thus, the Pareto optimal allocation x^* is the unique monopolistic equilibrium, for all $s > 0$ and all $\bar{w} > 0$, which coincides with the competitive equilibrium.

With normalization on the simplex $\Delta(\alpha)$ one has from (2.10) and (3.9)

$$(3.11) \quad \Pi_{\alpha}(x) = (1+\alpha) \frac{\pi(x)}{\alpha + D(x)} = (1+\alpha) \frac{rx - sc(x)}{\alpha(s+x) + r + c(x)}.$$

Since

$$(3.12) \quad \frac{\pi(x^*)}{\alpha + D(x^*)} \leq \max_{x \in X} \frac{\pi(x)}{\alpha + D(x)} \leq \frac{\max_{x \in X} \pi(x)}{\alpha + \min_{x \in X} D(x)} = \frac{\pi(x^*)}{\alpha + D(x^*)}$$

the Pareto optimal allocation $x^* > 0$ is also the unique monopolistic equilibrium for all $\alpha \geq 0$, i.e. it is independent of the normalization, for all $\alpha \geq 0$, and coincides with the competitive equilibrium. Thus, there exist convex economies with a unique monopolistic allocation whose supporting prices are independent of normalization and determined only up to an arbitrary scale factor. The net value of its allocation and of its associated profit are indeterminate.

The example confirms the indeterminacy of nominal values for monopolistic and for competitive equilibria when they coincide under varying normalizations for a class of convex economies with convex-concave preferences (quasi-concave and quasi-convex utility functions) and technologies satisfying the convex Inada conditions. On the one hand, this seems to be a strong result for independence of allocations under strategic producer behavior. On the other hand, it also appears as a weak property of indeterminacy of prices and values since it holds for the situation of the *coincidence* of competitive and monopolistic allocations, i.e. when global monopoly power implies no gain. Unfortunately, however, the result provides little additional insight into answering the question when or why normalization has allocative effects under monopolistic competition. Evidently, there are characteristics of economies or normalizations for which monopolistic equilibria are immune or not immune to normalization.

A different question begging an answer seems to be whether characteristics of economies can be identified for which monopolistic equilibria, which *differ* from the competitive one, are immune to all or to some normalizations. It would be useful to find an analytical method applicable to as many models as possible to examine the effects of classes of normalizations so as to verify or exclude dependencies for types of equilibria and for their characteristics, an approach pursued further below.

Fixed costs and decreasing variable returns

Before proceeding to duopolies it may be informative to examine the role of nonconvexities induced by fixed costs or by increasing returns within the above example for which independence from normalization was demonstrated. This provides further insight into issues of allocative consequences from normalizations in case of natural monopolies. Let consumer characteristics be given as in (3.1) and (3.5) and assume that the real cost function with positive fixed costs $a > 0$ is given by

$$(3.13) \quad C(x, a) := a + c(x), \quad a > 0, \quad c(0) = 0.$$

Inverse demand and feasible utility become

$$(3.14) \quad D(x) := \frac{r + a + c(x)}{s + x} \quad \text{and} \quad V(x) := \frac{s + x}{r + a + c(x)}$$

with $V(x) = 1/D(x)$ as before. If variable costs $c(x)$ satisfy the convex Inada conditions D is again strictly convex, U-shaped, and with a global interior minimum at

$$(3.15) \quad x^*(a) = \arg \min_{x \in X} \frac{r + a + c(x)}{s + x} \quad \iff \quad x^*(a) = \arg \max_{x \in X} \frac{s + x}{r + a + c(x)}$$

for every $a \geq 0$. Thus, $x^*(a)$ is also the unique interior Pareto optimal allocation satisfying the first order condition

$$(3.16) \quad c'(x^*(a)) = \frac{r + a + c(x^*(a))}{s + x^*(a)} \equiv D(x^*(a))$$

equalizing real marginal costs to the real price. Thus, the Pareto optimal allocation is also the unique competitive equilibrium. Moreover, one finds

$$(3.17) \quad \frac{dx^*}{da} > 0, \quad \frac{dV(x^*(a))}{da} < 0.$$

When $r > s = 0$, nominal profits are

$$(3.18) \quad \Pi(p, w, x) = px - w(a + c(x)) = w(xD(x) - (a + c(x))) = wr$$

with real profits $\pi(x) = r > 0$ for all $x \in X$, as in the case without fixed costs.

When $s > 0$, real profit becomes

$$(3.19) \quad \pi(x) = \frac{x(r + a + c(x))}{x + s} - (a + c(x)) = \frac{xr - s(a + c(x))}{x + s} = \frac{sx}{x + s} \left(\frac{r}{s} - \frac{a + c(x)}{x} \right).$$

Given the convex Inada conditions, average cost are strictly increasing and bijective. Therefore, $\lim_{x \rightarrow \infty} c(x)/x = \infty$ and (3.17) implies that, for every $r > 0, s > 0$, there exists a unique critical level $\bar{a} > 0$ such that

$$(3.20) \quad \frac{r}{s} = \frac{\bar{a} + c(x^*)}{x^*} = c'(x^*) \iff \pi(x^*(\bar{a})) = \frac{sx^*}{x^* + s} \left(\frac{r}{s} - \frac{\bar{a} + c(x^*)}{x^*} \right) = 0$$

with $\pi(x^*(a)) < 0$ if $a > \bar{a}$. Nevertheless, for large fixed costs, x^* is the unique competitive equilibrium (an equilibrium relative to a price system) with a negative profit level.

Finally, it is straightforward to show that positive fixed costs cause no sensitivity to normalization for monopolistic equilibria $\tilde{x}(a)$. They coincide with the Pareto optimal allocation $x^*(a)$ which maximizes real profit $\pi(x)$. Equation (3.19) which is analogous to (3.9) implies that the inequalities (3.11) prevail also under positive fixed costs. Therefore, if variable costs $c(x)$ satisfy the convex Inada conditions, a monopolistic equilibrium with positive fixed costs $a > 0$ depends in the same manner on normalization as without fixed costs, i.e. (1) if the wage is taken as numeraire and $s = 0$, all feasible allocations are monopolistic equilibria with the same profit level $\bar{w}r$ among them the Pareto optimal allocation, which coincides with the competitive equilibrium; (2) for $s > 0$ and all other normalizations $\alpha \geq 0$, the Pareto optimal allocation is the unique monopolistic equilibrium, equalizing real marginal costs with the real price of output. Thus, under all normalizations $\alpha \geq 0$, a profit maximizing monopolist chooses the same equilibrium allocation as the competitive firm making nominal profit levels indeterminate.

Fixed costs and nondecreasing variable returns

To complete the example of this section with the convex preferences of (3.1), assume that variable costs $c(x)$ are concave and increasing with constant or decreasing average variable costs $c(x)/x$, i.e. $c(x) \geq xc'(x)$ for all $x \in X$. Under these conditions utility $V(x) := U(x, a + c(x))$ is increasing for all $x \in X$, i.e.

$$(3.21) \quad V'(x) = \frac{r + a - sc'(x) + c(x) - xc'(x)}{(r + a + c(x))^2} > 0 \quad \text{if} \quad r + a > sc'(x).$$

For linear variable costs satisfying $(r + a)/s > c'(0) = c'(x) = c(x)/x$, no Pareto optimal allocation and no competitive equilibrium exists since $V(x)$ is strictly increasing. Under the same assumption $r + a - sc'(0) > 0$ and concavity of $c(x)$ one has

$$(3.22) \quad \begin{aligned} \pi'(x) &= \frac{s}{(s + x)^2} (r + a - sc'(x) + c(x) - xc'(x)) > 0 \\ D'(x) &= -\frac{r + a - sc'(x) + c(x) - xc'(x)}{(s + x)^2} < 0. \end{aligned}$$

This implies that, for all normalizations $\alpha \geq 0$, nominal profit

$$(3.23) \quad \Pi_\alpha(x) = \frac{\pi(x)}{\alpha + D(x)}$$

has no maximum. Summarizing, one has:

1. if $s = 0$ and labor is taken as numeraire, all feasible allocations are monopolistic equilibria with the same profit level; for $\alpha \geq 0$, no monopolistic equilibrium exists;
2. for $s > 0$ with $r + a - sc'(0) > 0$ and for all other normalizations $\alpha \geq 0$, no monopolistic equilibrium exists since profit is increasing for all $x \in X$;
3. if $s > 0$ but $r + a - sc'(x) \leq 0$ holds for all $x \in X$, concavity of $c(x)$ implies $c(x) \geq xc'(x) \geq x(r + a)/s$ for all x ; therefore,

$$(3.24) \quad V(x) = \frac{x + s}{r + a + c(x)} \leq \frac{x + s}{r + a + x(r + a)/s} = \frac{s}{r + a} = V(0)$$

holds for all $x \in X$, i.e. the allocation $(x, \ell) = (0, a)$ is Pareto optimal; it is the unique monopolistic quasi-equilibrium at marginal cost prices $p = c'(0) \geq D(0)$ and real profit $\pi(0) = -a$; it is the unique competitive equilibrium if $c'(0) > D(0)$;

4. if the set of feasible allocations is compact the Pareto optimal allocation is the unique monopolistic equilibrium with its characteristics depending on the boundary of the set of feasible allocations, but not on the parameter $\alpha \geq 0$ of the normalization.

Therefore, under the chosen class of preferences, interior regular monopolistic equilibria fail to exist when fixed costs or increasing variable returns occur. This suggests that any sensitivity of such equilibria (if they exist) on normalization is due more to features determining the boundaries of feasible allocations rather than to effects of scaling between rules of normalization (contrary to common folklore as reported by Dehez, Drèze & Suzuki, 2003).

4 Cournot-Walras Oligopolies

4.1 Cournot-Walras Duopolies

Consider the economy of the leading example from Section 2.2 (the Robinson-Crusoe economy) now with two firms producing the same commodity and using the same type of labor of the consumer. If both firms employ the same technology one obtains an objective inverse demand function $D : X \rightarrow \mathbb{R}_+$ defined on the set of feasible allocations $X \subset \mathbb{R}_+^2$ given by

$$(4.1) \quad D(x_1, x_2) := -\frac{U_x(x_1 + x_2, c(x_1) + c(x_2))}{U_\ell(x_1 + x_2, c(x_1) + c(x_2))}.$$

Without loss of generality, cost and utility functions can be chosen again such that the unique competitive equilibrium $(x_1^* + x_2^*, c(x_1^*) + c(x_2^*))$ satisfies $D(x_1^*, x_2^*) = 1$.

For the normalization on the simplex $\Delta(\alpha) = \{(w, p) \in \mathbb{R}_+^2 \mid p + \alpha w = 1 + \alpha, \alpha \geq 0\}$ defined in (2.8), the profit functions of the duopolists are

$$(4.2) \quad \begin{aligned} \Pi_1(\alpha, x_1, x_2) &:= (1 + \alpha) \frac{x_1 D(x_1, x_2) - c(x_1)}{\alpha + D(x_1, x_2)} = \frac{(1 + \alpha)\pi_1(x)}{\alpha + D(x_1, x_2)} \\ \Pi_2(\alpha, x_1, x_2) &:= (1 + \alpha) \frac{x_2 D(x_1, x_2) - c(x_2)}{\alpha + D(x_1, x_2)} = \frac{(1 + \alpha)\pi_2(x)}{\alpha + D(x_1, x_2)}. \end{aligned}$$

This implies the best-response map $\xi : \mathbb{R}_+ \times X \rightarrow X$, $(\alpha, x_1, x_2) \mapsto (\xi_1(\alpha, x_2), \xi_2(\alpha, x_1))$ given by the pair of best response functions of the duopolists

$$(4.3) \quad \begin{aligned} \xi_1(\alpha, x_2) &:= \arg \max_{x'_1} (1 + \alpha) \frac{x'_1 D(x'_1, x_2) - c(x'_1)}{\alpha + D(x'_1, x_2)} \\ \xi_2(\alpha, x_1) &:= \arg \max_{x'_2} (1 + \alpha) \frac{x'_2 D(x_1, x'_2) - c(x'_2)}{\alpha + D(x_1, x'_2)}. \end{aligned}$$

A zero $\tilde{x}(\alpha) = (\tilde{x}_1(\alpha), \tilde{x}_2(\alpha)) \in X$ of the fixed point map $\xi(\alpha, x) - x$

$$(4.4) \quad \begin{aligned} \xi_1(\alpha, \tilde{x}_2) - \tilde{x}_1 &= 0 \\ \xi_2(\alpha, \tilde{x}_1) - \tilde{x}_2 &= 0 \end{aligned}$$

is called a *Cournot-Walras equilibrium*. It will be called a *proper Cournot-Walras equilibrium* if $\tilde{x}(\alpha) \neq x^*$, i.e. if

$$(4.5) \quad c'(\tilde{x}_1(\alpha)) \neq D(\tilde{x}_1(\alpha), \tilde{x}_2(\alpha)) \neq c'(\tilde{x}_2(\alpha)).$$

It is called *regular* at $\alpha \geq 0$ if the fixed point map (4.4) is locally invertible, i.e. when

$$(4.6) \quad \det \begin{pmatrix} -1 & \frac{\partial \xi_1}{\partial x_2} \\ \frac{\partial \xi_2}{\partial x_1} & -1 \end{pmatrix} = 1 - \frac{\partial \xi_1}{\partial x_2}(\alpha, \tilde{x}(\alpha)) \cdot \frac{\partial \xi_2}{\partial x_1}(\alpha, \tilde{x}(\alpha)) \neq 0.$$

Proposition 4.1. *Let $(\tilde{x}_1(\alpha), \tilde{x}_2(\alpha)) \in X$ be an interior, proper, and regular Cournot-Walras equilibrium with positive profits for each duopolist. Then, the equilibrium map $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) : \mathbb{R}_+ \rightarrow \mathbb{R}_+^2$, $\tilde{x} : \alpha \mapsto (\tilde{x}_1(\alpha), \tilde{x}_2(\alpha))$ is locally non-constant for all $\alpha \geq 0$, i.e.*

$$(4.7) \quad \frac{d(\tilde{x}_1, \tilde{x}_2)}{d\alpha}(\alpha) = \begin{pmatrix} \frac{d\tilde{x}_1}{d\alpha}(\alpha) \\ \frac{d\tilde{x}_2}{d\alpha}(\alpha) \end{pmatrix} = \begin{pmatrix} -\frac{\frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial \alpha} + \frac{\partial \xi_1}{\partial \alpha}}{1 - \frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial x_1}} \\ -\frac{\frac{\partial \xi_2}{\partial x_1} \frac{\partial \xi_1}{\partial \alpha} + \frac{\partial \xi_2}{\partial \alpha}}{1 - \frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial x_1}} \end{pmatrix} \neq 0.$$

Proof. Let $(\tilde{x}_1, \tilde{x}_2) \gg 0$ denote an interior equilibrium. Proposition 2.1 implies that the first order conditions for (4.3) using (2.13) satisfy

$$(4.8) \quad \pi_1 \frac{\partial D / \partial x_1}{\partial \pi_1 / \partial x_1} = (\alpha + D) = \pi_2 \frac{\partial D / \partial x_2}{\partial \pi_2 / \partial x_2} > 0.$$

Since $\alpha + D(\tilde{x}_1, \tilde{x}_2) > 0$ and profits are strictly positive, the ratio of the partial price effect and of the partial profit effect of both duopolists are positive and they have the same sign. Then, applying the reasoning from (2.13) to (4.8) implies that $(\tilde{x}_1, \tilde{x}_2) \gg 0$ is a proper equilibrium. From (2.13) one has $\partial \xi_1 / \partial \alpha$ and $\partial \xi_2 / \partial \alpha$ are both non-zero. Together with regularity the best response map (4.3) satisfies

$$(4.9) \quad \frac{\partial \xi_1}{\partial \alpha} \neq 0, \quad \frac{\partial \xi_2}{\partial \alpha} \neq 0, \quad 1 - \frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial x_1} \neq 0.$$

Applying the Implicit Function Theorem to (4.4) one finds

$$(4.10) \quad \frac{d(\tilde{x}_1, \tilde{x}_2)}{d\alpha}(\alpha) = \begin{pmatrix} \frac{d\tilde{x}_1}{d\alpha}(\alpha) \\ \frac{d\tilde{x}_2}{d\alpha}(\alpha) \end{pmatrix} = \begin{pmatrix} -1 & \frac{\partial \xi_1}{\partial x_2} \\ \frac{\partial \xi_2}{\partial x_1} & -1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial \xi_1}{\partial \alpha} \\ \frac{\partial \xi_2}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} \frac{-\frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial \alpha} + \frac{\partial \xi_1}{\partial \alpha}}{1 - \frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial x_1}} \\ \frac{-\frac{\partial \xi_2}{\partial x_1} \frac{\partial \xi_1}{\partial \alpha} + \frac{\partial \xi_2}{\partial \alpha}}{1 - \frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial x_1}} \end{pmatrix}.$$

Suppose $\frac{d(\tilde{x}_1, \tilde{x}_2)}{d\alpha}(\alpha) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e.

$$(4.11) \quad -\frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial \alpha} + \frac{\partial \xi_1}{\partial \alpha} = 0 \quad \text{and} \quad -\frac{\partial \xi_2}{\partial x_1} \frac{\partial \xi_1}{\partial \alpha} + \frac{\partial \xi_2}{\partial \alpha} = 0$$

hold. Then,

$$0 = -\frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial \alpha} + \frac{\partial \xi_1}{\partial \alpha} = -\frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial x_1} \frac{\partial \xi_1}{\partial \alpha} + \frac{\partial \xi_1}{\partial \alpha} = \frac{\partial \xi_1}{\partial \alpha} \left(1 - \frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial x_1} \right) \neq 0$$

(4.12) and

$$0 = -\frac{\partial \xi_2}{\partial x_1} \frac{\partial \xi_1}{\partial \alpha} + \frac{\partial \xi_2}{\partial \alpha} = -\frac{\partial \xi_2}{\partial x_1} \frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial \alpha} + \frac{\partial \xi_2}{\partial \alpha} = \frac{\partial \xi_2}{\partial \alpha} \left(1 - \frac{\partial \xi_1}{\partial x_2} \frac{\partial \xi_2}{\partial x_1} \right) \neq 0,$$

which contradicts regularity. Therefore, proper and regular monopolistic equilibria satisfy $\frac{d(\tilde{x}_1, \tilde{x}_2)}{d\alpha}(\alpha) \neq 0$ for every $\alpha \geq 0$. \square

Note that the force of the statement of Proposition 4.1 that dependence occurs at *all levels of* $\alpha \geq 0$ including the two boundary cases (labor or output as numeraire) requires that allocations as well as profits must be positive for both duopolists. Therefore, boundary equilibria and those with zero profit may be immune to changes of the normalization. They are not covered by the proposition. The methodology and the effects shown, however, have numerous implications and applications for many allocative issues.

1. At asymmetric equilibria of homogeneous duopolies aggregate production is inefficient even when firms use the same technology. Since duopolists produce at different marginal costs equal to individual marginal revenue at the same price production efficiency also varies with α (see Böhm & Eichberger, 2006).
2. Consumer welfare, i.e. indirect utility changes with the parameter α confirming an early conjecture by Ginsburgh (1994), raising the question of a second best solution to the normalization issue under imperfect competition.
3. Many oligopoly models use the so-called best-response dynamics to analyze dynamic situations of strategic interaction. For duopoly models it is known that such models may exhibit cycles and complex dynamic behavior (see Rand, 1978; Bischi, Mammana & Gardini, 2000, and others). None of these investigate the role of the normalization rule used for stability or cycles. It is evident from Bischi et. al. that in a parametrized version the parameter α would play a major role for the stability of fixed points, cycles, and their basins of attraction. For example, the eigenvalues of a cycle $((x_{1,1}, x_{2,1}), \dots, (x_{1,n}, x_{2,n}))$

are determined by products of the form

$$\prod_{j=1}^n \frac{d(\xi_1(\alpha, \xi_2(\alpha, x_{1,j})))}{dx_{1,j}} = \prod_{j=1}^n \frac{\partial \xi_1}{\partial x_2}(\alpha, x_{2j}) \cdot \frac{\partial \xi_2}{\partial x_1}(\alpha, x_{1j})$$

whose signs and size depend generically on the normalization parameter.

4.2 General Cournot-Walras Oligopolies

It seems apparent from the two propositions and from the analytical approach that the dependence of regular equilibria on the price-wage normalization for homogeneous duopolies with one input factor extends under suitable smoothness conditions to economies of general oligopolies with an arbitrary finite number of firms with heterogeneous outputs and inputs.

In order to reveal the cause for the dependence while keeping the readability under the generalizations as transparent as possible the following analysis is split into three cases: Corollaries 4.1 and 4.2 analyzing the extension for the commodity market separately from the factor markets to exhibit optimally the causes for the dependence through the analytical form. The result for the general case with simultaneous heterogeneous inputs and outputs is given in Corollary 4.3 which implies further extensions of the dependence property also to economies with heterogeneous consumers.

Homogeneous oligopolies with differential technologies

Consider an economy

- with one consumer and utility function $U : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$, $(x', \ell) \mapsto U(x', \ell)$ (C^2 and strictly quasi-concave, strictly increasing in x' , strictly decreasing in ℓ , $U_x(x', \ell) > 0$, $U_\ell(x', \ell) < 0$, as in section 2.1)
- with finitely many oligopolists $i = 1, \dots, n$ and real cost functions $\ell_i = c_i(x_i)$
- with objective inverse demand function of the consumer $D : X \rightarrow \mathbb{R}_+$, defined on the set of feasible allocations $x = (x_1, \dots, x_n) \in X \subset \mathbb{R}_+^n$, $(x_1, \dots, x_n) \mapsto D(x_1, \dots, x_n)$ given by

$$(4.13) \quad D(x) := - \frac{U_x(\sum x_i, \sum c_i(x_i))}{U_\ell(\sum x_i, \sum c_i(x_i))}$$

and real profit functions $\pi_i(x) := x_i D(x) - c_i(x_i)$, $i = 1, \dots, n$,

- with price-wage normalization $\Delta(\alpha) = \{(p, w) \in \mathbb{R}_+^2 \mid p + \alpha w = 1 + \alpha, \alpha \geq 0\}$ implying normalized profits as

$$(4.14) \quad \Pi_i(\alpha, x) := (1 + \alpha) \frac{\pi_i(x)}{\alpha + D(x)}, \quad i = 1, \dots, n.$$

- Without loss of generality, let the unique competitive equilibrium x^* satisfy $D(x^*) = 1$

Corollary 4.1. *If $(\tilde{x}_1(\alpha), \dots, \tilde{x}_n(\alpha)) \gg 0$ is an interior, proper, and regular Cournot-Walras equilibrium, then*

$$(4.15) \quad \tilde{x}'(\alpha) := \frac{d(\tilde{x}_1, \dots, \tilde{x}_n)}{d\alpha}(\alpha) \neq 0, \quad \text{for all } \alpha \geq 0.$$

Proof. For every $i = 1, \dots, n$ and every $x = (x_1, \dots, x_n) \in X \subset \mathbb{R}_+^n$, denote the vector x without the i -th component as $x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and write $x \equiv (x_i, x_{-i}) = (x_i, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Define the list of best response functions

$$(4.16) \quad \xi_i(\alpha, x_{-i}) := \arg \max_{x_i} \Pi(\alpha, x_i, x_{-i}), \quad i = 1, \dots, n$$

and the associated best response map $\xi = (\xi_1, \dots, \xi_n) : \mathbb{R}_+ \times X \rightarrow \mathbb{R}_+^n$, $(\alpha, x) \mapsto \xi(\alpha, x) \in \mathbb{R}_+^n$, where $\xi_i(\alpha, x) \equiv \xi_i(\alpha, x_{-i})$, for each $i = 1, \dots, n$. A zero $\tilde{x}(\alpha) = (\tilde{x}_1(\alpha), \dots, \tilde{x}_n(\alpha))$ of $\xi(\alpha, x) - x$ is a Cournot-Walras equilibrium. If $\tilde{x}(\alpha)$ is regular then

$$(4.17) \quad \begin{pmatrix} \frac{d\tilde{x}_1}{d\alpha}(\alpha) \\ \vdots \\ \frac{d\tilde{x}_n}{d\alpha}(\alpha) \end{pmatrix} = \tilde{x}'(\alpha) = \begin{pmatrix} \tilde{x}'_1(\alpha) \\ \vdots \\ \tilde{x}'_n(\alpha) \end{pmatrix} = \left((Id)_n - \left(\frac{\partial \xi}{\partial x} \right) \right)^{-1} \left(\frac{\partial \xi}{\partial \alpha} \right)$$

Since \tilde{x} is proper and not a competitive equilibrium for each oligopolist it follows from Proposition 2.1 that $\partial \xi_i / \partial \alpha \neq 0$, for all $i = 1, \dots, n$. Therefore, $\tilde{x}'(\alpha) \neq 0$ \square

Homogeneous oligopolies with heterogeneous factor inputs

Consider next an economy

- with finitely many oligopolists $i = 1, \dots, n$, with real cost functions $\ell_i = c_i(x_i)$ using $n \geq 1$ different types of labor $\ell = (\ell_1, \dots, \ell_n)$ as inputs to produce the same homogeneous output,
- with one consumer and utility function $U : \mathbb{R}_+^{1+n} \rightarrow \mathbb{R}$, $(x', \ell) \mapsto U(x', \ell)$ (C^2 and strictly quasi-concave, strictly increasing in x' , strictly decreasing in ℓ , $U_x(x', \ell) > 0$, $U_{\ell_i}(x', \ell) < 0$, $i = 1, \dots, n$),
- with objective inverse demand function of the consumer $W : X \rightarrow \mathbb{R}_+$, defined on the set of feasible allocations $x = (x_1, \dots, x_n) \in X \subset \mathbb{R}_+^n$, $(x_1, \dots, x_n) \mapsto W(x_1, \dots, x_n)$ given by the list of real wages supporting $x \in X$ as an optimal consumption-labor plan under income consistency, i.e.

$$(4.18) \quad \begin{aligned} \left(\frac{1}{p} \right) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} &\stackrel{!}{=} W(x) \equiv \begin{pmatrix} W_1(x) \\ \vdots \\ W_n(x) \end{pmatrix} \\ &:= - \frac{1}{U_x(\sum x_i, c_1(x_1), \dots, c_n(x_n))} \begin{pmatrix} U_{\ell_1}(\sum x_i, c_1(x_1), \dots, c_n(x_n)) \\ \vdots \\ U_{\ell_n}(\sum x_i, c_1(x_1), \dots, c_n(x_n)) \end{pmatrix}, \end{aligned}$$

- and real profit functions with output as numeraire $\pi_i : X \rightarrow \mathbb{R}$ for each oligopolist, $i = 1, \dots, n$ given by

$$(4.19) \quad \pi_i(x) := x_i - W_i(x)c_i(x_i), \quad i = 1, \dots, n;$$

- let $\alpha = (\alpha_1, \dots, \alpha_n) \geq 0$ denote a list of normalization weights and define the linear normalization map $F_\alpha : \mathbb{R}_+^{1+n} \rightarrow \mathbb{R}_+$, $(p, w) \mapsto F_\alpha(p, w) = p + \sum \alpha_i w_i$ with price-wage set

$$(4.20) \quad \Delta(\alpha) := \left\{ (p, w) \in \mathbb{R}_+^{1+n} \mid p = 1 + \sum_i \alpha_i (1 - w_i), \alpha \geq 0 \right\}$$

implying a normalized commodity price p and wages $w = (w_1, \dots, w_n)$

$$(4.21) \quad p = \frac{1 + \alpha \cdot \mathbf{1}}{1 + \alpha \cdot W(x)}, \quad w = \frac{1 + \alpha \cdot \mathbf{1}}{1 + \alpha \cdot W(x)} W(x), \quad \mathbf{1} := \underbrace{(1, \dots, 1)}_{n\text{-times}}$$

and normalized profits as

$$(4.22) \quad \Pi_i(\alpha, x) := \frac{1 + \alpha \cdot \mathbf{1}}{1 + \alpha \cdot W(x)} (x_i - W_i(x)c_i(x_i)), \quad i = 1, \dots, n.$$

Corollary 4.2. *If $(\tilde{x}_1, \dots, \tilde{x}_n) \gg 0$ is an interior, proper, and regular Cournot-Walras equilibrium, then*

$$(4.23) \quad \left(\frac{\partial(\tilde{x}_1, \dots, \tilde{x}_n)}{\partial(\alpha_1, \dots, \alpha_n)}(\alpha) \right) := \left(\frac{\partial \tilde{x}_i}{\partial \alpha_j}(\alpha) \right)_{i,j} \neq 0, \quad \text{for all } \alpha \geq 0, \quad \text{for every } i = 1, \dots, n.$$

Proof. For every $i = 1, \dots, n$ and every $x = (x_1, \dots, x_n) \in X \subset \mathbb{R}_+^n$, denote the vector x without the i -th component as $x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and write $x \equiv (x_i, x_{-i}) = (x_i, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Define the list of best response functions

$$(4.24) \quad \xi_i(\alpha, x_{-i}) := \arg \max_{x_i} \Pi_i(\alpha, x_i, x_{-i}), \quad i = 1, \dots, n$$

and the associated best response map $\xi = (\xi_1, \dots, \xi_n) : \mathbb{R}_+^n \times X \rightarrow \mathbb{R}_+^n$, $(\alpha, x) \mapsto \xi(\alpha, x) \in \mathbb{R}_+^n$, where $\xi_i(\alpha, x) \equiv \xi_i(\alpha, x_{-i})$, for each $i = 1, \dots, n$. A zero $\tilde{x}(\alpha) = (\tilde{x}_1(\alpha), \dots, \tilde{x}_n(\alpha))$ of $\xi(\alpha, x) - x$ is a Cournot-Walras equilibrium. If $\tilde{x}(\alpha)$ is regular the Implicit Function Theorem implies

$$(4.25) \quad \left(\frac{\partial(\tilde{x}_1, \dots, \tilde{x}_n)}{\partial(\alpha_1, \dots, \alpha_n)}(\alpha) \right) := \left(\frac{\partial \tilde{x}_i}{\partial \alpha_j}(\alpha) \right)_{i,j} = \left((Id)_n - \left(\frac{\partial \xi}{\partial x} \right) \right)^{-1} \left(\frac{\partial \xi}{\partial \alpha} \right).$$

Since $\tilde{x}(\alpha)$ is proper and not a competitive equilibrium for each oligopolist it follows from Proposition 2.1 that, for all j , $\partial \xi_i / \partial \alpha_j \neq 0$ for all $i = 1, \dots, n$. Therefore,

$$(4.26) \quad \left(\frac{\partial \tilde{x}}{\partial \alpha_j}(\alpha) \right) \neq 0, \quad \text{for all } j = 1, \dots, n$$

□

A typical example in this class of economies and of the one following are models of international trade with multi-country or multi-regional structures without factor mobility. Cornwall (1977, 1984) seems to have been one of the first to indicate that equilibria with monopolistic commodity markets between countries are sensitive to the choice of the numeraire for the factor price. Apart from the awareness of the dependence in GE pure trade models (as in Srinivasan & Kletzer, 1999), few studies examine the effects of different normalizations to show, for example, implications of Nash trade equilibria or tariff games between countries for allocations, welfare, or other characteristics (see one recent exception Chang, Chen & Saito, 2021, and references therein).

A second class of economies with dependence of allocations on the choice of normalization is the one proposed by Blanchard & Kiyotaki (1987) describing monopolistic competition on labor markets as a possible source for unemployment and demand effects in the widely used so-called New Keynesian macroeconomic models. Their choice of the CES-type price index function for aggregate output represents a particular normalization rule, for which Corollary 4.2 and 4.3 apply.

Heterogeneous oligopolies with heterogeneous inputs

Consider an economy

- with finitely many oligopolists $i = 1, \dots, n$, producing n different commodities with real cost functions $\ell_i = c_i(x_i)$ using n different types of labor $\ell = (\ell_1, \dots, \ell_n)$ as inputs¹;
- with one consumer and utility function $U : \mathbb{R}_+^{2n} \rightarrow \mathbb{R}$, $(x', \ell') \mapsto U(x', \ell')$ (C^2 and strictly quasi-concave, strictly increasing in x' , strictly decreasing in ℓ' , $U_x(x', \ell') > 0$, $U_\ell(x', \ell') < 0$, $i = 1, \dots, n$);
- with objective inverse demand function of the consumer defined by two mappings $D : X \rightarrow \mathbb{R}_+^n$, $W : X \rightarrow \mathbb{R}_+^n$, on the set of feasible allocations $x = (x_1, \dots, x_n) \in X \subset \mathbb{R}_+^n$, $(x_1, \dots, x_n) \mapsto (D(x_1, \dots, x_n), W(x_1, \dots, x_n))$ given by the list of supporting prices and wages for $(x, c(x)) \in X \times \mathbb{R}_+^n$ as an optimal consumption-labor plan under income consistency, i.e. for any $\lambda > 0$

$$(4.27) \quad \begin{pmatrix} p_1 \\ \vdots \\ p_n \\ w_1 \\ \vdots \\ w_n \end{pmatrix} = \lambda \begin{pmatrix} D(x) \\ W(x) \end{pmatrix} \equiv \lambda \begin{pmatrix} D_1(x) \\ \vdots \\ D_n(x) \\ W_1(x) \\ \vdots \\ W_n(x) \end{pmatrix} := \lambda \begin{pmatrix} U_{x_1}(x, c_1(x_1), \dots, c_n(x_n)) \\ \vdots \\ U_{x_n}(x, c_1(x_1), \dots, c_n(x_n)) \\ U_{\ell_1}(x, c_1(x_1), \dots, c_n(x_n)) \\ \vdots \\ U_{\ell_n}(x, c_1(x_1), \dots, c_n(x_n)) \end{pmatrix};$$

- and scaled real profit $\pi_i : X \rightarrow \mathbb{R}$, $\lambda > 0$, for each oligopolist, $i = 1, \dots, n$, is given by

$$(4.28) \quad \lambda \pi_i(x) := \lambda(D_i(x)x_i - W_i(x)c_i(x_i)), \quad i = 1, \dots, n;$$

- let $a, b \in \mathbb{R}_{++}^n$ denote two lists of strictly positive weights for prices and wages and define the linear normalization map $F_\alpha : \mathbb{R}^{2n} \rightarrow \mathbb{R}_+$, $(p, w) \mapsto F_\alpha(p, w) = a \cdot p + \alpha(b \cdot w)$ with price-wage set

$$(4.29) \quad \Delta(\alpha) := \left\{ (p, w) \in \mathbb{R}_+^{2n} \mid a \cdot p + \alpha(b \cdot w) = 1 + \alpha, \alpha \geq 0 \right\};$$

for any feasible allocation $x \in X$, normalized price-wage pairs $(p, w) \in \Delta(\alpha)$ satisfy $a \cdot p = \lambda(a \cdot D(x))$ and $\alpha(b \cdot w) = \lambda\alpha(b \cdot W(x))$ so that the price-wage scale under F_α becomes

$$(4.30) \quad \lambda(\alpha, x) = \frac{1 + \alpha}{a \cdot D(x) + \alpha(b \cdot W(x))}.$$

This yields nominal profits for oligopolists as

$$(4.31) \quad \Pi_i(\alpha, x) := \frac{1 + \alpha}{a \cdot D(x) + \alpha(b \cdot W(x))} (D_i(x)x_i - W_i(x)c_i(x_i)), \quad i = 1, \dots, n.$$

The equation reveals the generalized multiplicative form of nominal profit being equal to the scale $\lambda(\alpha, x)$ times 'real profit' $\pi_i(x)$, a function of the allocation alone (analogous to (2.10) and (4.2)). Since the profit functions (4.31) depend in the same structural way on the scalar parameter α any change induces changes of the maximizers and of equilibria generically.

¹The mildly restrictive assumption of one input per output is used for ease of presentation and transparency and is made 'without loss of generality', i.e. without affecting the general validity of the statement on the influence of the normalization factor α on regular oligopolistic equilibria. It can be generalized further to cases with multi-output or multi-input firms.

Corollary 4.3. *If $(\tilde{x}_1(\alpha), \dots, \tilde{x}_n(\alpha)) \gg 0$ is an interior, proper, and regular Cournot-Walras equilibrium, then*

$$(4.32) \quad \tilde{x}'(\alpha) := \frac{d(\tilde{x}_1, \dots, \tilde{x}_n)}{d\alpha}(\alpha) \neq 0, \quad \text{for all } \alpha \geq 0.$$

This result is completely parallel to the one given in Corollary 4.1 with an identical proof for the scalar parameter α . Notice that the normalization function F_α embodies two additional explicit potential variations with respect to the parameters $((a_i), (b_i))_{i=1}^n$. Their impact on equilibria can be derived in the same fashion using the Implicit Function Theorem.

In the above analysis price normalization was shown to impact on equilibrium allocations with profit maximizing firms under monopolistic competition modeled as non-cooperative Nash equilibria. The methodology could be used as well for a different and wider strand of issues arising from the role of normalization when other non-competitive equilibria under specific choices of numeraire are compared for the same real economy, for example, when

- Cournot-Walras equilibria are compared with those under Bertrand competition (as in Böhm, 1990; Gaube, 1996),
- economies representable as market games are evaluated (as in Shapley & Shubik, 1977, and others),
- the role of large numbers of firms relative to the competitive outcome are studied (as in Codognato & Gabszewicz, 1993; FitzRoy, 1974; Roberts, 1980),
- optimal tariffs or trade blocs in models of international trade are introduced (as in Chang, Chen & Saito, 2021, and the related literature),
- equilibria of economies with incomplete markets are examined (as by Dierker, Dierker & Grodal, 2002; Zierhut, 2020).

Little is known as to whether and to what extent the choice of a numeraire matters

- for equilibria under second best policies within market economies (as for example in Boiteux, 1956; Gaube, 1997, for Ramsey-Boiteux pricing) or for general pricing rules (as in Dierker, Guesnerie & Neufeind, 1985; Böhm, 1986),
- for economies with externalities, public goods, or for models of mechanism design.

References

- ARROW, K. J. & F. H. HAHN (1971): *General Competitive Analysis*. Holden Day, San Francisco, Cali.
- BISCHI, G., C. MAMMANA & L. GARDINI (2000): “Multistability and cyclic attractors in duopoly games”, *Chaos, Solitons and Fractals*, 11(4), 543–564.
- BLANCHARD, O. J. & N. KIYOTAKI (1987): “Monopolistic Competition and the Effects of Aggregate Demand”, *American Economic Review*, 77, 647–666.
- BÖHM, V. (1986): “Existence of Equilibria with Price Regulation”, in *Contributions to Mathematical Economics – In Honor of Gérard Debreu*, ed. by W. Hildenbrand & A. Mas-Colell. North-Holland Publishing Company, Amsterdam a.o.
- (1990): “Bertrand, Cournot, Edgeworth, Walras - A Simple General Equilibrium Analysis”, *Beiträge zur Angewandten Wirtschaftsforschung* 437-90, University of Mannheim, Mannheim.
- (1994): “The Foundation of the Theory of Monopolistic Competition Revisited”, *Journal of Economic Theory*, 63, 208–218.

- BÖHM, V. & A. BARTEN (1982): “Consumer Theory”, in *Handbook of Mathematical Economics*, ed. by K. Arrow & M. Intrilligator, Vol. 2. North-Holland Publishing Company, Amsterdam a.o.
- BÖHM, V. & J. EICHBERGER (2006): “Production Efficiency and Imperfect Competition”, in *Institutions in Perspective*, ed. by U. Bindseil, J. Haucap & C. Wey, pp. 155–173. Mohr Siebeck, Tübingen.
- BOITEUX, M. (1956): “Sur la Gestion des Monopoles Publics astreints à l’Equilibre Budgetaire ; translated as: “On the Management of Public Monopolies Subject to Budgetary Constraints,” *Journal of Economic Theory*, 3(1971), 219–240.”, *Econometrica*, 24, 22–40.
- CHANG, W. W., T.-L. CHEN & T. SAITO (2021): “Formation of Symmetric Free-Trade Blocs, Optimal Tariff Structure, and World Welfare”, *Journal of Asian Economics*, 77, 10173.
- CODOGNATO, G. & J. J. GABSZEWICZ (1993): “Cournot-Walras Equilibria in Markets with a Continuum of Traders”, *Economic Theory*, 3(3), 453–464.
- CORNWALL, R. R. (1977): “The Concept of General Equilibrium in a Market Economy with Imperfectly Competitive Producers”, *Metroeconomica*, 29, 55–72.
- (1984): *Introduction to the Use of General Equilibrium Analysis*. North-Holland Publishing Company, Amsterdam a.o.
- DEBREU, G. (1960): “Topological methods in cardinal utility theory”, in *Mathematical methods in the social sciences*, ed. by K. J. Arrow, S. Karlin & P. Suppes, pp. 12–26. Stanford University Press.
- DEHEZ, P., J. H. DRÈZE & T. SUZUKI (2003): “Imperfect competition à la Negishi, also with fixed costs”, *Journal of Mathematical Economics*, 39, 219–237.
- DIERKER, E., H. DIERKER & B. GRODAL (2002): “Nonexistence of Constrained Efficient Equilibria When Markets Are Incomplete”, *Econometrica*, 70(3), 1245–1251.
- DIERKER, E. & B. GRODAL (1999): “The price normalization problem in imperfect competition and the objective of the firm”, *Economic Theory*, 14(2), 257–284.
- DIERKER, E., R. GUESNERIE & W. NEUEFEIND (1985): “General Equilibrium when Some Firms Follow Special Pricing Rules”, *Econometrica*, 53(6), 1369–1393.
- DIERKER, H. & B. GRODAL (1986): “Non-Existence of Cournot-Walras Equilibrium in a General Equilibrium Model with Two Oligopolists”, in *Contributions to Mathematical Economics. In honour of Gérard Debreu*, ed. by W. Hildenbrand & A. Mas-Colell, Chap. 10. North-Holland Publishing Company, Amsterdam a.o.
- FITZROY, F. R. (1974): “Monopolistic Equilibrium, Non-Convexity and Inverse Demand”, *Journal of Economic Theory*, 7(1), 1–16.
- GABSZEWICZ, J. J. & J.-P. VIAL (1972): “Oligopoly " à la Cournot " in general equilibrium analysis”, *Journal of Economic Theory*, 4, 381–400.
- GAUBE, T. (1996): “Price and Quantity Competition in a Duopoly Model with Product Differentiation”, Mimeo, University of Mannheim, Department of Economics.

- (1997): “Numéraire und Second-Best-Gleichgewicht: Eine Untersuchung allokativer Preisnormierungseffekte in Ramsey-Boiteux-Modellen”, Inauguraldissertation, Universität Mannheim, Fakultät für Volkswirtschaftslehre.
- GINSBURGH, V. (1994): “In the Cournot-Walras model, there may be more to gain by changing price normalization, than by eliminating imperfections. A two-goods example”, in *Applied General Equilibrium and Economic Development*, ed. by J. Mercenier & T. N. Srinivasan, pp. 217–224. Wisconsin University Press.
- HOUTHAKKER, H. S. (1960): “Additive Preferences”, *Econometrica*, 33, 797–801.
- LEONTIEF, W. (1947): “Introduction to a theory of the internal structure of function relationship”, *Econometrica*, 15, 361–373.
- MARSCHAK, T. & R. SELTEN (1974): *General Equilibrium with Price-Making Firms*. Springer-Verlag, Heidelberg a.o.
- RAND, D. (1978): “Exotic Phenomena in Games and Duopoly Models”, *Journal of Mathematical Economics*, 5, 173–184.
- ROBERTS, D. J. & H. SONNENSCHNEIN (1976): “On the Existence of Cournot Equilibrium without Concave Profit Functions”, *Journal of Economic Theory*, 8, 112–117.
- (1977): “On the Foundation of the Theory of Monopolistic Competition”, *Econometrica*, 45(1), 101–114.
- ROBERTS, K. W. (1980): “The Limit Points of Monopolistic Competition”, *Journal of Economic Theory*, 22(2), 256–278.
- SAMUELSON, P. A. (1947): *Foundations of economic analysis*. Harvard University Press, Cambridge, Mass.
- SHAPLEY, L. S. & M. SHUBIK (1977): “Trade Using One Commodity as a Means of Payment”, *Journal of Political Economy*, 85(5), 937–968.
- SRINIVASAN, T. N. & K. KLETZER (1999): “The importance of price normalization for models of international trade under imperfect competition”, in *Trade, theory and econometrics: essays in honor of John S. Chipman*, ed. by J. R. Melvin, J. C. Moore & R. Riezman, pp. 65–75. Routledge.
- ZIERHUT, M. (2020): “Indeterminacy of Cournot–Walras equilibrium with incomplete markets”, *Economic Theory*, 71(1), 81–114.