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Inequality, non-linear consumption behaviour, and monetary growth imperatives

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The debate about monetary growth imperatives centres around the question of whether a zero-growth economy is compatible with positive interest rates. An aspect that is mostly disregarded in this debate is under which circumstances a zero-growth economy can be accompanied by a reduction in wealth inequality, which is a prerequisite for the latter to be politically feasible. The paper analyses this question using a stock–flow consistent macro model including multiple households with heterogeneous wealth levels and a non-linear, concave consumption function. The results show an example of the theoretical conditions under which wealth inequality is reduced in a zero-growth economy. However, the conditions for stability of the zero-growth trajectory are found to be more restrictive in the case of a non-linear consumption function compared to the linear case.

Keywords: *inequality, ecological macroeconomics, zero growth, growth imperative*

JEL codes: *D31, Q01, O44, E12, E21*

1 INTRODUCTION

Transitioning to environmental sustainability requires large-scale and rapid reduction in resource use, greenhouse gas (GHG) emissions, and other environmental impacts. Whether this transition would be possible while maintaining ongoing exponential GDP growth is very much in doubt (Jackson 2009; Parrique et al. 2019). Even though there is empirical research around the environmental Kuznets curve that has found indications for some decoupling of environmental degradation and economic activity (Shafik 1994), this decoupling does not happen fast enough to meet the current global environmental challenges (Wiedenhofer et al. 2020; Haberl et al. 2020). Thus, if positive economic growth rates were indeed incompatible with environmental sustainability, then effective environmental measures would eventually lead to a zero-growth economy. In light of the urgency of the climate crisis (Ming et al. 2021) and other already-crossed planetary boundaries (Steffen et al. 2015), it is crucial to expand the toolbox of politically feasible projects beyond green-growth policies and in particular to analyse barriers to the feasibility of a zero-growth economy.

For a zero-growth economy to be politically feasible, its social consequences need to be clarified. Under which conditions does inequality rise or decline in a zero-growth economy? Over past decades, there has been a polarizing trend between low- and high-income groups (OECD 2011; Alvaredo 2018). In line with the saying *a rising tide lifts all boats*, it is said that a growing economy would offset the drawbacks from an unequal wealth and

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income distribution. However, in a zero-growth economy, an increase in inequality means that the absolute level of income and wealth in low-income groups decreases, causing people to fall further below poverty thresholds. Therefore, rising inequality is especially threatening in a zero-growth economy compared to an economy with positive growth rates.

This paper contributes to the connection of two areas of research, on the one hand inequality in a zero-growth economy, and on the other the debate on monetary growth imperatives (for example, Binswanger 2015; Jackson/Victor 2015). While not giving a comprehensive answer, the results of this paper show an example of theoretical conditions under which inequality is declining over time in a zero-growth economy while interest rates are positive.

The debate on monetary growth imperatives deals with the question of whether a zero-growth economy would be compatible with positive interest rates. While some authors claim that there is an inherent incompatibility of interest-bearing debt with a zero-growth economy (Binswanger 2009; 2015; Farley et al. 2013), other authors have found conditions under which positive interest rates are compatible with zero economic growth (Berg et al. 2015; Jackson/Victor 2015; Cahen-Fourot/Lavoie 2016). Furthermore, an overview of stability conditions of a zero-growth economy is provided in Richters/Siemonieit (2017).

However, in most of these models, wealth and income inequality are not considered. As mentioned by Richters/Siemonieit (2017: 122) as a direction for future research, ‘effects of distribution should be studied’ in models of a zero-growth economy. Rising inequality could pose a serious challenge in a zero-growth economy with positive interest rates, as pointed out by Piketty (2014), who shows empirical evidence for rising inequality in cases where the growth rate is below the rate of return on capital.

There are very few modelling approaches that deal with inequality in a zero-growth economy. Jackson/Victor (2016) analyse this question with a stock–flow consistent macro model, and find that, depending on the parameter values, inequality can either rise or decline in the process of lowering growth rates. Jackson/Victor (2016) measure inequality by looking at the difference between income from capital and income from wage labour.

An important mechanism for modelling inequality is the tendency for wealthier households to consume a smaller fraction of their wealth and income compared to poorer households (Keynes 1936: 31; Carroll/Kimball 1996). Mathematically, this implies that the consumption function must not be linear in income and wealth, but have a concave form. Even though this is a key insight of post-Keynesian economics (Menz 2010) and empirically well substantiated (Dynan et al. 2004; Mian et al. 2020), most models formulated in the tradition of Godley/Lavoie (2007) and all models compared in Richters/Siemonieit (2017) use consumption functions that are linear in income and wealth and might therefore not be well suited to analysing impacts of inequality on macroeconomic phenomena such as the stability of non-growing economies.

This paper addresses this research gap by showing conditions in which inequality declines in a zero-growth economy while interest rates are positive and agents have a concave consumption function. Furthermore, a stability analysis is conducted to find the parameter combinations for which the zero-growth trajectory is stable or unstable, respectively. This is meant as a theoretical illustration of such conditions and is not directly applicable to real economic systems, as discussed in Section 4. However, it can be used as a basis for further research in which more realistic economic mechanisms are considered.

Inequality is introduced in the model by introducing multiple households instead of an aggregated household sector, as well as by using a non-linear, concave consumption function instead of a linear one. As a baseline model, an existing linear, aggregated macro model is used, which was adapted from the book by Godley/Lavoie (2007) and analysed by Richters/Siemonieit (2017).

The key assumptions for this model are that income from wages is distributed equally among all households, and that all households invest a proportion of their wealth in government bonds. Government bonds are the only financial assets considered in this model. All households receive the same interest rate, irrespective of their current wealth level, not including differential saving rates or differential rates of return (Ederer et al. 2021).

The paper is structured as follows. In Section 2, the baseline model is described. In Section 3, the time development of wealth inequality is analysed, using linear and concave consumption functions. Section 4 concludes.

2 THE BASELINE MODEL

In order to analyse the time development of inequality as well as the impact on stability conditions of using a concave consumption function, the following model is used as a baseline model. Richters/Siemoneit (2017) adapt this model from the textbook *Monetary Macroeconomics* by Godley/Lavoie (2007), and conduct a stability analysis of the zero-growth economy in that model. Here, a linear consumption function is used, that is, consumption out of wealth depends linearly on the wealth level. In Section 3, this model is extended by introducing a disaggregated household sector as well as a non-linear consumption function.

2.1 Model description

The baseline model consists of three sectors: one household, one government and one firm sector. No external trade or banking system is considered. The system is described by the variables household wealth, aggregated wages, disposable income from households, and consumption. As computed in Richters/Siemoneit (2017), the model can be reduced to a single equation, namely the time development of households' wealth, which depends on wealth from the previous time-step as well as the model parameters – see equation (5).

Households In each time-step $t = 1, 2, 3, \dots$, households invest a fraction $b \in (0, 1)$ of their wealth $V(t)$ in government bonds. The income of households consists of interest payments on the previous investment and wages from the firm. The disposable income of households $Y_d(t) \geq 0$ is the income after taxation. It is assumed that households consume a fraction $c_Y \in (0, 1)$ of their disposable income $Y_d(t)$ and a fraction $c_V \in (0, 1)$ of their previous wealth or savings $V(t-1)$. Since invested money is not available for consumption, we have the restriction $b + c_V \leq 1$.

Government The government collects taxes as a constant fraction θ of the household's income. Furthermore, the government pays an interest rate $r_b \in (0, 1)$ on the previously invested money $bV(t-1)$. For simplicity, the two variables b and r_b are summarized to $r = b \cdot r_b \in (0, 1)$, so, each time-step, the government pays $rV(t-1)$ to the households. Government spending is modelled by a constant flow $G > 0$ to the firm.

Firm The firm's income consists of the payments from consumption $C(t)$ and constant government payments G . All the firm's income is paid to the household as wages, so the firm does not earn profit. Wages are denoted by $Y(t)$, so we have $Y(t) = C(t) + G$.

Summarizing the information above, the time dynamics are fully described by the following equations, $t = 0, 1, 2, \dots$,

$$\text{wages} \quad Y(t) = C(t) + G, \quad (1)$$

$$\text{disposable income} \quad Y_d(t) = (1 - \theta) \left(Y(t) + rV(t-1) \right), \quad (2)$$

$$\text{consumption} \quad C(t) = c_Y Y_d(t) + c_V V(t-1), \quad (3)$$

$$\text{wealth} \quad V(t) = V(t-1) + Y_d(t) - C(t), \quad (4)$$

where $\theta, c_V, c_Y \in (0, 1)$ and $r > 0$ such that $r \leq 1 - c_V$, which is needed to ensure $r_b < 1$ and $b + c_V \leq 1$. All variables Y, Y_d, C and V as well as the parameter G have a monetary unit. This system of equations can be reduced to one equation by inserting equations (2) and (3) into the first line (1) and solving for $Y(t)$. As calculated in Richters/Simoneit (2017), this results in a linear time dynamics for the wealth $V(t)$, $t = 0, 1, 2, \dots$,

$$V(t) = \underbrace{\left(1 - \frac{\theta c_V - r(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)} \right)}_{\equiv a} V(t-1) + \underbrace{\frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)}}_{\equiv c} G, \quad (5)$$

with some non-negative initial value $V(0) = V_0$. Table 1 shows a summary of the parameters of the model as well as their admissible range. Since the model is linear, the parameter G can be set to $G = 1$.

2.2 Stability analysis

The equation (5) has the form:

$$V(t) = aV(t-1) + c, \quad (6)$$

with constants a and c as defined in (5). Due to the chosen parameter ranges in Table 1 we always have $a > 0$. Furthermore, we have $c \in (0, G)$. A fixed point $V^* \in \mathbb{R}$, that is, a point with

$$V^* = aV^* + c \quad (7)$$

exists as long as a is unequal to 1. Suppose V^* is a fixed point, that is, $V^* = \frac{c}{1-a}$. Then wealth $V(t)$ converges to the fixed point V^* if $a < 1$, and diverges if $a > 1$. In more detail, this means that if $a > 1$, the fixed point is negative and the wealth of the household $V(t)$ is growing exponentially, since we only consider non-negative starting points. For $a < 1$, the fixed point V^* is positive. Starting below, $V_0 < V^*$, $V(t)$ is growing and converges to V^*

Table 1 Overview of model parameters and their admissible range

	Range	Meaning
c_V	$(0, 1)$	Consumption out of wealth
c_Y	$(0, 1)$	Consumption out of income
θ	$(0, 1)$	Tax rate on income
r	$(0, 1)$	Interest rate
G	$(0, \infty)$	Government spending (excl. interest payments)

from below. For a starting point above V^* , that is, $V_0 > V^*$, the wealth level is decreasing and $V(t)$ converges to V^* from above. For $a = 1$, the wealth level is growing by the constant amount c per time-step. Since the sum of all monetary stocks in the systems stays constant over time, any growth in V is balanced by negative government equity and in the case of a decline in V , the government equity rises by the same amount. As shown in Richters/Siemonit (2017), the stability condition $a < 1$ is equivalent to

$$c_V > r(1 - c_Y) \left(\frac{1}{\theta} - 1 \right), \quad (8)$$

which means that either the consumption out of savings c_V , the consumption out of income c_Y , or the tax rate θ have to be above a certain threshold or interest rates r have to be below a certain threshold to ensure convergence to the stationary state.

Obviously, very low interest rates $r \approx 0$ would ensure stability of the fixed point in (8). However, also with higher interest rates and the accordingly adjusted higher consumption and taxation parameters, a stable stationary state is possible. From equations (1) to (4) we get that a stationary state for V also means constant wages Y and constant returns from investment over time, which is equivalent to a zero-growth economy. Therefore, if condition (8) is fulfilled, the economy converges to a non-growing state. If condition (8) is not fulfilled, the wealth level $V(t)$ grows exponentially. In terms of a growth imperative, Richters/Siemonit (2017) conclude that in case of a stable stationary state, there is no growth imperative in the system.

3 INEQUALITY AND THE CONCAVE CONSUMPTION FUNCTION

In the model described in the previous section, as well as in the other stock–flow consistent models analysed in the overview in Richters/Siemonit (2017), sufficiently high consumption and taxation parameters ensure stability in a zero-growth economy. Stability here is defined in the sense that small perturbations from the constant output economy would not get amplified. The question in this paper is whether these conditions still hold when disaggregating the household sector as well as introducing a concave instead of a linear consumption function.

A concave consumption function is seen as more realistic, since wealthier people tend to consume a smaller proportion of their wealth than less wealthy people (Keynes 1936; Carroll/Kimball 1996).

3.1 Wealth inequality

Wealth inequality is modelled by considering N households with individual – that is, different – initial wealth levels V_0^k , $k = 1, \dots, N$. Apart from the initial wealth levels, the households are identical, that is, they have the same behavioural rules and earn the same wages. The rest of the economy is modelled using the baseline model described in Section 2.

Some comments on the choice of assumptions are required here. One straightforward way to introduce inequality into the model in Section 2 would be to just allow one group of households to earn income from interest payments, and another group to only rely on wages. This is implicitly done for example by Jackson/Victor (2016) by looking at the development of the labour and capital share of income. The choice in the present model, in which all households earn income from financial markets, can be justified by the fact that many people rely on income from financial markets, for example through

pension funds, even though they do not directly trade in the financial markets. However, still in that case, real rates of return on investment differ. Thus the presented modelling choice of equal interest rates for all households needs to be seen as a model simplification. The results for this model version with multiple households and a linear consumption function show a decrease in wealth inequality for all parameter choices, and in particular in a zero-growth economy.

3.1.1 Model description

The model still consists of one government and one firm, but instead of one household sector, we consider N households. Let $V^k(t) \in [0, \infty)$ denote the wealth of household k at time $t = 0, 1, 2, \dots$, $k = 1, \dots, N$. As above, $Y(t) \geq 0$ denotes the sum of all wages paid by the firm to the households. The main difference from the model above is in the equation for disposable income. The total amount of wages is distributed equally among all households, thus each household receives the wage payment $\frac{1}{N} Y(t)$. Let $Y_d^k \geq 0$ be the individual disposable income and $C^k \geq 0$ the individual consumption for $k = 1, \dots, N$. Then the model dynamics are determined by:

$$Y(t) = \sum_{k=1}^N C^k(t) + G \quad (9)$$

$$Y_d^k(t) = (1 - \theta) \left(\frac{1}{N} Y(t) + r V^k(t-1) \right), \quad (10)$$

$$C^k(t) = c_Y Y_d^k(t) + c_V V^k(t-1), \quad (11)$$

$$V^k(t) = V^k(t-1) + Y_d^k(t) - C^k(t), \quad (12)$$

for $k = 1, \dots, N$ and with the same conditions on the parameters as before, namely $\theta, c_V, c_Y \in (0, 1)$, $r \geq 0$, such that $r \leq 1 - c_V$ and $G > 0$.

In the same way as in the baseline model, we can now insert equation (10) into (11) and the result into (9), which is a linear implicit equation for $Y(t)$. Solving for $Y(t)$ yields

$$Y(t) = \frac{\left(c_Y r (1 - \theta) + c_V \right) \sum_{k=1}^N V^k(t-1) + G}{1 - c_Y (1 - \theta)}. \quad (13)$$

Next, equations (10) and (11) are used in equation (12) to get a formula for the individual wealth levels $V^k(t)$ that depends on the state variables $V^k(t-1)$ and output $Y(t)$. Thus, with the explicit formula for $Y(t)$ in (13), we get the linear dynamics for individual wealth levels:

$$\begin{aligned} V^k(t) &= \underbrace{\left(1 - c_V + r(1 - c_Y)(1 - \theta) \right)}_{=:a} V^k(t-1) \\ &\quad + \underbrace{\frac{1}{N} \left(r c_Y (1 - \theta) + c_V \right) \frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)}}_{=:b} \sum_{j=1}^N V^j(t-1) \\ &\quad + \underbrace{\frac{1}{N} \frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)} G}_{=:c}. \end{aligned} \quad (14)$$

In short, the model has the form

$$V^k(t) = aV^k(t-1) + \frac{b}{N} \sum_{i=1}^N V^i(t-1) + \frac{c}{N}, \quad t = 0, 1, 2, \dots, \quad k = 1, 2, \dots, N, \quad (15)$$

where a, b and c are the constants specified in equation (14). Note, that a, b and c are positive. In matrix form, this can be written as

$$\mathbf{V}(t) = A\mathbf{V}(t-1) + \mathbf{C}, \quad (16)$$

where $\mathbf{V}(t) = (V^1(t), \dots, V^N(t))^T$ and $\mathbf{C} = (\frac{c}{N}, \dots, \frac{c}{N})^T$. The superscript T denotes the vector being transposed. The matrix A has the form

$$A = \begin{bmatrix} a + \frac{b}{N} & \frac{b}{N} & \frac{b}{N} & \dots & \frac{b}{N} \\ \frac{b}{N} & a + \frac{b}{N} & \frac{b}{N} & \dots & \frac{b}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b}{N} & \frac{b}{N} & \frac{b}{N} & \dots & a + \frac{b}{N} \end{bmatrix}. \quad (17)$$

3.1.2 Model analysis

For a full model analysis, please see Appendix 1. Here, a summary is provided. Due to the linearity of the model, its long-term behaviour for all parameter values and initial values can be determined by performing a linear stability analysis – see Martelli (2011) or Strogatz (2014).

The matrix (17) has the eigenvalues $a + b$ and a . The eigenspace of eigenvalue $a + b$ is spanned by the vector $\mathbf{v} = (1, 1, \dots, 1)^T$, which is used for determining the time development of wealth inequality. The two relevant parameter areas are $|a + b| > 1$ and $|a + b| < 1$, since $a + b$ is the leading eigenvalue and thus $|a + b|$, being greater or smaller than one, determines the long-term behaviour of the system (Martelli 2011; Strogatz 2014). In this model, the specified admissible parameter space in Table 1 results in $a + b > 0$, hence we can omit taking the absolute value. In both cases $a + b > 1$ and $a + b < 1$ there is a unique stationary state of the system, that is, a vector of individual wealth levels $\mathbf{V}^* = (V^{*,1}, \dots, V^{*,N})^T$ that does not change under the dynamics of equations (9)–(12). For $a + b < 1$ the individual wealth levels in the stationary state are positive and equal, namely for $k = 1, 2, \dots, N$,

$$V^{*,k} = \frac{1}{1 - a - b} \cdot \frac{c}{N}. \quad (18)$$

For $a + b > 1$, the individual wealth levels in the stationary state would be negative, that is, $V^{*,k} < 0$.

3.1.3 Results and discussion

A stability analysis is conducted, which is described in detail in Appendix 1. Since we have a linear system of equations, the stability analysis provides insight into the long-term

behaviour of household wealth for all choices of initial values and parameters. The area of the parameter space in which the fixed point is stable (attracting) is the same as in the one-dimensional model in Section 2. The fixed point is attracting, if the following holds for consumption, taxation and interest parameters:

$$c_V > r(1 - c_Y) \left(\frac{1}{\theta} - 1 \right). \quad (19)$$

Thus, for sufficiently high consumption and taxation parameters or for sufficiently low interest payments, the wealth levels do not explode but converge to a stationary, positive level. Furthermore, an attracting fixed point means that all wealth levels $V^k(t)$ converge to the wealth level of the stationary state $V^{*,k}$ over time, for all households $k = 1, \dots, N$. Thus, since in the fixed point all wealth levels are equal, over time, all household wealth levels converge to the same level defined in formula (18). In the unstable case, that is, when $a + b > 1$, or equivalently when the consumption out of wealth parameter c_V is smaller than the right-hand side of equation (19), wealth levels diverge exponentially from the level specified in (18), $V^{*,k} = \frac{1}{1-a-b} \cdot \frac{c}{N}$. However, also in the unstable case, all wealth levels approach the same trajectory, that is, inequality decreases over time. (A formal proof is given in Appendix 1.)

Some remarks on negative values for wealth levels are required. Economically, wealth levels below zero would mean that the households are in debt. However, when starting with non-negative wealth levels in time $t = 0$, it follows from the construction of the model, and the fact that the matrix A and the vector C in (16) have only positive entries, that wealth cannot become negative. Thus, even though negative fixed points theoretically exist, only non-negative fixed points can actually be approached. Furthermore, if the wealth levels of all households diverge to infinity, this means that government equity goes to minus infinity. Government equity is not part of the system variables in (22)–(25), however it can be calculated by considering the expenditures for interest payments and G , as well as the income from taxes. On the other hand, decreasing wealth levels to 0 would mean that all household wealth is transferred from the household to the government. The latter case is economically rather unrealistic, whereas the first case can be interpreted as money supply in the economy by the government – see, for example, Godley/Lavoie (2007).

3.2 The concave consumption function

Wealthier households tend to consume a smaller proportion of their wealth and income compared to poorer households. It is seen as more realistic to model households with a diminishing marginal propensity to consume out of wealth (Keynes 1936; Carroll/Kimball 1996). Therefore, the above one-dimensional model in Section 2 is modified using a concave consumption function in the wealth term.

3.2.1 Model description

In the following model, the household sector is again aggregated, that is, no differences in wealth are considered between different households. However, compared to the baseline model, here households have a diminishing marginal propensity to consume out of wealth. This is modelled by using the following consumption function, where

consumption out of wealth no longer depends linearly on the wealth in the previous period $V(t-1)$, but using a concave function F mapping the wealth level V to a value between zero and V . We have:

$$C(t) = c_Y Y_d(t) + c_V F(V(t-1)), \quad (20)$$

where the function F is twice differentiable, is strictly monotonously increasing, that is, $\frac{d}{dV}F(V) > 0$, for all $V > 0$, and concave, that is, $\frac{d^2}{dV^2}F(V) < 0$, for all $V > 0$. For specific calculations, the function

$$F(V) = ((V+1)^\beta - 1), \quad V \geq 0, \quad \beta \in (0,1) \quad (21)$$

is used, which has the above properties. In particular, using the function defined in (21) instead of the simpler function $V \mapsto V^\beta$ ensures that consumption out of wealth never exceeds the household's current wealth $V(t-1)$, which is possible for small values of V in V^β . The parameter β influences the wealth elasticity of consumption in the sense that for large values of V , wealth elasticity of consumption $\frac{\partial C}{\partial V} \frac{V}{C}$ is approximately equal to $c_V \beta$. As before, we only consider non-negative values for initial wealth levels $V(0) = V_0$. The dynamics of the model are now fully described by the following discrete time system,

$$Y(t) = C(t) + G, \quad (22)$$

$$Y_d(t) = (1-\theta)(Y(t) + rV(t-1)), \quad (23)$$

$$C(t) = c_Y Y_d(t) + c_V F(V(t-1)), \quad (24)$$

$$V(t) = V(t-1) + Y_d(t) - C(t), \quad (25)$$

where again $\theta, c_V, c_Y \in (0,1)$, and the interest rate $r > 0$ such that the condition $r \leq 1 - c_V$ is fulfilled.

Analogously to the calculations in Section 3.1, the time dynamic for the aggregated wealth level is computed by first calculating the explicit formula for $Y(t)$, which is then used together with equations (23)–(25) to arrive at the following dynamic for the aggregated wealth level:

$$\begin{aligned} V(t) = & \left(1 + r \frac{(1-c_Y)(1-\theta)}{1-c_Y(1-\theta)}\right) V(t-1) \\ & - c_V \left(1 - \frac{(1-c_Y)(1-\theta)}{1-c_Y(1-\theta)}\right) F(V(t-1)) \\ & + \frac{(1-c_Y)(1-\theta)}{1-c_Y(1-\theta)} G. \end{aligned} \quad (26)$$

With the parameter

$$\alpha := \frac{(1-c_Y)(1-\theta)}{1-c_Y(1-\theta)}, \quad (27)$$

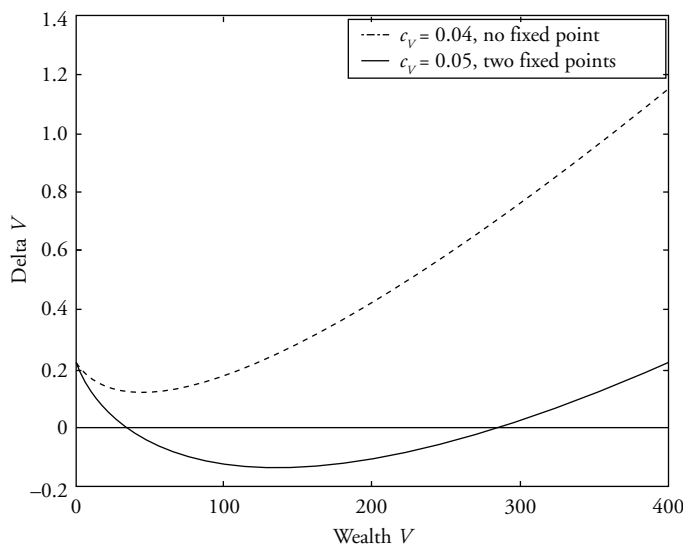
we can simplify the equation for the change in wealth $\Delta V(t) := V(t) - V(t-1)$ depending on the wealth level in the previous time-step $V(t-1)$ as follows:

$$\Delta V(t) = \alpha r \cdot V(t-1) - c_V(1-\alpha) \cdot F(V(t-1)) + \alpha G, \quad t = 0, 1, 2, \dots, \quad (28)$$

with initial value for household wealth $V(0) = V_0 \geq 0$. Note that, by definition, the parameter α is strictly between 0 and 1, since it is assumed that $0 < c_V < 1$ and $0 < \theta < 1$. Thus, there are two terms in which the change in wealth $\Delta V(t)$ depends on the previous wealth level, one linear term having a positive impact on the change in wealth, and one non-linear term having a negative impact on $\Delta V(t)$. Depending on the previous wealth level, wealth can increase, decrease, or stay the same. In the latter case, the wealth level is a fixed point. Due to the non-linearity of equation (28), depending on the model parameters, zero, one or two fixed points are possible (see Appendix 2 for a formal proof). In Figure 1, the dependence of ΔV on the previous wealth level V for two different choices of parameters is plotted, to illustrate the case without a fixed point and a case with two fixed points. One can see that the concavity of F results in the right-hand side of (28) being convex, and thus two fixed points are possible.

3.2.2 Results and discussion

Comparing the extended model with a concave consumption function (28) to the baseline model, we still have the possibility of a stable stationary state. However, this stationary state is no longer globally stable. Furthermore, the conditions for local stability for



Note: The parameters that are used are $c_V = 0.04$ in the case of no fixed point, and $c_V = 0.05$ in the case of two fixed points. Other parameters are $c_Y = 0.8$, $\theta = 0.4$, $r = 0.05$, $G = 1$, and $\beta = 0.8$.

Figure 1 Plot of ΔV versus V from equation (28)

Table 2 Conditions for local stability of the fixed point

Linear consumption function	Concave consumption function
$r(1 - c_Y)\left(\frac{1}{\theta} - 1\right) < c_V$	$r(1 - c_Y)\left(\frac{1}{\theta} - 1\right) < c_V \cdot \underbrace{F'(V^*)}_{<1}$

the parameters r , c_V , c_Y and θ are more restrictive than the baseline model described in Section 2. This means that lower interest rates, higher consumption parameters or higher taxation parameters are required for local stability in the case of a concave consumption function compared to a linear consumption function. Table 2 shows a comparison of both conditions for stability of the fixed point. Note that, for the function F as specified in equation (21), the smaller is β , the smaller is the derivative $F'(V^*)$. Furthermore, we have $F'(V^*) < \beta < 1$. Economically, a smaller β results in a smaller marginal propensity to consume out of wealth, all else being equal.

Furthermore, it is important to note that the size of the basin of attraction of the stable fixed point depends on the parameters r , c_V , c_Y and θ ; see Figure 1 for an illustration. If initial values of the households' wealth are above the unstable fixed point, the wealth level will diverge to infinity, even though the conditions for local stability are met. As a conclusion, making the model more realistic in terms of consumption behaviour leads to the stability of a zero-growth economy becoming more difficult to reach compared to the linear model.

3.3 The concave consumption function and wealth inequality

Now both effects of multiple households from Section 3.1 and the concave consumption function in Section 3.2 are analysed together. The question here is whether and under what circumstances a concave consumption function leads to an increase or a decrease in wealth inequality over time.

3.3.1 Model description

The same concave consumption function is used as defined in Section 3.2.1. The equations describing this system are:

$$Y(t) = \sum_{k=1}^N C^k(t) + G, \quad (29)$$

$$Y_d^k(t) = (1 - \theta) \left(\frac{1}{N} Y(t) + rV^k(r - 1) \right), \quad (30)$$

$$C^k(t) = c_Y Y_d^k(t) + c_V F(V^k(t - 1)), \quad (31)$$

$$V^k(t) = V^k(t - 1) + Y_d^k(t) - C^k(t), \quad k = 1, \dots, N. \quad (32)$$

As in the previous sections, computing an explicit formula for $Y(t)$ and using equations (30)–(32), we get the following system of equations describing the dynamics of individual wealth levels V^k , $k = 1, 2, \dots, N$:

$$\begin{aligned}
V^k(t) &= \left(1 + r(1 - c_Y)(1 - \theta)\right) V^k(t-1) \\
&+ \frac{1}{N} \frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)} c_Y r(1 - \theta) \sum_{j=1}^N V^j(t-1) \\
&+ \frac{1}{N} \frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)} c_V \sum_{j=1}^N F\left(V^j(t-1)\right) \\
&+ \frac{1}{N} \frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)} G \\
&- c_V F\left(V^k(t-1)\right).
\end{aligned} \tag{33}$$

Here we can again introduce the parameter α that summarizes the term that appears three times:

$$\alpha := \frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)} \in (0, 1). \tag{34}$$

With this definition, the above equation can be simplified as follows:

$$\begin{aligned}
V^k(t) &= \underbrace{\left(1 + r\alpha(1 - c_Y(1 - \theta))\right)}_{=:a} V^k(t-1) \\
&+ \frac{1}{N} \underbrace{r\alpha c_Y(1 - \theta)}_{=:b} \sum_{j=1}^N V^j(t-1) \\
&- c_V F\left(V^k(t-1)\right) \\
&+ \frac{1}{N} \alpha c_V \sum_{j=1}^N F\left(V^j(t-1)\right) \\
&+ \frac{1}{N} \alpha G, \quad k = 1, \dots, N.
\end{aligned} \tag{35}$$

We can rewrite (35) in matrix form by

$$\mathbf{V}(t) = \mathbf{A}\mathbf{V}(t-1) + \mathbf{B}\mathbf{F}\left(\mathbf{V}(t-1)\right) + \mathbf{C}, \tag{36}$$

where the matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ is again

$$\mathbf{A} = \begin{bmatrix} a + \frac{b}{N} & \frac{b}{N} & \frac{b}{N} & \dots & \frac{b}{N} \\ \frac{b}{N} & a + \frac{b}{N} & \frac{b}{N} & \dots & \frac{b}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b}{N} & \frac{b}{N} & \frac{b}{N} & \dots & a + \frac{b}{N} \end{bmatrix}; \tag{37}$$

with $a > 0$ and $b \in (0, 1)$ defined in (35), the matrix $B \in \mathbb{R}^{N \times N}$ is

$$B = \begin{bmatrix} -c_V + \frac{\alpha}{N}c_V & \frac{\alpha}{N}c_V & \frac{\alpha}{N}c_V & \dots & \frac{\alpha}{N}c_V \\ \frac{\alpha}{N}c_V & -c_V + \frac{\alpha}{N}c_V & \frac{\alpha}{N}c_V & \dots & \frac{\alpha}{N}c_V \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha}{N}c_V & \frac{\alpha}{N}c_V & \frac{\alpha}{N}c_V & \dots & -c_V + \frac{\alpha}{N}c_V \end{bmatrix}, \quad (38)$$

and the vector $C \in \mathbb{R}^N$ is again a vector with equal entries, namely

$$C := \frac{\alpha}{N} G \cdot (1, \dots, 1)^T. \quad (39)$$

The function $F : [0, \infty)^N \rightarrow [0, \infty)^N$ is defined by $F(V) := (F(V^1), \dots, F(V^N))^T$, where F is the concave function defined in Section 3.2.1. Thus, wealth levels of households evolve in time according to equation (36).

3.3.2 Existence of fixed points and stability analysis

Let $V^* \in [0, \infty)$. The condition for a vector of the form $V^* = V^* \cdot (1, \dots, 1)^T \in \mathbb{R}^N$ to be a fixed point is

$$V^* = (1 + r\alpha)V^* - c_V(1 - \alpha)F(V^*) + \frac{\alpha}{N}G. \quad (40)$$

The existence of solutions to this equation follows in the same way as in the one-dimensional case in (28). Thus, under the same condition as in (28), which is

$$(1 - c_Y) \left(\frac{1}{\theta} - 1 \right) (rV^* + G) = c_V F(V^*), \quad (41)$$

we have one or two existing positive fixed points in which the wealth levels for all households are equal.

To conduct a stability analysis, the non-linear system is linearized around the fixed points (see Appendix 2 for the details). The same stability conditions are found as in the one-dimensional case, that is:

$$r(1 - c_Y) \left(\frac{1}{\theta} - 1 \right) < c_V \underbrace{F'(V^*)}_{< 1}. \quad (42)$$

We can conclude that, in the case of a fixed point with equal entries, that is, a multiple of the vector $(1, \dots, 1)^T$, the wealth vector becomes more equally distributed over time, as in Section 3.1. In this case, however, the stability analysis only gives us information about the local behaviour around the fixed point. Numerical simulations in the next section show that, in most cases, wealth becomes more equally distributed also in the long run.

3.3.3 Numerical simulations and discussion

Short-term and long-term behaviour of inequality is tested for a grid of parameters and initial values, specified in Table 3. The goal of the numerical simulations is to get an

Table 3
 Parameter values for numerical simulations

Parameter	Values
c_V	0.01, 0.04, 0.05, 0.1, 0.2, 0.5
c_Y	0.1, 0.2, 0.4, 0.8, 0.9
θ	0.2, 0.6, 0.8
r	0.05, 0.1, 0.4, 0.7
β	0.2, 0.5, 0.9
$V^1(0)$	0, 0.5, 1, 1.5, ..., 3 and 5, 55, 105, 155, ..., 955
$V^2(0)$	1, 1.5, 2.5, ..., 4 and 6, 56, 106, 156, ..., 956

overview of all the trajectories of the system, not only the ones captured by the local stability analysis in the previous section but also the trajectories that move away from the fixed points. Inequality is measured using the Gini coefficient G , which is defined by:

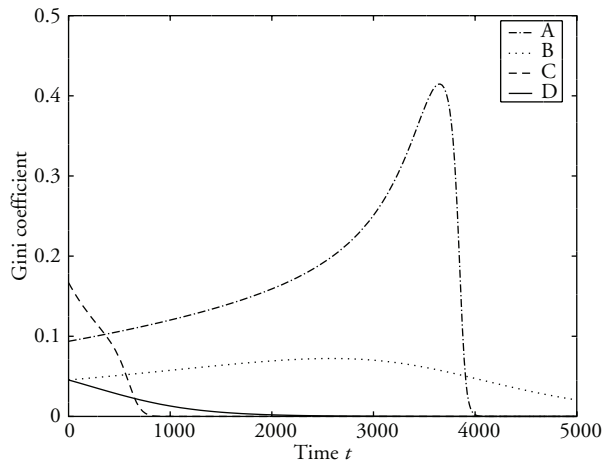
$$G = \frac{\sum_{i=1}^N \sum_{j=1}^N |V_i - V_j|}{2N \sum_{i=1}^N V_i}, \tag{43}$$

where (V_1, \dots, V_N) is the vector of wealth levels. The minimal number of households to test inequality is two. Since the agents are identical except for their wealth levels, there is no qualitatively different dynamics expected for more than two households. Thus, the numerical simulations are conducted for two individuals.

Results show that in all tested parameter combinations inequality eventually decreases. In most parameter combinations, inequality declines right away. In only 6 per cent of the tested parameter combinations from Table 3 does inequality rise after the first time-step. Testing only the parameter combinations with increasing inequality for more time-steps, the results show that eventually the inequality of all cases decreases again. The parameter combination that took the longest time to converge was $c_V = 0.04$, $c_Y = 0.9$, $\theta = 0.8$, $r = 0.05$ and $\beta = 0.5$, with over 5000 time steps. Note that the time-step in the simulations does not refer to a specific length of time, due to the stylized nature of the model. While the model can be used to explore theoretical possibilities of time dynamics, it should not be used to conclude quantitative results.

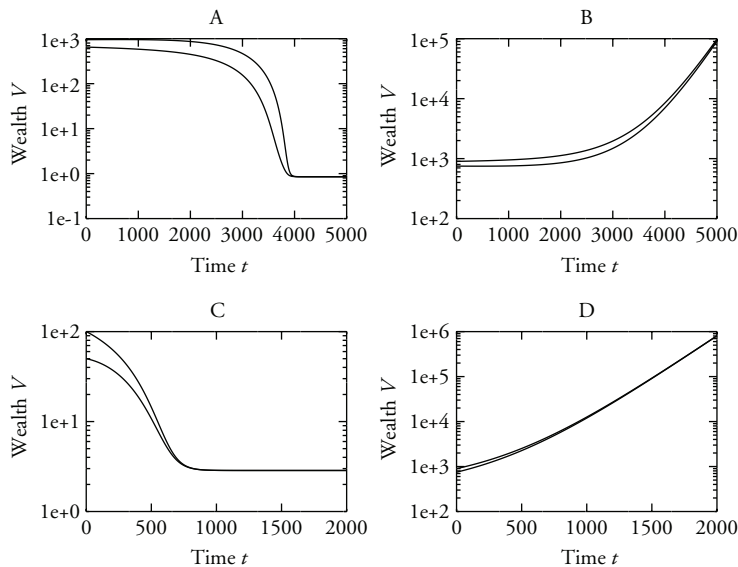
The typical dynamics of the model is for all individuals either to diverge to plus infinity or to converge to a constant wealth level. These typical cases are illustrated in Figures 2 and 3 and Table 4. In cases A and C of Figure 3 and Table 4, you can see that the dynamic converges to the lower stable fixed point while the trajectories for B and D lead to unlimited growth in wealth. Note that whether individual wealth levels all grow to infinity or converge to a constant level not only depends on the model parameters r , c_V , c_Y and θ , but also and especially on the initial values. As already discussed in the previous sections of this paper, the non-linearity of the model allows for the existence of two fixed points. All else being equal, a small change in initial values can turn a converging dynamic into a diverging one, as illustrated by the simulation runs A and B. This critical dependence of the outcome on initial conditions and parameters is a typical result in non-linear models (Strogatz 2014).¹

1. Another observation is that a reduction of the tax rate θ by 10 per cent from simulation run from B to D and from D to C leads to, all else being equal, a reduction of the basin of attraction between the fixed points V_1^* and V_2^* by more than one-half.



Notes: The parameters for the simulation are summarized in Table 4. Dynamics A and B use the same parameters but different initial values. In both cases A and B, inequality is first increasing and then decreasing in the end. In cases C and D, inequality decreases directly. As can be seen in Figure 3, in A and D the dynamics converges, whereas in B and C the dynamics is divergent.

Figure 2 Gini coefficient for two individuals



Notes: The parameters used in the simulations are listed in Table 4. Graphs A, B, C and D show typical behaviour of the dynamics. In A and C we have convergence, in B and D divergence. In plots A and B, inequality is first increasing and then decreasing over time. In plots C and D, inequality is decreasing from the beginning; see Figure 2.

Figure 3 Wealth of two individuals over time

Table 4 Parameter values and initial values for Figures 2 and 3

	A	B	C	D
c_V	0.10	0.10	0.10	0.10
c_Y	0.90	0.90	0.90	0.90
θ	0.60	0.60	0.40	0.50
r	0.05	0.05	0.05	0.05
β	0.50	0.50	0.50	0.50
$V^1(0)$	650	750	50	750
$V^2(0)$	950	900	100	900
$V^1(5000)$	0.855	–	2.860	–
$V^2(5000)$	0.855	–	2.860	–
V_1^*	0.855	0.855	2.860	1.480
V_2^*	819	819	128	339

Note: The table shows the final individual wealth levels $V^1(5000)$ and $V^2(5000)$ for the converging cases A and C, as well as the two stationary wealth levels V_1^* (stable) and V_2^* (unstable) for all the cases A, B, C and D. The wealth levels V_1^* and V_2^* are computed according to equation (40).

Moreover, simulation runs A and B are examples of the special cases in which inequality increases at first before it starts decreasing again. In cases C and D, inequality decreases right away. Note that the initial values in A, B and C are chosen such that the unstable stationary wealth level V_2^* is in between the initial values for the two individuals. One can expect that a repellent fixed point leads the two wealth levels to diverge in different directions at first, which could be seen as a reason for the initial increase in inequality in cases A and B. However, more detailed analysis is required to fully determine the conditions and in particular the economic meaning for the special cases in which inequality increases at the beginning before it starts decreasing. This is left for future work.

4 CONCLUSION

Tendencies for rising inequality could make a zero-growth economy socially unstable and politically unfeasible. Thus, system properties need to be investigated that allow for declining inequality in a zero-growth economy. This paper contributes to this question by providing an example of theoretical conditions in which inequality declines in a zero-growth economy while maintaining some aspects of current economic mechanisms, namely positive interest rates and concave consumption functions. Thereby, the present paper contributes to the debate about the (in)compatibility of positive interest rates and a zero-growth economy (see, for example, Binswanger 2015; Jackson/Victor 2015; Richters/Siemoneit 2017).

In the present paper, three versions of a stock–flow consistent model are analysed, a first one which considers multiple households with heterogeneous initial wealth levels, a second one in which the aggregated household sector has a non-linear, concave consumption function, and finally a third version which considers both mechanisms in one model. These three versions are analysed via analytical and numerical methods with respect to stability of fixed points and time development of inequality. In all three models, the conditions for the zero-growth economy to be stable are: a small interest rate, a large tax rate, and large marginal

propensities to consume out of income and wealth. In the non-linear model versions, the initial values must not be too far away from the stable stationary state to ensure convergence to the stable fixed point. Initial wealth inequality is shown to decline in the long run in all analysed model versions.

In more detail, in the linear model version with multiple households in Section 3.1, it is proven analytically that the wealth distribution becomes equally distributed over time except in some special cases that can be disregarded, in both the growing and the non-growing economy. Numerical simulations in Section 3.3 show that in the case of a more realistic concave consumption function, inequality declines in the long run as well.

However, when using a concave consumption function, all else being equal, the conditions on the parameters to reach the stable stationary state are more restrictive than in the case of a linear consumption function. For example, a higher level of consumption parameters or taxation or lower levels of interest rates would be needed to ensure stability compared to the linear model. Moreover, this stationary state is no longer globally stable in the non-linear case. Even in the case of an existing locally stable stationary state, households' wealth is growing to infinity if the initial values for wealth are above a certain level. In the case in which the conditions for a stable stationary state are not fulfilled, wealth levels grow to infinity as well. It needs to be noted that the term stability in this paper is used in the sense as the fixed point analysis of dynamical systems (see, for example, Strogatz 2014), which is not equivalent to economic or financial stability.

Although the main purpose of this paper is to illustrate system dynamics in which a zero-growth economy with positive interest rates is stable and has decreasing inequality, many mechanisms that can cause inequality are not included in the model. Out of the many possibilities to extend the presented analysis, I would like to point out the following:

First, in the presented model, all households receive the same wage, which is obviously a simplification and an extreme assumption. It needs to be investigated how the results would change when including an unequal income from wages.

Second, in the present paper only the non-linearity in consumption out of wealth is analysed. However, non-linearity in other mechanisms might be relevant. For example, differential savings rates and differential rates of return are found to be crucial for questions of distribution (Ederer et al. 2021). Furthermore, financial market returns are not deterministic and losses have a heavy tailed distribution. Investors with more wealth are able to hedge better against losses, for example by a diversified portfolio, and thus can realize higher average returns compared to households with lower wealth which might refrain from investing altogether due to loss aversion.

Third, not all households receive net positive interest payments. While some earn a positive income, some have to pay more interest on debt than they receive on their assets. This mechanism is not accounted for here, since households cannot be in debt. Furthermore, net debtor and net creditor dynamics play a crucial role when investigating global flows of money, and is a relevant direction of research about zero-growth economies with a global perspective.

Fourth, different asset classes should be considered. The distributional effects of dividend payments on investment in companies are likely to be different compared to the ones when considering only investment in government bonds.

Thus, as a summary, the present paper shows theoretical conditions for decreasing inequality in a zero-growth economy, however it also opens up many directions for further research, the most immediate being: 'What are the minimal requirements for the structure of the economic system that lead to declining inequality in a zero-growth economy?'

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APPENDIX 1 WEALTH INEQUALITY

Here, additional details for the model described in Section 2 are given. Recall that the linear dynamics for individual wealth levels $V^k(t)$, $k = 1, 2, \dots, N$ is defined by:

$$\begin{aligned} V^k(t) = & \underbrace{\left(1 - c_V + r(1 - c_Y)(1 - \theta)\right)}_{=a} V^k(t-1) \\ & + \underbrace{\frac{1}{N} \left(r c_Y (1 - \theta) + c_V \right) \frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)}}_{=b} \sum_{j=1}^N V^j(t-1) \\ & + \underbrace{\frac{1}{N} \frac{(1 - c_Y)(1 - \theta)}{1 - c_Y(1 - \theta)}}_{=c} G. \end{aligned} \quad (A1)$$

A1.1 Matrix form and stability analysis

In matrix form, the model equations in (A1) can be written as

$$\mathbf{V}(t) = A\mathbf{V}(t-1) + \mathbf{C}, \quad (A2)$$

where $\mathbf{V}(t) = (V^1(t), \dots, V^N(t))^T$ and $\mathbf{C} = (\frac{c}{N}, \dots, \frac{c}{N})^T$. The matrix A has the form

$$A = \begin{bmatrix} a + \frac{b}{N} & \frac{b}{N} & \frac{b}{N} & \dots & \frac{b}{N} \\ \frac{b}{N} & a + \frac{b}{N} & \frac{b}{N} & \dots & \frac{b}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b}{N} & \frac{b}{N} & \frac{b}{N} & \dots & a + \frac{b}{N} \end{bmatrix}. \quad (A3)$$

Matrix A has the eigenvalues $a + b$ and a . This can be seen by considering the vectors $\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^N \in \mathbb{R}^N$ defined by

$$\mathbf{v}^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \mathbf{v}^2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{v}^3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{v}^{N-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}^N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}. \quad (A4)$$

The vector \mathbf{v}^1 is an eigenvector of A with respect to the eigenvalue $a + b$, and all the other vectors $\mathbf{v}^2, \dots, \mathbf{v}^N$ are eigenvectors with respect to the eigenvalue a .

Since the vectors $\mathbf{v}^2, \dots, \mathbf{v}^N$ are linearly independent, the eigenspace of the eigenvalue a is $N - 1$ dimensional, and, hence, the eigenspace of $a + b$ is one-dimensional. Even though, here, the parameters a and b are positive, note that the statement above holds for any $a, b \in \mathbb{R}$.

Remark The fact that the eigenspace of the eigenvalue $a + b$ is the one-dimensional space $\{\gamma(1, 1, \dots, 1)^T : \gamma \in \mathbb{R}\}$, that is, it consists only of vectors with equal entries, is used later in the discussion about the time evolution of wealth inequality.

A1.1.1 Existence of fixed points

First we have to check the existence of a fixed point. A point $\mathbf{V}^* \in \mathbb{R}^N$ is a fixed point, if and only if

$$\mathbf{V}^* = A\mathbf{V}^* + \mathbf{C}, \quad (\text{A5})$$

or equivalently

$$(I - A)\mathbf{V}^* = \mathbf{C}, \quad (\text{A6})$$

where I is the identity matrix in \mathbb{R}^N . There exists a unique fixed point, if and only if the matrix $(I - A)$ is invertible, which is equivalent to the condition $\det(A - I) \neq 0$. The latter means that $\lambda = 1$ is not an eigenvalue of matrix A . Since we have already determined all eigenvalues of A which are $a + b$ and a , we have a unique fixed point if and only if $a + b \neq 1$ and $a \neq 1$. From (A6) we get that the fixed point $\mathbf{V}^* = (V_1^*, \dots, V_N^*)^T$ must satisfy

$$(1 - a)V_k^* - b \frac{1}{N} \sum_{j=1}^N V_j^* = \frac{c}{N}, \quad k = 1, \dots, N.$$

Thus, for any $a = 1$, any vector \mathbf{V}^* with $\sum_{j=1}^N V_j^* = -\frac{c}{b}$ is a fixed point and we have non-uniqueness. For $a \neq 1$, the vector \mathbf{V}^* must have equal entries, so for $a + b = 1$ no fixed point exists, since otherwise, we would have

$$0 = (1 - a - b)V_k^* = \frac{c}{N} > 0, \quad k = 1, \dots, N.$$

For $1 - a - b \neq 0$, the constant vector

$$\mathbf{V}^* = \frac{1}{1 - a - b} \cdot \frac{c}{N} \cdot (1, 1, \dots, 1)^T$$

solves the fixed point equation (A5). Thus for $a + b < 1$ we have positive equal wealth levels in the stationary state. For $a + b > 1$, the vector \mathbf{V}^* has negative entries.

A1.1.2 Stability analysis of fixed points

Let \mathbf{V}^* be a fixed point of the system (A2), that is, $\mathbf{V}^* = A\mathbf{V}^* + \mathbf{C}$. Then we have:

$$\begin{aligned} \mathbf{V}(t) - \mathbf{V}^* &= A\mathbf{V}(t-1) + \mathbf{C} - \mathbf{V}^* \\ &= A(\mathbf{V}(t-1) - \mathbf{V}^*). \end{aligned} \quad (\text{A7})$$

Since the matrix A is symmetric, it is diagonalizable. Moreover, there is a real orthogonal matrix Q , that is, $Q^{-1} = Q^T$, such that $Q^T A Q = D$, where D is a diagonal matrix with the entries $a + b$ and $N - 1$ times a . The columns of Q , $\mathbf{q}^1, \dots, \mathbf{q}^N \in \mathbb{R}^N$ are the eigenvectors of A . Without loss of generality, let \mathbf{q}^1 be the eigenvector corresponding to eigenvalue $a + b$, which means that \mathbf{q}^1 is in the linear span of the vector $(1, \dots, 1)^T$.

Let us consider the dynamics (A7) starting at a point $\mathbf{V}(0) = \mathbf{V}_0 \in \mathbb{R}^N$ and let us define $(\gamma_1, \dots, \gamma_N)^T := Q^T(\mathbf{V}_0 - \mathbf{V}^*)$. Then, we get iteratively from (A7) for $t = 1, 2, 3, \dots$

$$\begin{aligned} \mathbf{V}(t) - \mathbf{V}^* &= A^t(\mathbf{V}_0 - \mathbf{V}^*) \\ &= QD^tQ^T(\mathbf{V}_0 - \mathbf{V}^*) \\ &= (a+b)^t\gamma_1\mathbf{q}^1 + \sum_{k=2}^N a^t\gamma_k\mathbf{q}^k. \end{aligned} \quad (\text{A8})$$

Recall that, here, the parameters a and b are positive. So for $a+b < 1$ we have $(a+b)^t \rightarrow 0$ and $a^t \rightarrow 0$ for $t \rightarrow \infty$. Thus, the stationary state is globally attracting, which means that $\mathbf{V}(t)$ converges to \mathbf{V}^* as $t \rightarrow \infty$ for any starting point $\mathbf{V}_0 \in [0, \infty)^N$. The condition $a+b < 1$ is equivalent to the stability condition in the one-dimensional case

$$c_V > r(1-c_Y)\left(\frac{1}{\theta} - 1\right), \quad (\text{A9})$$

which can be seen by the following calculations, using the definitions for a , b and c in (A1),

$$\begin{aligned} a+b &= 1-c_V + r(1-c_Y)(1-\theta) + (rc_Y(1-\theta) + c_V)\frac{(1-c_Y)(1-\theta)}{1-c_Y(1-\theta)} \\ &= 1-c_V + \frac{(r+c_V)(1-c_Y)(1-\theta)}{1-c_Y(1-\theta)} \\ &< 1 \\ &\Leftrightarrow c_V > r\frac{(1-c_Y)(1-\theta)}{\theta}. \end{aligned}$$

For $a+b > 1$ and $\gamma_1 \neq 0$, we have divergence from the fixed point. The dynamics becomes dominated by the term $(a+b)^t\gamma_1\mathbf{q}^1$, as $t \rightarrow \infty$, therefore, $\mathbf{V}(t) - \mathbf{V}^*$ becomes more constant over time, even in the divergent case. More specifically, since $b > 0$, this means that

$$\begin{aligned} \frac{1}{(a+b)^t}(\mathbf{V}(t) - \mathbf{V}^*) &= \gamma_1\mathbf{q}^1 + \sum_{k=2}^N \left(\frac{a}{a+b}\right)^t \gamma_k\mathbf{q}^k \\ &\rightarrow \gamma_1\mathbf{q}^1, \text{ as } t \rightarrow \infty. \end{aligned} \quad (\text{A10})$$

So the term $(a+b)^{-t}(\mathbf{V}(t) - \mathbf{V}^*)$ converges to a multiple of the vector $(1, \dots, 1)^T$. Subtracting $\mathbf{V}^*/(a+b)^t$ on both sides in (A10) gives

$$\frac{1}{(a+b)^t}\mathbf{V}(t) = \frac{1}{(a+b)^t}\mathbf{V}^* + \gamma_1\mathbf{q}^1 + \sum_{k=2}^N \left(\frac{a}{a+b}\right)^t \gamma_k\mathbf{q}^k \rightarrow \gamma_1\mathbf{q}^1, \text{ as } t \rightarrow \infty. \quad (\text{A11})$$

If $\sum_{j=1}^N V_j^* \neq 0$, which is the case when $a \neq 1$, we can divide (A11) by its sum and conclude that the entries of $\mathbf{V}(t)$ converge to its average, that is,

$$\frac{V^k(t)}{\sum_{j=1}^N V^j(t)} \rightarrow \frac{1}{N}, \text{ as } t \rightarrow \infty, \quad (\text{A12})$$

for all $k = 1, \dots, N$. For $\sum_{j=1}^N V_j^* = 0$, the above convergence (A12) is true as well. In the special case $a + b > 1$ and $\gamma_1 = 0$, the initial configuration $V_0 - V^*$ would be exactly orthogonal to $(1, \dots, 1)^T$. However, since only non-negative V_0 are allowed, and V^* has only negative entries for $a \neq 1$, $V_0 - V^*$ cannot be orthogonal to $(1, \dots, 1)^T$. For $a = 1$, we can see directly from (A2) that for $V_0 \in [0, \infty)^N$, wealth goes to infinity, that is, $\|V(t)\| \rightarrow \infty$, as $t \rightarrow \infty$. In summary, for $a + b > 1$, the wealth levels grow.

A1.2 Summary of results for linear consumption function and N households

A1.2.1 Summary

The conditions for stability are the same in the version of the model with N households compared to the version with aggregated households. The important thing to note is that both in the case when the fixed point is attracting and when it is repelling, the wealth vector $V(t)$ becomes more equal over time. This means, for all $k = 1, \dots, N$

$$\frac{V^k(t)}{\sum_{j=1}^N V^j(t)} \rightarrow \frac{1}{N}, \text{ as } t \rightarrow \infty.$$

The following two propositions summarize the findings above. The definition, which is adapted from Martelli (2011) and Strogatz (2014), is used for the following propositions as well as the next sections.

Definition A1 Let (X, ρ) be a metric space and $\phi : X \rightarrow X$ a function. Then, a point $x^* \in X$ is called fixed point of the iterative system

$$x_n = \phi(x_{n-1}), x_0 \in X, n \in \mathbb{N}$$

if

$$\phi(x^*) = x^*.$$

- (a) The fixed point x^* is called attracting, if there is an open neighbourhood $U \ni x^*$ such that $\phi^n(x) = \underbrace{\phi \circ \dots \circ \phi}_{n \text{ times}} \rightarrow x^*$, as $n \rightarrow \infty$, for all $x \in U$.
- (b) The fixed point is called repelling, if there exists a open neighbourhood $V \ni x^*$ such that for all starting points $x_0 \in V$, $x_0 \neq x^*$, there exists an $n \in \mathbb{N}$ such that $x_n \notin V$.
- (c) The set of all points $\{x \in X : \phi^n(x) \rightarrow x^*, n \rightarrow \infty\}$ is called stable set of x^* .
- (d) For ϕ invertible, the set of all points $\{x \in X : \phi^{-n}(x) \rightarrow x^*, n \rightarrow \infty\}$ is called unstable set of x^* .

This proposition summarizes the different cases for the existence of fixed points of the system (A2).

Proposition A1 (Existence of fixed points) Let a, b and c be defined as in (A1) and let the matrix $A \in \mathbb{R}^{N \times N}$ and the vector $C \in \mathbb{R}^N$ be defined as in (A3). Then $a, b, c > 0$. Consider the discrete time system

$$V(t) = AV(t-1) + C, t \in \mathbb{N}, V(0) = V_0 \in [0, \infty)^N. \quad (A13)$$

Then,

- (a) if $a + b < 1$ there exists a unique fixed point with equal positive entries given by $V^* = \frac{c}{N(1-a-b)} > 0$;
- (b) if $a + b > 1$ and $a \neq 1$, there exists a unique fixed point with equal negative entries given by $V^* = \frac{c}{N(1-a-b)} < 0$, which is outside of the state space;
- (c) if $a + b > 1$ and $a = 1$, there are multiple fixed points characterized by $\sum_{j=1}^N V_j^* = -\frac{c}{b}$;
at least one entry of the fixed point is negative, so the fixed point is outside of the state space;
- (d) if $a + b > 1$, no fixed point exists.

The next proposition summarizes the dynamic behaviour of the system (A2). The norm $\|\cdot\|$ denotes the Euclidean distance in \mathbb{R}^N .

Proposition A2 (Dynamics) *Let V^* be a fixed point of the system*

$$V(t) = AV(t-1) + C, t \in \mathbb{N}, V(0) = V_0 \in [0, \infty)^N, \quad (A14)$$

which is defined as in (A2).

- (a) If $a + b < 1$, then $\|V(t) - V^*\| \rightarrow 0$ as $t \rightarrow \infty$, for all $V_0 \in [0, \infty)^N$.
- (b) If $a + b > 1$, then wealth levels diverge, that is, $\|V(t)\| \rightarrow \infty$ for $t \rightarrow \infty$.
- (c) Furthermore, we get in both cases

$$\frac{V^k(t)}{\sum_{j=1}^N V^j(t)} \rightarrow \frac{1}{N}, \text{ as } t \rightarrow \infty, \text{ for all } k = 1, \dots, N.$$

APPENDIX 2 THE CONCAVE CONSUMPTION FUNCTION AND AGGREGATED HOUSEHOLDS

The equation for the change in wealth $\Delta V(t) := V(t) - V(t-1)$ depending on the wealth level in the previous time-step $V(t-1)$ is:

$$\Delta V(t) = \alpha r \cdot V(t-1) - c_V(1-\alpha) \cdot F(V(t-1)) + \alpha G, \quad t = 0, 1, 2, \dots \quad (A15)$$

Existence of stationary state For the difference between one time-step and the next, that is, $\Delta V := V(t) - V(t-1)$, for all $t \in \mathbb{N}$ in (A15), we get

$$\Delta V = \alpha r \cdot V - c_V(1-\alpha) \cdot F(V) + \alpha G, \quad (A16)$$

where the time dependence of V can be omitted because of time homogeneity. A function $\mathcal{F} : [0, \infty) \rightarrow \mathbb{R}$, $V \mapsto \mathcal{F}(V)$ is introduced to denote the right-hand side of (A16):

$$\mathcal{F}(V) := \alpha r \cdot V - c_V(1-\alpha) \cdot F(V) + \alpha G. \quad (A17)$$

In Figure 1, two realizations of \mathcal{F} for different levels of consumption out of wealth parameter c_V are plotted.

Consider $V = 0$. Then, the right-hand side of (A16) is positive, that is, $\mathcal{F}(0) > 0$. Thus, we need $\mathcal{F}'(0) < 0$, as a necessary condition for the existence of a positive fixed point. Dividing by α and using the definition of the parameter α , we get

$$\begin{aligned}\mathcal{F}'(0) &< 0 \\ \Leftrightarrow \quad r \frac{\alpha}{1-\alpha} &< c_V F'(0) \\ \Leftrightarrow \quad r(1-c_Y) \left(\frac{1}{\theta} - 1 \right) &< c_V F'(0).\end{aligned}$$

In the specific case $F(V) = (V+1)^\beta - 1$ we have $F'(0) = \beta < 1$. This means that the smaller is β , the larger are the values for c_V , c_Y or θ must be, or the smaller the value r has to be to counteract the effect from concavity. It follows from the strict concavity of F and thus the strict convexity of \mathcal{F} that the equation $\Delta V = 0$ has either zero, one or two solutions. Dividing (A16) by $\alpha \neq 0$ gives

$$\begin{aligned}V &= 0 \\ \Leftrightarrow \quad rV^* + G &= c_V \frac{(1-\alpha)}{\alpha} \cdot F(V^*) \\ \Leftrightarrow \quad (1-c_Y) \left(\frac{1}{\theta} - 1 \right) (rV^* + G) &= c_V F(V^*) \\ &< c_V V^*,\end{aligned}$$

since $V^* > 0$ if it exists. Again, we see that, compared to the linear case where $F(V) = V$, larger values for c_V , c_Y and θ or a smaller value for r are necessary to ensure the existence of a positive fixed point.

Local stability of fixed point Let V^* be a positive fixed point. Then, using (A15) locally around the fixed point we get

$$\begin{aligned}V(t) - V^* &= (1 + \alpha r) \left(V(t-1) - V^* \right) - c_V (1 - \alpha) \left(F(V(t-1)) - F(V^*) \right) \\ &\approx \left(1 + \alpha r - c_V (1 - \alpha) F'(V^*) \right) \left(V(t-1) - V^* \right).\end{aligned}$$

So if $|1 + \alpha r - c_V (1 - \alpha) F'(V^*)| < 1$, we have local stability of that fixed point, which means that the fixed point is attracting. This is equivalent to

$$\frac{r(1-c_Y)(1-\theta)}{\theta} < c_V \underbrace{F'(V^*)}_{< 1}. \quad (\text{A18})$$

For the specific example of F with parameter β , we get

$$\frac{r(1-c_Y)(1-\theta)}{\theta} \cdot \frac{1}{\beta} (V^* + 1)^{1-\beta} < c_V. \quad (\text{A19})$$

Using the convexity of the function \mathcal{F} , we can even say something about the global behaviour of the fixed points, which leads to the following proposition:

Proposition A3 *Let the function $\mathcal{F} : [0, \infty) \rightarrow \mathbb{R}$ be defined as in (A17). Then the discrete time dynamics (A15)*

$$V(t) = \mathcal{F}(V(t-1)) + V(t-1), t \in \mathbb{N}, V(0) = V_0 \in [0, \infty) \quad (\text{A20})$$

has zero, one or two fixed points. Furthermore,

- (a) *the dynamics of V over time in (A20) is monotone, that is, either $V(t) > V(t-1)$ for all $t \in \mathbb{N}$, or $V(t) < V(t-1)$ for all $t \in \mathbb{N}$, or $V(t) = V_0$ for all $t \in \mathbb{N}$;*
- (b) *in the case of two fixed points V_1^* and V_2^* , $V_1^* < V_2^*$, the smaller fixed point V_1^* is attracting and the larger fixed point V_2^* is repelling; the stable set of V_1^* is $[0, V_2^*)$, and the unstable set of V_2^* is (V_1^*, ∞) ; for $V_0 > V_2^*$, we have $V(t) \rightarrow \infty$, as $t \rightarrow \infty$.*

Proof *The function \mathcal{F} , (A17), is strictly convex per definition. Thus, there cannot be more than two points $V^* \in [0, \infty)$ with $\mathcal{F}(V^*) = V^*$. For zero or two fixed points, see the example in Figure 1. Since $\mathcal{F}(V)$ is continuous in the parameter c_V for all $V \geq 0$, there is some intermediate parameter value for c_V with one fixed point.*

- (a) *Let $V^* > 0$ be a fixed point of (A20) and let $V_0 \in [0, \infty)$ with $V_0 \neq V^*$. Then, using differentiability of \mathcal{F} and the mean value theorem in (*), we get*

$$\begin{aligned} V(1) - V^* &= \mathcal{F}(V_0) + V_0 - V^* \\ &= \mathcal{F}(V_0) - \mathcal{F}(V^*) + V_0 - V^* \\ &\stackrel{(*)}{=} (\mathcal{F}'(\xi) + 1)(V_0 - V^*), \end{aligned} \quad (\text{A21})$$

for some value $\xi \in (V_0, V^)$ or $\xi \in (V^*, V_0)$ respectively. However, we have $\mathcal{F}'(V) + 1 > 0$ for all $V > 0$, by*

$$\begin{aligned} \mathcal{F}'(V) + 1 &= 1 + \alpha r - c_V(1 - \alpha)F'(V) > 0 \\ &\Leftrightarrow 1 + \underbrace{\alpha r}_{>0} > \underbrace{c_V}_{<1} \underbrace{(1 - \alpha)}_{<1} \underbrace{F'(V)}_{<1}. \end{aligned}$$

Thus, it follows from (A21) that $V(1) - V^$ and $V_0 - V^*$ have the same sign. This means that from one time-step to the next, $V(t)$ cannot jump across a fixed point. Since \mathcal{F} is continuous and only changes its sign in a fixed point, the dynamics of V is monotone.*

- (b) *Suppose that V_1^* and $V_2^* > 0$ are two fixed points, $V_1^* < V_2^*$. Since $\mathcal{F}(0) > 0$ and \mathcal{F} is continuous, we know that $\mathcal{F}(V) > 0$ for both $V < V_1^*$ and $V > V_2^*$ and $\mathcal{F}(V) < 0$ for $V \in (V_1^*, V_2^*)$. Let $V_0 \in (V_1^*, V_2^*)$. Then we know, from (a), that the sequence $(V(t))_t$ is strictly decreasing and bounded by V_1^* . Suppose that $(V(t))_t$ converges to a point $V^* > V_1^*$. Then, this would mean $V(t+1) - V(t) = \mathcal{F}(V(t)) \rightarrow 0$ as $t \rightarrow \infty$ and by continuity $\mathcal{F}(V^*) = 0$, which is a contradiction to the fact that the only zeros of \mathcal{F} are V_1^* and V_2^* . Thus $(V(t))_t$ converges to V_1^* . A similar argument can be made for $V_0 < V_1^*$.*

Let $V_0 > V_2^*$. Suppose that $(V(t))$ does not go to infinity. Since $(V(t))$ is monotone, this means that $(V(t))$ is bounded. A bounded, monotone sequence in \mathbb{R} is convergent, thus there is a point $V^* > V_2^*$ with $\mathcal{F}(V(t)) \rightarrow \mathcal{F}(V^*) = 0$ as $t \rightarrow \infty$ which is again a contradiction. \square

APPENDIX 3 THE CONCAVE CONSUMPTION FUNCTION AND WEALTH INEQUALITY

Recall that the system of equations in matrix form is

$$\mathbf{V}(t) = A\mathbf{V}(t-1) + B\mathbf{F}(\mathbf{V}(t-1)) + C. \quad (\text{A22})$$

Let us assume we have a fixed point $\mathbf{V}^* \in [0, \infty)^N$. Then we subtract \mathbf{V}^* on both sides of (A22) and linearize around the fixed point. That gives

$$\begin{aligned} \mathbf{V}(t) - \mathbf{V}^* &= A\mathbf{V}(t-1) + B\mathbf{F}(\mathbf{V}(t-1)) + C - \mathbf{V}^* \\ &= A(\mathbf{V}(t-1) - \mathbf{V}^*) + B(\mathbf{F}(\mathbf{V}(t-1)) - \mathbf{F}(\mathbf{V}^*)) \\ &\approx \underbrace{(A + B\mathbf{F}'(\mathbf{V}^*))}_{=:M} (\mathbf{V}(t-1) - \mathbf{V}^*), \end{aligned}$$

where \mathbf{F}' is the derivative of \mathbf{F} , that is,

$$\mathbf{F}'(\mathbf{V}) = \begin{bmatrix} F'(V_1) & 0 & 0 & \dots & 0 \\ 0 & F'(V_2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F'(V_N) \end{bmatrix},$$

where $\mathbf{V} = (V_1, V_2, \dots, V_N)^T \in \mathbb{R}^N$. Locally, the dynamics around the fixed point can be described by the equation

$$\mathbf{V}(t) - \mathbf{V}^* = M(\mathbf{V}(t-1) - \mathbf{V}^*),$$

where the matrix M was defined as $M := A + B\mathbf{F}'(\mathbf{V}^*)$. In the case that \mathbf{V}^* is a vector with equal entries, the matrix M is symmetric and has the same form as the matrix A in the linear, N -dimensional case; see (A3).

Stationary state with equal entries Let us consider the case where the fixed point \mathbf{V}^* has equal entries in more detail (that is, \mathbf{V}^* is a multiple of the vector $(1, \dots, 1)^T$). The matrix $M = A + B\mathbf{F}'(\mathbf{V}^*)$ has the element

$$a + \frac{b}{N} + \left(-c_V + \frac{\alpha}{N}c_V\right) \cdot F'(V^*)$$

on the diagonal and

$$\frac{b}{N} + \frac{\alpha}{N} c_V F'(V^*)$$

in every other entry. Thus, the same analysis applies as to the linear case (A3) and we get that M has the eigenvalues

$$\begin{aligned}\lambda_1 &= a + b - (1 - \alpha) c_V F'(V^*) \\ &= 1 + r\alpha - (1 - \alpha) c_V F'(V^*) \\ \lambda_2 &= a - c_V F'(V^*) \\ &= 1 + r\alpha(1 - c_Y(1 - \theta)) - c_V F'(V^*).\end{aligned}\tag{A23}$$

As before, λ_1 is the eigenvalue with one-dimensional eigenspace and all eigenvectors being multiples of $(1, 1, \dots, 1)^T \in \mathbb{R}^N$. Note that, since the derivative of F is less or equal to 1, and the other parameters lie in the interval $(0, 1)$, the eigenvalue λ_1 is positive. The condition $|\lambda_1| < 1$ is then equivalent to

$$\begin{aligned}r\alpha &< (1 - \alpha) c_V F'(V^*) \\ \Leftrightarrow r &< \frac{\theta}{(1 - c_Y)(1 - \theta)} c_V F'(V^*),\end{aligned}$$

which is the same condition as in the one-dimensional case with the concave consumption function, (A18).

Again, we have that the eigenvector $(1, \dots, 1)^T$ of M corresponds to the largest eigenvalue. The matrix M can be diagonalized with $M = QDQ^T$, where Q is an orthogonal matrix with the eigenvectors $\mathbf{q}^1, \dots, \mathbf{q}^N$ as columns. The matrix D is a diagonal matrix, with λ_1 and $N-1$ times λ_2 on the diagonal. Locally, the dynamics (A22) can be approximated by the linearization as shown in Martelli (2011). As in the linear case (A8), we get locally around the fixed point,

$$\begin{aligned}\mathbf{V}(t) - \mathbf{V}^* &\approx M^t(\mathbf{V}_0 - \mathbf{V}^*) \\ &= QD^tQ^{-1}(\mathbf{V}_0 - \mathbf{V}^*) \\ &= \lambda_1^t \gamma_1 \mathbf{q}^1 + \sum_{k=2}^N \lambda_2^t \gamma_k \mathbf{q}^k,\end{aligned}\tag{A24}$$

where again, \mathbf{q}^1 is a vector with equal entries and $(\gamma_1, \dots, \gamma_N)^T := Q^T(\mathbf{V}_0 - \mathbf{V}^*)$ the coordinates of $(\mathbf{V}_0 - \mathbf{V}^*)$ with respect to the basis of eigenvectors of M . Following the same analysis as in (A8) and Proposition A3 we get that for $\lambda_1 < 1$, the fixed point is attracting. For $\lambda_1 > 1$ there are again several cases. Let $\lambda_1 > 1$. Then, unless $\mathbf{V}_0 - \mathbf{V}^*$ is orthogonal to \mathbf{q}^1 , the fixed point is repelling, which means that the difference $\|\mathbf{V}(t) - \mathbf{V}^*\|$ becomes bigger locally around \mathbf{V}^* , with increasing t . In the orthogonal case, that is, $\gamma_1 = 0$, the dynamics depends on the second eigenvalue λ_2 as in Proposition A3, and the dynamics can converge, diverge, or stay the same. However, this is only the case sufficiently close to the fixed point \mathbf{V}^* .