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Finite Approximations of the Sion-Wolfe Game

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Finite Approximations of the Sion-Wolfe Game*

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Abstract. As pointed out by Sion and Wolfe (1957), a non-cooperative game on the unit square need not admit a Nash equilibrium, neither in pure nor in randomized strategies. In this paper, we consider finite approximations of the Sion-Wolfe game. For all parameter constellations relevant for the limit consideration, we characterize the set of Nash equilibria in iteratively undominated strategies. Values of finite approximations of the Sion-Wolfe game are shown to accumulate around three values that do not correspond in a simple way to the majorant and minorant values of the continuous game. To understand why this is happening, we apply the iterated elimination of weakly dominated strategies to the continuous game as well. The existence of ε -equilibrium, however, does not seem to be related to the properties of finite approximations.

Keywords. Two-person zero-sum games · Sion-Wolfe game · Existence of Nash equilibrium · Finite approximations · Iterated elimination of dominated strategies · ε -equilibrium · Colonel Blotto games

MSC-Codes. 90C – Mathematical Programming; 91A – Game Theory

JEL-Codes. C62 – Existence and Stability Conditions of Equilibrium; C72 – Noncooperative Games; D72 – Political Processes: Rent-Seeking, Lobbying, Elections, Legislatures, and Voting Behavior

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1. Introduction

In the early years of game-theoretic research, fundamental contributions established the existence of mixed-strategy solutions for noncooperative games in increasing generality. Notable results have been achieved, in particular, for finite two-person zero-sum games (von Neumann, 1928; von Neumann and Morgenstern, 1945), finite n -player games (Nash, 1950, 1951), and games with infinite strategy spaces (Glicksberg, 1952; Fan, 1952; Debreu, 1952). This line of research came to a sudden halt, however, when Sion and Wolfe (1957) presented an example of a two-person zero-sum game on the unit square that does not have a value. What that example means for the modern theory of games that is not restricted by the zero-sum condition is that a game with compact strategy spaces and bounded payoff functions need not possess a Nash equilibrium in mixed strategies.

In this paper, we consider finite approximations of the Sion-Wolfe game. Players are assumed to choose strategies from a finite equidistant grid. Payoff functions are adjusted correspondingly. Applying the iterated elimination of weakly dominated strategies, and subsequently characterizing the set of Nash equilibria in the reduced game, we determine the value of each finite approximation. Our main finding is that, as the number of grid points goes to infinity and the payoff function approaches that of the continuous game, the values of the finite approximations accumulate around *three* values. Moreover, those values do not correspond in a simple way to the majorant (minimax) and minorant (maximin) values of the continuous game.

As Sion and Wolfe (1957) noted, their game admits an interpretation as a Colonel Blotto game with two battlefields in which one player has a head start in the form of additional troops in one battlefield. In a standard Colonel Blotto game *without* a head start (Borel, 1921), each player allocates a budget of one unit of a perfectly divisible resource across several equivalued battlefields where, on each battlefield separately, the highest bidder wins (and the winner is drawn randomly in the case of a tie). Moreover, the resource is either used or lost. In a Colonel Blotto game *with* a head start, however, the bid of the privileged player in one battlefield is raised by the head start. The analysis of Sion and Wolfe (1957) assumed that the head start corresponds to precisely one half of the unit budget.

This interpretation carries over to the finite approximation. Denoting by n the number of

soldiers available for allocation (assumed to be the same for each player), and by k the head start consisting of additional soldiers for player 1 on one of the battlefields, we study the set of mixed-strategy Nash equilibria in the corresponding Colonel Blotto game, which will be denoted by $\mathfrak{B}(n, k)$. We restrict attention to those cases that matter in the limit consideration, i.e., n large and $k \approx n/2$. The analysis naturally leads to the consideration of three cases:

- (A) The head start is strictly larger than half of the budget, i.e., $k > n/2$;
- (B) the head start is precisely equal to half of the budget, i.e., $k = n/2$ with n even;
- (C) the head start is strictly smaller than half of the budget, i.e., $k < n/2$.

In each case, we identify a Nash equilibrium in mixed strategies and thereby characterize the unique pair of equilibrium payoffs (values).

It turns out that, even though $\mathfrak{B}(n, k)$ is, in general, not dominance solvable (in the sense of Moulin, 1979), the identification of equilibrium candidates in the finite approximations is largely simplified by the *iterated elimination of dominated strategies*.¹ In particular, it is shown that, if the approximation is sufficiently close to the continuous game, the finite game $\mathfrak{B}(n, k)$ admits an essentially unique Nash equilibrium in iteratively undominated strategies. In fact, in the most natural approximation captured by case (B), the equilibrium turns out to be unique even without the prior elimination of dominated strategies.

To understand the gap between the finite and continuous variants of the game, we apply iterated weak dominance to the continuous model as well. This exercise not only leads to a potentially more transparent proof of the main result of Sion and Wolfe (1957), but also provides valuable intuition. Specifically, the restriction to a finite grid prohibits players' mutual undercutting that is ultimately the crucial point underlying the non-existence problem. As a result, the strategic interaction in the discrete game is not representative of the strategic interaction in the continuous game. The very same intuition is supported by the analysis of an extension in which only one player is restricted to choose from a discrete set, while the other chooses from the unit interval. In contrast, there does not seem to be a tight link between the existence of ε -equilibria on the one hand and the gap between the finite and the continuous model on the other.

¹Throughout the paper, we will use term *dominance* for what is commonly understood as weak dominance between pure strategies. See Section 2 for details.

Related literature. More than a century ago, Borel (1921) proposed the study of Colonel Blotto games. Since then, a sizable literature on Colonel Blotto games has emerged. Notable contributions concerning continuous strategy spaces include Borel and Ville (1938), Gross and Wagner (1950), Friedman (1958), and Roberson (2006), in particular.² Related to the present study is work by Macdonell and Mastronardi (2015) who solved the two-battlefield case with heterogeneous budgets. Washburn (2013, Sec. 5.1.3) constructed equilibria for the Colonel Blotto game with arbitrary head starts subject to a playability constraint (which amounts to the existence of a suitable copula that satisfies the budget constraint *ex post*). Vu and Loiseau (2021, Sec. 5) studied Colonel Blotto games with three or more battlefields, allowing for pre-allocated resources and different effectiveness across players. Extending Roberson’s approach, they derived exact equilibria in special cases. Further, they obtained approximate equilibria in more general classes of Colonel Blotto games with favoritism. They achieved this by first drawing realizations of the unilaterally optimal strategies for a sufficiently large number of battlefields, and subsequently rescaling the realizations to ensure the budget constraint.

Colonel Blotto games are particularly appealing if units of the budget are indivisible. E.g., Borel considered an example with three battlefields and a budget of $n = 7$ soldiers. The number seven was chosen because it is “the smallest integer for which the game does not have simple manners of playing superior to all others” (Borel, 1921, p. 100).³ An informative review of the early literature, some of which apparently is still classified, may be found in Beale and Heselden (1962). More recently, Hart (2008) derived optimal strategies in discrete Colonel Blotto games from optimal strategies in so-called General Lotto games. In a General Lotto game, each player chooses a one-dimensional distribution that, if applied to all battlefields in an i.i.d. fashion, satisfies the budget constraint in expectation. Then, provided that an optimal strategy in a General Lotto game may be represented as a mixed strategy in a Colonel Blotto game that is symmetric across all battlefields, the mixed strategy in a Colonel Blotto game is optimal as well, i.e., an equilibrium is found in the Colonel Blotto game. That method delivers optimal strategies for Colonel Blotto games in the case of homogeneous budgets, and a variety of partial results in the

²For a survey, see Kovenock and Roberson (2010).

³Similarly, the analysis below considers the simplest non-trivial case of a Colonel Blotto game with a head start, viz. the case of two battlefields.

case of heterogeneous budgets.⁴ Note, however, that the introduction of a head start creates an asymmetry between battlefields. Therefore, the “Lotto approach” just described does not extend in an obvious way to the setup considered in the present paper.⁵ Hortala-Vallve and Llorente-Saguer (2012) studied the properties of pure-strategy equilibria in Colonel Blotto games, allowing for battlefield valuations that are heterogeneous both across battlefields and across players. The paper most closely related to the present analysis is Liang et al. (2019), who used brute force to characterize the equilibrium set of the Colonel Blotto game with two battlefields and heterogeneous budgets (yet no head start).⁶ In contrast to the present study, however, they did not apply dominance arguments.

The remainder of the paper is structured as follows. Section 2 introduces the model. Section 3 concerns the equilibrium analysis. In Section 4, we derive implications for the Sion-Wolfe example. Section 5 offers additional discussion. Section 6 concludes. Technical proofs have been relegated to an Appendix.

2. Preliminaries

2.1 Set-up and notation

Two players, called player 1 and player 2, are competing on two battlefields, called battlefield A and battlefield B. Each player commands an integer number n of indivisible soldiers, where we assume $n \geq 2$ throughout. Players compete by allocating soldiers across battlefields. We denote by x_i the number of soldiers allocated by player $i \in \{1, 2\}$ to battlefield A. Then, without loss of generality, the remainder $y_i = n - x_i$ is the number of soldiers allocated by player i to battlefield B. Thus, strategy spaces for player 1 and 2 may be specified (in reduced form) as $X_1 = \{x_1 = 0, \dots, x_1 = n\}$ and $X_2 = \{x_2 = 0, \dots, x_2 = n\}$, respectively.

⁴Cf. the discussion in Dziubiński (2013, Sec. 5).

⁵Beale and Heselden (1962), using an approach that might be considered a predecessor of Hart’s (2008), allowed for asymmetries between battlefields. While interesting, that approach leads in general to approximations of optimal strategies only. Therefore, we do not expect this approach to be useful for the general identification of equilibria in the case of asymmetries across battlefields. Cf. Washburn (2014, Sec. 6.3) whose discussion suggests a similar conclusion.

⁶A class of games related to those considered above are finite Colonel Blotto games with heterogeneous budgets yet no head start. That type of model was first considered in some generality by Hart (2008) who established, in particular, general bounds on the values. The precise characterization of values has been accomplished by Liang et al. (2019).

Player 1 is assumed to have a head start on battlefield A in the form of a nonnegative integer number k of additional soldiers. Player 2, in contrast, does not have any additional soldiers. Throughout the analysis, it will be assumed that $k \in \{2, \dots, n-1\}$.⁷ If a player has, in total, more soldiers on a battlefield than her opponent, then she wins on that battlefield and receives a reward of 1, while the loser suffers a penalty of -1 from that battlefield. In the case of a tie on a battlefield, both players get a payoff of 0 from that battlefield. Note that, after both players have deployed their troops, player 1 has a total of $x_1 + k$ soldiers on battlefield A, while player 2 has a total of x_2 soldiers there. Player 1's payoff from battlefield A is therefore given as

$$\Pi_1^A(x_1, x_2) = \text{sgn}((x_1 + k) - x_2), \quad (1)$$

where the *sign function* $\text{sgn}(\cdot)$ is defined as usual by $\text{sgn}(\xi) = 1$ if $\xi > 0$, by $\text{sgn}(\xi) = 0$ if $\xi = 0$, and by $\text{sgn}(\xi) = -1$ if $\xi < 0$. Moreover, due to the zero-sum condition, $\Pi_2^A(x_1, x_2) = -\Pi_1^A(x_1, x_2)$. On battlefield B, player 1 has a total of $y_1 = n - x_1$ soldiers, while player 2 has a total of $y_2 = n - x_2$ soldiers. From the accounting identity

$$y_1 - y_2 = (n - x_1) - (n - x_2) = x_2 - x_1, \quad (2)$$

player 1's payoff from battlefield B is therefore seen to be given as

$$\Pi_1^B(x_1, x_2) = \text{sgn}(x_2 - x_1). \quad (3)$$

Again, we have $\Pi_2^B(x_1, x_2) = -\Pi_1^B(x_1, x_2)$. Payoffs are assumed additively separable across battlefields. Consequently, player i 's total payoff is $\Pi_i(x_1, x_2) = \Pi_i^A(x_1, x_2) + \Pi_i^B(x_1, x_2)$, for $i = 1, 2$. The two-person zero-sum game just defined will be referred to as the *Colonel Blotto game with budget n and head start k* , in short $\mathfrak{B}(n, k)$.

The general structure of the payoff matrix is illustrated in Figure 1.⁸ Only player 1's payoffs are shown. Below the main diagonal, all entries vanish (player 1 wins on battlefield A, but loses on battlefield B). On the main diagonal, all entries equal 1 (player 1 wins on battlefield A, but ties with player 2 on battlefield B). The entries of the neighboring $k - 1$ upper off-diagonals

⁷While the cases $k = 0$ and $k \geq n$ are trivial, the case $k = 1$ is complicated. More importantly, none of the cases excluded by this assumption is needed for our later limit consideration.

⁸Strictly speaking, the matrix represents payoffs in the case $n > 2k + 2$ only, with (straightforward) adaptations necessary to cover other cases.

equal 2 (player 1 wins on both battlefields), while the entries of the k -th upper off-diagonal equal one (player 1 ties with player 2 on battlefield A, but wins on battlefield B). The entries in the remaining upper off-diagonals vanish (player 1 loses on battlefield A, but wins on battlefield B).

$x_1 \backslash x_2$	0	1	2	...	$k-1$	k	$k+1$...	$n-k$	$n-k+1$	$n-k$...	$n-2$	$n-1$	n
0	1	2	2	...	2	1	0	...	0	0	0	...	0	0	0
1	0	1	2	...	2	2	1	...	0	0	0	...	0	0	0
2	0	0	1	...	2	2	2	...	0	0	0	...	0	0	0
...
$k-1$	0	0	0	...	1	2	2	0	0	0
k	0	0	0	...	0	1	2	0	0	0
$k+1$	0	0	0	...	0	0	1	0	0	0
...
$n-k-1$	0	0	0	1	2	2	...	2	1	0
$n-k$	0	0	0	0	1	2	...	2	2	1
$n-k+1$	0	0	0	0	0	1	...	2	2	2
...
$n-2$	0	0	0	...	0	0	0	...	0	0	0	...	1	2	2
$n-1$	0	0	0	...	0	0	0	...	0	0	0	...	0	1	2
n	0	0	0	...	0	0	0	...	0	0	0	...	0	0	1

Figure 1: Player 1's payoffs in $\mathfrak{B}(n, k)$.

The set of probability distributions over the finite set X_i will be denoted by $\Delta(X_i)$, and any element $\mu_i \in \Delta(X_i)$ will be referred to as a *mixed strategy* for player i . Given any pure strategy realization $\nu \in \{0, \dots, n\}$, the *Dirac distribution* δ_ν is the probability distribution on $\{0, \dots, n\}$ that gives all probability weight to the realization ν , i.e., $\delta_\nu(\{\nu\}) = 1$, and $\delta_\nu(\{\hat{\nu}\}) = 0$ for any $\hat{\nu} \neq \nu$. Then, any probability distribution $\mu_i \in \Delta(X_i)$ may be expressed in a unique way as a weighted sum $\mu_i = \sum_{\nu=0}^n p_\nu \delta_\nu$, where $p_\nu \in [0, 1]$ is the probability that player i 's mixed strategy μ_i realizes as $x_i = \nu$, so that $\sum_{\nu=0}^n p_\nu = 1$. For any $\mu_i \in \Delta(X_i)$, we call the set $\text{supp}\{\mu_i^*\} = \{\nu : p_\nu > 0\} \subseteq X_i$ the *support* of μ_i . As usual, players' payoff functions are extended to mixed strategies by taking expectations. We will use notation such as $E_{\mu_i}[\cdot]$ and $E_{\mu_1, \mu_2}[\cdot]$ for the respective expected values. A mixed-strategy Nash equilibrium is a pair $(\mu_1^*, \mu_2^*) \in \Delta(X_1) \times \Delta(X_2)$ such that $E_{\mu_1^*, \mu_2^*}[\Pi_1(x_1, x_2)] \geq E_{\mu_1, \mu_2^*}[\Pi_1(x_1, x_2)]$ for any $\mu_1 \in \Delta(X_1)$, and $E_{\mu_1^*, \mu_2^*}[\Pi_2(x_1, x_2)] \geq E_{\mu_1^*, \mu_2}[\Pi_2(x_1, x_2)]$ for any $\mu_2 \in \Delta(X_2)$.

Given that the considered game is finite, a mixed-strategy equilibrium exists by Nash's theo-

rem. Moreover, $\mathfrak{B}(n, k)$ is zero-sum, so that all equilibria are payoff-equivalent. We will refer to player 1's equilibrium payoff $v_1 = E_{\mu_1^*, \mu_2^*}[\Pi_1(x_1, x_2)]$ as the *value* of the game. Since player 1 has an advantage over player 2, we (correctly) anticipate that $v_1 \in [0, 1]$.

2.2 Dominated strategies

The consideration of dominance relationships between strategies (Farquharson, 1969; Brams, 1975; Moulin, 1979) turns out to be a useful tool for identifying equilibria in the finite approximations of the Sion-Wolfe game. We recall the definitions.

Let $\hat{X}_1 \subseteq X_1$ and $\hat{X}_2 \subseteq X_2$ be nonempty sets of strategies for both players. We will say that $x_1 \in \hat{X}_1$ is *dominated by* $\hat{x}_1 \in \hat{X}_1$ for player 1 in $\hat{X}_1 \times \hat{X}_2$ if the following two conditions are simultaneously satisfied. First, for any pure strategy $x_2 \in \hat{X}_2$, we have $\Pi_1(x_1, x_2) \leq \Pi_1(\hat{x}_1, x_2)$. Second, there exists a strategy $\hat{x}_2 \in \hat{X}_2$ such that $\Pi_1(x_1, \hat{x}_2) < \Pi_1(\hat{x}_1, \hat{x}_2)$. Similarly, we will say that a pure strategy $x_2 \in \hat{X}_2$ is dominated by a strategy $\hat{x}_2 \in \hat{X}_2$ for player 2 in $\hat{X}_1 \times \hat{X}_2$ if analogous conditions hold with the roles of players 1 and 2 exchanged. We will say that $x_i \in \hat{X}_i$ is *dominated in* $\hat{X}_1 \times \hat{X}_2$ if there exists a strategy $\hat{x}_i \in \hat{X}_i$ such that x_i is dominated by \hat{x}_i for player i in $\hat{X}_1 \times \hat{X}_2$.⁹

We are interested in undominated strategies. Let, therefore, $X_1^{(1)}$ and $X_2^{(1)}$ denote the sets of strategies for player 1 and 2, respectively, that are *not* dominated in $X_1 \times X_2$. The following result characterizes these sets in the Colonel Blotto game with budget n and head start k .

Lemma 1. *The respective sets of undominated strategies in $\mathfrak{B}(n, k)$ are given by $X_1^{(1)} = \{0, \dots, n - k + 1\}$ and $X_2^{(1)} = \{0\} \cup \{k + 1, \dots, n\}$, respectively.*

Proof. As can be seen from Figure 1, $\hat{x}_1 = n - k + 1$ dominates all $x_1 \in \{n - k + 2, \dots, n\}$ for player 1. There are no other dominated strategies for player 1. Similarly, $\hat{x}_2 = 0$ dominates all $x_2 \in \{1, \dots, k\}$ for player 2, while no other strategy is dominated for player 2. \square

We provide some intuition. In the considered case, strategy $\hat{x}_1 = n - k + 1$ dominates all higher strategies for player 1. Indeed, any strategy $x_1 \geq n - k + 1$ secures a win on battlefield A, but

⁹In general, a pure strategy may be dominated by a mixed strategy even though it is not dominated by any pure strategy (cf. Pearce, 1984). However, the additional mileage gained by using the more powerful definition is limited in our present application. Hence, we stick to the basic concept.

$\hat{x}_1 = n - k + 1$ is the most parsimonious in doing so, i.e., it leaves the largest number of soldiers for battlefield B. Similarly, for player 2, bidding any $x_2 \in \{1, \dots, k - 1\}$ is dominated by $\hat{x}_2 = 0$, because out of all strategies that imply a certain loss on battlefield A, $\hat{x}_2 = 0$ brings most soldiers to battlefield B. In fact, also $x_2 = k$ is dominated by $\hat{x}_2 = 0$ because the only scenario in which player 2 achieves a tie on battlefield A with $x_2 = k$ entails that player 1 sends n soldiers to battlefield B, so that player 2 loses on battlefield B rather than achieving a tie there with $\hat{x}_2 = 0$.

Starting from players' unrestricted strategy spaces, we may recursively define

$$X_i^{(0)} = X_i \quad (i \in \{1, 2\}), \quad (4)$$

$$X_i^{(t)} = \left\{ x_i \in X_i^{(t-1)} \text{ s.t. } x_i \text{ is not dominated in } X_1^{(t-1)} \times X_2^{(t-1)} \right\} \quad (i \in \{1, 2\}; t \in \{1, 2, \dots\}). \quad (5)$$

Needless to say, this extends our earlier definition of $X_1^{(1)}$ and $X_2^{(1)}$. A simple induction argument shows that $X_i^{(t)} \neq \emptyset$ for any i and t . Since $\mathfrak{B}(n, k)$ is a finite game, no further eliminations take place for sufficiently high t , so that $X_1^{(t)} \times X_2^{(t)} = X_1^{(t-1)} \times X_2^{(t-1)}$. In this case, we refer to $X_i^{(\infty)} \equiv X_i^{(t)}$ as the set of strategies surviving the *IEDS* (iterated elimination of dominated strategies) procedure.

Two well-known facts about dominated strategies should be recalled. First, the elimination of dominated strategies may eliminate Nash equilibria.¹⁰ Second, and more importantly for the identification of equilibrium strategies, a Nash equilibrium found in a reduced game obtained by iteratively eliminating dominated strategies remains an equilibrium in the original game.¹¹

We will say that a mixed-strategy equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$ in $\mathfrak{B}(n, k)$ is an *equilibrium in iteratively undominated strategies* if $\text{supp}\{\mu_i^*\} \subseteq X_i^{(\infty)}$ for $i \in \{1, 2\}$. Using the second fact recalled above, the set of equilibria in iteratively undominated strategies is isomorphic to the set of equilibria in the reduced game obtained via the IEDS procedure from $\mathfrak{B}(n, k)$.

3. Equilibrium analysis

In this section, we will identify Nash equilibria in the finite Colonel Blotto game with budget n and head start k . As discussed in the Introduction, there are three cases.

¹⁰The analogous statement for strictly dominated strategies is not true, of course.

¹¹For a formal argument, see the proof of Proposition 1 in Moulin (1979).

3.1 The case $k > n/2$

Suppose first that the head start is strictly larger than half of the budget. In that case, dominance arguments turn out to have a lot of bite and quickly guide to an equilibrium. The following example illustrates this fact.

(a)

x_1	x_2	0	1	2	3	4	5
0	1	2	2	1	0	0	
1	0	1	2	2	1	0	
2	0	0	1	2	2	1	
3	0	0	0	1	2	2	
4	0	0	0	0	1	2	
5	0	0	0	0	0	1	

(b)

x_1	x_2	0	4	5
0	1	0	0	
1	0	1	0	
2	0	2	1	
3	0	2	2	

(c)

x_1	x_2	0	5
0	1	0	
3	0	2	

Figure 2: Elimination of dominated strategies in $\mathfrak{B}(5, 3)$.

Example 1. Let $n = 5$ and $k = 3$. The payoff matrix of $\mathfrak{B}(5, 3)$ is shown in panel (a) of Figure 2, where dominated strategies are marked in grey. Panels (b) and (c) exhibit the reduced game after one and two rounds of elimination, respectively. As can be seen, the reduced game obtained after application of IEDS is represented by a simple two-by-two matrix. Player 1 either sends $x_1 = 3$ soldiers to battlefield A (the minimum necessary to win with certainty there), or she sends all her troops to battlefield B (if $x_1 = 0$ so that $y_1 = 5$). Similarly, Player 2 throws all her troops either on battlefield A (if $x_2 = 5$) or on battlefield B (if $x_2 = 0$, so that $y_2 = 5$). We conclude that the strategies $\mu_1^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_3$ and $\mu_2^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_5$ form the unique equilibrium in iteratively undominated strategies.¹²

¹²For this, we make use of the aforementioned fact that a Nash equilibrium in a given game remains a Nash equilibrium if a dominated strategy is added to the game. Starting from the mixed equilibrium in the two-by-two game, and iteratively adding strategies in reversed order of the elimination procedure shows that the Nash equilibrium in the reduced form is a Nash equilibrium also in $\mathfrak{B}(5, 3)$.

The general analysis in the case $k > n/2$ is entirely analogous. We arrive at our first result.

Proposition 1. *Suppose that $k > n/2$. Then, there is a unique equilibrium in iteratively undominated strategies in $\mathfrak{B}(n, k)$, which is given by $\mu_1^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_{n-k+1}$ and $\mu_2^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_n$. The value is $v_1 = \frac{2}{3}$.*

Proof. See the Appendix. \square

Admitting strategies eliminated by the elimination procedure, the equilibrium is no longer unique. In Example 1, any strategy for player 2 of the form $\mu_2^\lambda = \frac{2}{3}\delta_0 + \frac{\lambda}{3}\delta_4 + \frac{1-\lambda}{3}\delta_5$, with $\lambda \in [0, 1]$ is optimal. However, only for $\lambda = 0$, the corresponding equilibrium (μ_1^*, μ_2^λ) is in iteratively undominated strategies.

3.2 The case $k = n/2$, with n even

In this case, the iterated elimination of dominated strategies is equally effective as before. In fact, the equilibrium turns out to be unique.

Proposition 2. *Suppose that $k = n/2$, with n even. Then, there is a unique Nash equilibrium in $\mathfrak{B}(n, k)$, which is given by $\mu_1^* = \frac{3}{5}\delta_0 + \frac{1}{5}\delta_k + \frac{1}{5}\delta_{k+1}$ and $\mu_2^* = \frac{3}{5}\delta_0 + \frac{1}{5}\delta_{k+1} + \frac{1}{5}\delta_n$. The value of $\mathfrak{B}(n, k)$ is $v_1 = \frac{3}{5}$.*

Proof. See the Appendix. \square

To prove uniqueness, we show that pure strategies outside of the equilibrium support are never a best response to the equilibrium identified by the IEDS procedure. We then exploit interchangeability of Nash equilibria in two-person constant-sum games. It suffices to note that the payoff matrix, cut back to represent only strategies in the equilibrium support, is invertible.

3.3 The case $n = 2k + r$, with $r \in \{1, \dots, k-1\}$

The structure of equilibrium changes slightly compared to the previous cases. We start again with an example.

Example 2. Let $n = 5$ and $k = 2$. The elimination procedure simplifies the game, as shown in panels (a) through (c) of Figure 3. The unique equilibrium in iteratively undominated strategies is given by $\mu_1^* = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_2 + \frac{1}{4}\delta_4$ and $\mu_2^* = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_3 + \frac{1}{4}\delta_5$. The value is $v_1 = \frac{1}{2}$.

(a)

	x_2	0	1	2	3	4	5
x_1	0	1	2	1	0	0	0
	1	0	1	2	1	0	0
	2	0	0	0	2	1	0
	3	0	0	0	1	2	1
	4	0	0	0	0	1	2
	5	0	0	0	0	0	1

(b)

	x_2	0	3	4	5
x_1	0	1	0	0	0
	1	0	1	0	0
	2	0	2	1	0
	3	0	1	2	1
	4	0	0	1	2

(c)

	x_2	0	3	4	5
x_1	0	1	0	0	0
	2	0	2	1	0
	3	0	1	2	1
	4	0	0	1	2

Figure 3: Elimination of dominated strategies in $\mathfrak{B}(5, 2)$.

The following result identifies an equilibrium in this case.

Proposition 3. *Let $n = 2k + r$ with $r \in \{1, \dots, k - 2\}$ and k sufficiently large.¹³ Then, an equilibrium in $\mathfrak{B}(n, k)$ is given by $\mu_1^* = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_k + \frac{1}{4}\delta_{n-k+1}$ and $\mu_2^* = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_{k+r} + \frac{1}{4}\delta_n$. The equilibrium is unique in iteratively undominated strategies unless $r = 1$, in which case it has one degree of freedom (for player 1). The corresponding value is always $v_1 = \frac{1}{2}$.*

Proof. See the Appendix. \square

Thus, the equilibrium in iteratively undominated strategies is unique for $r \neq 1$. For $r = 1$, however, player 1 may substitute a fair draw between $x_1 = k$ and $x_1 = n - k + 1 = k + 2$ by the pure strategy $x_1 = k + 1$ without affecting the equilibrium.

4. Implications for the Sion-Wolfe example

Sion and Wolfe (1957, Sec. 2) considered a Colonel Blotto game with two battlefields and homogeneous budgets in which one player has a head start in one battlefield corresponding in size to half of the budget. In contrast to the model considered above, however, they worked with continuous

¹³The cases where either $r = k - 1$ or k is small are not needed for the limit consideration and, therefore, omitted.

strategy spaces. In this section, we will review their main result and relate it to our analysis of finite approximations.

4.1 The Sion-Wolfe game

The continuous model is defined in analogy to the finite case. Given a head start parameter $\kappa \geq 0$, let the two-person zero-sum game \mathfrak{B}_κ on the unit square be given by payoff functions

$$\pi_1(x_1, x_2) = \text{sgn}((x_1 + \kappa) - x_2) + \text{sgn}(x_2 - x_1), \quad (6)$$

and $\pi_2(x_1, x_2) = -\pi_1(x_1, x_2)$, respectively. In the mixed extension, players choose arbitrary probability distributions on the unit interval, i.e., probability measures defined on the Borel sets of $[0, 1]$. We denote by $\tilde{\mu}_i$ the mixed strategy of player $i \in \{1, 2\}$. The main result in Sion and Wolfe (1957) may now be summarized as follows.

Lemma 2. (Sion and Wolfe, 1957) *Suppose that players choose probability distributions $\tilde{\mu}_1$ and $\tilde{\mu}_2$, respectively, on the unit interval. Then,*

$$\underline{v}^{\text{SW57}} \equiv \sup_{\tilde{\mu}_1} \inf_{\tilde{\mu}_2} \iint_{[0,1] \times [0,1]} \pi_1(x_1, x_2) d\tilde{\mu}_1(x_1) d\tilde{\mu}_2(x_2) = \frac{4}{7} \simeq 0.571, \quad (7)$$

$$\bar{v}^{\text{SW57}} \equiv \inf_{\tilde{\mu}_2} \sup_{\tilde{\mu}_1} \iint_{[0,1] \times [0,1]} \pi_1(x_1, x_2) d\tilde{\mu}_1(x_1) d\tilde{\mu}_2(x_2) = \frac{2}{3} \simeq 0.667. \quad (8)$$

In particular, $\mathfrak{B}_{0.5}$ has no value.

Proof. See the Appendix. \square

The maximin value (or minorant value) \underline{v} of $\mathfrak{B}_{0.5}$ is defined through equation (7). As usual, it is defined from player 1's perspective. Intuitively, this value is the expected payoff level that player 1 can guarantee herself in a sequential setting in which a first-moving player 1's mixed strategy (but not its pure-strategy realization) is observed by a second-moving player 2. The minimax value (or majorant value) \bar{v} of $\mathfrak{B}_{0.5}$ is, in turn, defined through equation (8). One way to interpret this number in a game-theoretic setting is to consider it as the negative of the maximin value from player 2's perspective, i.e., $\bar{v}_1 = -\underline{v}_2$, where \underline{v}_2 is defined in analogy to (7) with the roles of players 1 and 2 exchanged. The remarkable point about Lemma 2 is that $\underline{v}_1 < \bar{v}_1$, i.e., the

maximin value is strictly smaller than the corresponding minimax value. As a result, the game $\mathfrak{B}_{0.5}$ has no value, and there is no Nash equilibrium in mixed strategies.¹⁴

We briefly survey existing approaches to resolving the non-existence problem. First, as pointed out by Parthasarathy (1970), if one of the players is restricted to use an absolutely continuous strategy, then the resulting game has a value. Second, Kindler (1983) proposed the use of finitely additive measures. While that approach is intuitive, it is not straightforward to give the resulting solution an economic interpretation.¹⁵ Next, Boudreau and Schwartz (2019) proposed to modify the Sion-Wolfe game at points of discontinuity. Even though this approach is consistent with Simon and Zame’s (1990) idea of endogenizing the payoff implications of ties, it does not resolve the nonexistence problem for the original game. Finally, one might resort to ε -equilibria (Radner, 1980), as done by Vu and Loiseau (2021) in closely related games. However, for $\varepsilon > 0$ small enough, even ε -equilibrium will not exist, as noted by Dasgupta and Maskin (1986).¹⁶

4.2 Finite approximations

As discussed in the Introduction, we try to shed light on the result above by considering discrete-grid approximations of $\mathfrak{B}_{0.5}$. Thus, we consider sequences of finite Colonel Blotto games $\mathfrak{B}(n, k)$, where $n, k \rightarrow \infty$ and $n/k \simeq 2$. By the results obtained so far, each such $\mathfrak{B}(n, k)$, for n sufficiently large and for n/k not too far away from 2, admits an essentially unique Nash equilibrium in iteratively undominated strategies. Taking the respective limits of these sequences, we arrive at

¹⁴It may be noted that we departed from Sion and Wolfe’s (1957) original notation, which leads to the potentially unfamiliar values for the maximin and minimax values shown in equations (7) and (8). Details on the transformation applied may be found in the Appendix. In a nutshell, we swapped the roles of the two players, and likewise the roles of the two battlefields. Moreover, we normalized payoffs so that a tie on a battlefield implies a payoff of zero from the battlefield. In fact, that same payoff normalization is proposed in the original article as well when the abstract game is interpreted as a Colonel Blotto game with head start.

¹⁵Finitely additive measures are a more flexible notion of probability distribution, where σ -additivity is replaced by finite additivity (Yanovskaya, 1970). Intuitively, this means that it is feasible to define a bid that exceeds a given (deterministic) bid by an arbitrarily small amount. However, from an economic perspective, finitely additive measures suffer from the problem that Fubini’s theorem does not in general hold. In plain English, this means that ex-ante expected payoffs need not be well-defined. The reason is that, should both players make use of an arbitrarily small increment over an existing bid, then there would be ambiguity regarding who wins with what probability.

¹⁶We will resume the discussion of ε -equilibria further below.

the three mixed strategy profiles

$$\tilde{\mu}^{\text{Huey}} = \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_{1/2}, \frac{1}{2}\delta_0 + \frac{1}{4}\delta_{1/2} + \frac{1}{4}\delta_1 \right), \quad (9)$$

$$\tilde{\mu}^{\text{Dewey}} = \left(\frac{3}{5}\delta_0 + \frac{2}{5}\delta_{1/2}, \frac{3}{5}\delta_0 + \frac{1}{5}\delta_{1/2} + \frac{1}{5}\delta_1 \right), \quad (10)$$

$$\tilde{\mu}^{\text{Louie}} = \left(\frac{2}{3}\delta_0 + \frac{1}{3}\delta_{1/2}, \frac{2}{3}\delta_0 + \frac{1}{3}\delta_1 \right), \quad (11)$$

where the Dirac distributions are defined in straightforward extension of the finite case. For example, in the profile $\tilde{\mu}^{\text{Dewey}}$, derived using Proposition 2 in the case where $n = 2k$, the privileged player 1 sends all of her troops to battlefield B with probability $3/5$, and splits her troops equally across the two battlefields with probability $2/5$. Player 2, however, sends all of her troops to battlefield B with probability $3/5$, splits her troops equally with probability $1/5$, and send all of her troops to battlefield A with probability $1/5$. The other profiles admit analogous interpretations.

Using this notation, we can make the following observation.

Corollary 1. *Consider a sequence of discrete Colonel Blotto games $\mathfrak{B}(n, k)$ with $n, k \rightarrow \infty$ such that $n/k \rightarrow 2$. Then, the accumulation points of the corresponding sequence of unique Nash equilibria in iteratively undominated strategies are contained in the set*

$$\mathfrak{M}_3 = \{\tilde{\mu}^{\text{Huey}}, \tilde{\mu}^{\text{Dewey}}, \tilde{\mu}^{\text{Louie}}\}. \quad (12)$$

Conversely, there exists a sequence of discrete Colonel Blotto games with $n, k \rightarrow \infty$ such that $n/k \rightarrow 2$ and such that the accumulation points of the corresponding sequence of unique Nash equilibria in iteratively undominated strategies are precisely the elements of \mathfrak{M}_3 .

Proof. Consider a sequence of discrete Colonel Blotto games $\mathfrak{B}(n, k)$ with $n, k \rightarrow \infty$, $n/k < 2$, and $n - 2k = o(n)$.¹⁷ Then, the corresponding sequence of iteratively undominated Nash equilibria characterized in Proposition 1 converges weakly to the limit profile $\tilde{\mu}^{\text{Louie}}$. Next, by Proposition 2, any sequence of unique Nash equilibria in $\mathfrak{B}(n, k)$ with $n, k \rightarrow \infty$ such that $n/k = 2$ converges weakly to the limit profile $\tilde{\mu}^{\text{Dewey}}$. Consider, finally, the sequence of discrete Colonel Blotto games $\mathfrak{B}(n, k)$ with $n, k \rightarrow \infty$, $n/k > 2$, and $n - 2k = o(n)$. Then, the corresponding sequence of Nash equilibria characterized in Proposition 3, which have been seen to be essentially unique in

¹⁷As usual, the notation $f(n) = o(n)$ means that $\lim_{n \rightarrow \infty} f(n)/n = 0$.

iteratively undominated strategies for n large enough, converges weakly to the limit profile $\tilde{\mu}^{\text{Huey}}$. This proves the claim. \square

The situation does not become clearer when one compares the values. The corresponding limit values are given by

$$v^{\text{Huey}} = \frac{2}{3} \simeq 0.667 \quad (13)$$

$$v^{\text{Dewey}} = \frac{3}{5} \simeq 0.600 \quad (14)$$

$$v^{\text{Louie}} = \frac{1}{2} \simeq 0.500. \quad (15)$$

What is a bit strange about Corollary 1 is that the accumulation points of values of finite discretizations of the Sion-Wolfe game do not correspond in a simple way to the minimax and maximin values of the continuous game shown in Lemma 2. Specifically, $\underline{v}^{\text{SW57}}$ lies strictly between v^{Louie} and v^{Dewey} , while \bar{v}^{SW57} equals v^{Huey} . Moreover, $v^{\text{Dewey}} = \frac{3}{5}$ is the limit of v_1 as $n = 2k \rightarrow \infty$, but this value does not appear in the statement of Lemma 2.

Using our notation, players' maximin strategies identified by Sion and Wolfe (1957) read

$$\mu_1^{\text{SW57}} = \frac{4}{7}\delta_0 + \frac{2}{7}\delta_{1/2} + \frac{1}{7}\delta_{3/4}, \quad (16)$$

$$\mu_2^{\text{SW57}} = \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{1/2} + \frac{1}{3}\delta_1. \quad (17)$$

These strategies are not components of any element of \mathfrak{M}_3 , neither do they relate to the members \mathfrak{M}_3 in any transparent way. In fact, player 1's respective component strategies $\tilde{\mu}_1^{\text{Huey}}$, $\tilde{\mu}_1^{\text{Dewey}}$, and $\tilde{\mu}_1^{\text{Louie}}$ fail to be maximin strategies in the continuous game.¹⁸ For player 2, it turns out that

¹⁸To see this for $\tilde{\mu}_1^{\text{Louie}}$, for example, it suffices to check that

$$\begin{aligned} E_{\tilde{\mu}_1^{\text{Louie}}}[\pi_1(x_1, 1)] &= \frac{2}{3}\pi_1(0, 1) + \frac{1}{3}\pi_2(\frac{1}{2}, 1) \\ &= \frac{2}{3}(\text{sgn}((0 + \frac{1}{2}) - 1) + \text{sgn}(1 - 0)) + \frac{1}{3}(\text{sgn}((\frac{1}{2} + \frac{1}{2}) - 1) + \text{sgn}(1 - \frac{1}{2})) \\ &= \frac{1}{3} < \underline{v}_1. \end{aligned}$$

Analogous strict inequalities, likewise for $x_2 = 1$, hold for $\tilde{\mu}_1^{\text{Dewey}}$ and $\tilde{\mu}_1^{\text{Huey}}$.

$\tilde{\mu}_2^{\text{Louie}}$ is actually a maximin strategy since

$$\begin{aligned} & \inf_{x_1 \in [0,1]} E_{\tilde{\mu}_2^{\text{Louie}}}[\pi_2(x_1, x_2)] \\ &= \inf_{x_1 \in [0,1]} \frac{2}{3}\pi_2(x_1, 0) + \frac{1}{3}\pi_2(x_1, 1) \end{aligned} \quad (18)$$

$$= \inf_{x_1 \in [0,1]} -\frac{2}{3} \left(\text{sgn}(x_1 + \frac{1}{2}) - \text{sgn}(x_1) \right) - \frac{1}{3} \left(\text{sgn}(x_1 - \frac{1}{2}) + \text{sgn}(1 - x_1) \right) \quad (19)$$

$$= -\frac{2}{3} = \underline{v}_2. \quad (20)$$

However, $\tilde{\mu}_2^{\text{Dewey}}$ and $\tilde{\mu}_2^{\text{Huey}}$ fail to be maximin strategies for player 2.¹⁹ Further, none of the six strategies appearing as components of a profile in \mathfrak{M}_3 plays any role in the analysis of Sion and Wolfe (1957). Thus, there does not seem to be any straightforward connection between the limits of minimax/maximin values and strategies in the finite approximations and the minimax/maximin values and strategies in the continuous game.

4.3 Dominated strategies

To understand what is going on, we apply the iterated elimination of weakly dominated strategies to the Sion-Wolfe game.²⁰

We will need some definitions. Let $G = (\mathfrak{X}_1, \mathfrak{X}_2, u_1, u_2)$ be an arbitrary two-person zero-sum game, i.e., \mathfrak{X}_1 and \mathfrak{X}_2 are nonempty sets of strategies of players 1 and 2, respectively, and $u_1 : \mathfrak{X}_1 \times \mathfrak{X}_2 \rightarrow \mathbb{R}$ and $u_2 : \mathfrak{X}_1 \times \mathfrak{X}_2 \rightarrow \mathbb{R}$ are the respective payoff functions, where $u_1 \equiv -u_2$. Let

$$\underline{v}_G = \sup_{\xi_1 \in \mathfrak{X}_1} \inf_{\xi_2 \in \mathfrak{X}_2} u_1(\xi_1, \xi_2) \quad (21)$$

and

$$\bar{v}_G = \inf_{\xi_2 \in \mathfrak{X}_2} \sup_{\xi_1 \in \mathfrak{X}_1} u_1(\xi_1, \xi_2), \quad (22)$$

respectively, denote the *minorant* and the *majorant* of the game G .²¹ Then, $\bar{v}_G \geq \underline{v}_G$.²² As

¹⁹Indeed, we have

$$\begin{aligned} E_{\tilde{\mu}_2^{\text{Dewey}}}[\pi_2(0, x_2)] &= \frac{3}{5}\pi_2(0, 0) + \frac{1}{5}\pi_2(0, \frac{1}{2}) + \frac{1}{5}\pi_2(0, 1) \\ &= -\frac{3}{5} \left(\text{sgn}((0 + \frac{1}{2}) - 0) + \text{sgn}(0 - 0) \right) - \frac{1}{5} \left(\text{sgn}((0 + \frac{1}{2}) - \frac{1}{2}) + \text{sgn}(\frac{1}{2} - 0) \right) \\ &\quad - \frac{1}{5} \left(\text{sgn}((0 + \frac{1}{2}) - 1) + \text{sgn}(1 - 0) \right) \\ &= -\frac{4}{5} < \underline{v}_2. \end{aligned}$$

A similar inequality holds for $\tilde{\mu}_2^{\text{Huey}}$, again with $x_1 = 0$.

²⁰We are indebted to Marco Serena for suggesting taking that route.

²¹Both minorant and majorant are guaranteed to be finite if u_1 is a bounded function.

²²Clearly, $u_1(\xi_1, \xi_2) \geq \inf_{\hat{\xi}_2 \in \mathfrak{X}_2} u_1(\xi_1, \hat{\xi}_2)$, for any $\xi_1 \in \mathfrak{X}_1$ and $\xi_2 \in \mathfrak{X}_2$. Hence, $\sup_{\xi_1 \in \mathfrak{X}_1} u_1(\xi_1, \xi_2) \geq \underline{v}_G$, for any $\xi_2 \in \mathfrak{X}_2$. Taking the infimum over all $\xi_2 \in \mathfrak{X}_2$ yields the claim.

before, G is said to possess a *value* if $\bar{v}_G = \underline{v}_G \equiv v_G$. In that case, v_G is referred to as the value of G . If G admits a Nash equilibrium (ξ_1^*, ξ_2^*) , then G has a value.²³ Thus, as mentioned in the Introduction, a game without value does not possess a Nash equilibrium.²⁴

The following observation, for which we could not find a reference, will be our main tool for providing an intuition underlying the Sion-Wolfe game.

Lemma 3. *Suppose that G' results from G by eliminating a set of strategies $\mathfrak{S}_i \subseteq \mathfrak{X}_i$ for each player $i \in \{1, 2\}$ that are dominated by some fixed strategy $\hat{\xi}_i \in \mathfrak{X}_i$. Then $\bar{v}_{G'} = \bar{v}_G$ and $\underline{v}_{G'} = \underline{v}_G$. In particular, G has a value if and only if G' has a value.*

Proof. See the Appendix. \square

Thus, both the majorant and the minorant are unchanged if, for each player, a set of weakly dominated strategies is eliminated, provided that the dominating strategies are kept. In infinite games, it is indeed crucial that the dominating strategy is kept. Otherwise, the elimination may lead to different values for the majorant and minorant.²⁵ As will become clear below, the game used by Sion and Wolfe (1957) is an example illustrating this possibility. The way in which Lemma 3 is formulated, however, makes it immune to this problem.

A straightforward variant of Lemma 3 assumes that G is the mixed extension of the Sion-Wolfe game, and that weak dominance is applied to pure strategies. Using that variant, we arrive at the following result.

Proposition 4. *Let $\varepsilon > 0$ be small. Then, the Sion-Wolfe game may be simplified by the iterated elimination of weakly dominated strategies (keeping the dominating strategies in each step) so that spaces of pure strategies in the reduced game are given by $X_1(\varepsilon) = \{0\} \cup \{\frac{1}{2}\} \cup (\frac{1}{2}, \frac{1}{2} + \varepsilon]$ and $X_2(\varepsilon) = \{0\} \cup (\frac{1}{2}, \frac{1}{2} + \varepsilon] \cup \{1\}$, respectively.*

Proof. We execute two rounds of elimination. In the first round, we note that, in analogy to Lemma 1, any strategy $x_1 \in (\frac{1}{2} + \varepsilon, 1]$ is weakly dominated by $\hat{x}_1 = \frac{1}{2} + \varepsilon$, and that any strategy

²³Since (ξ_1^*, ξ_2^*) is a Nash equilibrium, $u_1(\xi_1^*, \xi_2^*) = \sup_{\xi_1 \in \mathfrak{X}_1} u_1(\xi_1, \xi_2^*) \geq \bar{v}_G$. Moreover, from the zero-sum property, $u_1(\xi_1^*, \xi_2^*) = \inf_{\xi_2 \in \mathfrak{X}_2} u_1(\xi_1^*, \xi_2) \leq \underline{v}_G$. The claim follows.

²⁴For mixed extensions of finite two-person zero-sum games, the converse of Lemma 2 holds as well (Osborne and Rubinstein, 1994). The converse of Lemma 2 does not hold in general, however, as we discuss in Section 4.

²⁵In general, the elimination of strategies for player 1 (player 2) may raise (lower) either the minorant or the majorant (or both). This is immediate from the definitions.

$x_2 \in (0, \frac{1}{2}]$ is weakly dominated by $\hat{x}_2 = 0$. This leaves us with strategy spaces $X'_1 = [0, \frac{1}{2} + \varepsilon]$ and $X'_2 = \{0\} \cup (\frac{1}{2}, 1]$.²⁶ In a second round of elimination, we see that any $x_1 \in (0, \frac{1}{2})$ is weakly dominated by $\hat{x}_1 = \frac{1}{2}$, and that any $x_2 \in (\frac{1}{2} + \varepsilon, 1)$ is weakly dominated by $\hat{x}_2 = 1$. This proves the claim. \square

Proposition 4 sheds light on the Sion-Wolfe game by showing that each player has essentially three strategic options in the reduced game. Indeed, in the reduced game, player 1 chooses between three options, either $x_1 = 0$, or $x_1 = \frac{1}{2}$, or some number slightly above one half, i.e., $x_1 = \frac{1}{2} + \varepsilon_1$, where $\varepsilon_1 \in (0, \varepsilon]$. The second-moving player 2 chooses between $x_2 = 0$, or a bid slightly above one half, i.e., $x_2 = \frac{1}{2} + \varepsilon_2$, where $\varepsilon_2 \in (0, \varepsilon]$, or $x_2 = 1$.

		x_2		
		0	$0.5 + \varepsilon_2$	1
x_1	0	1	0	0
	0.5	0	2	1
	$0.5 + \varepsilon_1$	0	z	2

Figure 4: Elimination of dominated strategies in $\mathfrak{B}_{0.5}$.

This particular structure plays out in the case where moves are sequential, as implicit in the definition of minorant and majorant. A first-moving player 1 bidding $\frac{1}{2} + \varepsilon_1$ cannot avoid being undercut by player 2. Therefore, the minorant \underline{v}_G may be determined from the game in Figure 4 with $z = 0$. In that case, an optimal strategy for player 1 is $\mu_1^* = \frac{4}{7}\delta_0 + \frac{2}{7}\delta_{0.5} + \frac{1}{7}\delta_{0.5+\varepsilon}$. For $\varepsilon = 0.25$, this is indeed equal to μ_1^{SW57} . In fact, the corresponding minorant is $\underline{v} = \frac{4}{7}$ equals $\underline{v}^{\text{SW57}}$ regardless of the choice of ε .

If player 2 moves first, however, then the bidding game allows player 1 to undercut player 2, which means that the majorant may be determined from the game in Figure 4 with $z = 2$. In that case, an optimal strategy for player 2 is $\mu_2^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_1$. The mixed strategy μ_2^{SW57} is equivalent

²⁶The eliminations applied in the proof of Proposition 4 do not eliminate all weakly dominated strategies for player 1 in the first round. Doing this would take away also dominating strategies and, hence, be inconsistent with the assumptions of Lemma 3. Indeed, eliminating all weakly dominated strategies in each round, it is easy to check that $X_1^{(1)} = [0, \frac{1}{2}]$ and $X_2^{(1)} = \{0\} \cup (\frac{1}{2}, 1]$, so that $X_1^{(2)} = \{0, \frac{1}{2}\}$ and $X_2^{(2)} = \{0, 1\}$. The resulting two-by-two game has value $v = \frac{1}{2}$, i.e., both minorant and majorant are affected.

in terms of expected payoffs but uses the weakly dominated pure strategy $x_2 = \frac{1}{2}$ without good reason. The corresponding majorant is $\bar{v} = \frac{2}{3} = \bar{v}^{\text{SW57}}$.

The logic of mutual undercutting breaks down when continuous strategy spaces are replaced by finite approximations. For example, if $k = n/2$ with n even, as in Proposition 2, then there is a lowest bid strictly above k , which is $k + 1$. Then, regardless of the order of moves, the payoff matrix with player 1 choosing $x_1 \in \{0, k, k + 1\}$ and player 2 choosing $x_2 \in \{0, k + 1, n\}$ would be the one shown in Figure 4 with the tie-breaking payoff $z = 1$. This is why, even in that most natural approximation, neither the limit strategy profile $\tilde{\mu}^{\text{Dewey}}$ nor the corresponding limit $v^{\text{Dewey}} = \frac{3}{5}$ have no simple interpretation in $\mathfrak{B}_{0.5}$.

Similar considerations can be made in the other two cases considered above. If $k > n/2$, as in Proposition 1, then the payoff matrix with player 1 choosing $x_1 \in \{0, k, n - k + 1\}$ and player 2 choosing $x_2 \in \{0, k + 1, n\}$ would be the one shown in Figure 4 with $z = 2$. This explains why $v^{\text{Huey}} = \frac{2}{3} = \bar{v}^{\text{SW57}}$. Finally, if $k < n/2$, as in Proposition 3, then for the same set of pure strategies, we would have $z = 0$, but also the payoff from playing $x_1 = k$ against $x_2 = n$ would be 0 rather than 1, which explains why, again, neither $\tilde{\mu}^{\text{Louie}}$ nor $v^{\text{Louie}} = \frac{1}{2}$ appear in the analysis of $\mathfrak{B}_{0.5}$.

5. Further discussion

This section elaborates on several additional topics. These include ε -equilibria, the possibility that only one player is restricted to a discrete grid, Colonel Blotto games with a head start, and the use of discretization and IEDS in other games.

5.1 ε -equilibria

Is there a connection between the limiting values not corresponding with the continuous case and the lack of ε -equilibria for Sion and Wolfe's game? By definition, in an ε -equilibrium, profitable deviations may exist, but the player must not gain more than a given $\varepsilon > 0$ by deviating (Radner, 1980). As mentioned above, an ε -equilibrium does not exist in the Sion-Wolfe game for $\varepsilon > 0$ sufficiently small. This, however, is a general property of two-person zero-sum games that do not have a value, as may be seen from the following general observation.

Proposition 5. *A two-person zero-sum game G has a value if and only if it admits an ε -equilibrium for any $\varepsilon > 0$.*

Proof. See the Appendix. \square

The proposition above, which might be of independent interest, says that a two-person zero-sum game has a value if and only if it admits ε -equilibria for all $\varepsilon > 0$. In particular, ε -equilibria, for $\varepsilon > 0$ small, do not exist in the Sion-Wolfe game and similar examples of games without value. However, there is no obvious link of this fact to the anomaly discussed in the previous section.²⁷

In view of Proposition 5, it is natural to ask for the set of $\varepsilon > 0$ for which an ε -equilibrium exists in a given two-person zero-sum game. Given that an ε -equilibrium is trivially also an ε' -equilibrium for any $\varepsilon' > \varepsilon$, it is clear that there exists, provided that payoffs are bounded, a smallest $\varepsilon_{\min} \geq 0$ such that, for any $\varepsilon > \varepsilon_{\min}$, an ε -equilibrium exists. From the proof of Proposition 5, it follows that

$$\frac{\bar{v} - \underline{v}}{2} \leq \varepsilon_{\min} \leq \bar{v} - \underline{v}. \quad (23)$$

For example, for the Sion-Wolfe game, $\varepsilon_{\min} \in [\frac{1}{21}, \frac{2}{21}]$.

5.2 Only one player's strategy is on a finite grid²⁸

Liang et al. (2019) considered a setting where one of the players chooses from the continuum, while the other player chooses from a finite grid. Below, we follow this approach. For convenience, we restrict attention to the case where n is even. Moreover, we stick to the convention followed throughout the paper that the player choosing from the continuum chooses from the unit interval.

Proposition 6. *Let $k = \frac{n}{2}$ with $n \geq 2$ even. (i) Suppose that player 1 chooses her pure strategy from a finite grid $X_1 = \{0, \dots, n\}$, while player 2 chooses her pure strategy from the unit interval $X_2 = [0, 1]$. Then, there is an essentially unique equilibrium in iteratively undominated strategies, which is given by $\mu_1^* = \frac{4}{7}\delta_0 + \frac{2}{7}\delta_k + \frac{1}{7}\delta_{k+1}$ and $\mu_2^* = \frac{4}{7}\delta_0 + \frac{1}{7}\delta_{\frac{1}{2}+\varepsilon} + \frac{2}{7}\delta_1$, and the corresponding value is $v_1 = \frac{4}{7} = \underline{v}^{SW57}$. (ii) Suppose that player 1 chooses her pure strategy from the unit interval*

²⁷For completeness, we mention that, even if G admits an ε -equilibrium for any $\varepsilon > 0$, a Nash equilibrium still need not exist. To see this, it suffices to consider a mixed extension of a matching-pennies game in which fair randomization is forbidden for the players.

²⁸We are grateful to one of the referees for suggesting this extension.

$X_1 = [0, 1]$, while player 2 chooses her pure strategy from a finite grid $X_2 = \{0, \dots, n\}$. Then, there is a unique equilibrium in iteratively undominated strategies, which is given by $\mu_1^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_{\frac{1}{2}+\varepsilon}$ and $\mu_2^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_n$, the corresponding value is $v_1 = \frac{2}{3} = \bar{v}^{SW57}$.

Proof. See the Appendix. \square

This exercise proves to be instructive. If player 1’s choice is restricted to the finite grid, while player 2’s choice is unrestricted, then the value of the resulting game is the minorant of the Sion-Wolfe game. If, however, player 2’s choice is restricted to the finite grid, while player 1’s choice is unrestricted, then the value of the game is the majorant of the Sion-Wolfe game. Compared to the analysis of the finite games, allowing one player to choose from the continuum keeps the concept of undercutting the opponent alive, but gives this possibility to only one of the two players, so that a Nash equilibrium is feasible.

5.3 Colonel Blotto games with a head start

The purpose of the present paper has been the analysis of discrete approximations of the Sion-Wolfe game. As a by-product, however, we have characterized equilibria in the discrete Colonel Blotto game with a head start. This part of the analysis generalizes to arbitrary integer values for $n \geq 1$ and $k \geq 0$. As illustrated by Figure 3(c), the iterated elimination of weakly dominated strategies leads to reduced-form payoff matrices closely related to symmetric Toeplitz matrices (Gohberg and Semencul, 1972; Rodman and Shalom, 1992).²⁹ Applying conditions sufficient for the invertibility of such matrices, one can show that the equilibrium in iteratively undominated strategies is often unique in $\mathfrak{B}(n, k)$, with players deploying “units” of k soldiers rather than individual soldiers (Aspect and Ewerhart, 2022). As a result of the restricted flexibility of the head start compared to a larger budget, however, the value of $\mathfrak{B}(n, k)$ is always weakly lower, and sometimes strictly so, than in a setting where k soldiers are added to player 1’s budget, as in Liang et al. (2019).

Vu and Loiseau (2021) considered a general class of Colonel Blotto games with head starts. They allowed for three or more battlefields, heterogeneity of valuations across battlefields and players, and heterogeneous effectiveness of players’ resources in each battlefield. One special case

²⁹As will be discussed below, Toeplitz matrices arise similarly in the analysis of Silverman’s game.

for which equilibria can be obtained in their approach is the case of homogeneous battlefields and homogeneous favouritism across the battlefields (for instance, in the case of equal effectiveness, this would require equal head starts for the two players across the battlefields). Notably, homogeneous valuations of the battlefields is not enough. This point is important, because if there was an equilibrium existence result for three battlefields with homogeneous values of battlefields, it would mean that the case of two battlefields is special in that it does not allow equilibrium existence in general. Further, although Vu and Loiseau (2021) do not provide equilibrium existence, they do provide a characterization of ε -Nash equilibria with $\varepsilon \in \tilde{O}(1/\sqrt{m})$, where m is the number of battlefields.³⁰ This tells us that, when the number of battlefields grows beyond any finite bound, the margin between the majorant and the minorant values in the continuous game goes to zero. In this sense, the two-battlefield variant may be considered the most acute in terms of this margin.

5.4 Use of discretization and weak dominance in other games

Silverman's game is a particular two-person zero-sum game (Evans, 1979; Heuer and Leopold-Wildburger, 2012). In a simple version with continuous strategy spaces, each player selects a number from a given set $S \subseteq \mathbb{R}_{>0}$. The player who selected the larger number wins 1, unless the larger number is at least $c > 1$ times as large, in which case the player who selected the smaller number wins $b > 0$. In the case of a tie, payoffs are zero. The variety and depth of the game-theoretic analysis of Silverman's game contrasts with the elementary nature of the present analysis. For example, Evans and Heuer (1992) work with polynomial conditions that are absent from our analysis. However, dominance arguments and Toeplitz matrices naturally arise in the analysis of both classes of games. Similarly, motivated by Silverman's game that partitions the square into four subsets with equal payoffs, Heuer (2001) studies a general class of zero-sum games on the square that partitions the square into three subsets with equal payoffs. This class includes a variant of the Sion-Wolfe game as a special case. In terms of results, however, the conclusions are often similar. I.e., continuous variants of Silverman's game need not possess a value, and discrete variants of it may often possess an essentially unique equilibrium.³¹

³⁰Thus, $\varepsilon > 0$ is bounded from above by a product of $1/\sqrt{m}$ and other factors that grow at most logarithmically in m .

³¹Similar examples of zero-sum games on the square appear in Carmona (2005), Duggan (2007), Montero and Page (2007), Prokopovych and Yannelis (2014), and Boudreau and Schwarz (2019), for instance. Those papers,

6. Conclusion

The research efforts documented in this paper have been motivated by the desire to better understand the nature of the Sion-Wolfe example. It is simply hard to accept that noncooperative game theory fails to deliver any prediction for the outcome of a simple game with straightforward economic interpretation. In an attempt to resolve the issue, we have characterized the values of discrete-grid approximations of the Sion-Wolfe game. In the cases relevant for the approximation, the equilibrium was shown to be essentially unique in iteratively undominated strategies. Linking the findings back to the game with continuous strategies, two main observations have been made. First, the set of limit points of values in approximating finite games consists of three points. Second, one of the three limit points lies outside of the interval formed by the maximin and minimax values in the continuous Blotto game. Given that the existence of a value may be a question hard to settle in specific games, we conclude that the analysis of finite approximations is potentially misleading about the nature of interaction in games with continuous strategy spaces.

The contribution of Sion and Wolfe (1957) was it to formally show that an economically relevant game with compact strategy spaces and bounded payoff functions need not possess an equilibrium point. However, as seen above, the iterated elimination of weakly dominated strategies is an alternative and potentially more transparent way to establish that the majorant and minorant values in the Sion-Wolfe game do not coincide.

The iterated elimination of weakly dominated strategies is natural in the context of games considered in the present paper. This technique might prove useful also in the study of other variants of conflicts with not too many battlefields, for example, with heterogeneous budgets or heterogeneous valuations. However, it might be less effective in Blotto games without head starts or too many battlefields.

however, pursue the more ambitious objective of characterizing the condition of better-reply security (Reny, 1999) in the mixed extension.

Appendix

This Appendix contains technical material omitted from the body of the paper. Lemma A.1 prepares the proof of Proposition 1.

Lemma A.1 *If $k > n/2$, then the sets of iteratively undominated strategies in $\mathfrak{B}(n, k)$ are given by $X_1^{(\infty)} = X_1^{(2)} = \{0, n - k + 1\}$ and $X_2^{(\infty)} = X_2^{(2)} = \{0, n\}$, respectively.*

Proof. By Lemma 1, the respective sets of undominated strategies after the first round of elimination are given by $X_1^{(1)} = \{0, \dots, n - k + 1\}$ and $X_2^{(1)} = \{0\} \cup \{k + 1, \dots, n\}$. See panel (a) of Figure A.1 for illustration. In the second round, $\hat{x}_1 = n - k + 1$ dominates any $x_1 \in \{1, \dots, n - k\}$ for player 1, and $\hat{x}_2 = n$ dominates any $x_2 \in \{k + 1, \dots, n - 1\}$ for player 2, as is evident from panel (b). The game obtained after two rounds of elimination is, consequently, represented by the two-by-two payoff matrix shown in panel (c). Clearly, further eliminations are not feasible. \square

(a)	x_1	x_2	0	1	2	...	$k - 1$	k	$k + 1$...	n
	0		1	2	2	...	2	1	0	...	0
	1		0	1	2	...	2	2	1	...	0
	2		0	0	1	...	2	2	2	...	0
	1
	$n - k + 1$		0	0	0	...	1	2	2	...	2
	$n - k + 2$		0	0	0	...	0	1	2	...	2

	$n - 1$		0	0	0	...	0	0	0	...	2
	n		0	0	0	...	0	0	0	...	1

(b)	x_1	x_2	0	$k + 1$...	n
	0		1	0	...	0
	1		0	1	...	0
	2		0	2	...	0
	1
	$n - k + 1$		0	2	...	2

(c)	x_1	x_2	0	n
	0		1	0
	$n - k + 1$		0	2

Figure A.1: Iterated elimination of dominated strategies in the case $k > n/2$.

Proof of Proposition 1. The game shown in Figure A.1(c) admits the unique equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$, where $\mu_1^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_{n-k+1}$ and $\mu_2^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_n$. As discussed in the body of

the paper, this implies that μ^* is the unique equilibrium in iteratively undominated strategies in $\mathfrak{B}(n, k)$. Obviously, player 1's equilibrium payoff is $v_1 = \frac{2}{3}$. \square

Lemma A.2 is used in the proof of Proposition 2.

Lemma A.2 *Suppose that $k = n/2$, with n even. Then, the sets of iteratively undominated strategies are given by $X_1^{(\infty)} = X_1^{(2)} = \{0, k, k+1\}$ and $X_2^{(\infty)} = X_2^{(2)} = \{0, k+1, n\}$, respectively.*

Proof. By Lemma 1 and using $n-k = k$, the respective sets of undominated strategies are given by $X_1^{(1)} = \{0, \dots, k+1\}$ and $X_2^{(1)} = \{0\} \cup \{k+1, \dots, n\}$. See Figure A.2 for illustration. In the second round, $\hat{x}_1 = k$ dominates any $x_1 \in \{1, \dots, k-1\}$ for player 1, while $\hat{x}_2 = n$ dominates any $x_2 \in \{k+2, \dots, n-1\}$ for player 2. The reduced payoff matrix is, consequently, the three-by-three matrix shown in panel (c). Further eliminations are obviously not feasible. \square

(a)

$x_1 \backslash x_2$	0	1	...	$k-1$	k	$k+1$	$k+2$...	n
0	1	2	...	2	1	0	0	...	0
1	0	1	...	2	2	1	0	...	0
...
$k-1$	0	0	...	1	2	2	2	...	0
k	0	0	...	0	1	2	2	...	1
$k+1$	0	0	...	0	0	1	2	...	2
$k+2$	0	0	...	0	0	0	1	...	2
...
n	0	0	...	0	0	0	0	...	1

(b)

$x_1 \backslash x_2$	0	$k+1$	$k+2$...	n
0	1	0	0	...	0
1	0	1	0	...	0
...
k	0	2	2	...	1
$k+1$	0	1	2	...	2

(c)

$x_1 \backslash x_2$	0	$k+1$	n
0	1	0	0
k	0	2	1
$k+1$	0	1	2

Figure A.2: Iterated elimination in the case $n = 2k$.

Proof of Proposition 2. (*Equilibrium property*) The reduced game shown in panel (c) of Figure A.2 admits a unique equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$, given by strategies $\mu_1^* = \frac{3}{5}\delta_0 + \frac{1}{5}\delta_k + \frac{1}{5}\delta_{k+1}$ and

$\mu_2^* = \frac{3}{5}\delta_0 + \frac{1}{5}\delta_{k+1} + \frac{1}{5}\delta_n$. Thus, μ^* is the unique equilibrium in iteratively undominated strategies in $\mathfrak{B}(n, k)$. Clearly, player 1's equilibrium payoff in μ^* is $v_G = \frac{3}{5}$. (*Uniqueness*) We check that no strategy outside of player 1's equilibrium support is a best response. Player 1's expected payoff against μ_2^* is given by

$$E_{\mu_2^*}[\Pi_1(x_1, x_2)] = \frac{3}{5} \cdot \Pi_1(x_1, 0) + \frac{1}{5} \cdot \Pi_1(x_1, k+1) + \frac{1}{5} \cdot \Pi_1(x_1, n), \quad (24)$$

where player 1's ex-post payoffs are taken from Figure A.2(a). This needs to be compared to $\Pi_1^* = \frac{3}{5}$. There are four cases. First, if $x_1 = 1$, then $E_{\mu_2^*}[\Pi_1(x_1, x_2)] = \frac{1}{5} \cdot 1 < \Pi_1^*$. Second, if $x_1 \in \{2, \dots, k-1\}$, then $E_{\mu_2^*}[\Pi_1(x_1, x_2)] = \frac{1}{5} \cdot 2 < \Pi_1^*$. Third, if $x_1 = k+r$ for some $r \in \{2, \dots, k-1\}$, then $E_{\mu_2^*}[\Pi_1(x_1, x_2)] = \frac{1}{5} \cdot 2 < \Pi_1^*$. Fourth and finally, if $x_1 = n$, then $E_{\mu_2^*}[\Pi_1(x_1, x_2)] = \frac{1}{5} \cdot 1 < \Pi_1^*$. Thus, no pure strategy outside player 1's equilibrium support is a best response. In particular, μ_1^* is the unique best response to μ_2^* in $\mathfrak{B}(n, k)$. We now show analogously that no pure strategy outside of player 2's equilibrium support is a best response to μ_1^* . Player 2's expected payoff is

$$E_{\mu_1^*}[\Pi_2(x_1, x_2)] = \frac{3}{5} \cdot \Pi_2(0, x_2) + \frac{1}{5} \cdot \Pi_2(k, x_2) + \frac{1}{5} \cdot \Pi_2(k+1, x_2), \quad (25)$$

and this needs to be compared to $\Pi_2^* = -\frac{3}{5}$. There are three cases. First, if $x_2 \in \{1, \dots, k-1\}$, then $E_{\mu_1^*}[\Pi_2(x_1, x_2)] = \frac{3}{5} \cdot (-2) < \Pi_2^*$. Next, if $x_2 = k$, then $E_{\mu_1^*}[\Pi_2(x_1, x_2)] = \frac{3}{5} \cdot (-1) + \frac{1}{5} \cdot (-1) = -\frac{4}{5} < \Pi_2^*$. Third and finally, if $x_2 = k+r$ for some $r \in \{2, \dots, k-1\}$, then $E_{\mu_1^*}[\Pi_2(x_1, x_2)] = \frac{1}{5} \cdot (-2) + \frac{1}{5} \cdot (-2) = -\frac{4}{5} < \Pi_2^*$. Thus, as above, we have shown that μ_2^* is the unique best response to μ_1^* in $\mathfrak{B}(n, k)$. Since no other strategy than the ones played in μ^* are best responses, and Nash equilibria in zero-sum games like $\mathfrak{B}(n, k)$ are interchangeable (see, e.g., Osborne and Rubinstein, 1994, Section 2.5), there cannot be any other equilibrium in $\mathfrak{B}(n, k)$. \square

The following lemma is needed in the proof of Proposition 3.

Lemma A.3 *Suppose that $n = 2k + r$, for some $r \in \{1, \dots, k-1\}$, and k sufficiently large.*

- (i) *If $r = 1$, then $X_1^{(\infty)} = X_1^{(2)} = \{0, k, k+1, k+2\}$ and $X_2^{(\infty)} = X_2^{(3)} = \{0, k+1, n\}$;*
- (ii) *if $r \in \{2, \dots, k-2\}$, then $X_1^{(\infty)} = X_1^{(4)} = \{0, k, n-k+1\}$ and $X_2^{(\infty)} = X_2^{(4)} = \{0, k+1, n\}$.*

Proof. (i) In this case, $n - k + 1 = k + 2$. Hence, the respective sets of undominated strategies are given by $X_1^{(1)} = \{0, \dots, k+2\}$ and $X_2^{(1)} = \{0\} \cup \{k+1, \dots, n\}$, as illustrated in Figure

A.3. In the second round, any $x_1 \in \{1, \dots, k-1\}$ is dominated by $\hat{x}_1 = k$ for player 1, so that $X_1^{(2)} = \{0, k, k+1, k+2\}$. Similarly, any $x_2 \in \{k+3, \dots, n-1\}$ is dominated by $\hat{x}_2 = n$ for player 2, so that $X_2^{(2)} = \{0, k+1, k+2, n\}$. In a third round of elimination, we see that $x_2 = k+2$ is dominated by $\hat{x}_2 = k+1$ for player 2, so that $X_2^{(3)} = \{0, k+1, n\}$. Since no further eliminations are feasible, the claim follows. (ii) In this case, $n-k+1 = k+r+1$, so that $X_1^{(1)} = \{0, \dots, k+r+1\}$ and $X_2^{(1)} = \{0\} \cup \{k+1, \dots, n\}$. See Figure A.4. In the second round, any $x_1 \in \{1, \dots, k-1\}$ is dominated by $\hat{x}_1 = k$ for player 1, and any $x_2 \in \{k+r+2, \dots, n-1\}$ is dominated by $\hat{x}_2 = n$ for player 2. Therefore, in analogy with the previously considered case, $X_1^{(2)} = \{0\} \cup \{k, \dots, k+r+1\}$ and $X_2^{(2)} = \{0\} \cup \{k+1, \dots, k+r+1\} \cup \{n\}$. In the third round, $x_1 \in \{k+1, \dots, k+r-1\}$ is dominated by $\hat{x}_1 = k$ for player 1, and $x_2 \in \{k+2, \dots, k+r+1\}$ is dominated by $\hat{x}_2 = k+1$ for player 2, so that $X_1^{(3)} = \{0, k, k+r, k+r+1\}$ and $X_2^{(3)} = \{0, k+1, n\}$. In the fourth and final round, $x_1 = k+r$ is dominated by $\hat{x}_1 = k+r+1$ for player 1. Thus, $X_1^{(4)} = \{0, k, k+r+1\}$. Clearly, further eliminations are not feasible. \square

(a)	x_1	x_2	0	$k+1$	$k+2$	$k+3$...	$n-1$	n
	0		1	0	0	0	...	0	0
	1		0	1	0	0	...	0	0
	2		0	2	1	0	...	0	0

	$k-1$		0	2	2	2	...	0	0
	k		0	2	2	2	...	1	0
	$k+1$		0	1	2	2	...	2	1
	$k+2$		0	0	1	2	...	2	2

(b)	x_1	x_2	0	$k+1$	$k+2$	n
	0		1	0	0	0
	k		0	2	2	0
	$k+1$		0	1	2	1
	$k+2$		0	0	1	2

(c)	x_1	x_2	0	$k+1$	n
	0		1	0	0
	k		0	2	0
	$k+1$		0	1	1
	$k+2$		0	0	2

Figure A.3: The case $r = 1$.

Proof of Proposition 3. If $r = 1$, then the reduced payoff matrix, shown in Figure A.3(c), leaves four strategies for player 1 and three strategies for player 2, which makes it straightforward to verify the claim. If $r \in \{2, \dots, k-2\}$, the reduced payoff matrix is even a diagonal matrix. Multiplying its inverse with a constant vector and normalizing delivers the unique equilibrium strategy. \square

(a)		x_2	0	$k+1$...	$k+r+1$	$k+r+2$...	$n-1$	n
x_1										
0		1	0	...	0	0	...	0	0	
1		0	1	...	0	0	...	0	0	
2		0	2	...	0	0	...	0	0	
...		
$k-1$		0	2	...	2	1	...	0	0	
k		0	2	...	2	2	...	0	0	
...		
$k+r$		2	2	...	2	1	
$k+r+1$		0	0	...	1	2	...	2	2	

(b)		x_2	0	$k+1$	$k+2$...	$k+r$	$k+r+1$	n
x_1									
0		1	0	0	0
k		0	2	2	...	2	2	2	0
$k+1$		0	1	2	2	0
...	
$k+r-1$		0	0	0	...	2	2	2	0
$k+r$		0	0	0	...	1	2	2	1
$k+r+1$		0	0	0	...	0	1	1	2

(c)		x_2	0	$k+1$	n
x_1					
0		1	0	0	
k		0	2	0	
$k+r$		0	0	1	
$k+r+1$		0	0	2	

(d)		x_2	0	$k+1$	n
x_1					
0		1	0	0	
k		0	2	0	
$k+r+1$		0	0	2	

Figure A.4: The case $r \in \{2, \dots, k-2\}$.

Proof of Lemma 2. Sion and Wolfe (1957) consider a two-person zero-sum game on the square, with player 1 choosing \mathbf{x} and player 2 choosing \mathbf{y} , where player 1's payoff function is given as

$$\mathbf{K}(\mathbf{x}, \mathbf{y}) = \begin{cases} -1 & \text{if } \mathbf{x} < \mathbf{y} < \mathbf{x} + \frac{1}{2} \\ 0 & \text{if } \mathbf{x} = \mathbf{y} \text{ or } \mathbf{y} = \mathbf{x} + \frac{1}{2} \\ 1 & \text{otherwise.} \end{cases} \quad (26)$$

In comparison, one notes that in $\mathfrak{B}_{0.5}$,

$$\pi_1(x_1, x_2) = \begin{cases} 2 & \text{if } x_1 < x_2 < x_1 + \frac{1}{2} \\ 1 & \text{if } x_2 = x_1 \text{ or } x_2 = x_1 + \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

It is now easy to check that, by letting $\mathbf{x} = 1 - x_2$ and $\mathbf{y} = 1 - x_1$, we obtain $\pi_1(x_1, x_2) = 1 - \mathbf{K}(\mathbf{x}, \mathbf{y})$. Thus, exchanging the roles of players 1 and 2, and applying the linear transformations of strategy spaces and payoff functions, the Sion-Wolfe game is indeed strategically equivalent to $\mathfrak{B}_{0.5}$. Moreover, as the maximin value in the Sion-Wolfe game is $\sup \inf \mathbf{K} = \frac{1}{3}$ and the minimax value is $\inf \sup \mathbf{K} = \frac{3}{7}$, we obtain

$$\sup_{\tilde{\mu}_1} \inf_{\tilde{\mu}_2} \iint_{[0,1] \times [0,1]} \pi_1(x_1, x_2) d\tilde{\mu}_1(x_1) d\tilde{\mu}_2(x_2) = 1 - \inf \sup \mathbf{K} = \frac{4}{7}, \quad (28)$$

$$\inf_{\tilde{\mu}_2} \sup_{\tilde{\mu}_1} \iint_{[0,1] \times [0,1]} \pi_1(x_1, x_2) d\tilde{\mu}_1(x_1) d\tilde{\mu}_2(x_2) = 1 - \sup \inf \mathbf{K} = \frac{2}{3}, \quad (29)$$

as claimed. \square

Proof of Lemma 3. By the definition of weak dominance, $\widehat{\xi}_i \in \mathfrak{X}_i \setminus \mathfrak{S}_i$, for $i \in \{1, 2\}$. Hence, player i 's strategy set $\mathfrak{X}'_i = \mathfrak{X}_i \setminus \mathfrak{S}_i$ is non-empty, for $i \in \{1, 2\}$. By assumption, $u_1(\xi_1, \xi_2) \leq u_1(\widehat{\xi}_1, \xi_2)$ for any $\xi_1 \in \mathfrak{S}_1$ and $\xi_2 \in \mathfrak{X}_2$. Therefore, $\inf_{\xi_2 \in \mathfrak{X}_2} u_1(\xi_1, \xi_2) \leq \inf_{\xi_2 \in \mathfrak{X}_2} u_1(\widehat{\xi}_1, \xi_2)$ for any $\xi_1 \in \mathfrak{S}_1$. Hence, $\sup_{\xi_1 \in \mathfrak{X}_1} \inf_{\xi_2 \in \mathfrak{X}_2} u_1(\xi_1, \xi_2) \leq \sup_{\xi_1 \in \mathfrak{X}'_1} \inf_{\xi_2 \in \mathfrak{X}_2} u_1(\xi_1, \xi_2)$. Since the reverse inequality is obvious, it follows that $\sup_{\xi_1 \in \mathfrak{X}_1} \inf_{\xi_2 \in \mathfrak{X}_2} u_1(\xi_1, \xi_2) = \sup_{\xi_1 \in \mathfrak{X}'_1} \inf_{\xi_2 \in \mathfrak{X}_2} u_1(\xi_1, \xi_2)$. Next, we note that $\sup_{\xi_2 \in \mathfrak{X}_2} u_2(\xi_1, \xi_2) = \sup_{\xi_2 \in \mathfrak{X}'_2} u_2(\xi_1, \xi_2)$ for any $\xi_1 \in \mathfrak{X}_1$. By the zero-sum property, this implies $\inf_{\xi_2 \in \mathfrak{X}_2} u_1(\xi_1, \xi_2) = \inf_{\xi_2 \in \mathfrak{X}'_2} u_1(\xi_1, \xi_2)$ for any $\xi_1 \in \mathfrak{X}_1$. Combining the last two insights, it follows that $\bar{v}_{G'} = \bar{v}_G$, as claimed. Exchanging the roles of player 1 and 2, this also proves the second claim and, hence, the lemma. \square

Proof of Proposition 5. (If) Take some $\varepsilon > 0$. By the definition of the minorant, there exists $\xi_1^* \in \mathfrak{X}_1$ such that

$$\inf_{\xi_2} u_1(\xi_1^*, \xi_2) \geq \underline{v} - \frac{\varepsilon}{2}, \quad (30)$$

and therefore

$$u_1(\xi_1^*, \xi_2) \geq \underline{v} - \frac{\varepsilon}{2} \quad (\xi_2 \in \mathfrak{X}_2). \quad (31)$$

Similarly, by the definition of the majorant, there exists $\xi_2^* \in \mathfrak{X}_2$ such that

$$\sup_{\xi_1} u_1(\xi_1, \xi_2^*) \leq \bar{v} + \frac{\varepsilon}{2}, \quad (32)$$

so that

$$u_1(\xi_1, \xi_2^*) \leq \bar{v} - \frac{\varepsilon}{2} \quad (\xi_1 \in \mathfrak{X}_1). \quad (33)$$

Evaluating inequality (31) at $\xi_2 = \xi_2^*$, and subsequently subtracting the result from inequality (33), one obtains

$$u_1(\xi_1, \xi_2^*) - u_1(\xi_1^*, \xi_2^*) \leq \bar{v} - \underline{v} + \varepsilon \quad (\xi_1 \in \mathfrak{X}_1). \quad (34)$$

In complete analogy, one shows that

$$u_2(\xi_1^*, \xi_2) - u_2(\xi_1^*, \xi_2^*) \leq \bar{v} - \underline{v} + \varepsilon \quad (\xi_2 \in \mathfrak{X}_2). \quad (35)$$

Thus, if G has a value, then (ξ_1^*, ξ_2^*) is an ε -equilibrium. (*Only if*) Suppose that (ξ_1^*, ξ_2^*) is an ε -equilibrium, for some $\varepsilon > 0$. Then,

$$u_2(\xi_1^*, \xi_2^*) \geq u_2(\xi_1^*, \xi_2) - \varepsilon \quad (\xi_2 \in \mathfrak{X}_2). \quad (36)$$

Since $u_2 = -u_1$, this implies

$$u_1(\xi_1^*, \xi_2^*) \leq u_1(\xi_1^*, \xi_2) + \varepsilon \quad (\xi_2 \in \mathfrak{X}_2). \quad (37)$$

Hence,

$$u_1(\xi_1^*, \xi_2^*) \leq \inf_{\xi_2} u_1(\xi_1^*, \xi_2) + \varepsilon \leq \sup_{\xi_1} \inf_{\xi_2} u_1(\xi_1, \xi_2) + \varepsilon = \underline{v} + \varepsilon. \quad (38)$$

In particular, $\underline{v} > -\infty$. Analogously, one shows that $u_2(\xi_1^*, \xi_2^*) \leq -\bar{v} + \varepsilon$, or equivalently, that $u_1(\xi_1^*, \xi_2^*) \geq \bar{v} - \varepsilon$. In particular, $\bar{v} < \infty$. Using that $\underline{v} \leq \bar{v}$, it follows that both \bar{v} and \underline{v} are finite, and hence, $\bar{v} - \underline{v} \leq 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $\bar{v} - \underline{v} = 0$, as claimed. \square

Proof of Proposition 6. (i) We execute two rounds of elimination. In the first round, we note that any strategy $x_1 \in \{k+2, \dots, n\}$ is weakly dominated by $\hat{x}_1 = k+1$, and that any strategy $x_2 \in (0, \frac{1}{2}]$ is weakly dominated by $\hat{x}_2 = 0$. This leaves us with strategy sets $\mathfrak{X}'_1 = \{0, \dots, k+1\}$ for player 1 and $X'_2 = \{0\} \cup (\frac{1}{2}, 1]$ for player 2. In the second round, we note that any $x_1 \in \{1, \dots, k-1\}$ is weakly dominated by $\hat{x}_1 = k$, and that any strategy $x_2 \in [\frac{k+1}{n}, 1)$ is weakly dominated by $\hat{x}_2 = 1$. This leaves us with strategy sets $X''_1 = \{0, k, k+1\}$ for player 1 and $X''_2 = \{0\} \cup (\frac{1}{2}, \frac{k+1}{n}) \cup \{1\}$ for player 2. The resulting payoff matrix, with three options for each player, is again that shown in Figure 4, where $z = 0$. Solving that game leads to the strategies and the value, as claimed.³² (ii)

As in the previous case, one executes two rounds of elimination. In the first round, we note that any $x_1 \in (\frac{1}{2} + \varepsilon, 1]$ is weakly dominated by $\hat{x}_1 = \frac{1}{2} + \varepsilon$, and that any $x_2 \in \{0, \dots, k\}$ is weakly dominated by $\hat{x}_2 = 0$. This leaves us with strategy sets $X'_1 = [0, \frac{1}{2} + \varepsilon]$ and $X'_2 = \{0\} \cup \{k+1, \dots, n\}$, respectively. In the second round, we see that any $x_1 \in (0, \frac{1}{2} + \varepsilon)$ is weakly dominated by $\hat{x}_1 = 0$, and that any $x_2 \in \{k+1, \dots, n\}$ is weakly dominated by $\hat{x}_2 = n$. This leaves us with strategy sets $X''_1 = \{0, \frac{1}{2} + \varepsilon\}$ for player 1 and $X''_2 = \{0, n\}$ for player 2. The resulting payoff matrix, with two options for each player, is that shown in Figure 2(c). Solving that game leads to the strategies and the value, as claimed. \square

³²There is a multiplicity in so far that the ε maybe chosen over the interval $(0, \frac{1}{n})$.

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