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## Computing and comparing measures of rationality

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# **Computing and Comparing Measures of Rationality**

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# Computing and Comparing Measures of Rationality

Lasse Mononen<sup>†</sup>

## Abstract

The rationality of choices is one of the most fundamental assumptions of traditional economic analysis. Yet, substantial evidence has documented that choices often cannot be rationalized by utility maximization. Several measures of rationality have been introduced in the literature to quantify the size of rationality violations. However, it is not clear which of these measures should be used in applications, and many measures are computationally very demanding, which has restricted their widespread use. First, we introduce novel variations of the measures that allow us to establish connections between the different measures. Second, we develop methods to compute the most-used measures of rationality. Exploiting this computational progress, we offer simulation-based comparisons of the accuracy of the measures. These simulations show that a new type of measure that combines the size of rationality violations with the number of rationality violations outperforms other measures. Finally, we offer a method to calculate statistical significance levels for rationality violations.

## 1 Introduction

One of the most fundamental assumptions in economics is that choices are rational, i.e., that they maximize a utility function. Yet, there is substantial evidence that choices are not always rational (e.g. Echenique et al., 2011, Choi et al., 2014, Dean and Martin, 2016). In order to understand the reasons for non-rational choices and to evaluate and improve the quality of choices, it is fundamental to understand how close choices are to maximizing

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A supplementary computational toolbox for Matlab implementing the discussed programs is available at <https://github.com/lmononen/measures-of-rationality-toolbox>

utility in different settings. This helps us understand if people make better choices in some markets than others or if some groups make better choices than others.

Over the past decades, several measures of rationality have been proposed to quantify the quality of choices. However, it is not clear which of these measures should be used in empirical applications. Additionally, most of the measures are computationally very demanding which has restricted their widespread use (Smeulders et al., 2014). Due to these computational constraints, most of the empirical literature has focused on a single, easy-to-compute measure even though it has been argued to not be a good choice (Echenique, 2022). Later on, in Section 1.1, we will show that this choice of measure might have impacted conclusions about the quality of choices in empirical applications.

To address these issues, we provide novel efficient computation methods for the most common measures of rationality. Exploiting this computational progress, we perform a simulation-based comparison of the accuracy of the different measures of rationality. Additionally, we consider novel variations of the measures and show how these variations provide connections between the existing measures. Especially, our simulation-based comparison suggests a new type of rationality measure that combines measures capturing the size of rationality violations with measures counting the number of rationality violations.

Before discussing the results, we highlight the importance of the rationality measures with two applications. First, consider a government that wants to improve the citizens' welfare by regulating markets where the citizens make bad choices. These bad choices might be due to impulsive purchases, limited attention, complexity of the markets from advertising, price shrouding, or complexity of the alternatives as with financial derivatives. However, the government only observes the purchases that people make in this market but not the preferences of the people, so it cannot judge if any given purchase was good or bad. Here, measures of rationality offer a way of quantifying the quality of the choices without making any assumptions about the citizens' preferences or about the source of mistakes, simply by comparing how close the observed choices are to rational choices. This allows the government to evaluate the complexity of the markets and choose the correct markets to regulate.

Second, measures of rationality offer a method for assessing the suitability of economic analysis. Virtually every study of consumption behavior assumes rationality in the form of

utility maximization. Hence it is fundamental to understand how close people’s behavior is to the assumption of rationality. The measures of rationality allow us to assess if some markets are more suitable for economic analysis than others, or if the assumption of rationality is a better approximation for some groups than for others.

We study the rationality of consumer choice. Afriat (1967) showed that a consumer’s choices are rationalizable by maximizing a utility function iff there are no revealed preference cycles among the observed choices, a condition known as the generalized axiom of revealed preference (GARP). Measures of rationality quantify the severity of GARP violations.

A well-known measure of rationality is Varian’s goodness-of-fit index (Varian, 1990). The idea of this measure is that the consumer has some underlying rational preferences. Then the distance of the observed choices to the underlying preferences is measured by the fraction of income that the consumer could have saved without a loss of welfare. Since we do not observe the consumer’s true preferences, this measure takes the minimum over all rational preferences of the average fraction of income that the consumer could have saved (see Halevy et al., 2018).

Here, the total size of rationality violations is measured in absolute deviations or the  $\ell^1$  norm. Alternatively, the total size of rationality violations could be measured using the sum of squared errors, or the  $\ell^2$  norm raised to the power of two. More generally, we consider variations of Varian’s index that use the  $\ell^p$  norm raised to the power of  $p$ , thereby generalizing the sum of squared errors to any power  $p$ . As  $p$  increases, we give more weight to the largest violation of rationality. As  $p$  decreases, we give more weight to the number of rationality violations.

First, we show that, as  $p \rightarrow \infty$ , this variation of Varian’s index converges to Afriat’s efficiency index (Afriat, 1972), which captures the single largest fraction of income that the consumer could have saved without a welfare loss. Second, we show that, as  $p \rightarrow 0$ , the variation converges to a refinement of the Houtman-Maks index (Houtman and Maks, 1985), which is the smallest fraction of observations that needs to be dropped to rationalize the choices. The limit refines the Houtman-Maks index by breaking its ties by the geometric average of rationality violations. This result shows that the Houtman-Maks index is a

counting version of Varian’s index that only counts the number of rationality violations but does not take into account their size.

Additionally, we consider the well-known swaps index (Apestequia and Ballester, 2015) that counts the smallest number of pairwise preference swaps required to rationalize the choices. We offer a novel alternative representation for the swaps index as the smallest number of revealed preference removals to make the revealed preferences acyclical. This alternative representation is computationally more tractable and allows us to apply the same computational methods as for Varian’s index to calculate the swaps index. In analogy to the connection between Varian’s index and the Houtman-Maks index, we show that the swaps index is a counting version of the normalized minimum cost index (Mononen, 2022), counting only the number of revealed preference removals but not taking into account the amount of money that the consumer could have saved.

In addition to deriving these theoretical results, we provide novel computational techniques to compute Varian’s index and its variations. We reformulate Varian’s index as a simple binary linear programming problem that we solve with a modified version of a depth-first search algorithm. Even though computing Varian’s index is known to be NP-hard, our algorithm works efficiently and fast on the datasets used in typical applications. The same technique can be used for the different variations of Varian’s index and the other rationality measures discussed above.

Using these computational advances, we offer a simulation-based comparison of the accuracy of different measures of rationality. We employ a calibrated model of choice mistakes and compare the accuracy of the measures in capturing the extent of these choice mistakes. We show that the best measure at capturing choice mistakes crucially depends on the fat-tailedness of the mistake distribution. With fat-tailed choice mistakes, variations of Varian’s index with a low  $p \approx 0.1$  perform the best. With concentrated mistakes, variations of Varian’s index with  $p \approx 0.5$  perform the best.

Especially, variations with  $p = 2$  or  $p = 1$  that have been suggested in the literature always perform worse than variations with  $p = 0.5$ . This suggests that measures that give more weight to the number of rationality violations work best in applications. The intuition for this result is that it is more likely that the data has a single large mistake than multiple small ones. To

illustrate this with a simple example, consider two positively truncated normally distributed variables  $X$  and  $Y$ . Here, we have that the probability of  $\max\{X, Y\} \geq 2 \min\{X, Y\}$  is above 0.5. That is, it is more likely to have one large mistake than two small mistakes. Now, if the observed data can be explained by two small rationality violations or one large rationality violation, then variations of measures with low  $p$  will rationalize the data with the single large violation, and variations with high  $p$  will rationalize the data with the two small violations. The measure with low  $p$  is more often correct. Our general approach based on simulated choice mistakes can be tailored to any specific choice environment, to give a recommendation for the best measure in that environment.

Finally, we offer a method to calculate statistical significance levels for violations of rationality. This test is based on testing if the observed measure of rationality could have been generated by random choice with the same expenditures and prices. The previous literature has used a fixed measure of 0.05 as a cut-off for large violations of rationality. In contrast, our resulting significance thresholds depend on the choice situation and on how difficult it is to observe violations of rationality in that choice situation. We find that in the experimental data from Choi et al. (2014), 63% of people are statistically significantly more rational than random behavior at 95% significance level, even though only 17% of the choices are rationalizable by a utility function.

This paper contributes to the literature on computing measures of rationality. Smeulders et al. (2014) showed that most measures of rationality are computationally complex and NP-hard (except for Afriat's efficiency index, which is only polynomially complex). We highlight how this computational complexity captures only the worst-case scenario for large problems. Instead, for the data sets often used in applications, the measures can be efficiently calculated. Recently, Heufer and Hjertstrand (2019) and Demuyneck and Rehbeck (2023a) provided computational methods for calculating Varian's goodness-of-fit, the Houtman-Maks index, and the minimum cost index, based on reformulating these indices as mixed linear programming problems. In Section 6.1 we will show that the methods presented here compute these indices substantially faster and more reliably in existing data sets. Our results also contribute to the literature that applies measures of rationality in empirical and experimental studies (e.g., Choi et al., 2014, Dean and Martin, 2016, Drichoutis and Nayga, 2020). Our

results offer guidance as to which measure to choose in applications and provide a method to calculate the significance levels for violations of rationality.

The remainder of the paper proceeds as follows. We begin, in Section 1.1, with a simple empirical application highlighting that the different variations of measures of rationality matter for empirical analysis. Next, in Section 2, we introduce the formal setting and discuss the computation of measures of rationality. Here, we focus on Varian’s goodness-of-fit index and provide extensions for other measures. In Section 3, we consider variations of the measures and show that existing measures are the limits of the variations that weigh the size and the number of rationality violations differently. In Section 4, we provide the simulation-based comparisons and we offer a method to calculate statistical significance levels. Section 5 concludes. The proofs are included in the appendix.

## 1.1 Empirical Motivation

We begin with a simple empirical application highlighting that the choice of the rationality measure matters in practice. We focus on Drichoutis and Nayga’s (2020) recent experiment on the effect of cognitive load on the rationality of risky choice. In the experiment, subjects were asked to memorize numbers while they allocated a budget between two risky accounts where one of them was paid with 50% probability. The authors found no effect of cognitive load on the degree of rationality. We show that, by using the measures suggested by our simulation-based comparisons, cognitive load has a significant effect on the quality of choice.

In Table 1, we report the effect of cognitive load in the experiment on different measures of rationality (capturing distance to rationalization by a utility function that respects first-order stochastic dominance). A higher score means less rational choices. The first two columns show the results from Drichoutis and Nayga (2020), according to which cognitive load does not affect Afriat’s efficiency index or the Houtman-Maks index. The next two columns show that cognitive load does also not affect Varian’s goodness-of-fit or a variation of this index that gives more weight to the size of rationality violations. The last two columns show, however, that cognitive load does affect the variations of Varian’s index that give more weight to the existence of rationality violations and are suggested by our simulation as the



	AEI	HM	Var	Var2	Var0.5	Var0.1
High Cognitive load treatment	.018 (.022)	.027 (.017)	.0038 (.0041)	.00039 (.0014)	.013* (.008)	.03** (.014)
Age	.0026 (.0078)	-.00039 (.0057)	-.00015 (.0014)	-.0000097 (.00048)	-.00039 (.0028)	-.00094 (.0049)
Female	.019 (.023)	.018 (.017)	.0048 (.0041)	.0014 (.0013)	.01 (.0081)	.021 (.015)
Household size	-.0079 (.011)	.0067 (.0079)	-.0005 (.0024)	-.0001 (.00086)	-.000084 (.0045)	.0044 (.0072)
Reference income	.0097 (.0078)	.0075 (.0063)	.0018 (.0015)	.00064 (.00053)	.0035 (.0029)	.0063 (.0054)
Raven’s test score	-.02*** (.0077)	-.0079 (.0056)	-.0042*** (.0014)	-.001** (.00047)	-.009*** (.0028)	-.012** (.0049)
Experience with Raven’s test	.057** (.024)	.0096 (.017)	.0068 (.0044)	.0024 (.0015)	.012 (.0085)	.012 (.015)
Total decision time	-.012*** (.0032)	-.0054** (.0027)	-.0026*** (.00068)	-.00076*** (.00024)	-.0056*** (.0013)	-.0079*** (.0023)
Observations	343	343	343	343	343	343

Standard errors in parentheses. \*  $p < 0.1$ , \*\*  $p < 0.05$ , \*\*\*  $p < 0.01$

**Table 1.** The table shows a regression of rationality on cognitive load and demographics, using the experimental data of Drichoutis and Nayga (2020). AEI, HM, and Var stand for Afriat’s efficiency index, the Houtman-Maks index, and Varian’s goodness-of-fit, respectively.  $Varp$  stands for Varian’s index when the total size of adjustments is measured in  $\ell^p$  norm raised to the power of  $p$ . A higher measure always signifies less rational choices.

best measures. This shows that cognitive load might affect the quality of choices, contrary to the authors’ conclusion.

## 2 Computing Measures of Rationality

We first introduce the standard consumer choice setting and when the choices can be rationalized. Then, we move on to considering non-rational choices and to measuring how close the choices are to being rational, using Varian’s goodness-of-fit index. Next, we move on to computing this index by providing a novel computational method. Finally, we show how this computational method can be applied to other indices. The paper is accompanied by an easy-to-use toolbox for Matlab that implements all procedures.

## 2.1 Setting

We study the rationality of consumer choice. At each time period  $t$ , the consumer makes non-negative purchases  $x_t$  at positive prices  $p_t$  for  $G$  goods over  $T$  time periods (or observations). These prices  $p_t$  with the purchases  $x_t$  form data  $D$

$$D = \left( (p_1, x_1), \dots, (p_T, x_T) \right).$$

We say that data are rationalizable with a utility function if the purchases can be explained as maximizing a non-satiated utility function.

**Definition** Data  $D = \left( (p_1, x_1), \dots, (p_T, x_T) \right)$  is *rationalizable* if there exists a non-satiated utility function  $u : \mathbb{R}_+^G \rightarrow \mathbb{R}$  such that for all  $1 \leq t \leq T$ ,

$$x_t \in \arg \max \left\{ u(x) \mid p_t \cdot x \leq p_t \cdot x_t \right\}.$$

As is well known, the rationalizability of choice data is characterized by the generalized axiom of revealed preference (GARP) stating that there are no choice cycles (Afriat, 1967). That is, the revealed preference is acyclical as defined next.

**Definition**  $(R, P)$  is the *revealed preference* of data  $D$  if for all  $1 \leq t, t' \leq T$ ,

$$x_t R x_{t'} \iff p_t \cdot x_t \geq p_t \cdot x_{t'} \quad \text{and}$$

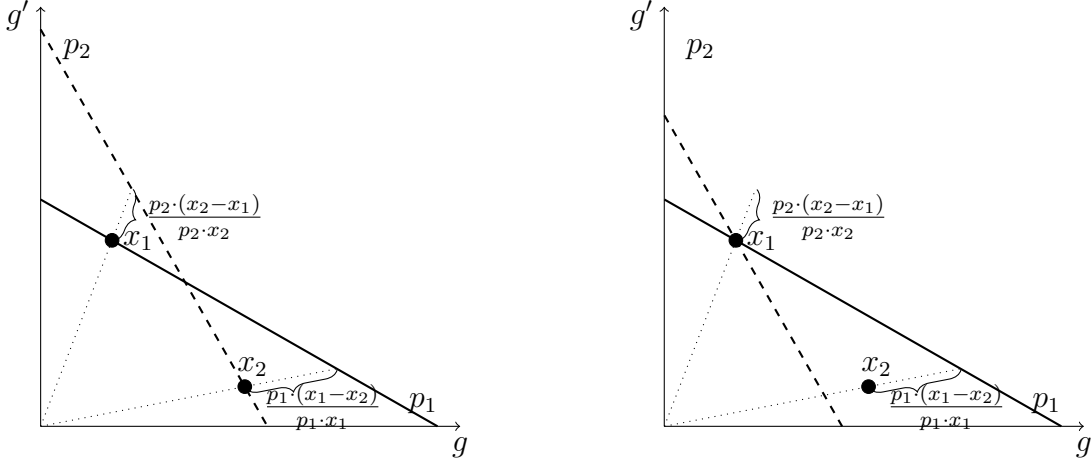
$$x_t P x_{t'} \iff p_t \cdot x_t > p_t \cdot x_{t'}.$$

**Definition**  $(x_{t_1}, \dots, x_{t_n})$  is a *cycle* if for all  $1 \leq i \leq n$ , when  $x_{t_{n+1}} := x_{t_1}$ ,

$$x_{t_i} R x_{t_{i+1}} \quad \text{and for some } 1 \leq j \leq n, x_{t_j} P x_{t_{j+1}}.$$

$(R, P)$  is *acyclical* if it does not have cycles.

Here each cycle constitutes a money pump cycle, i.e., a sequence of trades where the consumer is always willing to pay a non-negative amount of money to do each of the trades but at the end of the cycle is left with the starting bundle.



**Figure 1.** An example of Varian's index. The data is  $D = ((p_1, x_1), (p_2, x_2))$  with a cycle between  $x_1$  and  $x_2$ . Each budget is individually adjusted downwards until there are no cycles left. The smallest average size of budget adjustments to break all the cycles is the value of Varian's index. In this example, we adjust the income of at observation  $(p_2, x_2)$  until  $x_1$  is not anymore in the budget set to break the cycle.

## 2.2 Varian's Goodness-of-Fit Index

Next, we consider data with cycles and measure how close the data is to rational choice by using *Varian's goodness-of-fit index* (Varian, 1990). This index relaxes the revealed preference with observation-specific adjustment factors  $(e_t) \in [0, 1]^T$  as follows

$$x_t R^{(e_t)} x_{t'} \iff (1 - e_t)p_t \cdot x_t \geq p_t \cdot x_{t'} \quad \text{and}$$

$$x_t P^{(e_t)} x_{t'} \iff (1 - e_t)p_t \cdot x_t > p_t \cdot x_{t'}.$$

Varian's index is the least average of adjustments required to rationalize the data

$$I_{\text{Var}}(D) = \frac{1}{T} \inf_{(e_t) \in [0, 1]^T} \sum_{t=1}^T e_t \text{ such that } (R^{(e_t)}, P^{(e_t)}) \text{ is acyclical.}$$

The idea is that Varian's index is the most conservative estimate for the per observation average fraction of income that the decision maker could have saved without affecting the welfare. Each budget is adjusted downwards by a fraction  $e_t$  until there are no cycles left. This is illustrated in Figure 1. In the figure, we have two goods and our data consists of two observations:  $x_1$  at price  $p_1$  and  $x_2$  at price  $p_2$ . At the first observation, everything under the budget line defined by  $p_1$  is available and similarly at the second observation. Especially, there is a cycle between  $x_1$  and  $x_2$ . Varian's index is calculated by pulling each budget line individually towards the origin until there are no cycles left. The minimal average of adjustment factors is the value of Varian's index.

Next, we move on to computing Varian's index. Our computation method proceeds in two steps. First, we reformulate the index as a binary linear programming problem of removing all the cycles. Second, we provide a novel algorithm for finding a critical set of cycles that needs to be removed.

### 2.2.1 Reformulating Varian's Goodness-of-Fit Index

Varian's index requires finding the optimal levels of adjustment factors  $e_t \in [0, 1]$  for each observation. However, the problem can be reformulated to a simple binary linear programming problem with a binary variable for each revealed preference.

First, we simplify the variables  $e_t \in [0, 1]$  to binary variables. At the infimum solution, each  $e_t$  is  $\frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}$  for some  $t'$  or 0. When the value of  $e_t$  is at  $\frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t} + \varepsilon$  for any  $\varepsilon > 0$ , then this adjustment  $e_t$  removes the revealed preference  $x_t R x_{t'}$  and all the "cheaper" revealed preferences  $x_t R x_{t^*}$  such that

$$p_t \cdot (x_t - x_{t^*}) \leq p_t \cdot (x_t - x_{t'}).$$

This observation allows us to reformulate Varian's index as a problem of removing sets of cheaper trades for revealed preference  $x_t R x_{t'}$  at the cost  $\frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}$  until there are no cycles left.

Next, we simplify the constraint. In the previous reformulation, a revealed preference  $x_t R x_{t^*}$  is removed if a more expensive trade  $x_t R x_{t'}$  is removed. Formally, for a revealed preference pair  $(x_t, x_{t^*})$ , denote the set of more expensive trades than  $x_{t^*}$  at  $x_t$  as

$$U(x_t, x_{t^*}) = \left\{ (x_t, x_{t'}) \mid p_t \cdot (x_t - x_{t^*}) \leq p_t \cdot (x_t - x_{t'}) \right\}.$$

Now, the revealed preference  $x_t R x_{t^*}$  is removed if the set of cheaper trades for a revealed preference in  $U(x_t, x_{t^*})$  is removed. The constraint of removing all the cycles can be equivalently stated as that for each cycle  $(x_{t_1}, \dots, x_{t_n})$ , there exists  $1 \leq i \leq n$  such that a set of cheaper trades for a revealed preference in  $U(x_{t_i}, x_{t_{i+1}})$  is removed.

Finally, since the cost of removing revealed preferences that are not strict is zero, we can focus on the strict revealed preference  $P$  and strict cycles as defined next.

**Definition**  $(x_{t_1}, \dots, x_{t_n})$  is a *strict cycle* if for all  $1 \leq i \leq n$ , when  $x_{t_{n+1}} := x_{t_1}$ ,

$$x_{t_i} P x_{t_{i+1}}.$$

The collection of strict cycles up to different rotations is denoted  $\mathcal{C}^s$ .

The next result formalizes the reformulation of Varian's index.

**Theorem 1** Varian's index is the value of the following binary linear programming problem

$$\min_{\theta \in \{0,1\}^{\mathcal{P}}} \sum_{(x_t, x_{t'}) \in \mathcal{P}} \theta(x_t, x_{t'}) \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}$$

subject to

$$\text{for all } (x_{t_1}, \dots, x_{t_n}) \in \mathcal{C}^s, \quad \sum_{i=1}^n \sum_{(x_{t_i}, x_{t_{i+1}}) \in U(x_{t_i}, x_{t_{i+1}})} \theta(x_{t_i}, x_{t_{i+1}}) \geq 1,$$

where the summation of indices is taken modulo the length of the sequence and  $x_{t_{n+1}} = x_{t_1}$ .

### 2.2.2 Row Generation: Finding Critical Cycles

Solving the binary linear programming problem in Theorem 1 involves finding all the cycles and finding an optimal method to remove all of them, both of which are NP-hard problems. However, the problem of optimally removing all the cycles can be efficiently solved by standard methods. The problem of finding all the cycles is more demanding in practice. However, we can simplify this problem since, generally, only a few cycles are critical cycles that are binding constraints at the optimal solution. Then an optimal solution for removing only these critical cycles will give the optimal solution to the full problem in Theorem 1. This reduces the number of constraints in the problem significantly. We will find these critical cycles by solving the problem in Theorem 1 for a subset of cycles and finding new critical cycles that the optimal solution did not remove by using a greedy method.

Our method for finding critical cycles is a novel variation of the depth-first search. The algorithm is described in Algorithm 1. It starts from an optimal solution  $\theta^*$  to a subproblem of Theorem 1 for a set of cycles  $\mathcal{C}^*$  and finds new critical cycles that this solution missed. It follows in several steps. First, we compute the additional cost of removing the revealed preference  $(x_t, x_u)$  from the current solution  $\theta^*$ :

$$\frac{p_t \cdot (x_t - x_u)}{p_t \cdot x_t} - \sum_{(x_t, x_v) \in \mathcal{P}} \theta^*(x_t, x_v) \frac{p_t \cdot (x_t - x_v)}{p_t \cdot x_t}$$

where the first term is the original cost of removing  $(x_t, x_u)$  and the second term is the cost of the current optimal removals of revealed preferences involving  $x_t$  on the left-hand side. In the optimal solution, there is at most one removal of this type. Second, we implement the

current solution  $\theta^*$  for the relaxed revealed preference and remove all the revealed preferences with an additional cost lower than 0. Third, we implement a variation of depth-first search to find cycles. This is a standard method to find cycles in a graph. In our variation, when a cycle is found, we rollback the depth-first search to the beginning of the cycle and remove the cycle greedily by removing the revealed preference with the lowest additional cost. After this, we continue the depth-first search from the beginning of the cycle without the additionally removed revealed preference. The cycles that the search finds give new critical cycles that the previous solution did not remove.

Finally, our algorithm to calculate Varian’s index is described in Algorithm 2. As a first step, we find all the cycles of length 2. After this, we iteratively solve for the optimal removals for a subproblem of Theorem 1 for the current set of cycles and find new critical cycles for the current optimal solution until no more critical cycles are found. When there are no more critical cycles that the current optimal solution missed, the current optimal solution gives the optimal solution to the full problem in Theorem 1.

Finally, Varian’s goodness-of-fit can be solved separately within each strong component of the revealed preference graph to further simplify the problem. This was noted by Dean and Martin (2016) in the context of the minimum cost index.

### 2.3 Swaps Index and Normalized Minimum Cost Index

Our computation methods generalize to the *swaps index* (Apesteguia and Ballester, 2015) after providing a novel alternative representation for it. The swaps index captures the smallest number of better alternatives in the choice sets than the chosen alternatives for any preference. This measure is defined for random data in an abstract choice setting with a finite set of alternatives. An observation  $(x, A)$  is choosing  $x$  from the choice set  $A$ . The set of all the possible observations is denoted by  $\mathcal{O}$ . Random data  $f$  is a distribution of observations,  $f \in \Delta(\mathcal{O})$ . Denote the set of strict preferences (without indifference) by  $\mathcal{P}$ . The swaps index is the smallest number of preference swaps required to rationalize the data by a strict preference,

$$I_{\text{Swaps}}(f) = \min_{\succ \in \mathcal{P}} \sum_{(x,A) \in \mathcal{O}} f(x, A) \left| \{y \in A \mid y \succ x\} \right|.$$

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**Algorithm 1: Critical Cycles Search**

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**Input:**  $(p_t)_{t=1}^\top, (x_t)_{t=1}^\top, \theta^* \in \{0, 1\}^P$   
**Global variables:**  $\mathcal{C}$ , /\* The set of new critical cycles that the program returns \*/  
 $B$ , /\* The set of greedily removed revealed preferences to break new cycles \*/  
 $(\text{visited}(t))_{t=1}^\top$ , /\* Visited observations \*/  
 $(\text{stacked}(t))_{t=1}^\top$  /\* Visited observations that part of currently searched cycles \*/

**Denote:**  $\text{AddCost}(x_t, x_u) := \frac{p_t \cdot (x_t - x_u)}{p_t \cdot x_t} - \sum_{(x_t, x_v) \in P} \theta^*(x_t, x_v) \frac{p_t \cdot (x_t - x_v)}{p_t \cdot x_t}$

**begin**

- for**  $t = 1$  **to**  $T$  **do**
  - if**  $\text{visited}(t) = \text{false}$  **then** /\* Start DFS from  $x_t$  if unvisited \*/
    - $\text{visited}(t) := \text{stacked}(t) := \text{true}$
    - $\text{Critical\_Cycles\_Search\_Sub}(t)$
    - $\text{stacked}(t) := \text{false}$  /\* No more cycles found that include  $x_t$  \*/
- return**  $\mathcal{C}$  /\* Return the new critical cycles that found \*/

**Function**  $\text{Critical\_Cycles\_Search\_Sub}(t)$ : /\* Search for cycles from  $x_t$  and update  $\mathcal{C}$  and  $B$  for the found cycles. Exits if found a cycle that includes a previous observation in the search or no more cycles that include  $x_t$ . \*/

- for**  $u = 1$  **to**  $T$  **do** /\* Move the search on the revealed preference from  $x_t$  to  $x_u$  \*/
  - if**  $\text{AddCost}(x_t, x_u) > 0 \ \& \ (x_t, x_u) \notin B$  **then** /\*  $x_t \succ x_u$  and not removed \*/
    - if**  $\text{visited}(u) = \text{false}$  **then** /\* Move to  $x_u$  if unvisited \*/
      - while**  $\text{visited}(u) = \text{false} \ \& \ (x_t, x_u) \notin B$  **do** /\* Search until no cycles found that include  $x_u$  or remove the revealed preference  $x_t \succ x_u$  \*/
        - $\text{visited}(u) := \text{stacked}(u) := \text{true}$
        - $\text{cycle} := \text{Critical\_Cycles\_Search\_Sub}(u)$  /\* Search  $x_u$  \*/
        - if**  $\text{cycle} \neq 0$  **then** /\* cycle  $\neq 0$  iff found a cycle that includes  $x_t$  &  $x_u$  \*/
          - $\text{visited}(u) := \text{stacked}(u) := \text{false}$  /\* Rollback the search to  $x_t$  \*/
          - if**  $\text{visited}(t) = \text{true}$  **then** /\* Not at the beginning of the cycle \*/
            - return**  $(x_t, \text{cycle})$  /\* Return the cycle collected so far \*/
          - else** /\* At the beginning of the cycle \*/
            - $\text{visited}(t) := \text{true}$  /\* Unmark the first element of the cycle \*/
            - $i^* = \arg \min \{ \text{AddCost}(\text{cycle}_i, \text{cycle}_{i+1}) \mid 1 \leq i \leq |\text{cycle}| \}$
            - $B := B \cup \{ (\text{cycle}_{i^*}, \text{cycle}_{i^*+1}) \}$  /\* Greedily break the cycle \*/
            - $\mathcal{C} := \mathcal{C} \cup \{ \text{cycle} \}$  /\* Add the found cycle & search  $x_u$  again \*/
          - else** /\* No more cycles found that include  $x_u$  \*/
            - $\text{stacked}(u) := \text{false}$  /\* Return the search to  $x_t$  \*/
        - else if**  $\text{stacked}(u) = \text{true}$  **then** /\* The last visited elements until  $x_u$  form a cycle \*/
          - $\text{visited}(u) := \text{false}$  /\* Mark  $u$  & rollback the search until  $t = u$  \*/
          - return**  $(x_t, x_u)$  /\* Return the cycle collected so far \*/
      - return**  $0$  /\* If no cycles found, return  $0$  \*/

---

**Algorithm 2:** Varian's Goodness of Fit
 

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**Input:**  $(p_t)_{t=1}^\top, (x_t)_{t=1}^\top,$

**Notation:**  $\text{Varian\_argmin}(p, x, \mathcal{C})$ : the optimal solution to a subproblem of Theorem 1 for the set of cycles  $\mathcal{C}$ .

**begin**

$\theta^* := 0$

**for**  $t = 1$  **to**  $\top$  **do** /\* Search for two-cycles \*/

**for**  $u = t$  **to**  $\top$  **do**

**if**  $p_t \cdot (x_t - x_u) > 0$  &  $p_u \cdot (x_u - x_t) > 0$  **then**

$\mathcal{C}_{\text{new}} := \mathcal{C}_{\text{new}} \cup \{(x_t, x_u)\}$

**if**  $\mathcal{C}_{\text{new}} = \emptyset$  **then** /\* If no two-cycles, then search for any cycles \*/

$\mathcal{C}_{\text{new}} := \text{Critical\_Cycles\_Search}(p, x, \theta^*)$

**while**  $\mathcal{C}_{\text{new}} \neq \emptyset$  **do** /\* Solve the optimal revealed preference removals if found new critical cycles \*/

$\mathcal{C} := \mathcal{C} \cup \mathcal{C}_{\text{new}}$  /\* Update the set of cycles \*/

$\theta^* := \text{Varian\_argmin}(p, x, \mathcal{C})$  /\* Optimally break the current set of cycles \*/

$\mathcal{C}_{\text{new}} := \text{Critical\_Cycles\_Search}(p, x, \theta^*)$  /\* Search for new critical cycles that the current optimal solution missed \*/

**return**  $\sum_{(x_t, x_{t'}) \in \mathcal{P}} \theta^*(x_t, x_{t'}) \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}$  /\* Return the value of the optimal solution that removed all the cycles \*/

---

We provide a novel alternative representation for the swaps index based on a revealed preference on random data. We say that  $x$  is revealed to be preferred over  $y$  if  $x$  is chosen when  $y$  was available,

$$x \text{ P}^\Delta y \iff \text{there exists } A \supseteq \{x, y\} \text{ such that } f(x, A) > 0.$$

The swaps index focuses on rationalization by a strict preference and so here we do not distinguish between weak and strict revealed preferences.

Using this revealed preference, the next result shows that the swaps index has a representation as the lowest cost of removing revealed preferences to make the revealed preference rationalizable by a utility function when the cost of removing a revealed preference  $x \text{ P}^\Delta y$  is the probability that  $x$  is chosen over  $y$  in the data.

**Theorem 2** For all random data  $f \in \Delta(\mathcal{O})$ ,

$$I_{\text{Swaps}}(f) = \min_{B \subseteq \mathcal{P}^\Delta} \sum_{(x,y) \in B} \sum_{A \supseteq \{x,y\}} f(x, A) \text{ such that } \mathcal{P}^\Delta \setminus B \text{ is acyclical.}$$



Using this alternative representation, we can compute the swaps index analogously to Varian’s index when we remove revealed preferences instead of sets of cheaper trades. Formally, we use a trivial set of more expensive trades  $U(x_t, x_{t'}) = \{(x_t, x_{t'})\}$ . With this definition, the swaps index can be computed using Algorithms 1 and 2.

Moving back to the consumer setting, the *normalized minimum cost index* (NMCI) (Mononen, 2022) removes revealed preferences similarly to the swaps index and our computational methods generalize symmetrically to this index.<sup>1</sup> The normalized minimum cost index is defined in the consumer choice setting as

$$I_{\text{NMCI}}(D) = \frac{1}{|D|} \min_{B \subseteq \mathbb{R}} \sum_{(x_t, x_{t'}) \in B} \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t} \text{ such that } (\mathbb{R} \setminus B, P \setminus B) \text{ is acyclical.}$$

Here, NMCI removes revealed preferences until the revealed preference is rationalizable by a utility function, where the cost of removing a revealed preference is the fraction of income that could be extracted from that trade.<sup>2</sup> This index captures conservatively the per observation fraction of income that the decision maker could have saved without affecting welfare from all the possible trades. It can be computed analogously to the swaps index using the same trivial upper contour sets using Algorithms 1 and 2. In Section 3, we elaborate on the connection between the swaps index and NMCI and show that the swaps index is a counting version of the normalized minimum cost index that only counts the number of revealed preference removals but does not take into account the amount of money that the consumer could have saved.

## 2.4 Afriat’s Efficiency Index

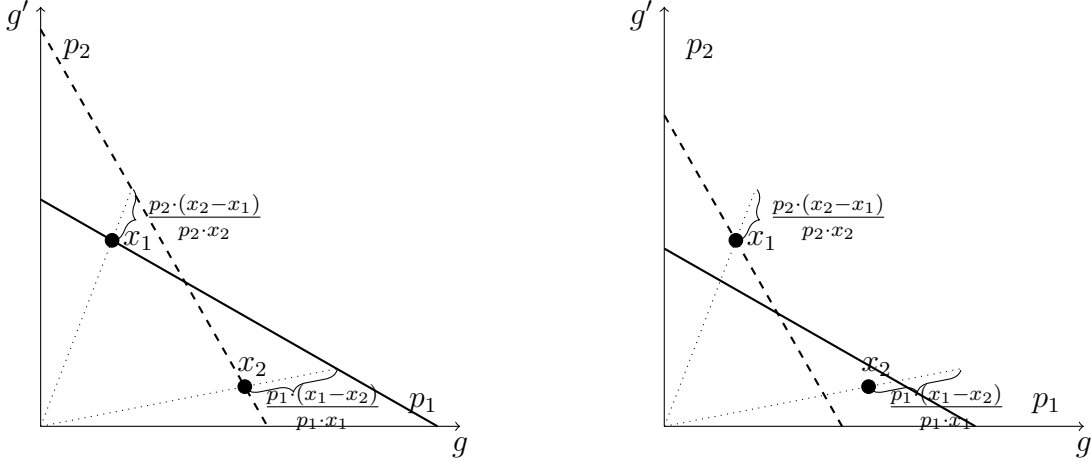
Next, we move on to *Afriat’s efficiency index* (Afriat, 1967; 1972). This is the most often used measure of rationality in the literature. This index relaxes the revealed preference with a common adjustment factor  $e \in [0, 1]$ ,

$$\begin{aligned} x_t R^e x_{t'} &\iff (1 - e)p_t \cdot x_t \geq p_t \cdot x_{t'} \quad \text{and} \\ x_t P^e x_{t'} &\iff (1 - e)p_t \cdot x_t > p_t \cdot x_{t'}. \end{aligned}$$

---

<sup>1</sup>The computational methods extend also to the *minimum cost index* (Dean and Martin, 2016).

<sup>2</sup>In contrast, Varian’s index removes observations, as in Theorem 1, instead of revealed preferences.



**Figure 2.** An example of Afriat's efficiency index. We have two goods  $g$  and  $g'$ . Prices are represented by lines defining the budget sets. The data is  $D = ((p_1, x_1), (p_2, x_2))$  with a cycle between  $x_1$  and  $x_2$ . The budgets are adjusted downwards until there are no cycles left. The size of the common budget adjustment to break all the cycles is the value of Afriat's index.

Afriat's efficiency index is the least amount of adjustment required to rationalize the data

$$I_{\text{AEI}}(D) = \inf_{e \in [0,1]} e \text{ such that } (R^e, P^e) \text{ is acyclical.}$$

This index captures conservatively the largest fraction of income that the decision maker could have saved without affecting the welfare. This is graphically illustrated in Figure 2. In contrast to Varian's index, here all the budget lines are moved towards zero at a common speed until there are no cycles left. The fraction that the budget lines are moved towards zero is Afriat's index.

A variation of our critical cycles search provides a novel algorithm to compute Afriat's efficiency index that is more efficient than the current algorithms in the literature. Our algorithm is based on increasing an estimate of Afriat's index until there are no cycles left. This differs from the algorithm by Smeulders et al. (2014) that estimates the index from above by decreasing the estimate for Afriat's index until the data does not satisfy GARP. Our algorithm is described in Algorithm 3. Here, we always remove all revealed preferences with a lower cost than the current estimate for Afriat's index. We search for cycles as in the critical cycles search, Algorithm 1. However, when the search finds a cycle, instead of removing the revealed preference with the cheapest cost, we update the estimate for Afriat's index by the cheapest cost. When the search finishes, there are no cycles left and the estimate is Afriat's index.

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**Algorithm 3:** Afriat's Depth-First Search
 

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**Input:**  $(p_t)_{t=1}^\top, (x_t)_{t=1}^\top$ ,  
**Global variables:**  $e$ ,  $(\text{visited}(t))_{t=1}^\top$ ,  $(\text{stacked}(t))_{t=1}^\top$

$e$   $\quad$  /\* The current estimate for Afriat's index \*/  
 $(\text{visited}(t))_{t=1}^\top$   $\quad$  /\* Visited observations \*/  
 $(\text{stacked}(t))_{t=1}^\top$   $\quad$  /\* Visited observations that part of currently searched cycles \*/

**begin**  
 $e := 0$   $\quad$  /\* Set starting point at 0 \*/  
**for**  $t = 1$  **to**  $T$  **do**  
  **if**  $\text{visited}(t) = \text{false}$  **then**  $\quad$  /\* Start DFS from  $x_t$  if unvisited \*/  
  |  $\text{visited}(t) := \text{stacked}(t) := \text{true}$   
  |  $\text{AEI\_DFS\_sub}(t)$   
  |  $\text{stacked}(t) := \text{false}$   $\quad$  /\* No more cycles found that include  $x_t$  \*/  
**return**  $e$   $\quad$  /\* The final estimate is Afriat's index \*/

**Function**  $\text{AEI\_DFS\_sub}(t)$ : /\* Search for cycles from  $x_t$  and update  $e$  for the found cycles. Exits if found a cycle that includes previous observations in the search or no more cycles that include  $x_t$ . \*/  
**for**  $u = 1$  **to**  $T$  **do**  $\quad$  /\* Move the search on the revealed preference from  $x_t$  to  $x_u$  \*/  
  **if**  $\frac{p_t \cdot (x_t - x_u)}{p_t \cdot x_t} > e$  **then**  $\quad$  /\*  $x_t P x_u$  & not removed by  $e$  \*/  
  | **if**  $\text{visited}(u) = \text{false}$  **then**  $\quad$  /\* Move to  $x_u$  if unvisited \*/  
  | | **while**  $\text{visited}(u) = \text{false}$  &  $\frac{p_t \cdot (x_t - x_u)}{p_t \cdot x_t} > e$  **do**  $\quad$  /\* Search until no cycles found that include  $x_u$  or remove the revealed preference  $x_t P x_u$  \*/  
  | | |  $\text{visited}(u) := \text{stacked}(u) := \text{true}$   
  | | |  $\text{cost} := \text{AEI\_DFS\_sub}(u)$   $\quad$  /\* Search  $x_u$  \*/  
  | | | **if**  $\text{cost} \neq 0$  **then**  $\quad$  /\*  $\text{cost} \neq 0$  iff found a cycle that includes  $x_t$  &  $x_u$  \*/  
  | | | |  $\text{cost} := \min(\text{cost}, \frac{p_t \cdot (x_t - x_u)}{p_t \cdot x_t})$   $\quad$  /\* Update the cheapest trade of the cycle \*/  
  | | | |  $\text{visited}(u) := \text{stacked}(u) := \text{false}$   $\quad$  /\* Rollback the search to  $x_t$  \*/  
  | | | | **if**  $\text{visited}(t) = \text{true}$  **then**  $\quad$  /\* Not at the beginning of the cycle \*/  
  | | | | | **return**  $\text{cost}$   $\quad$  /\* Return the cheapest trade of the cycle so far \*/  
  | | | | | **else**  $\quad$  /\* At the beginning of the cycle \*/  
  | | | | | |  $\text{visited}(t) := \text{true}$   $\quad$  /\* Unmark the first element \*/  
  | | | | | |  $e := \max(e, \text{cost})$   $\quad$  /\* Update the estimate  $e$  & search  $x_u$  again \*/  
  | | | | **else**  $\quad$  /\* No more cycles found that include  $x_u$  \*/  
  | | | | |  $\text{stacked}(u) := \text{false}$   $\quad$  /\* Return the search to  $x_t$  \*/  
  | **else if**  $\text{stacked}(u) = \text{true}$  **then**  $\quad$  /\* The last visited elements until  $x_u$  form a cycle \*/  
  | |  $\text{visited}(u) := \text{false}$   $\quad$  /\* Mark  $u$  & rollback the search until  $t = u$  \*/  
  | | **return**  $\frac{p_t \cdot (x_t - x_u)}{p_t \cdot x_t}$   $\quad$  /\* Return the cheapest trade of the cycle found so far \*/  
**return**  $0$   $\quad$  /\* If no cycles found, return 0 \*/

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## 2.5 Other Measures

Another commonly used measure of rationality is the *Houtman-Maks index* (Houtman and Maks, 1985) that is the smallest fraction of observations that needs to be dropped to rationalize the data. This corresponds to Varian's index when the cost of removing any lower contour set is 1.<sup>3</sup> In Section 3, we formalize this connection and show that the Houtman-Maks index is a counting version of Varian's index that only counts the number of budget adjustments but does not take into account the size of the budget adjustments.

Next, instead of decreasing the income as in Varian's index, we could increase the price of the bundles until the revealed preference is rationalizable. This is the motivation for *inverse Varian's goodness-of-fit index*. Here, again, we relax preferences by adjustments  $(e_{t'}) \in [0, 1]^T$  for each observation but this time we scale up the price of the bundles:

$$\begin{aligned} x_t R^{(e_{t'})} x_{t'} &\iff p_t \cdot x_t \geq p_t \cdot \frac{x_{t'}}{1 - e_{t'}} \quad \text{and} \\ x_t P^{(e_{t'})} x_{t'} &\iff p_t \cdot x_t > p_t \cdot \frac{x_{t'}}{1 - e_{t'}}. \end{aligned}$$

Inverse Varian's index is the least amount of adjustment required to rationalize the data.

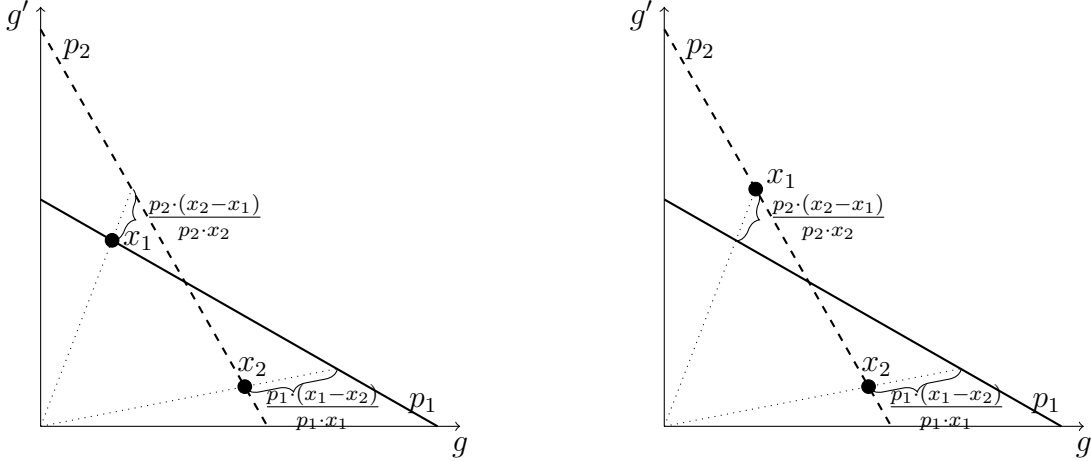
$$I_{\text{InvVar}}(D) = \frac{1}{T} \inf_{(e_{t'}) \in [0, 1]^T} \sum_{t=1}^T e_{t'} \text{ such that } (R^{(e_{t'})}, P^{(e_{t'})}) \text{ is acyclical.}$$

The idea of this measure is illustrated in Figure 3. Here, instead of pulling the budget lines inwards to break the cycles, we are increasing the costs of bundles and pushing the bundles outwards away from the origin to break the revealed preference cycles.

The next result shows that this measure corresponds to Varian's index after modifying the observed data by transposing the matrix  $[\frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}]_{t, t'}$ . This means that if we observed that when  $x_t$  was purchased the bundle  $x_{t'}$  was available at a fraction  $\frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}$  cheaper cost, then in the modified data set  $D'$  when  $x'_{t'}$  was purchased also  $x'_t$  was available at a fraction  $\frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}$  cheaper cost. Especially, this result shows that inverse Varian's index can be computed using Varian's index after reversing the direction of the revealed preference graph.

---

<sup>3</sup>When the weak and strict revealed preferences differ and there are  $t \neq t'$  such that  $p_t \cdot x_t = p_t \cdot x_{t'}$ , Algorithm 1 looking only for strict critical cycles does not find all the critical GARP cycles for the Houtman-Maks index. In this case, we use a modification of Johnson's (1975) algorithm for finding all the critical GARP cycles.



**Figure 3.** An example of inverse Varian's index. The data is  $D = ((p_1, x_1), (p_2, x_2))$  with a cycle between  $x_1$  and  $x_2$ . Each bundle is individually scaled upwards until there are no cycles left. The smallest total size of bundle adjustments to break all the cycles is the value of inverse Varian's index. In this example, we adjust  $x_1$  until it is not anymore in the budget set at  $(p_2, x_2)$  to break the cycle.

**Theorem 3** Assume that the data set is  $D = ((p_1, x_1), \dots, (p_T, x_T))$ . If a data set  $D' = ((p'_1, x'_1), \dots, (p'_T, x'_T))$  is such that for all  $t, t'$

$$\frac{p'_t \cdot (x'_t - x'_{t'})}{p'_t \cdot x'_t} = \frac{p_{t'} \cdot (x_{t'} - x_t)}{p_{t'} \cdot x_{t'}},$$

then

$$I_{\text{InvVar}}(D) = I_{\text{Var}}(D').$$

Finally, we discuss methods that require finding all the revealed preference cycles. Swoford and Whitney (1987) and Famulari (1995) suggest the *number of cycles* to measure rationality. Echenique et al. (2011) suggest *the money pump index* that is the average cost of each cycle. Formally,

$$I_{\text{MPI}}(D) = \frac{1}{|\mathcal{C}|} \sum_{(x_{t_1}, \dots, x_{t_n}) \in \mathcal{C}} \sum_{i=1}^n \frac{p_{t_i} \cdot (x_{t_i} - x_{t_{i+1}})}{\sum_{j=1}^n p_{t_j} \cdot x_{t_j}}$$

where  $\mathcal{C}$  denotes the set of cycles.

These measures are computationally demanding. However, the problem of finding all the cycles can be parallelized to make these measures computationally easier. There are multiple methods to find all the cycles, such as Johnson (1975) that we focus on here. This method searches for cycles based on a depth-first search of the revealed preference graph. This can be parallelized by combining it first with a breadth-first search for the first  $n$ -elements of the cycles and then continuing the depth-first search starting from these first  $n$ -elements. Here

all the searches with different starting sequences will find different cycles since the sequence of the first  $n$ -elements of the cycle will be different.<sup>4</sup> So these searches can be performed in parallel using the standard algorithm. Due to computational constraints, we do not use these measures in our following applications.

### 3 Variations of Measures

We now introduce variations of Varian's index, inverse Varian's index, and the normalized minimum cost index by using different norms to measure the total size of the adjustments. These variations provide a connection to Afriat's index, the Houtman-Maks index, and the swaps index. The variations follow Varian's (1990) suggestion of using the sum of squared errors or the Euclidean norm to measure the total size of the adjustments. With the sum of squared errors, a single large fraction of income that the consumer could have saved without a loss of welfare is more irrational than consistently making small mistakes with the same total fraction of income that the consumer could have saved. Additionally, we consider concave norms that consider consistent small mistakes as more irrational than a single large mistake.

Varian's index measures the total size of adjustment factors  $(e_t) \in [0, 1]^T$  in the  $\ell^1$  norm. We generalize this by measuring the total size of adjustment factors in the  $\ell^p$  norm for  $p \in \mathbb{R}_{++}$ . As is standard, we raise the norm to the power of  $p$  and focus on the sum of errors. Formally, Varian's index of degree  $\alpha$  is

$$I_{\text{Var}\alpha}(D) = \frac{1}{T} \inf_{(e_t) \in [0, 1]^T} \sum_{t=1}^T e_t^\alpha \text{ such that } (R^{(e_t)}, P^{(e_t)}) \text{ is acyclical.}$$

For  $\alpha = 1$ , this is Varian's index. For  $\alpha = 2$ , the mistakes are measured in the Euclidean norm with the sum of squared mistakes. Here, the idea is that bigger mistakes are worse and should be weighted more relative to small mistakes. For  $0 < \alpha < 1$ , this gives a concavification of Varian's index. The idea of the concavification is that we give more weight to the existence of mistakes relative to the size of the mistakes.

We define analogously the inverse Varian's index of degree  $\alpha$ ,  $\text{InvVar}\alpha$ , and the normalized minimum cost index of degree  $\alpha$ ,  $\text{NMCI}\alpha$ . These variations of measures can be

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<sup>4</sup>Additionally, the algorithm guarantees that the first element of the cycle is the lowest element in the cycle. This guarantees that we do not find different permutations of the same cycle.

computed symmetrically to the original measures. For Varian’s index, these variations can be reformulated as in Theorem 1 with the only difference that the objective function is

$$\min_{\theta \in \{0,1\}^P} \sum_{(x_t, x_{t'}) \in P} \theta(x_t, x_{t'}) \left( \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t} \right)^\alpha.$$

### 3.1 Connections Between Measures

We now provide connections between the different measures of rationality using the new variations. When  $\alpha \rightarrow \infty$ , we give more weight to the largest mistake. The next theorem shows that in the limit, after a normalization, we obtain Afriat’s index.

**Theorem 4** For all data  $D$ ,

$$\lim_{\alpha \rightarrow \infty} \left( I_{\text{Var}\alpha}(D) \right)^{\frac{1}{\alpha}} = \lim_{\alpha \rightarrow \infty} \left( I_{\text{InvVar}\alpha}(D) \right)^{\frac{1}{\alpha}} = \lim_{\alpha \rightarrow \infty} \left( I_{\text{NMCI}\alpha}(D) \right)^{\frac{1}{\alpha}} = I_{\text{AEI}}(D).$$

When  $\alpha \rightarrow 0$ , we give more weight to the number of mistakes. The next theorem shows that in the limit, for Varian’s index and inverse Varian’s index, we obtain the Houtman-Maks index. For the normalized minimum cost index, we obtain the swaps index.<sup>5</sup> These limits apply, however, only to data sets where the strict and the weak revealed preferences coincide, since the cost of removing weak revealed preferences that are not strict is 0 for the limit measures but 1 for the counting measures.

**Theorem 5** For all data  $D$  such that for all  $t \neq t'$ ,  $p_t \cdot (x_t - x_{t'}) \neq 0$ ,

$$\lim_{\alpha \rightarrow 0} I_{\text{Var}\alpha}(D) = \lim_{\alpha \rightarrow 0} I_{\text{InvVar}\alpha}(D) = I_{\text{HM}}(D)$$

and

$$\lim_{\alpha \rightarrow 0} I_{\text{NMCI}\alpha}(D) = I_{\text{Swaps}}(D).$$

The Houtman-Maks index provides a very coarse ranking for the rationality of choices and many different datasets  $D$  and  $D'$  have the same rationality index. However, it is often the case that, for any small  $\alpha$ , Varian’s index provides a strict ranking for the datasets,  $I_{\text{Var}\alpha}(D) \neq I_{\text{Var}\alpha}(D')$ , even if  $I_{\text{HM}}(D) = I_{\text{HM}}(D')$ . Thus, by focusing on the ranking of Varian’s index for small  $\alpha$ , we obtain a refinement of the Houtman-Maks index. We show next that

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<sup>5</sup>Here, we interpret deterministic data as random data with a uniform probability. Netzer and Lu (2022) offer an alternative extension of the swaps index to consumer choice based on Lebesgue measure of non-chosen alternatives that are better than the chosen alternative.

this refinement is captured by breaking the ties of Houtman-Maks with the geometric average of budget adjustments. Later on, in Section 4.1, measures of rationality based on these limiting orders are highlighted as the best variations in some cases.

Before stating the result, we need some notation. First, we denote the support of  $(e_t) \in [0, 1]^T$  as the positive indices,

$$\text{supp}(e_t) = \{t | e_t > 0\}.$$

Second, we break the ties of Houtman-Maks by using the geometric average of budget adjustments required to rationalize the data when a fraction of  $I_{\text{HM}}(D)$  budgets are adjusted:

$$I_{\text{Geom.Avg}}^{\text{Var}}(D) = \inf_{(e_t) \in [0, 1]^T} \left( \prod_{t \in \text{supp}(e_t)} e_t \right)^{\frac{1}{|\text{supp}(e_t)|}} \text{ s.t. } \frac{|\text{supp}(e_t)|}{T} = I_{\text{HM}}(D) \text{ and } (R^{(e_t)}, P^{(e_t)}) \text{ is acyclical.}$$

The next result formalizes that in the limit as  $\alpha \rightarrow 0$ , Varian's index firstly ranks the rationality of choices based on the Houtman-Mask index and secondly based on the geometric average of budget adjustments.

**Theorem 6** Assume that data sets  $D$  and  $D'$  are such that for all  $t \neq t'$ ,  $p_t \cdot (x_t - x_{t'}) \neq 0$  and  $p'_t \cdot (x'_t - x'_{t'}) \neq 0$ . If

$$I_{\text{HM}}(D) > I_{\text{HM}}(D')$$

or

$$I_{\text{HM}}(D) = I_{\text{HM}}(D') \text{ and } I_{\text{Geom.Avg}}^{\text{Var}}(D) > I_{\text{Geom.Avg}}^{\text{Var}}(D'),$$

then there exists  $\alpha^* > 0$  such that for all  $\alpha \in (0, \alpha^*)$ ,

$$I_{\text{Var}\alpha}(D) > I_{\text{Var}\alpha}(D').$$

This result shows that the appropriate way to break the ties of the Houtman-Maks index is by the geometric average of budget adjustments. Since budget adjustments are fractions of income, a natural way to capture the average budget adjustment is by the geometric average. Formally, the result follows from taking a first-order approximation of the  $\ell^p$  norm as  $p \rightarrow 0$ . In the appendix, we show that using higher-order approximations provides a full characterization for the limiting pairwise comparisons of rationality. Additionally, in the appendix, we extend the result to any data sets where the weak and strict revealed preferences might not coincide.



**Remark** Usually, in practice, data sets have the same number of observations. In this case, the lexicographic ranking of rationality of the data sets in Theorem 6 is captured by the measure of rationality

$$I_{\text{HM}}(D) + \frac{1}{T} I_{\text{Geom.Avg}}^{\text{Var}}(D).$$

This measure of rationality is included in the supplementary toolbox.

Lastly, we extend symmetrically the limiting order for NMCI and the swaps index and show that the appropriate way to break the ties of the swaps index is by the geometric average of the removed revealed preferences. Define

$$I_{\text{Geom.Avg}}^{\text{NMCI}}(D) = \min_{B \subseteq \mathbb{R}} \left( \prod_{(x_t, x_{t'}) \in B} \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t} \right)^{\frac{1}{|B|}} \text{ s.t. } \frac{|B|}{T} = I_{\text{Swaps}}(D) \text{ and } (R \setminus B, P \setminus B) \text{ is acyclical.}$$

**Theorem 7** Assume that data sets  $D$  and  $D'$  are such that for all  $t \neq t'$ ,  $p_t \cdot (x_t - x_{t'}) \neq 0$  and  $p'_t \cdot (x'_t - x'_{t'}) \neq 0$ . If

$$I_{\text{Swaps}}(D) > I_{\text{Swaps}}(D')$$

or

$$I_{\text{Swaps}}(D) = I_{\text{Swaps}}(D') \text{ and } I_{\text{Geom.Avg}}^{\text{NMCI}}(D) > I_{\text{Geom.Avg}}^{\text{NMCI}}(D'),$$

then there exists  $\alpha^* > 0$  such that for all  $\alpha \in (0, \alpha^*)$

$$I_{\text{NMCI}\alpha}(D) > I_{\text{NMCI}\alpha}(D').$$

## 4 Applications

### 4.1 Simulation Comparisons

In our first application, we compare the accuracy of different measures of rationality based on simulations with choice mistakes. We find that the variations of measures with a degree 0.5 always improve the performance. This suggests that consistent small mistakes are more indicative of lower rationality than a single large mistake.

More specifically, our comparisons show that the choice of the best measure depends on how fat-tailed the mistake distribution is, but it does not depend on the number of periods or the number of goods. Especially, with fat-tailed mistake distributions, the best variations have a degree of approximately 0.

### 4.1.1 Choice Model

We model a decision maker (DM) who makes mistakes at purchasing the optimal quantities. Specifically, the DM makes choices with a trembling hand around the optimal bundle on the budget line.<sup>6</sup>

We assume that the DM has a constant income  $w$  of 8. We assume that the prices are independent draws from a normal distribution with mean 4 and variance 1 that is truncated to be above 0.1. The number of goods  $G$  is varied in different specifications.

The decision maker is assumed to have a symmetric additive log utility over the goods. The utility from a bundle  $(x_g)_{g=1}^G \in \mathbb{R}_+^G$  is

$$u((x_g)_{g=1}^G) = \sum_{g=1}^G \log(x_g).$$

The optimal demand is  $(x_g^*)_{g=1}^G$  given the income  $w$  and prices  $(p_g)_{g=1}^G$ . The decision maker makes choice mistakes  $(\varepsilon_g)_{g=1}^G$  in the income shares of goods and scales uniformly the income shares to consume the whole income. That is, the observed choice with mistakes is

$$\frac{x_g^* + \frac{w}{p_g} \varepsilon_g}{1 + \sum_{g'=1}^G \varepsilon_{g'}}.$$

We test for different mistake distributions. The distributions are chosen to have a mode at 0 and are truncated to be above  $-\frac{p_g}{w} x_g^*$ , for the bundles to be non-negative. Here we focus on two very different mistake distributions: normal and Cauchy distributions. The normal distribution is a concentrated standard distribution and the Cauchy distribution is a standard example of a fat-tailed distribution. The results are robust to using log-normal, log-Cauchy, uniform, Weibull, Fréchet, Gumbel, or Gamma distributions.

### 4.1.2 Ordinal Measure Comparisons

The mistake model gives an objective size of mistakes from the variance or scaling of the distributions. We study how well the measures of rationality capture these objectively sized mistakes. The first difficulty is that these objective mistakes might be measured in different units from the measures of rationality and that values of different measures of rationality

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<sup>6</sup>Our results are robust to very different types of mistakes: randomizing uniformly on the budget plane, random utility with an additive utility error for logit choices on the budget plane, or making the utility function observation-dependent with variance in the parameters of the utility function.

might not be directly comparable. To avoid these unit problems, we focus on ordinal comparisons and focus on finding out if different measures rank the irrationality of different data sets correctly according to the objective criterion.

We compare pairwise the rationality of data sets drawn from two different populations. These two populations have different mistake levels  $e_1$  and  $e_2$  such that  $e_1 > e_2$ . We draw data sets  $D_1$  and  $D_2$  with different prices from these two populations using the previous section's choice model. Now  $D_1$  is objectively more irrational than  $D_2$  based on the mistake levels in the population. We focus on non-trivial comparisons and redraw these data sets until they have violations of rationality.<sup>7</sup> We then estimate the probability that the measure of rationality  $I$  ranks the data sets correctly:

$$P(I(D_1) > I(D_2) | D_1, D_2 \text{ not rational}),$$

where the probability is taken over the price distribution and the mistake distributions for  $D_1$  and  $D_2$ . With these "hit rates" for correct rankings for each measure, we can compare the performance of the different measures.

### 4.1.3 Calibration

We calibrate the size of mistakes in our simulations using the share of rational people in the population. For a mistake level  $e$ , we estimate the share of rational people,

$$P(D \text{ is rationalizable})$$

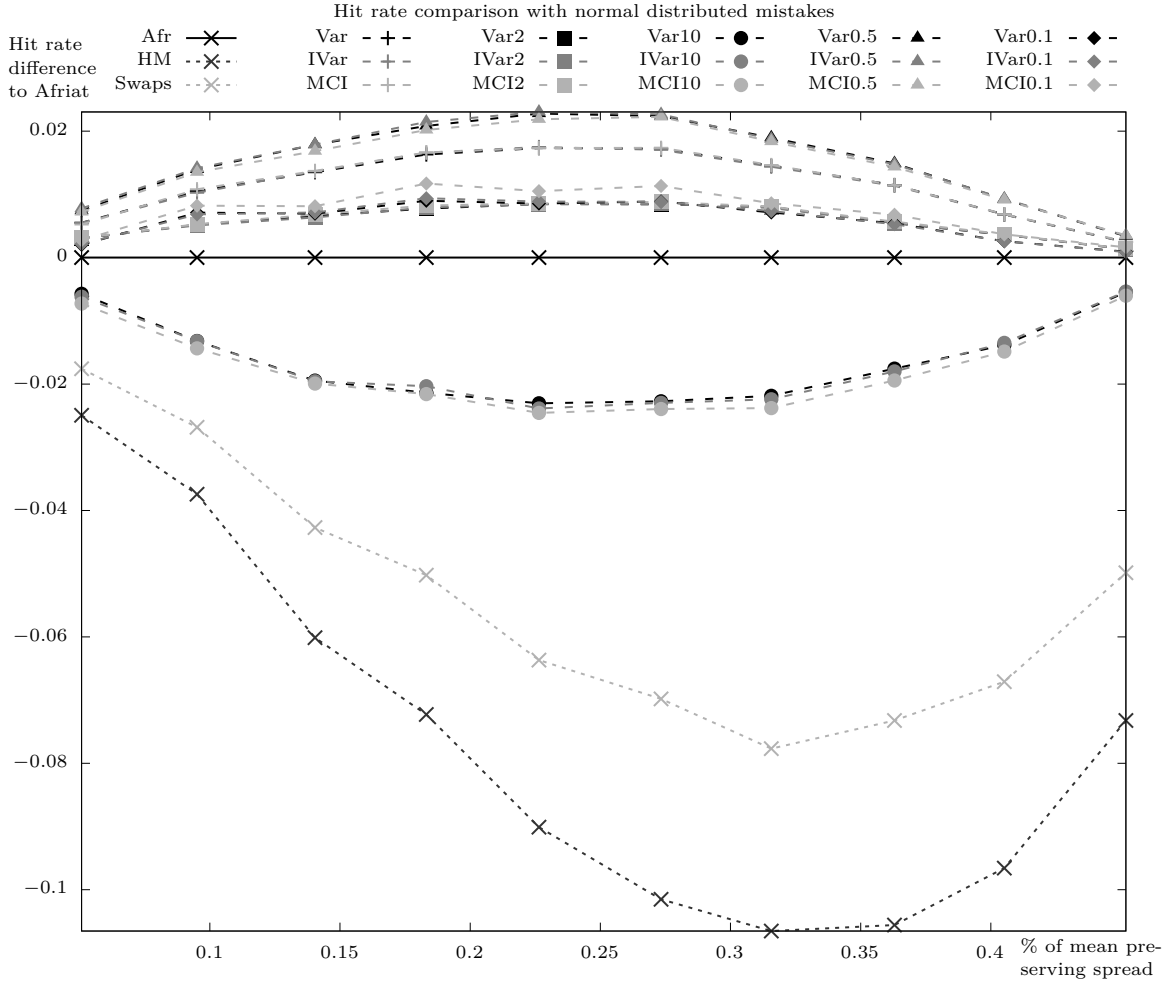
where  $D$  is a random data set from a population with mistake level  $e$ . We calibrate the mistake level  $e$  based on this estimate using a simulated method of moments.<sup>8</sup>

In the calibration, we keep the average share of rational people across populations constant at 50% and compare two populations that differ in their share of rational people. Formally, we start from two populations that consist of 50% rational individuals and increase the level of mistakes of one population and decrease the level of mistakes of the other population in such a way that the average share of rational people is kept at 50%. That is,

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<sup>7</sup>Without violations of rationality, all the different measures would agree on the ranking of the data sets and the comparison would be trivial.

<sup>8</sup>Since the mistake levels are unidimensional, we implement this simulated method of moments as a root-finding problem. We solve the root-finding problem using a combination of Brent's method (Brent, 1973) and Algorithm 748 (Alefled et al., 1995) that combines inverse cubic and quadratic interpolation with double length secant and bisection methods.

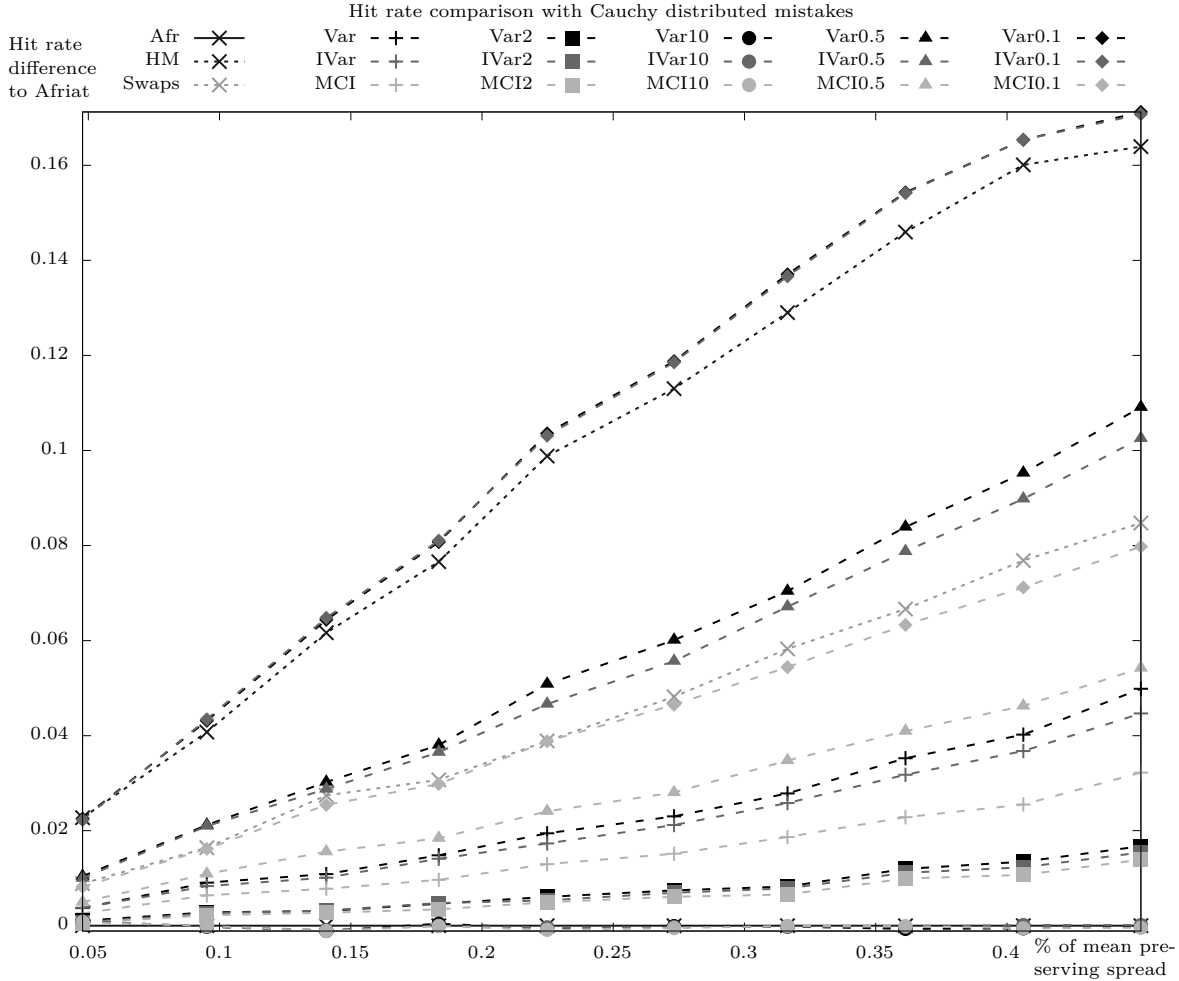


**Figure 4.** Measures' hit rate differences to Afriat's index's hit rate when mistakes are drawn from normal distribution. The two populations share of rational people are calibrated to be a mean preserving spread from 50%. The x-axis changes the percentage level of mean preserving spread.

we take mean preserving spreads of rationality for these two populations. We vary the level of these mean preserving spreads from 5% to 45%. This changes the difficulty of predicting which data set has lower mistakes. We have chosen the average share of rationality to be 50% in order to have as many mean preserving spreads for comparison as possible.

#### 4.1.4 Comparisons of Measures

We focus on 6 goods with 45 time periods in our simulations. The low number of goods follows empirical studies on the measures of rationality that aggregate goods into a few categories. The results are robust to changing the number of goods or time periods. The simulation sample size for each point is 100,000 pairwise comparisons.



**Figure 5.** Measures' hit rate differences to Afriat's index's hit rate when mistakes are drawn from Cauchy distribution. The two populations share of rational people are calibrated to be a mean preserving spread from 50%. The x-axis changes the percentage level of mean preserving spread.

Figures 4 and 5 compare the hit rates of different measures to Afriat's index's hit rate when mistakes are normal and Cauchy distributed, respectively.

We start with normally distributed mistakes in Figure 4. First, this graph shows that Varian's index, inverse Varian's index, and the normalized minimum cost index are performing better than Afriat's index. However, the differences are small. Second, the counting measures of the Houtman-Maks index and the swaps index are performing up to 10 percentage points worse than Afriat's index. Third, variations of measures with  $\alpha > 1$  decrease the performance. Especially, the measures perform worse than Afriat's index for high  $\alpha$ . Fourth, variations of measures with  $\alpha < 1$  first improve the performance of the measures for  $\alpha$  close to one and then decrease the performance of the measure for lower value of  $\alpha$ . The peak performance is around  $\alpha = 0.5$ .

The normally distributed mistakes illustrate a concentrated mistake distribution. Other mistake distributions and mistake models with concentrated mistake distributions create the same results: logit-choice mistakes, trembling hand mistakes, or time-dependent utility function parameters with concentrated distributions of log-normal, uniform, Gumbel, or reversed Weibull.

Next, we move on to Cauchy distributed mistakes in Figure 5. This graph shows that Varian's index, inverse Varian's index and the normalized minimum cost index are performing better than Afriat's index. The performance improvement is up to 4 percentage points. Second, the counting measures the Houtman-Maks index and the swaps index are performing much better than Afriat's index. The Houtman-Maks index improves the performance by up to 16 percentage points and the swaps index by up to 8 percentage points. Third, variations of the measures with  $\alpha > 1$ , again, decrease the performance of the measures. For  $\alpha = 10$ , the measures are performing equal to Afriat's index. Variations of the measures with  $\alpha < 1$  increase the performance of the measures. For  $\alpha = 0.1$ , Varian's index and Inverse Varian's index are performing better than the Houtman-Maks index and improve the performance by up to 16.5 percentage points. For  $\alpha = 0.1$ , the normalized minimum cost index is performing slightly worse than the swaps index. This highlights that for Varian's and inverse Varian's index, using the geometric average of the size of the adjustments for tie-breaking improves the performance, but for the normalized minimum cost index it decreases the performance.

The Cauchy distributed mistakes illustrate a fat-tailed mistake distribution. Other mistake distributions and mistake models with fat-tailed mistake distributions create the same results: uniform random choice on the budget plane, trembling hand mistakes, or observation dependent utility function parameters with fat-tailed distributions of log-Cauchy or Fréchet.

These results highlight the connection between measures counting the number of rationality violations, such as the Houtman-Maks index and the swaps index, and measures capturing the intensity of rationality violations, such as Varian's index and NMCI. This connection was highlighted theoretically in Theorems 5 to 7. Our simulation results show how overweighting small budget adjustments always improves the performance of the measures. However, the optimal amount of overweighting depends on how fat-tailed the mistake distribution is. If the mistakes are concentrated, then the intensities of rationality violations are valuable at

capturing the size of choice mistakes and it is optimal to overweight the mere existence of budget adjustments only slightly. On the other hand, if the mistakes are fat-tailed, then the size of the budget adjustments is less useful at capturing the size of choice mistakes. Instead, it is optimal to prioritize the existence of budget adjustments and use the size of the budget adjustments only to break ties.

The intuition for this distinction is that with fat-tailed mistakes there are big mistakes even at low mistake levels, so the size of the budget adjustments is not useful at distinguishing between different mistake levels anymore. Instead, the rate of mistakes should be used for identifying the extent of mistakes. This highlights how the choice of the measure of rationality depends on the application and different measures are better at capturing different types of mistakes depending on how concentrated the mistakes are.

## 4.2 Significance Levels for Measures of Rationality

In our second application, we calculate statistical significance levels for the measures of rationality. We show that these significance levels give a different picture for the prevalence of irrational choices in empirical applications than the current approaches used in the literature. Especially, these significance levels depend on the choice situation and how difficult it is to observe violations of rationality in the choice situation. This situation-dependency in the power of measures of rationality has been noted in the literature (Andreoni and Harbaugh, 2008; Andreoni et al., 2013).

Our significance levels are based on testing if the measure of rationality observed in the data could have been generated by a person choosing randomly on the budget line. Formally, the observed data is  $((p_1, x_1), \dots, (p_T, x_T))$ . Denote the income  $w_t = p_t \cdot x_t$  and the budget line

$$B(p_t, w_t) = \{x \in \mathbb{R}_+^G \mid p_t \cdot x = w_t\}.$$

For a measure of rationality  $I$ , we are testing the null hypothesis

$$H_0 : I((p_1, x_1), \dots, (p_T, x_T)) \sim I((p_1, \text{Uni}(B(p_1, w_1))), \dots, (p_T, \text{Uni}(B(p_T, w_T))))).$$

Here,  $\text{Uni}(B(p_t, w_t))$  denotes the uniform distribution with a constant density function on  $B(p_t, w_t)$ .

We apply this approach to the experimental data from Choi et al. (2014). In this experiment, the subjects allocated income between two risky accounts. The accounts were equally likely and one of them paid off. The experiment had 1182 participants. We focus on one-sided tests of the null hypothesis at 95% significance level. We estimate the cumulative density function for  $I\left((p_1, \text{Uni}(B(p_1, w_1))), \dots, (p_T, \text{Uni}(B(p_T, w_T)))\right)$  by Monte Carlo simulations.

The first column of Table 2 shows that, in this experimental data, 56% of people are rational using Afriat’s index and 61% using Varian’s index. The swaps index classifies 59% of people as rational and is close to Varian’s index and the normalized minimum cost index. Inverse Varian’s index of degree 0.5 classifies the most people as rational (63%). Variations of the measures with a degree above one distinguish less people as more rational than random behavior. Additionally, performing one-sided test for more irrational choices than random behavior shows that 2% of the participants were significantly more irrational than random behavior.

Second, we apply this method to the empirical data set from Dean and Martin (2016). This is scanner level data for 977 representative households from Denver metropolitan area over 2 years. The data is aggregated to monthly purchases and to three aggregate products of beverages, meals, and snacks. The second column of Table 2 shows that, in this empirical data the shares of rational people are approximately halved compared to the previous experimental data set. A major difference to the experimental data set is that the Houtman-Maks index and the swaps index classify fewer people as rational than Afriat’s index due to many random choices having the same values for these indices.

To get a perspective into these numbers, we contrast them to the standard methods used in the literature to classify choices as rational. The first method is if the observed choices are rationalizable and satisfy GARP. The second method uses a cut-off value of 0.05 for Afriat’s index to classify the choices as rational. This method has been used e.g. in Varian (1990), Choi et al. (2007), Lanier et al. (2018), and Polisson et al. (2020).

The first column of Table 2 shows that, in the experimental data, satisfying GARP is too stringent a condition. Up to 45% of statistically significantly rational people do not satisfy GARP. Additionally, Afriat’s index cut-off of 0.05 is also too demanding and misclassifies



Share Significantly	Choi et al. (2014)		Dean & Martin (2016)	
	Rational	Irrational	Rational	Irrational
Afriat's Index	0.563	0.020	0.277	0.011
Houtman-Maks	0.511	0.003	0.184	0.002
Swaps	0.589	0.017	0.207	0.002
Varian	0.608	0.022	0.287	0.008
InvVarian	0.613	0.016	0.288	0.008
NMCI	0.600	0.021	0.286	0.006
Varian2	0.583	0.020	0.284	0.010
InvVarian2	0.585	0.019	0.282	0.009
NMCI2	0.583	0.019	0.282	0.009
Varian10	0.568	0.022	0.252	0.011
InvVarian10	0.568	0.021	0.256	0.011
NMCI10	0.569	0.021	0.255	0.011
Varian0.5	0.622	0.018	0.288	0.005
InvVarian0.5	0.632	0.015	0.288	0.005
NMCI0.5	0.608	0.020	0.288	0.005
Varian0.1	0.611	0.013	0.277	0.005
InvVarian0.1	0.608	0.014	0.278	0.004
NMCI0.1	0.621	0.019	0.278	0.004
Satisfy GARP	0.173		0.300	
Afriat's Index $\leq 0.05$	0.453		0.955	

**Table 2.** Shares of statistically significantly rational and irrational people according to different measures of rationality in the experimental data in Choi et al. (2014) and the empirical data in Dean and Martin (2016). Additionally, the share of rational people according to criteria that the choices are rationalizable or Afriat's index is below 0.05.

11% as irrational when the choices are statistically significantly more rational than random behavior according to Afriat's index.

On the contrary, the second column shows that, in the empirical data satisfying GARP is an excellent proxy for classifying people as rational.<sup>9</sup> On the other hand, Afriat's index cut-off at 0.05 is too lenient and misclassifies at least 65% of the sample. This highlights that the significance level of the measure of rationality depends on the choice situation and the observed budget sets.

## 5 Conclusion

This paper has provided novel, more accurate methods to evaluate the severity of nonrational choices. These methods are easy-to-use and computationally feasible. Our empirical example

<sup>9</sup>Note that there might be people who satisfy GARP but are not statistically significantly more rational than random choice behavior since more than 95% of the random choices satisfy GARP.

showed that using more accurate measures of rationality for studying the determinants of rationality of choices can give a substantially different picture.

Our analysis compared the measures of rationality in a general environment. However, our general approach offers a recipe for comparisons that can be tailored to any specific experimental environment. First, we focused on different types of choice mistakes. However, in other environments, different types of mistakes could be more reasonable, such as (rational) inattention in observing the budget slope, probability of temptation, or choice framing. Second, the budget environment can be tailored to be the same as in any experiment. The tailored comparisons then give a ranking of the accuracy of the different measures in the specific experimental environment.

## 6 Appendix

### 6.1 A Comparison to Demuynck and Rehbeck (2023a)

Recently, Demuynck and Rehbeck (2023a; 2023b) provided computational methods for calculating Varian’s goodness-of-fit, the Houtman-Maks index, and the minimum cost index. Their method is based on reformulating the indices as a mixed linear programming problem. However, this method is prone to reporting incorrect values due to evaluating strict inequality constraints in the linear programming problem with weak inequalities. The following example illustrates this point.

**Example 1** The data set consists of two goods and three periods.

$$D = \left( (p_1, x_1), (p_2, x_2), (p_3, x_3) \right) = \left( \left( \begin{bmatrix} 74 \\ 68 \end{bmatrix}, \begin{bmatrix} 47 \\ 23 \end{bmatrix} \right), \left( \begin{bmatrix} 86 \\ 13 \end{bmatrix}, \begin{bmatrix} 10 \\ 57 \end{bmatrix} \right), \left( \begin{bmatrix} 21 \\ 82 \end{bmatrix}, \begin{bmatrix} 2 \\ 72 \end{bmatrix} \right) \right).$$

Here,  $I_{\text{Var}}(D) = 0.0877$ . The computational toolbox of Demuynck and Rehbeck (2023b) claims erroneously that  $I_{\text{Var}}(D) = 0.0595$ .

However, by adjusting the fixed parameter in Demuynck and Rehbeck (2023b) that makes the strict inequalities weak, their method can be corrected.

In Table 3, we compare the computation time for Varian’s goodness-of-fit, the Houtman-Maks index, and the normalized minimum cost index between our method and the method of Demuynck and Rehbeck (2023a; 2023b) using Drichoutis and Nayga’s (2020) experiment from Section 1.1. We compare the total time of computing the measures for all 343 subjects in the experiment. We compare our method implemented in Matlab to the original method of Demuynck and Rehbeck and to an adjusted version that reports correct values (but increases computation time). We do this for both an implementation in Matlab for a comparable comparison and an implementation using the proprietary Gurobi solver to solve the mixed linear programming problem of Demuynck and Rehbeck. The comparison was run on a desktop computer with an Intel(R) Core(TM) i9-10850K CPU 3.60GHz processor. This comparison shows that the algorithms presented here perform substantially better than the methods in Demuynck and Rehbeck (2023a). Additionally, with the unadjusted method of Demuynck and Rehbeck, 15% of the values of Varian’s index and 62% of NMCI were

	Varian	Houtman-Maks	NMCI
Algo. 1-2 in Matlab	5.83	4.30	12.14
D&R in Matlab	65820	163.19	311.94
Adjusted D&R in Matlab	571119	37287	347515
D&R w/ Gurobi	312.98	83.95	160.34
Adjusted D&R w/ Gurobi	442.00	112.58	741.63

**Table 3.** Comparison of computation time in seconds for Algorithms 1 and 2 when implemented in Matlab and Demuyck and Rehbeck’s (2023b) (D&R) method when implemented in Matlab and when using Gurobi solver and when adjusted to report the correct values. The comparison is for Varian’s goodness-of-fit, the Houtman-Maks index, and the normalized minimum cost index using the experiment from Drichoutis and Nayga (2020). The computation time is the total time to compute the index for 343 subjects in the experiment.

incorrect by more than  $\frac{0.01}{T}$  where  $T = 60$  is the number of budget choices that normalizes the indices.

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# Online Appendix to “Computing and Comparing Measures of Rationality”

Not intended for publication

## S.1 Proof for Theorem 1

We prove a more general version of Theorem 1 that shows the result also for all the variations of Varian’s index.

**Theorem 8** Assume that  $f : [0, 1] \rightarrow \mathbb{R}_+$  is a strictly increasing and continuous function. Define the index

$$I_{\text{FVar}}(D) = \inf_{(e_t) \in [0, 1]^T} \sum_{t=1}^T f(e_t) \text{ such that } (\mathbf{R}^{(e_t)}, \mathbf{P}^{(e_t)}) \text{ is acyclical.}$$

We have,

$$I_{\text{FVar}}(D) = \min_{\theta \in \{0, 1\}^{\mathbf{P}}} \sum_{(x_t, x_{t'}) \in \mathbf{P}} \theta(x_t, x_{t'}) f\left(\frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}\right) \quad \text{subject to}$$

$$\text{for all } (x_{t_1}, \dots, x_{t_n}) \in \mathcal{C}^s, \quad \sum_{i=1}^n \sum_{(x_{t_i}, x_{t_{i+1}}) \in U(x_{t_i}, x_{t_{i+1}})} \theta(x_{t_i}, x_{t_{i+1}}) \geq 1,$$

where the summation of indices is taken modulo the length of the sequence and  $x_{t_{n+1}} = x_{t_1}$ .

*Proof.* Denote the value of the binary linear programming problem by  $I^*$  and the optimal removals by  $\theta^*$ .

We show first that for all  $t$ , there is at most one  $t'$  such that  $\theta^*(x_t, x_{t'}) = 1$ . Assume, per contra, that there exist  $t^1, t^2$  such that  $t^1 \neq t^2$  and  $\theta^*(x_t, x_{t^1}) = \theta^*(x_t, x_{t^2}) = 1$ . Assume, w.l.o.g. that

$$p_t \cdot (x_t - x_{t^1}) \geq p_t \cdot (x_t - x_{t^2}).$$

Now we have for all  $(x, y) \in \mathbf{P}$ , if  $(x_t, x_{t^2}) \in U(x, y)$ , then  $(x_t, x_{t^1}) \in U(x, y)$ . Thus  $\tilde{\theta}$  defined by  $\tilde{\theta}(x, y) = \theta^*(x, y)$  by for all  $(x, y) \in \mathbf{P} \setminus (x_t, x_{t^2})$  and  $\tilde{\theta}(x_t, x_{t^2}) = 0$  is a solution to the binary linear integer programming problem. Additionally, this solution has a lower value since

$$p_t \cdot (x_t - x_{t^2}) > 0 \text{ and so } f\left(\frac{p_t \cdot (x_t - x_{t^2})}{p_t \cdot x_t}\right) > 0$$

contradicting the optimality of  $\theta^*$ .

Next, we show that  $I^* \geq I_{\text{fVar}} - \varepsilon$  for any  $1 > \varepsilon > 0$ . Since  $[0, 1]$  is a compact set and  $f$  is continuous,  $f$  is uniformly continuous. There exist  $\delta > 0$  such that for all  $x, y \in [0, 1]$  with  $|x - y| < 2\delta$ ,  $|f(x) - f(y)| < \frac{\varepsilon}{T}$ . Define  $(e_t^\theta) \in [0, 1]^T$  by the following. For all  $t$  such that there exist  $t'$  with  $\theta(x_t, x_{t'}) = 1$ , define

$$e_t^\theta = \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}.$$

For all other  $t$ , define  $e_t^\theta = 0$ . Additionally, define  $(e_t^\delta) \in [0, 1]^T$  by for all  $t$

$$e_t^\delta = \min\{e_t^\theta + \delta, 1\}.$$

Now we have by the above observation

$$\sum_{t=1}^T f(e_t^\theta) = I^*$$

and so

$$\sum_{t=1}^T f(e_t^\delta) \leq \sum_{t=1}^T f(e_t^\theta) + \sum_{t=1}^T |f(e_t^\delta) - f(e_t^\theta)| \leq I^* + \varepsilon.$$

We show that  $(\mathbb{R}^{(e_i^\delta)}, \mathbb{P}^{(e_i^\delta)})$  is acyclical. We have  $\mathbb{R}^{(e_i^\delta)} \supseteq \mathbb{R}$  and  $\mathbb{P}^{(e_i^\delta)} \supseteq \mathbb{P}$ . Thus it suffices to show that any of the cycles of  $(\mathbb{R}, \mathbb{P})$  is not a cycle in  $(\mathbb{R}^{(e_i^\delta)}, \mathbb{P}^{(e_i^\delta)})$ . Let  $(x_{t_1}, \dots, x_{t_n})$  be a cycle of  $(\mathbb{R}, \mathbb{P})$ . First, assume that there exists  $1 \leq i \leq n$  such that  $(x_{t_i}, x_{t_{i+1}}) \in \mathbb{R} \setminus \mathbb{P}$ . Then  $p_{t_i} \cdot x_{t_i} = p_{t_i} \cdot x_{t_{i+1}}$  and since  $e_{t_i}^\delta \geq \delta$ ,  $(x_{t_i}, x_{t_{i+1}}) \notin \mathbb{R}^{(e_i^\delta)}$ . Thus  $(x_{t_1}, \dots, x_{t_n})$  is not a cycle of  $(\mathbb{R}^{(e_i^\delta)}, \mathbb{P}^{(e_i^\delta)})$ . Second, assume that there exists  $1 \leq i \leq n$  such that

$$\frac{p_{t_i} \cdot (x_{t_i} - x_{t_{i+1}})}{p_{t_i} \cdot x_{t_i}} = 1.$$

Now  $p_{t_i} \cdot x_{t_{i+1}} = 0$ . Since  $p_{t_i}$  is strictly positive and  $x_{t_{i+1}}$  is non-negative, this is only possible if  $x_{t_{i+1}} = 0$ . Since  $x_{t_{i+1}} \in \mathbb{R} \setminus x_{t_{i+2}}$ , we have  $0 = p_{t_{i+1}} \cdot x_{t_{i+1}} \geq p_{t_{i+1}} \cdot x_{t_{i+2}}$ . Thus we must have  $x_{t_{i+2}} = 0$ . Continuing this by induction, we have for all  $1 \leq j \leq n$ ,  $x_{t_j} = 0$  which is a contradiction since for some  $1 \leq j \leq n$ ,  $x_{t_j} \in \mathbb{P} \setminus x_{t_{j+1}}$ . Next, assume that for all  $1 \leq i \leq n$ ,  $(x_{t_i}, x_{t_{i+1}}) \in \mathbb{P}$  and

$$\frac{p_{t_i} \cdot (x_{t_i} - x_{t_{i+1}})}{p_{t_i} \cdot x_{t_i}} < 1.$$

Now  $(x_{t_1}, \dots, x_{t_n}) \in \mathcal{C}^s$ . Thus by the definition of  $\theta^*$ , there exist  $1 \leq i \leq n$  and  $(x_{t_i}, x_{t'}) \in U(x_{t_i}, x_{t_{i+1}})$  such that  $\theta(x_{t_i}, x_{t'}) = 1$ . First, if

$$\frac{p_{t_i} \cdot (x_{t_i} - x_{t'})}{p_{t_i} \cdot x_{t_i}} = 1,$$



then we have

$$e_t^\delta \geq \frac{p_{t_i} \cdot (x_{t_i} - x_{t'})}{p_{t_i} \cdot x_{t_i}} > \frac{p_{t_i} \cdot (x_{t_i} - x_{t_{i+1}})}{p_{t_i} \cdot x_{t_i}}.$$

Second, if

$$\frac{p_{t_i} \cdot (x_{t_i} - x_{t'})}{p_{t_i} \cdot x_{t_i}} < 1,$$

then

$$e_t^\delta > \frac{p_{t_i} \cdot (x_{t_i} - x_{t'})}{p_{t_i} \cdot x_{t_i}} \geq \frac{p_{t_i} \cdot (x_{t_i} - x_{t_{i+1}})}{p_{t_i} \cdot x_{t_i}}.$$

Thus, in both cases we have  $(x_{t_i}, x_{t_{i+1}}) \notin \mathbb{R}^{(e_t^\delta)}$ . Hence,  $(x_{t_1}, \dots, x_{t_n})$  is not a cycle of  $(\mathbb{R}^{(e_t^\delta)}, \mathbb{P}^{(e_t^\delta)})$ . This shows that  $(\mathbb{R}^{(e_t^\delta)}, \mathbb{P}^{(e_t^\delta)})$  is acyclical. So we have,

$$I^* + \varepsilon \geq \sum_{t=1}^T f(e_t^\delta) \geq I_{\text{fVar}}.$$

Since  $1 > \varepsilon > 0$  was arbitrary, this shows that

$$I^* \geq I_{\text{fVar}}.$$

Next, we show that

$$I^* \leq I_{\text{fVar}}.$$

By the definition of  $I_{\text{fVar}}$  and the continuity of  $f$ , there exist a convergent sequence  $(e_t^{*,j})_{j=1}^T \subseteq [0, 1]^T$  that converges to  $(e_t^*)$  such that for each  $j$ ,  $(\mathbb{R}^{(e_t^{*,j})}, \mathbb{P}^{(e_t^{*,j})})$  is acyclical and  $I_{\text{fVar}} = \sum_t f(e_t^*)$ .

We show first that for each  $t$  such that  $e_t^* > 0$ , there exist  $t'$  such that

$$e_t^* = \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}. \quad (1)$$

Assume, per contra, that there exist  $t^0$  with  $e_{t^0}^* > 0$  such that for all  $t'$

$$e_{t^0}^* \neq \frac{p_{t^0} \cdot (x_{t^0} - x_{t'})}{p_{t^0} \cdot x_{t^0}}.$$

Denote

$$v = \frac{1}{2} \min \left\{ \left| e_{t^0}^* - \frac{p_{t^0} \cdot (x_{t^0} - x_{t'})}{p_{t^0} \cdot x_{t^0}} \right| \mid 1 \leq t' \leq T \right\}.$$

Now, we have  $v > 0$  and for all  $1 \leq t' \leq T$

$$(1 - e_{t^0}^*)p_{t^0} \cdot x_{t^0} \geq p_{t^0} \cdot x_{t'} \iff (1 - e_{t^0}^* + v)p_{t^0} \cdot x_{t^0} \geq p_{t^0} \cdot x_{t'}$$

by the choice of  $v$ . Since  $f$  is uniformly continuous, there exist  $\delta \in (0, v)$  such that for all  $x, y \in [0, 1]$  with  $|x - y| < 2\delta$ , we have  $|f(x) - f(y)| < \frac{f(e_{t^0}^*) - f(e_{t^0}^* - v)}{T}$ . Define  $(\tilde{e}_t)$  by  $\tilde{e}_t = e_t^* + \delta$

for  $t \neq t_0$  and  $\tilde{e}_{t^0} = e_{t^0}^* - v$ . Since  $(e_t^{*,j}) \rightarrow (e_t^*)$  as  $j \rightarrow \infty$ , there exist  $k$  such that for each  $t$   $e_t^{*,k} < e_t^* + \delta$ . Now by the choice of  $v$ ,  $R^{(e_t^{*,k})} \supseteq R^{(\tilde{e}_t)}$  and  $P^{(e_t^{*,k})} \supseteq P^{(\tilde{e}_t)}$ . Since  $(R^{(e_t^{*,k})}, P^{(e_t^{*,k})})$  is acyclical, especially  $(R^{(\tilde{e}_t)}, P^{(\tilde{e}_t)})$  is acyclical. Additionally, we have

$$\begin{aligned} \sum_t f(\tilde{e}_t) &\leq \sum_{t \neq t_0} |f(\tilde{e}_t) - f(e_t^*)| + \sum_{t' \neq t_0} f(e_{t'}^*) + f(\tilde{e}_{t^0 v}) \\ &< f(e_{t^0}^*) - f(\tilde{e}_{t^0}) + \sum_{t \neq t_0} f(e_t^*) + f(\tilde{e}_{t^0}) = \sum_t f(e_t^*). \end{aligned}$$

This contradicts the optimality of  $(e_t^*)$ .

Now for each  $t$  such that  $e_t^* > 0$ , choose  $\tau(t)$  such that

$$e_t^* = \frac{p_t \cdot (x_t - x_{\tau(t)})}{p_t \cdot x_t}.$$

Define  $\theta^e \in \{0, 1\}^P$  by defining for each  $t$  such that  $e_t^* > 0$ ,  $\theta^e(x_t, x_{\tau(t)}) = 1$  and  $\theta^e(x_t, x_{t'}) = 0$  for all other  $(x_t, x_{t'}) \in P$ .

We show that  $\theta^e$  solves the binary linear optimization problem. Assume, per contra, that there exists  $(x_{t_1}, \dots, x_{t_n}) \in \mathcal{C}^s$  such that for all  $1 \leq i \leq n$  and  $(x_t, x_{t'}) \in U(x_{t_i}, x_{t_{i+1}})$ ,  $\theta(x_t, x_{t'}) = 0$ . Thus for all  $1 \leq i \leq n$ , by the definitions of  $U(x_{t_i}, x_{t_{i+1}})$  and  $\theta^e$  and (1),

$$e_{t_i}^* < \frac{p_{t_i} \cdot (x_{t_i} - x_{t_{i+1}})}{p_{t_i} \cdot x_{t_i}}. \quad (2)$$

Denote

$$\eta = \frac{1}{2} \min \left\{ \frac{p_{t_i} \cdot (x_{t_i} - x_{t_{i+1}})}{p_{t_i} \cdot x_{t_i}} - e_{t_i}^* \mid 1 \leq i \leq n \right\}.$$

Now  $\eta > 0$ . Define  $(e_t^\eta) \in [0, 1]^T$  by for each  $1 \leq i \leq n$   $e_{t_i}^\eta = e_{t_i}^* + \eta$  and for all other  $t$ ,  $e_t^\eta = e_t^*$ . By (1,2), for each  $1 \leq i \leq n$ ,  $(x_{t_i}, x_{t_{i+1}}) \in P^{(e_t^\eta)}$ . Thus  $(R^{(e_t^\eta)}, P^{(e_t^\eta)})$  is not acyclical. Since  $e^{*,j} \rightarrow e^*$  as  $j \rightarrow \infty$ , especially there exist  $j^*$  such that for each  $1 \leq i \leq n$ ,  $e_{t_i}^{*,j^*} \leq e_{t_i}^* + \eta$ , but for each  $1 \leq i \leq n$ ,  $(x_{t_i}, x_{t_{i+1}}) \in P^{(e_t^{*,j^*})}$  and so  $(R^{(e_t^{*,j^*})}, P^{(e_t^{*,j^*})})$  is not acyclical contradicting the definition of  $(e_t^{*,j^*})$ . Thus there does not exist  $(x_{t_1}, \dots, x_{t_n}) \in \mathcal{C}^s$  such that for all  $1 \leq i \leq n$  and  $(x_t, x_{t'}) \in U(x_{t_i}, x_{t_{i+1}})$ ,  $\theta(x_t, x_{t'}) = 0$ . This shows that  $\theta^e$  solves the binary linear optimization problem. Thus

$$I^* \leq \sum_{(x_t, x_{t'}) \in P} \theta^e(x_t, x_{t'}) f\left(\frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t}\right) = \sum_{t=1}^T f(e_t^*) = I_{\text{FVar}}.$$

□

## S.2 Proof of Theorem 2

**Lemma 9** For all  $f \in \Delta(\mathcal{O})$ ,

$$I_{\text{Swaps}}(f) = \min_{\succ \in \mathcal{P}} \sum_{(y,x) \in \succ} \sum_{A \supseteq \{x,y\}} f(x, A).$$

*Proof.* We have

$$\begin{aligned} I_{\text{Swaps}}(f) &= \min_{\succ \in \mathcal{P}} \sum_{(x,A) \in \Delta(\mathcal{O})} f(x, A) \left| \{y \in A \mid y \succ x\} \right| \\ &= \min_{\succ \in \mathcal{P}} \sum_{x \in X} \sum_{A \supseteq \{x\}} f(x, A) \sum_{y \in A} \mathbb{1}(y \succ x) \\ &= \min_{\succ \in \mathcal{P}} \sum_{(x,y) \in X \times X} \sum_{A \supseteq \{x\}} f(x, A) \mathbb{1}(y \in A) \mathbb{1}(y \succ x) \\ &= \min_{\succ \in \mathcal{P}} \sum_{(x,y) \in X \times X} \mathbb{1}(y \succ x) \sum_{A \supseteq \{x,y\}} f(x, A) \\ &= \min_{\succ \in \mathcal{P}} \sum_{(y,x) \in \succ} \sum_{A \supseteq \{x,y\}} f(x, A). \end{aligned}$$

□

**Theorem 2** For all random data  $f \in \Delta(\mathcal{O})$ ,

$$I_{\text{Swaps}}(f) = \min_{B \subseteq \mathcal{P}^\Delta} \sum_{(x,y) \in B} \sum_{A \supseteq \{x,y\}} f(x, A) \text{ such that } \mathcal{P}^\Delta \setminus B \text{ is acyclical.}$$

*Proof.* Denote for all  $f \in \Delta(\mathcal{O})$ ,

$$I^*(f) = \min_{B \subseteq \mathcal{P}^\Delta} \sum_{(x,y) \in B} \sum_{A \supseteq \{x,y\}} f(x, A) \text{ s.t. } \mathcal{P}^\Delta \setminus B \text{ is acyclical.} \quad (3)$$

Let  $B^*$  be an optimal solution to (3). Since  $\mathcal{P}^\Delta \setminus B^*$  is acyclical there exist an extension  $\succ^*$  of  $\mathcal{P}^\Delta \setminus B^*$  such that  $\succ^*$  is a linear order (Chambers & Echenique, 2016, Theorem 1.5). Especially, we have for all  $(x, y) \in B^*$ ,  $y \succ^* x$ . Additionally, we have for all  $(x, y) \notin B^*$  such that  $y \succ^* x$ ,  $\sum_{A \supseteq \{x,y\}} f(x, A) = 0$ .

By (3) and the above observations, we have

$$\begin{aligned} I^*(f) &= \sum_{(y,x) \in B^*} \sum_{A \supseteq \{x,y\}} f(y, A) = \sum_{\substack{(x,y) \in \succ^*, \\ (y,x) \notin B^*}} \sum_{A \supseteq \{x,y\}} f(y, A) + \sum_{\substack{(x,y) \in \succ^*, \\ (y,x) \in B^*}} \sum_{A \supseteq \{x,y\}} f(y, A) \\ &= \sum_{(x,y) \in \succ^*} \sum_{A \supseteq \{x,y\}} f(y, A). \end{aligned}$$

So, by Lemma 9,

$$I^*(f) \geq \min_{P \in \mathcal{P}} \sum_{(y,x) \in P} \sum_{A \supseteq \{x,y\}} f(x, A) = I_{\text{Swaps}}(f).$$

Finally, for each  $P^* \in \mathcal{P}$ , we have by denoting  $B^P = P^\Delta \setminus P^*$

$$\sum_{(x,y) \in P^*} \sum_{A \supseteq \{x,y\}} f(y, A) = \sum_{\substack{(x,y) \in P^*, \\ (y,x) \notin B^P}} \sum_{A \supseteq \{x,y\}} f(y, A) + \sum_{\substack{(x,y) \in P^*, \\ (y,x) \in B^P}} \sum_{A \supseteq \{x,y\}} f(y, A) = \sum_{(y,x) \in B^P} \sum_{A \supseteq \{x,y\}} f(y, A).$$

So by (3) and Lemma 9,

$$I^*(f) \leq \min_{P \in \mathcal{P}} \sum_{(y,x) \in P} \sum_{A \supseteq \{x,y\}} f(x, A) = I_{\text{Swaps}}(f).$$

□

### S.3 Proof of Theorem 3

**Theorem 3** Assume that the data set is  $D = ((p_1, x_1), \dots, (p_T, x_T))$ . If a data set  $D' = ((p'_1, x'_1), \dots, (p'_T, x'_T))$  is such that for all  $t, t'$

$$\frac{p'_t \cdot (x'_t - x'_{t'})}{p'_t \cdot x'_t} = \frac{p_{t'} \cdot (x_{t'} - x_t)}{p_{t'} \cdot x_{t'}},$$

then

$$I_{\text{InvVar}}(D) = I_{\text{Var}}(D').$$

*Proof.* Let  $(e_t) \in [0, 1]^T$ . Now we have for all  $t, t'$

$$p_t \cdot x_t \geq \frac{p_t \cdot x_{t'}}{1 - e_{t'}} \iff e_{t'} \leq \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t} = \frac{p'_{t'} \cdot (x'_{t'} - x'_t)}{p'_{t'} \cdot x'_{t'}} \iff (1 - e_{t'}) p'_{t'} \cdot x'_{t'} \geq p'_{t'} \cdot x'_t.$$

This shows that  $(R^{(e_{t'})}, P^{(e_{t'})})$  has a cycle iff for the modified data  $(R^{(e_t)}, P^{(e_t)})$  has a cycle going to the opposite direction. This shows the claim by the definitions of Inverse Varian's index and Varian's index. □

### S.4 Limits for Measure Variations

The next two results show Theorem 4.

**Proposition 10** For all data  $D$ ,

$$\lim_{\alpha \rightarrow \infty} \left( I_{\text{NMCI}_\alpha}(D) \right)^{\frac{1}{\alpha}} = I_{\text{AEI}}(D).$$

*Proof.* Let  $\varepsilon > 0, \alpha > 0$ , and denote

$$B^* = \left\{ (x_t, x_{t'}) \in \mathbb{R} \mid \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t} \leq I_{\text{AEI}}(D) + \varepsilon \right\}.$$

By the definition of Afriat's index,  $(\mathbb{R} \setminus B^*, \mathbb{P} \setminus B^*)$  is acyclical. Additionally,  $|B^*| \leq T(T-1)$ .

Thus, for all  $\alpha$

$$\left( I_{\text{NMCI}\alpha}(D) \right)^{\frac{1}{\alpha}} \leq \left( \frac{1}{T} \sum_{(x_t, x_{t'}) \in B^*} \left( \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t} \right)^\alpha \right)^{\frac{1}{\alpha}} \leq (T-1)^{\frac{1}{\alpha}} (I_{\text{AEI}}(D) + \varepsilon).$$

On the other hand,

$$\left( I_{\text{NMCI}\alpha}(D) \right)^{\frac{1}{\alpha}} \geq \left( \frac{1}{T} \inf_{e \in [0,1]} e^\alpha \text{ s.t. } (\mathbb{R}^e, \mathbb{P}^e) \text{ is acyclical} \right)^{\frac{1}{\alpha}} = T^{-\frac{1}{\alpha}} I_{\text{AEI}}(D)$$

by the definition of Afriat's index. Thus by taking  $\varepsilon \rightarrow 0$  and  $\alpha \rightarrow 0$ , shows the claim.  $\square$

**Proposition 11** For all data  $D$ ,

$$\lim_{\alpha \rightarrow \infty} \left( I_{\text{Var}\alpha}(D) \right)^{\frac{1}{\alpha}} = I_{\text{AEI}}(D).$$

*Proof.* For all data  $D$  and  $\alpha \in (0, 1)$ ,

$$\left( I_{\text{NMCI}\alpha}(D) \right)^{\frac{1}{\alpha}} \geq \left( I_{\text{Var}\alpha}(D) \right)^{\frac{1}{\alpha}} \geq \left( \frac{1}{T} \inf_{e \in [0,1]} e^\alpha \text{ s.t. } (\mathbb{R}^e, \mathbb{P}^e) \text{ is acyclical} \right)^{\frac{1}{\alpha}} = T^{-\frac{1}{\alpha}} I_{\text{AEI}}(D)$$

by the definitions of  $\text{NMCI}\alpha$  and Afriat's index. This shows the claim by Proposition 10.  $\square$

The claim for inverse Varian's index in Theorem 4 follows symmetrically and the proof is omitted.

Define for all data  $D$

$$I_{\text{Strict-HM}}(D) = \frac{1}{T} \inf_{(e_t) \in [0,1]^T} |\{t \mid e_t > 0\}| \text{ such that } \mathbb{P}^{(e_t)}_D \text{ is acyclical.}$$

Especially, if the data  $D$  is such that for all  $t \neq t', p_t \cdot (x_t - x_{t'}) \neq 0$ , then  $I_{\text{HM}}(D) = I_{\text{Strict-HM}}(D)$ .

The next result shows Theorem 5 for Varian's index.

**Proposition 12** For all data  $D$ ,

$$\lim_{\alpha \rightarrow 0} I_{\text{Var}\alpha}(D) = I_{\text{Strict-HM}}(D).$$

*Proof.* Let  $D = ((p_t, x_t))_{t=1}^T$  be the data. First, if  $I_{\text{Strict-HM}}(D) = 0$ , then  $\mathbb{P}$  is acyclical and so for all  $\alpha \in \mathbb{R}_{++}$ ,  $I_{\text{Var}\alpha}(D) = 0$ , which shows the claim.

Assume that  $I_{\text{Strict-HM}}(D) > 0$ . Denote

$$A = \frac{1}{2} \min \left\{ \frac{p_t \cdot (x_t - x_{t'})}{p_t \cdot x_t} \mid p_t \cdot (x_t - x_{t'}) > 0 \right\}$$

and

$$B = \frac{1}{2} + \frac{1}{2} \max \left\{ \frac{p_t \cdot (x_t - x_t)}{p_t \cdot x_t} \mid p_t \cdot (x_t - x_t) > 0, x_t \neq 0 \right\}.$$

Now there exist  $e^* \in \{0, B\}^T$  such that  $|\{t \mid e_t^* \neq 0\}| = I_{\text{Strict-HM}}(D)$  and  $P^{e^*}$  is acyclical. On the other hand, if  $P^e$  is acyclical, then there exist  $N \subseteq \{1, \dots, T\}$  such that  $|N| \geq I_{\text{Strict-HM}}(D)$  and for each  $t \in N$ ,  $e_t \geq A$ . Thus for all  $\alpha > 0$ ,

$$I_{\text{Strict-HM}}(D)B^\alpha \geq I_{\text{Var}\alpha}(D) \geq I_{\text{Strict-HM}}(D)A^\alpha.$$

Taking  $\alpha \rightarrow 0$  shows the claim. □

The rest of the limits in Theorem 5 follow symmetrically and the proof is omitted.

### S.4.1 Proof of Theorem 6

Before proving Theorem 6, we show some simple lemmas.

**Proposition 13** Assume that  $(x_i)_{i=1}^n, (a_i)_{i=1}^n \in \mathbb{R}_{++}^n$ . Then

$$\lim_{k \rightarrow \infty} \left( \sum_{i=1}^n a_i x_i^k \right)^{\frac{1}{k}} = \max_i x_i.$$

*Proof.* We have

$$\left( \sum_{i=1}^n a_i x_i^k \right)^{\frac{1}{k}} \leq \left( \max_l a_l \sum_{i=1}^n \max_j x_j^k \right)^{\frac{1}{k}} = \left( n \max_l a_l \right)^{\frac{1}{k}} \max_j x_j$$

and

$$\left( \sum_{i=1}^n a_i x_i^k \right)^{\frac{1}{k}} \geq \left( \min_l a_l \max_j x_j^k \right)^{\frac{1}{k}} = \left( \min_l a_l \right)^{\frac{1}{k}} \max_j x_j.$$

By taking  $k \rightarrow \infty$  shows the claim. □

**Proposition 14** Assume that  $(x_i)_{i=1}^n, (a_i)_{i=1}^n \in \mathbb{R}_{++}^n, (y_j)_{j=1}^m, (b_j)_{j=1}^m \in \mathbb{R}_{++}^m$  are such that for each  $k \in \mathbb{N}$

$$\sum_{i=1}^n a_i x_i^k = \sum_{j=1}^m b_j y_j^k,$$

then  $\max_i x_i = \max_j y_j$ .

*Proof.* Follows directly from Proposition 13 since for each  $k \in \mathbb{N}$ ,

$$\left( \sum_{i=1}^n a_i x_i^k \right)^{\frac{1}{k}} = \left( \sum_{j=1}^m b_j y_j^k \right)^{\frac{1}{k}}.$$

□

**Proposition 15** Assume that  $(x_i)_{i=1}^n, (y_j)_{j=1}^m \in \mathbb{R}_{++}$ ,  $a, b \in \mathbb{R}_{++}$  are such that for each  $k \in \mathbb{N}$

$$a \sum_{i=1}^n x_i^k = b \sum_{j=1}^m y_j^k,$$

then

$$\max_i x_i = \max_j y_j \text{ and } a|\{i \in \{1, \dots, n\} | x_i = \max_k x_k\}| = b|\{j \in \{1, \dots, m\} | y_j = \max_k y_k\}|.$$

*Proof.* Assume w.l.o.g. that for each  $i, j$   $x_i \leq x_{i+1}$  and  $y_j \leq y_{j+1}$ . First, by Proposition 14,  $x_n = y_m$ . Denote,

$$T = |\{i \in \{1, \dots, n\} | x_i = x_n\}| \text{ and } T' = |\{j \in \{1, \dots, m\} | y_j = y_m\}|.$$

Assume w.l.o.g. that  $bT' \geq aT$ . Now for each  $k \in \mathbb{N}$

$$a \sum_{i=1}^{n-T} x_i^k = b \sum_{j=1}^{m-T'} y_j^k + (bT' - aT)y_m^k.$$

By Proposition 14,  $bT' - aT = 0$  that shows the claim. □

**Proposition 16** Assume that  $(x_i)_{i=1}^n, (y_j)_{j=1}^m \in \mathbb{R}_{++}$ ,  $a, b \in \mathbb{R}_{++}$  are such that for each  $k \in \mathbb{N}$

$$a \sum_{i=1}^n x_i^k = b \sum_{j=1}^m y_j^k,$$

then for each  $c \in \mathbb{R}_{++}$

$$a|\{i \in \{1, \dots, n\} | x_i = c\}| = b|\{j \in \{1, \dots, m\} | y_j = c\}|.$$

*Proof.* Follows directly by applying Proposition 15 inductively. □

**Lemma 17** Assume that  $X$  is finite,  $\theta \in \mathbb{R}_+$ , and  $f : X \times \mathbb{R}_+$  is such that for each  $x \in X$   $f(x, \cdot)$  is  $\infty$ -times differentiable. For each  $\alpha \in \mathbb{R}_+$ , denote  $C^0(\alpha) = X$  and for each  $n \in \mathbb{N}$ ,

$$C^n(\alpha) = \arg \min_{x \in C^{n-1}(\alpha)} f_2^n(x, \alpha)$$

where  $f_2^n$  is the  $n$  derivative of  $f(x, \cdot)$ . Then for each  $n \in \mathbb{N} \cup \{0\}$ , there exist a function  $h : X \times \mathbb{R}_+$  such that for each  $x \in X$ ,

$$f(x, \alpha) = \sum_{i=0}^n \alpha^i f_2^i(y, 0) + \alpha^n h(x, \alpha) \text{ and } \lim_{\alpha \rightarrow 0^+} h(x, \alpha) = 0$$

and there exist  $\delta^n > 0$  such that for all  $\alpha \in [0, \delta^n)$ ,

$$\min_{x \in X} f(x, \alpha) = \sum_{i=0}^n \min_{y \in C^i(0)} \alpha^i f_2^i(y, 0) + \min_{z \in C^n(0)} \alpha^n h(z, \alpha).$$

*Proof.* We show the claim by induction on  $n$ . First, the case  $n = 0$  follows trivially since  $f$  is continuous for each  $x$ .

Assume that  $n \in \mathbb{N}$  and that the claim holds for  $n-1$ . Endow  $X$  with the discrete topology. By the maximum theorem, each  $C^n$  is upper hemicontinuous and so by the discrete topology there exist  $\delta^1 > 0$  such that for all  $\alpha \in [0, \delta^1)$ ,  $C^n(\alpha) \subseteq C^n(0)$ .

Denote

$$\varepsilon = \min \left\{ \left| f_2^n(x, 0) - f_2^n(y, 0) \right| \mid x, y \in C^n(0), f_2^n(x, 0) \neq f_2^n(y, 0) \right\},$$

where  $\min \emptyset = \infty$ .

By Taylor approximation, there exist a function  $h^n : X \times \mathbb{R}_{++}$  such that for each  $x \in X$ ,

$$f(x, \alpha) = \sum_{i=0}^n \alpha^i f_2^i(x, 0) + \alpha^n h^n(x, \alpha) \text{ and } \lim_{\alpha \rightarrow 0_+} h^n(x, \alpha) = 0.$$

Especially, by the induction assumption, for all  $x \in X$  and  $\alpha \in \mathbb{R}_+$ ,

$$h^{n-1}(x, \alpha) = \alpha f_2^n(x, 0) + \alpha h^n(x, \alpha).$$

Let  $\delta > 0$  be such that  $\delta < \delta^1$  and for all  $\alpha \in [0, \delta)$  and  $x \in X$ ,  $|h^n(x, \alpha)| < \varepsilon$ . Then we have for all  $\alpha \in [0, \delta)$ , by the induction assumption,

$$\begin{aligned} \min_{x \in X} f(x, \alpha) &= \sum_{i=0}^{n-1} \min_{y \in C^i(0)} \alpha^i f_2^i(y, 0) + \min_{z \in C^{n-1}(0)} \alpha^{n-1} h^{n-1}(z, \alpha) \\ &= \sum_{i=0}^{n-1} \min_{y \in C^i(0)} \alpha^i f_2^i(y, 0) + \min_{z \in C^{n-1}(0)} \alpha^n f_2^n(z, 0) + \alpha^n h^n(z, \alpha) \\ &= \sum_{i=0}^{n-1} \min_{y \in C^i(0)} \alpha^i f_2^i(y, 0) + \min_{z \in C^n(0)} \alpha^n f_2^n(z, 0) + \min_{z \in C^n(0)} \alpha^n h^n(z, \alpha) \end{aligned}$$

where the last equality follows from the assumption that for all  $\alpha \in [0, \delta)$  and  $x \in X$ ,  $|h^n(x, \alpha)| < \varepsilon$ .  $\square$

**Proposition 18** For each data  $D$ , the following set is closed

$$\{(e_t) \in [0, 1]^T \mid \text{such that } P^{(e_t)} \text{ is acyclical}\}.$$

*Proof.* For each  $n \in \mathbb{N}$ , let  $(e_t^n) \in [0, 1]^T$  be such that  $P^{(e_t^n)}$  is acyclical and  $(e_t^n) \rightarrow (e_t)$  as  $n \rightarrow \infty$ . Let  $t_1, \dots, t_m \subseteq \{1, \dots, T\}$  be a sequence. Denote

$$A = \bigcup_{i=1}^m \left\{ (a_t) \in [0, 1]^T \mid a_{t_i} \geq \frac{p_{t_i} \cdot (x_{t_i} - x_{t_{i+1}})}{p_{t_i} \cdot x_{t_i}} \right\}.$$



$A$  is closed as a finite union of closed sets. For each in  $n \in \mathbb{N}$ ,  $(e_t^n) \in A$ . Thus  $(e_t) \in A$ . Since the sequence  $t_1, \dots, t_m \subseteq \{1, \dots, T\}$  was arbitrary, this shows that  $P^{(e_t)}$  is acyclical.  $\square$

**Proposition 19** Assume that  $D$  is a data set,  $(e_t) \in [0, 1]^T$  is such that  $P^{(e_t)}$  is acyclical, and  $u \in \{1, \dots, T\}$  is such that  $e_u = 1$ . Then there exist  $(e'_t) \in [0, 1]^T$  such that  $P^{(e'_t)}$  is acyclical and for all  $t \in \{1, \dots, T\}$   $e_t \geq e'_t$  and  $e_u > e'_u$

*Proof.* First, if  $x_t = 0$ , the  $x_t$  cannot be a part of cycle for  $P$  since for all  $v \in \{1, \dots, T\}$ ,  $x_t \not\preceq x_v$ .

Define  $(e'_t) \in [0, 1]^T$  by for all  $t \neq u$ ,  $e'_t = e_t$  and

$$e'_u = \frac{1}{2} + \frac{1}{2} \max \left\{ \frac{p_u \cdot (x_u - x_t)}{p_u \cdot x_u} \mid t \in \{1, \dots, T\}, x_t \neq 0 \right\} \cup \{0\}$$

where in the case of  $x_u = 0$ , for all  $c \in \mathbb{R}$ ,  $\frac{c}{0} = 0$ .

Now, we have for all  $t \in \{1, \dots, T\}$  such that  $x_t \neq 0$ ,  $x_u P^{(e_t)} x_t$  iff  $x_u P^{(e'_t)} x_t$  that shows the claim by the first observation.  $\square$

Finally, we show a more general version of Theorem 6 that fully characterizes the limiting order.

**Proposition 20** For each  $n \in \mathbb{N}$ , denote

$$C^0(D) = \arg \min_{(e_t) \in [0, 1]^T} \frac{1}{T} |\{t | e_t > 0\}| \text{ such that } P^{(e_t)}_D \text{ is acyclical.}$$

$$V^0(D) = \min_{(e_t) \in [0, 1]^T} \frac{1}{T} |\{t | e_t > 0\}| \text{ such that } P^{(e_t)}_D \text{ is acyclical.}$$

$$C^n(D) = \arg \min_{(e_t) \in C^{n-1}(D)} \frac{1}{|\{t | e_t > 0\}|} \sum_{t | e_t > 0} \ln(e_t)^n,$$

$$V^n(D) = \min_{(e_t) \in C^{n-1}(D)} \frac{1}{|\{t | e_t > 0\}|} \sum_{t | e_t > 0} \ln(e_t)^n,$$

and

$$m = \min \left\{ n \in \mathbb{N} \mid V^n(D) \neq V^n(D') \right\}.$$

(1) If  $m < \infty$  and  $V^m(D) > V^m(D')$ , then there exists  $\alpha^* > 0$  such that for all  $\alpha \in (0, \alpha^*)$ ,

$$I_{\text{Var}\alpha}(D) > I_{\text{Var}\alpha}(D').$$

(2) If  $m = \infty$ , then there exists  $\alpha^* > 0$  such that for all  $\alpha \in (0, \alpha^*)$ ,  $I_{\text{Var}\alpha}(D) = I_{\text{Var}\alpha}(D')$ .

*Proof.* Endow  $[0, 1]^T$  with the standard order  $>$  in  $\mathbb{R}^T$ . Denote the set of minimizers of  $>$  of the set  $\{(e_t) \in [0, 1]^T \mid \text{such that } P^{(e_t)}_D \text{ is acyclical}\}$  by  $X(D)$ . By Proposition 18, this is

well-defined. Now if  $(e_t) \in [0, 1]^T$  is such that  $P^{(e_t)}_D$  is acyclical then for all  $(e'_t) \in [0, 1]^T$  with  $(e'_t) \geq (e_t)$ , we have  $P^{(e'_t)}_D$  is acyclical. Thus by Theorem 1

$$X(D) \subseteq \left\{ \frac{p_t \cdot (x_t - x_u)}{p_t \cdot x_t} \middle| t, u \in \{1, \dots, T\} \right\}$$

and so  $X(D)$  is a finite set.

Define

$$\begin{aligned} \tilde{C}^0(D) &= \arg \min_{(e_t) \in X(D)} \frac{1}{T} |\{t | e_t > 0\}| \\ \tilde{V}^0(D) &= \min_{(e_t) \in X(D)} \frac{1}{T} |\{t | e_t > 0\}| \\ \tilde{C}^n(D) &= \arg \min_{(e_t) \in C^{n-1}(D)} \frac{1}{T} \sum_{t | e_t > 0} \ln(e_t)^n, \\ \tilde{V}^n(D) &= \min_{(e_t) \in C^{n-1}(D)} \frac{1}{T} \sum_{t | e_t > 0} \ln(e_t)^n. \end{aligned}$$

For each  $n \in \mathbb{N}$ ,  $\tilde{V}^n(D) = V^0(D)V^n(D)$ . Define the function  $f^D : X(D) \times \mathbb{R}_+ \rightarrow \mathbb{R}$  by

$$f^D((e_t), \alpha) = \frac{1}{T} \sum_{t | e_t > 0} e_t^\alpha.$$

For each  $\alpha > 0$ , we have

$$I_{\text{Var}\alpha}(D) = \min_{(e_t) \in X(D)} f^D((e_t), \alpha)$$

and

$$f_2^{D,n}((e_t), 0) = \ln(e_t)^n.$$

Define  $X(D')$ ,  $\tilde{C}^n(D')$ ,  $\tilde{V}^n(D')$ , and  $f^{D'}$  similarly for  $D'$ .

First, if  $m = 0$ , then the claim follows from Proposition 12. Next, assume that  $0 < m < \infty$  and  $V^m(D) > V^m(D')$ . By Lemma 17 and since  $X(D)$  and  $X(D')$  are finite, there exist  $h^D : \mathbb{R}_+ \rightarrow \mathbb{R}$ ,  $h^{D'} : \mathbb{R}_+ \rightarrow \mathbb{R}$ , and  $\alpha^0 > 0$  such that  $\lim_{\alpha \rightarrow 0_+} h^D(\alpha) = \lim_{\alpha \rightarrow 0_+} h^{D'}(\alpha) = 0$  and for all  $\alpha \in (0, \alpha^0)$

$$\begin{aligned} I_{\text{Var}\alpha}(D) &= \sum_{i=0}^m \min_{y \in C^i(D)} \alpha^i f_2^{D,i}(y, 0) + \alpha^m h^D(\alpha) \\ I_{\text{Var}\alpha}(D') &= \sum_{i=0}^m \min_{z \in C^i(D')} \alpha^i f_2^{D',i}(z, 0) + \alpha^m h^{D'}(\alpha). \end{aligned}$$

We have by the definition of  $m$ ,

$$I_{\text{Var}\alpha}(D) - I_{\text{Var}\alpha}(D') = \alpha^m \left( \min_{y \in C^m(D)} f_2^{D,m}(y, 0) - \min_{z \in C^m(D')} f_2^{D',m}(z, 0) + h^D(\alpha) - h^{D'}(\alpha) \right).$$

Since  $\min_{y \in C^m(D)} f_2^{D,m}(y, 0) > \min_{z \in C^m(D')} f_2^{D',m}(z, 0)$ , there exist  $\alpha^1 < \alpha^0$  such that for all  $\alpha \in (0, \alpha^1)$ ,

$$|h^D(\alpha) - h^{D'}(\alpha)| < \min_{y \in C^m(D)} f_2^{D,m}(y, 0) - \min_{z \in C^m(D')} f_2^{D',m}(z, 0).$$

Thus for all  $\alpha \in (0, \alpha^1)$ ,

$$I_{\text{Var}\alpha}(D) > I_{\text{Var}\alpha}(D').$$

Lastly, assume that  $m = \infty$ . Since  $X(D)$  and  $X(D')$  are finite, there exists  $n^0 \in \mathbb{N}$  such that  $\tilde{C}^{n^0}(D) \cap_{i=0}^{\infty} \tilde{C}^i(D)$  and  $\tilde{C}^{n^0}(D') \cap_{i=0}^{\infty} \tilde{C}^i(D')$ . Let  $(e_t) \in \cap_{i=0}^{\infty} \tilde{C}^i(D)$  and  $(e'_t) \in \cap_{i=0}^{\infty} \tilde{C}^i(D')$ .

Now for each  $n \in \mathbb{N}$ , we have

$$\frac{1}{T} \sum_{t|e_t > 0} \ln(e_t)^n = \frac{1}{T'} \sum_{t|e'_t > 0} \ln(e'_t)^n.$$

So since each  $e_t, e'_t \in (0, 1]$ , for each  $n \in \mathbb{N}$ ,

$$\frac{1}{T} \sum_{t|e_t > 0} |\ln(e_t)|^n = \frac{1}{T'} \sum_{t|e'_t > 0} |\ln(e'_t)|^n.$$

Thus by Proposition 16, for each  $c \in \mathbb{R}_{++}$

$$\frac{1}{T} \left| \left\{ t \in \{1, \dots, T\} \mid |\ln(e_t)| = c \right\} \right| = \frac{1}{T'} \left| \left\{ t \in \{1, \dots, T'\} \mid |\ln(e'_t)| = c \right\} \right|.$$

So by Proposition 19, we have for all  $c \in (0, 1]$

$$\frac{1}{T} \left| \left\{ t \in \{1, \dots, T\} \mid e_t = c \right\} \right| = \frac{1}{T'} \left| \left\{ t \in \{1, \dots, T'\} \mid e'_t = c \right\} \right|. \quad (4)$$

By Lemma 17, there exist  $\alpha^0 > 0$  such that for all  $\alpha \in (0, \alpha^0)$

$$\min_{x \in X(D)} f^D(x, \alpha) = \min_{x \in \tilde{C}^{n^0}(D)} f^D(x, \alpha)$$

and

$$\min_{x \in X(D')} f^{D'}(x, \alpha) = \min_{x \in \tilde{C}^{n^0}(D')} f^{D'}(x, \alpha).$$

By (4), for all  $x \in \tilde{C}^{n^0}(D), y \in \tilde{C}^{n^0}(D')$  and  $\alpha \in (0, \alpha^0)$ ,

$$f^D(x, \alpha) = f^{D'}(y, \alpha)$$

that shows the claim. □

The proof of Theorem 7 follows symmetrically and is omitted.