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Chen, Ting-Fu; Lin, Shih-kuei; Chang, An-Sing; Wang, Wei-Hao

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The pricing model of pension benefit guaranty corporation insurance with regime-switching processes

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Ting-Fu Chen^{1,*}, Shih-Kuei Lin², An-Sing Chang² and Wei-Hao Wang³

- ¹ Department of Applied Mathematics, Feng Chia University, Taichung 407, Taiwan
 - Department of Money and Banking, National Chengchi University, Taipei City 116, Taiwan;
- square@nccu.edu.tw (S.-K.L.); 100352506@nccu.edu.tw (A.-S.C.)
- ³ Cathay Life Insurance, Taipei City 106, Taiwan; h871412006@yahoo.com.tw
- Correspondence: tfchen@fcu.edu.tw

Abstract: This paper aims to evaluate Pension Benefit Guaranty Corporation (PBGC) insurance values through regime-switching models. We separate periods of the economy with faster growth from those with slower growth to observe long-term trends in the economy. We derive a fair PBGC insurance pricing formula under distress termination and intervention termination using regime-switching processes. We set parameters by estimating the S&P 500 index and one-year treasury bills via expectation maximization particle swarm optimization (EM-PSO)-Gradient, which is an extension of the EM-Gradient method. Then, we conduct sensitivity analysis to investigate the impact of model parameters on insurance values. According to the maximum likelihood estimation results, the Akaike information criterion (AIC) and Bayesian information criterion (BIC) estimators show that the regime-switching process has better goodness of fit than the geometric Brownian motion. Scenario analysis also supports the adequacy of our pricing formula.

Keywords: pension benefit; PBGC insurance; regime-switching; EM algorithm

1. Introduction

This paper considers two financial market states and proposes a regime-switching process model for pension benefits. We propose insurance pricing formulas for distress termination and intervention of contracts by Pension Benefit Guaranty Corporation (PBGC). The fair PBGC insurance value is derived considering the pension benefits, accrued benefits, fair value of a firm's assets, and liabilities, with respect to risk-neutral probability measures and forward probability measures.

The generous computational method for pensions and insufficient contribution rates have led to a growing gap in pension funds. This issue is coupled with medical advances that have brought longer life expectancy after retirement and declining fertility, both of which escalate the aging population problem, posing severe challenges to postretirement security.

Employer-sponsored retirement plans can be divided into defined benefit (DB) plans (DB) and defined contribution (DC) plans. With DB plans, the employer contributes a specific percentage of the employee's salary as pension benefits. The pension benefits are owned by the employer and will be paid to the employees by way of a lump-sum or annuity in accordance with the originally agreed scheme to ensure the employee's retirement life. With DC plans, the employer contributes a specific percentage of the employee's salary as pension benefits, but the employees may also contribute a portion of their salary to their pension benefits accounts. In contrast to DB plans, the pension benefits are owned by the employees, and the total amount of pension the employees are eligible to receive upon retirement is determined solely by the performance of the investment portfolio of the pension benefits.



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Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). Due to the adoption of the Employee Retirement Income Security Act (ERISA) in 1974, DB plans must be insured by the U.S. Pension Benefit Guaranty Corporation (PBGC) to safeguard the pension benefits provided by private institutions. This provides appropriate guarantees in the event of underfunded pension benefits, thereby making the pension benefits reach a statutory threshold that is established by law and adjusted on an annual basis. However, PBGC's financial statements indicate that its financial deficit continues to rise. Thus, situations of insolvent benefits have become increasingly urgent. In recent years, PBGC's premiums have gradually risen to alleviate PBGC's financial deficit.

As the proportion of the retired population continues to climb, pension insurance has received growing attention. Key studies on the evaluation of pension insurance include Marcus (1987), Hsieh et al. (1994), Pennacchi and Lewis (1994), and Lee and Yu (2006). However, these studies did not consider that the market will undergo structural transformation over the long term. Therefore, this paper attempts to estimate market parameters based on a regime-switching model and thus to propose a fair pricing formula for PBGC insurance.

The contributions of this study are three-fold. First, we use the S&P 500 index and one-year treasury bills from 1999 to 2013 and apply the expectation maximization (EM) components of expectation maximization particle swarm optimization (EM-PSO)-Gradient algorithms to estimate the model's parameters under the regime-switching process. Second, we derive insurance pricing formulas for distress termination and intervention of contracts by PBGC considering stochastic interest rates under the forward measure. Third, we conduct sensitivity analysis on the regime-switching model's evaluation of insurance value with the following results: the ratio of the present value of accrued benefits to the present value of pension benefits increases; the ratio of the present value of the firm's liabilities to the fair present value of the firm's assets increases; and the limit of intervention and the fluctuation degree of the model's increase.

The rest of this paper is organized as follows. Section 2 reviews the related literature. Section 3 establishes an evaluation model. First, we describe the conditions for termination of the PBGC contract and discuss the liabilities that PBGC will assume after termination. Next, we extend the architecture of Marcus (1987) to build a regime-switching model. Section 4 evaluates PBGC's insurance value. First, we convert the model measure to a risk-neutral measure and forward measures. Second, we assume that distress termination and intervention termination can only occur at a time point of one year, which is the maturity date. Thus, we consider distress termination and intervention termination as put options to evaluate the insurance value under regime-switching. Section 5 describes the processes for parameter estimation and model selection under different EM algorithms, and then describes the sensitivity analysis results. Finally, Section 6 concludes.

2. Literature

The continually developing pension insurance system is a topic of interest for many scholars. Bodie et al. (1988) assessed the advantages and disadvantages of the DB and DC pension systems according to the categorization of the ERISA. They first discussed how the plans operate under the two systems and then briefly explored the advantages and disadvantages of the two systems with respect to the performance of pension fund investments, growth, termination, and portability. Then, they adopted the model of social security presented by Merton (1983) to discuss the social welfare of individual employees. Finally, they suggested that the main advantage of the DB system is to provide employees a stable income replacement rate, and the DB retirement plan can be considered as providing a certain degree of protection against actual salary risk. The main advantages of the DC system are that employees can decide the present value of the pension during each period, and it is easier to calculate than the DB system.

Marcus (1987) used the evaluation method for contingent claims to generate PBGC insurance value under two conditions. In the first case, the possibility of company bankruptcy was not considered, and it only considered the scenarios whereby the company can obtain maximum value in the event of plan suspension. Then, the closed solution of the insurance value was adjusted, and the static analysis was carried out on optimal termination ratio and put option value. Then, considering that the net contribution rate of pension benefits plus up to 30 percent of the company's net worth will change with the financial situation. Under this assumption, the numerical solutions of the company's optimal contribution decision and insurance value were calculated. The second case considered instances wherein the plan was suspended due to company bankruptcy, and PBGC needed to cover the underfunded portion of the pension benefits. Then, considering that under the voluntary termination by the company, the net contribution rate of the pension benefits will change with the financial situation, the numerical solutions of the ratio of insurance value to liabilities was calculated.

Hsieh et al. (1994) applied the option pricing model proposed by Margrabe (1978) to deduct the put option value of PBGC pensions under uncertain maturity. Next, they examined the put option value of the pensions and the premium charged by PBGC at the time and assessed the fairness of pension premiums at the time. Moreover, they discussed the overall situation and the two subsets of overfunded and underfunded reserves and found that there were significant differences between the overall situation and underfunded subset in terms of the put option value of the pension and the premiums charged by PBGC, while the difference in the overfunded subset was not significant. Finally, the authors suggested that the variable premium rate should be increased to reduce the discrepancy.

Pennacchi and Lewis (1994) proposed the P-L model that considered PBGC's liabilities as put options and inferred the put option value for the four dynamic processes, namely, pension benefits, accrued benefits, the fair value of corporate assets, and corporate liabilities under the conditions of company bankruptcy. Then, they compared the ratio of pension security expenditure to pension benefits and accrued benefits, the ratio of pension security expenditure to the company's net worth, and the ratio of pension security expenditure to the net contribution rate of pension benefits, and found that the pension security expenditure decreased with the increase in the ratio of pension benefits to accrued benefits; in the event of underfunded pension benefits, the pension security expenditure decreased with the increase in the company's net worth; but more specifically, in the case of overfunded pension benefits, this relationship was not monotonic, and the pension security expenditure decreased with the increase in the net contribution rate of pension benefits. Finally, the authors compared the results of the P-L model with those of the Marcus (1987) model, and as expected, the pension security expenditure calculated by the P-L model was higher than that of Marcus' model, and moreover, the gap increased as the ratio of pension benefits to pension liabilities increases.

Karla and Jain (1997) observed that PBGC's liabilities rose from USD 12 million in 1975 to USD 2.9 billion in 1993, and according to projections from PBGC in 1992, situations in the future could be even bleaker. From PBGC's standpoint, the authors proposed an intervention policy of PBGC—evaluated through the put option method—and believed that PBGC should write a put option that viewed pension benefits as the target and pension as the strike price; however, PBGC may exercise the put option prior to the maturity. Therefore, in this article, the necessity of intervention was examined, and the continuous-time model was adopted to capture the behaviors of PBGC and observe the feasibility of the intervention policy. Finally, they believed that PBGC should only terminate the contract as a last resort so that it does not violate its social welfare purposes.

Lee and Yu (2006) mentioned that not all underfunded pension plans are terminated immediately. Instead, these plans will be asked to raise premiums and be able to continue. Therefore, from this perspective, the authors considered the pension insurance contract as a put option with random strike prices and extendable maturity, and that moral risk was likely to occur during the extension period. The authors thus used a multi-stage model that combined interest rate risk, plan suspension, and moral risk to calculate the cost of insurance for insurers. Since this model cannot find a closed-form solution, the authors used the Monte Carlo simulation method to conduct sensitivity analyses of insurance rates under different scenarios, namely, the different initial capitals of the company, different

reserve levels, and different coverage periods with and without moral risks. The final model showed that compared to the overfunded retirement plans, the underfunded ones will not adopt a high-risk investment strategy. In addition, capital forbearance will increase the insurance rate with the coverage period.

During the evaluation of pension insurance contracts, it is necessary to make assumptions about the dynamics of the firm's assets and liabilities. The most common and basic model is the stock price dynamics model developed by Black and Scholes (1973). The Black–Scholes model assumes that the stock prices follow a random walk and obey the normal distribution for a continuous period of time; under this framework, evaluation formulae for European put options can be deduced. While the Black–Scholes model is simple and easy to use, it does not provide a good explanation for changes in the market economy environment.

To make the model more suitable for the market, we use a Markov regime-switching model to calibrate the market economy environment. In this model, the environment is composed of some state, and the current state is a sequence of random variables that follow the Markov property. It is given that the current state, the future state, and the past state are independent. That is, given the current state and all past states of a variable, the conditional probability distribution of its future state depends only on the current state. We can use the Markov regime-switching process to describe different market structures in the financial markets. For example, a long-term market environment can be composed of a high-risk period of a volatile market and a period of stable development.

Previous studies have applied the regime-switching model to the economic environment and financial market. Hamilton (1989) analyzed the quarterly data of gross national product (GNP) in the postwar period in the U.S. and found that GNP returns change with the business prosperity cycle. Furthermore, he divided the business prosperity cycle into two states: a state with positive growth rates is considered normal, and a state with negative growth rates, known as a recession. As the state is not easily observed, Hamilton (1989) presented a Markov regime-switching model to describe the change in GNP data under the business prosperity cycle.

Hardy (2001) used monthly data of the S&P 500 index and TSE 300 index to estimate the parameters for the regime-switching model using the maximum likelihood estimation (MLE) method. Then, she used the likelihood ratio test to examine the estimates of the regime-switching model and other time series models. The results showed that the regime-switching model was better at describing the long-term dynamic process of stock prices.

State transition models have also been studied in this field. Costa and Kwon (2019) solved a risk parity optimization problem under a Markov regime-switching framework to improve estimation results. Türkvatan et al. (2020) used the regime-switching model for temperature modeling and derived a weather derivatives pricing formula. Wang et al. (2021) applied a Markov regime-switching model for asset pricing with a focus on empirically measuring ambiguity degrees in the Chinese mainland and Hong Kong stock markets. Yahya et al. (2021) discussed the relationship between the clean energy stock price and oil price under a two-regime-switching models are a preferred and suitable option to model a time-varying environment.

Marcus (1987) assumed that pension benefits, accrued benefits, the fair value of company assets, and corporate liabilities were part of the geometric Brownian motion model to calculate PBGC's insurance value. However, the value of the company assets is related to the economic prosperity status. According to the historical state of financial markets, it can be found in special events such as the dot-com bubble in 2000, the 911 terrorist attacks in the U.S. in 2001, and the Iraq War launched by the U.S. in 2003. With the bankruptcy of Lehman Brothers and the US presidential election in 2008, the returns of share indexes in the financial market fluctuated considerably. In other years, the indexes showed an upward trend, and their returns showed smaller fluctuations. Therefore, this paper extends the assumption of the model of Marcus (1987) and describes changes in asset value using

the Markov regime-switching model to evaluate insurance value in the event of PBGC's distress termination and intervention termination, in efforts to be close to the situation of regime-switching in the market.

3. The Model

3.1. PBGC Insurance

The insurance businesses undertaken by PBGC are single-employer or multi-employer pension schemes exercising the DB plan. Therefore, PBGC's funding sources are mainly the following. First, a PBGC charges reasonable premiums from companies protected by the PBGC in order to maintain the operation of the institution, and this is also the most important funding source for that PBGC. Second, when a company faces financial distress, it submits a termination notice of the pension plan, and after a PBGC approves this termination, the PBGC begins to take over its pension fund to guarantee employees' lives after retirement. Third, to protect the pension fund or retirement insurance system, a PBGC may take the initiative to terminate a pension plan and begin to take over the pension fund. Finally, a PBGC uses existing funds to invest and gain profits.

There are three forms of termination of PBGC's pension plans, including standard termination, distress termination, and intervention termination. A standard termination refers to the fact that a company can terminate its pension plan when the pension fund has sufficient amounts to cover all pension participants and beneficiaries of the plan. Therefore, standard termination does not generate any liabilities for PBGC. Distress terminates the test, and the pension fund is not sufficient to cover the pensions for participants and beneficiaries, thereby the pension plan is terminated. At this point, the employer must submit a financial statement to prove that it is unable to pay the pensions, and then a PBGC will take over its pension fund, become the account trustee, and use the pension fund to ensure that the current and future retirees are able to receive pensions under statutory restrictions.

Marcus (1987) assumed that the conditions for distress termination occur when the fair value of a firm's assets is less than its liabilities. *A* represents accrued benefits; *F* represents pension benefits; *V* represents the fair value of a firm's assets, and *D* represents its liabilities. Therefore, in the event of distress termination, a PBGC's liabilities are

$$(A-F) \cdot I_{\{V < D\}}$$

We assume that the maturity period is one year, the termination occurs only on the maturity date, and subscripts indicate the time points. Therefore, in the event of distress termination, PBGC's liabilities are

$$(A_1 - F_1) \cdot I_{\{V_1 < D_1\}}$$

The above formula may be considered as a put option with a strike price as A_1 and underlying asset as F_1 .

Intervention termination means that when a company's pension benefits do not meet the statutory minimum requirements, or the pension benefits are insufficient to cover the existing retirement benefits, or there are other conducts that may cause unforeseen losses or damages to the insurance system, PBGC may terminate the pension plan in accordance with the law even if the company does not take the initiative to terminate the pension plan.

Lee and Yu (2006) assume *A* as accrued benefits, *F* as pension benefits, *V* as the fair value of firm's assets, *D* as its liabilities, k' as the intervention threshold, and the time points are indicated by t_i . Therefore, in the event of intervention termination, a PBGC's liabilities are

$$[(A_{t_i} - F_{t_i}) - (V_{t_i} - D_{t_i})] \cdot I_{\{T_{V,D} > t_i, (A_{t_i} - F_{t_i}) \ge k'(V_{t_i} - D_{t_i})\}}$$

We assume that the time to maturity is one year. The termination occurs only on the maturity date, configure k as the intervention ratio, use subscripts to indicate time points, and modify the following formula as the profit function of intervention termination. In the event of intervention termination, a PBGC's liabilities are

$$(A_1 - F_1) \cdot I_{\{A_1 > F_1, 1 < \frac{V_1}{D_1} < k\}}$$

Finally, the above formula may be considered as a put option with a strike price as A_1 and underlying asset as F_1 . Section 3.2 will provide the closed-form formula of the insurance value of intervention termination under regime-switching.

3.2. The Regime-Switching Processes

This section describes the dynamic model of pension benefits, accrued benefits, and the firm's assets and liabilities. In this paper, accrued benefits are built on the geometric Brownian motion model, a dynamic regime-switching model consisting of the dynamics of pension benefits, firm's assets, and liabilities. The regime-switching model can better describe the difference between the rate of return and volatility under normal economic conditions and recession.

(1) Dynamic process of pension benefits under a regime-switching model

Following the regime-switching model proposed by Hamilton (1989), the stochastic differential equation for pension benefits over a continuous time is configured as

$$\frac{dF_t}{F_t} = \left(C_{F,q_t} + \alpha_{F,q_t}\right)dt + \sigma_{F,q_t}dW_F, \ q_t = 1,2$$

where q_t is the market regime at time point t, and it meets and obeys the first-order Markov chain. The probability of regime-switching is $P(q_t = j | q_{t-1} = i) = p_{ij}$ and $\Sigma_{j=1}^2 p_{ij} = 1$; F_t is the value of the pension benefits at time point t; C_{F,q_t} is the net contribution rate of the pension benefits under the regime of q_t ; α_{F,q_t} is the expected instantaneous return for the pension benefits under the regime of q_t ; σ_{F,q_t} is the standard deviation for the pension benefits under the regime of q_t ; σ_{F,q_t} is the standard deviation for the pension benefits under the regime of q_t ; α_{F,q_t} is the standard deviation for the pension benefits under the regime of q_t ; α_{F,q_t} is the standard deviation for the pension benefits under the regime of q_t ; α_{F,q_t} is the standard deviation for the pension benefits under the regime of q_t ; α_{F,q_t} is the standard deviation for the pension benefits under the regime of q_t ; α_{F,q_t} is the standard deviation for the pension benefits under the regime of q_t ; α_{F,q_t} is a Brownian motion.

In the model of this article, the interest rate is assumed to be random, so prices of zero-coupon bonds (ZCB) are set to obey the stochastic differential equation as

$$\frac{dB(t,T')}{B(t,T')} = \mu_{B,q_t}dt + \sigma_{B,q_t}dW_B, \ q_t = 1,2$$

B(t, T') is the price of ZCB on the maturity date of T' at time point t; μ_{B,q_t} is the average of instantaneous expected returns for ZCB under the regime of q_t ; σ_{B,q_t} is the standard deviation for ZCB under the regime of q_t ; and dW_B is a Brownian motion term, and its correlation coefficient with dW_F is ρ_{FB,q_t} .

Ito's lemma is used to deduct the dynamic process of pension benefits during the period of (t, T) under the condition that zero-coupon bond prices are the numeraire, and the process is

$$\frac{F(T)}{B(T,T')} = \frac{F(t)}{B(t,T')}, \exp\left\{ \begin{pmatrix} C_{F,q_t} + \alpha_{F,q_t} - \mu_{B,q_t} + \sigma_{B,q_t}^2 - \rho_{FB,q_t} \sigma_{F,q_t} \sigma_{B,q_t} - \frac{\sigma_{FB,q_t}}{2} \end{pmatrix} \\ (T-t) + \sigma_{F,q_t} W_F(T-t) - \sigma_{B,q_t} W_B(T-t) \end{pmatrix} \right\}$$

where $\sigma_{FB,q_t} = \sqrt{\sigma_{F,q_t}^2 + \sigma_{B,q_t}^2 - 2\rho_{FB,q_t}\sigma_{F,q_t}\sigma_{B,q_t}}$.

(2) Dynamic process of accrued benefits

Following the settings of Marcus (1987) and Pennacchi and Lewis (1994), we assume that the accrued benefits follow the geometric Brownian motion model, and the stochastic differential equation for accrued benefits over a continuous time is

$$\frac{dA_t}{A_t} = (C_A + \alpha_A)dt + \sigma_A dW_A$$

where A_t is the value of accrued benefits at time point t; C_A is the growth rate of accrued benefits due to demographic factors; α_A is the same expected return during the accrued benefit period; σ_A is the deviation of the mortality rate of pension participants; and W_A is a Brownian motion associated with accrued benefits.

Similarly, suppose stochastic interest rates are considered in the model. In that case, zero-coupon bonds should also be the numeraire for accrued benefits, so the relative price dynamics can be derived through Ito's lemma as

$$\frac{A(T)}{B(T,T')} = \frac{A(t)}{B(t,T')} \exp\left\{ \begin{pmatrix} C_{A,q_t} + \alpha_A - \mu_{B,q_t} + \sigma_{B,q_t}^2 - \rho_{AB,q_t} \sigma_A \sigma_{b,q_t} - \frac{\sigma_{AB,q_t}}{2} \end{pmatrix} \\ (T-t) + \sigma_A W_A(T-t) - \sigma_{B,q_t} W_B(T-t) \end{pmatrix} \right\},$$

where $\sigma_{AB,q_t} = \sqrt{\sigma_A^2 + \sigma_{B,q_t}^2 - 2\rho_{AB,q_t}\sigma_A\sigma_{B,q_t}}$.

(3) Dynamic process of firm's assets and liabilities under a regime-switching model

To highlight that the firm value has different growth trends and degrees of change with regime-switching in the market, this model assumes that the stochastic differential equations for a firm's assets and liabilities also have the characteristics of regime-switching.

$$\begin{aligned} \frac{dV_t}{V_t} &= \alpha_{V,q_1} dt + \sigma_{V,q_t} dW_V, \ q_t = 1, 2, \\ \frac{dD_t}{D_t} &= \alpha_{D,q_t} dt + \sigma_{D,q_t} dW_D, \ q_t = 1, 2, \end{aligned}$$

 V_t is the fair value of firm's assets at time point t; α_{V,q_t} is the expected return of a firm's assets under the regime of q_t ; σ_{V,q_t} is the standard deviation of the fair value of the firm's assets under the regime of q_t . D_t is the company liabilities at time t; α_{D,q_t} is the expected growth rate of company liabilities under the regime of q_t ; σ_{D,q_t} is the standard deviation of the company liabilities under the regime of q_t ; σ_{D,q_t} is the standard deviation of the company liabilities under the regime of q_t ; σ_{D,q_t} and W_D are Brownian motions.

Under the given dynamics of firm's assets and liabilities, Ito's lemma uses ZCB as the numeraire for the fair value of the firm's assets and liabilities, whose dynamic processes during the period of (t, T) are

$$\frac{V(T)}{B(T,T')} = \frac{V(t)}{B(t,T')} \exp\left\{ \begin{pmatrix} \alpha_{V,q_t} - \mu_{B,q_t} + \sigma_{B,q_t}^2 - \rho_{VB,qt}\sigma_{V,q_t}\sigma_{b,q_t} - \frac{\sigma_{VB,q_t}}{2} \\ (T-t) + \sigma_{V,q_t}W_V(T-t) - \sigma_{B,q_t}W_B(T-t) \end{pmatrix} \right\},\$$
$$\frac{D(T)}{B(T,T')} = \frac{D(t)}{B(t,T')} \exp\left\{ \begin{pmatrix} \alpha_{D,q_t} - \mu_{B,q_t} + \sigma_{B,q_t}^2 - \rho_{DB,q_t}\sigma_{D,q_t}\sigma_{b,q_t} - \frac{\sigma_{DB,q_t}}{2} \\ (T-t) + \sigma_{D,q_t}W_D(T-t) - \sigma_{B,q_t}W_B(T-t) \end{pmatrix} \right\},\$$

where

$$\sigma_{VB,q_t} = \sqrt{\sigma_{V,q_t}^2 + \sigma_{B,q_t}^2 - 2\rho_{VB,q_t}\sigma_{V,q_t}\sigma_{B,q_t}}$$
$$\sigma_{DB,q_t} = \sqrt{\sigma_{D,q_t}^2 + \sigma_{B,q_t}^2 - 2\rho_{DB,q_t}\sigma_{D,q_t}\sigma_{B,q_t}}$$

4. Pricing PBGC Insurance under the Regime-Switching Model

This section derives the dynamics of pension benefits, accrued benefits, and the firm's assets and liabilities under the risk-neutral and forward measures. We change the physical measure to the risk-neutral measure (refer to Appendix A), which without arbitrage

opportunity, for pricing the PBGC insurance. Then, we derive the fair price formula for the insurance contract under the distress termination and intervention termination.

4.1. Pricing PBGC Insurance

Given pension benefit dynamics under the risk-neutral measure in case of a constant interest rate. We also have dynamics under the forward measure for the stochastic interest rate assumption. With the dynamics of pension benefits, accrued benefits, and the firm's assets and liabilities, we can formulate the fair price of PBGC insurance under the distress termination and intervention termination.

First, we derive the fair price of PBGC insurance under the regime-switching model. We regard the event of distress termination as a one-year put option, and the strike price is the accrued benefits at maturity, and the pension benefit is underlying. Then, in the event of distress termination, PBGC's liabilities are

$$(A_1 - F_1) \cdot I_{\{V_1 < D_1\}}$$

Under the constant interest rate assumption, the current fair value of PBGC insurance, P_0 , can be expressed as the discounted value of the put option in case of distress termination.

$$P_0 = E_q^Q \left[E^Q \left[e^{-r} A_1 I_{\{A_1 > F_1, V_1 < D_1\}} | q = i \right] \right] - E_q^Q \left[E^Q \left[e^{-r} F_1 I_{\{A_1 > F_1, V_1 < D_1\}} | q = i \right] \right]$$

The E_q^Q denotes the expectation under the risk-neutral measure with respect to the state space *q*. To evaluate P_0 , we divide the put option formula into two parts of expectations to obtain the fair value. Then we have the formula

$$P_0 = \sum_{i=1}^{2} \gamma_i \Big(A_o e^{C_A} N_2(d_{11}, d_{12}, \rho_1) - F_0 e^{C_F, q_i} N_2(d_{13}, d_{14}, \rho_1) \Big)$$

where $\gamma_i = P(q_0 = 1)p_{1i} + P(q_0 = 2)p_{2i}$ is the probability in state *i* after 1 year; A_0 is the initial firm's asset; F_0 is the initial value of pension benefits; $N_2(\cdot)$ stands for the cumulated density function of bivariate normal distribution. Other notations are defined as

$$d_{11} = \frac{\ln \frac{A_0}{F_0} + \left(C_A + 0.5\sigma_A^2 - C_{F,q_t} - \rho_{AF,q_t}\sigma_A\sigma_{F,q_t} + 0.5\sigma_{F,q_t}^2\right)}{\sqrt{\sigma_A^2 + \sigma_{F,q_t}^2 - 2\rho_{AF,q_t}\sigma_A\sigma_{F,q_t}}},$$

$$d_{12} = \frac{\ln \frac{D_0}{V_0} - \left(\rho_{VA,q_t}\sigma_{V,q_t}\sigma_A - 0.5\sigma_{V,q_t}^2 - \rho_{DA,q_t}\sigma_{D,q_t}\sigma_A + 0.5\sigma_{D,q_t}^2\right)}{\sqrt{\sigma_{V,q_t}^2 + \sigma_{D,q_t}^2 - 2\rho_{VD,q_t}\sigma_{V,q_t}\sigma_{D,q_t}}},$$

$$d_{13} = \frac{\ln \frac{A_0}{F_0} + \left(C_A + \rho_{AF,q_t}\sigma_A\sigma_{F,q_t} - 0.5\sigma_A^2 - C_{F,q_t} - 0.5\sigma_{F,q_t}^2\right)}{\sqrt{\sigma_A^2 + \sigma_{F,q_t}^2 - 2\rho_{AF,q_t}\sigma_{A,q_t}\sigma_{F,q_t}}},$$

$$d_{14} = \frac{\ln \frac{D_0}{V_0} - \left(\rho_{VF,q_t}\sigma_{V,q_t}\sigma_{F,q_t} - 0.5\sigma_{V,q_t}^2 - \rho_{DF,q_t}\sigma_{D,q_t}\sigma_{F,q_t} + 0.5\sigma_{D,q_t}^2\right)}{\sqrt{\sigma_{V,q_t}^2 + \sigma_{D,q_t}^2 - 2\rho_{VD,q_t}\sigma_{V,q_t}\sigma_{D,q_t}}},$$

$$\rho_1 = \frac{\sigma_{F,q_t}\sigma_{V,q_t}\rho_{FV,q_t} - \sigma_{F,q_t}\sigma_{D,q_t}\sigma_{F,q_t}}{\sqrt{\sigma_A^2 + \sigma_{F,q_t}^2 - 2\rho_{AF,q_t}\sigma_{A,q_t}\sigma_{F,q_t}} - 2\rho_{VD,q_t}\sigma_{V,q_t}\rho_{AV,q_t} + \sigma_A\sigma_{D,q_t}\rho_{AD,q_t}}}{\sqrt{\sigma_A^2 + \sigma_{F,q_t}^2 - 2\rho_{AF,q_t}\sigma_{A,q_t}\sigma_{F,q_t}} - 2\rho_{VD,q_t}\sigma_{V,q_t}\rho_{AV,q_t} + \sigma_A\sigma_{D,q_t}\rho_{AD,q_t}}},$$

When considering the assumption of stochastic interest rate in the current model, we adopt the ZCB numeraire and derive the fair price under the forward measure. In this case, the present value of PBGC insurance can be expressed as the discounted value relative to the ZCB as

$$\frac{P_0}{B(0,T')} = E_q^{Q^T} \left[E^{Q^T} \left[\frac{A_1 I_{\{A_1 > F_1, V_1 < D_1\}}}{B(1,T')} | q = i \right] \right] - E_q^{Q^T} \left[E^{Q^T} \left[\frac{F_1 I_{\{A_1 > F_1, V_1 < D_1\}}}{B(1,T')} | q = i \right] \right]$$

Then we apply the dynamics under the forward measure as shown in Section 4.1 into the expectation. Then we can derive the pricing formula of PBGC insurance under stochastic interest rate

$$P_{0} = E_{q}^{Q} \left\{ A_{0} e^{C_{A}} N_{2}(d'_{11}, d'_{12}, \rho'_{1}) - F_{0} e^{C_{F,q_{t}}} N_{2}(d'_{13}, d'_{14}, \rho'_{1}) \right\}$$

= $\Sigma_{i=1}^{2} \gamma_{i} \left(A_{0} e^{C_{A}} N_{2}(d'_{11}, d'_{12}, \rho'_{1}) - F_{0} e^{C_{F,q_{t}}} N_{2}(d'_{13}, d'_{14}, \rho'_{1}) \right)$

where

$$\begin{split} d_{11}^{\prime} &= \frac{\ln \frac{A_{0}}{k_{0}} + \left(C_{A} + \alpha_{A} - h_{A,q_{1}} + \sigma_{A}\sigma_{AB,q_{1}} - 0.5\sigma_{A}^{2} - C_{F,q_{1}} - \alpha_{F,q_{1}} + h_{F,q_{1}} - \rho_{AFB,q_{1}}\sigma_{AB,q_{1}}\sigma_{F,q_{1}} + 0.5\sigma_{F,q_{1}}^{2}\right)}{\sqrt{\sigma_{A}^{2} + \sigma_{F,q_{1}}^{2} - 2\rho_{AFB,q_{1}}\sigma_{A}\sigma_{F,q_{1}}}} \\ d_{12}^{\prime} &= \frac{\ln \frac{A_{0}}{V_{0}} - \left(\alpha_{V,q_{1}} - h_{V,q_{1}} + \rho_{AVB,q_{1}}\sigma_{AB,q_{1}}\sigma_{V,q_{1}} - 0.5\sigma_{V,q_{1}}^{2} - \alpha_{D,q_{1}} + h_{D,q_{1}} - \rho_{ADB,q_{1}}\sigma_{AB,q_{1}}\sigma_{D,q_{1}} + 0.5\sigma_{D,q_{1}}^{2}\right)}{\sqrt{\sigma_{V,q_{1}}^{2} + \sigma_{D,q_{1}}^{2} - 2\rho_{VD,q_{1}}\sigma_{V,q_{1}}\sigma_{D,q_{1}}}} \\ d_{13}^{\prime} &= \frac{\ln \frac{A_{0}}{V_{0}} + \left(C_{A} + \alpha_{A} - h_{A,q_{1}} + \rho_{AFB,q_{1}}\sigma_{A}\sigma_{FB,q_{1}} - 0.5\sigma_{A}^{2} - C_{F,q_{1}} - \alpha_{F,q_{1}} + h_{F,q_{1}} - \sigma_{T}\sigma_{FB,q_{1}} + 0.5\sigma_{D,q_{1}}^{2}\right)}{\sqrt{\sigma_{A}^{2} + \sigma_{F,q_{1}}^{2} - 2\rho_{AF,q_{1}}\sigma_{A,q_{1}}\sigma_{F,q_{1}}}} \\ d_{14}^{\prime} &= \frac{\ln \frac{D_{0}}{V_{0}} - \left(\alpha_{V,q_{1}} - h_{V,q_{1}} + \rho_{FVB,q_{1}}\sigma_{V,q_{1}}\sigma_{FB,q_{1}} - 0.5\sigma_{V,q_{1}}^{2} - \alpha_{D,q_{1}} + h_{D,q_{1}} - \rho_{DFB,q_{1}}\sigma_{D,q_{1}}\sigma_{FB,q_{1}} + 0.5\sigma_{D,q_{1}}^{2}}\right)}{\sqrt{\sigma_{V,q_{1}}^{2} + \sigma_{F,q_{1}}^{2} - 2\rho_{VD,q_{1}}\sigma_{V,q_{1}}\sigma_{D,q_{1}}}} \\ d_{14}^{\prime} &= \frac{\ln \frac{D_{0}}{V_{0}} - \left(\alpha_{V,q_{1}} - h_{V,q_{1}} + \rho_{FVB,q_{1}}\sigma_{A}\sigma_{FB,q_{1}} - 0.5\sigma_{V,q_{1}}^{2} - \alpha_{D,q_{1}} + h_{D,q_{1}} - \rho_{DFB,q_{1}}\sigma_{D,q_{1}}\sigma_{FB,q_{1}} + 0.5\sigma_{D,q_{1}}^{2}}\right)}{\sqrt{\sigma_{V,q_{1}}^{2} + \sigma_{D,q_{1}}^{2} - 2\rho_{VD,q_{1}}\sigma_{V,q_{1}}\sigma_{D,q_{1}}}} \\ d_{14}^{\prime} &= \frac{\ln \frac{D_{0}}{V_{0}} - \left(\alpha_{V,q_{1}} - h_{V,q_{1}}} + \rho_{FVB,q_{1}}\sigma_{A}\sigma_{F}\sigma_{A}} + 0.5\sigma_{V,q_{1}}^{2} + \sigma_{D,q_{1}}^{2} - 2\rho_{VD,q_{1}}\sigma_{V,q_{1}}\sigma_{D,q_{1}}} \right)}{\sqrt{\sigma_{V,q_{1}}^{2} + \sigma_{E,q_{1}}^{2} - 2\rho_{VD,q_{1}}\sigma_{V,q_{1}}} + \sigma_{D,q_{1}}^{2} - 2\rho_{VD,q_{1}}\sigma_{V,q_{1}}\sigma_{D,q_{1}}} \right)} \\ d_{14}^{\prime} &= \frac{\sigma_{A}\sigma_{D}\sigma_{D,q}\rho_{A}\rho_{A}\rho_{A}\rho_{A} - \sigma_{A}\sigma_{A}\rho_{A}\rho_{A}\rho_{A}} + \sigma_{E,q_{1}}^{2} - 2\rho_{VD,q_{1}}\sigma_{V,q_{1}}} + \sigma_{A}^{2}\rho_{A}\rho_{A}\rho_{A}} + \sigma_{A}^{2}\rho_{A}} \right)} \\ d_{14}^{\prime} &= \frac{\sigma_{A}\sigma_{D}\sigma_{D,q}\rho_{A}\rho_{A}\rho_{A}\rho_{A}} + \sigma_{A}\rho_{A}\rho_{A}\rho_{A}\rho_{A}\rho_{A}\rho_{A}} + \sigma_{A}\rho_{A}\rho_{A}\rho_{A}\rho_{A}} + \sigma_{A}\rho_{A$$

ee that when $\mu_{B,q_t} = r$ and $\sigma_{B,q_t} = 0$, the formula will degenerate to the constant interest rate case. Another contract evaluated in this study is the PBGC insurance under intervention termination. The fair value of the insurance contract can also be regarded as a one-year

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put option with a strike price A_1 and the underlying asset F_1 . In the event of intervention termination, PBGC's liabilities are

$$(A_1 - F_1) \cdot I_{\{A_1 > F_1, 1 < \frac{V_1}{D_1} < k\}}$$

Therefore, under the constant interest rate and regime-switching model, the intervention termination insurance value can be obtained by calculating the expected present value.

$$P_{0} = E_{q}^{Q} \left[E^{Q} \left[e^{-r} (A_{1} - F_{1}) I_{\{A_{1} > F_{1}, 1 < \frac{V_{1}}{D_{1}} < k\}} | q = i \right] \right] \\ = \Sigma_{i=1}^{2} \gamma_{i} \left\{ \begin{array}{c} \left[A_{0} e^{C_{A}} N_{2}(d_{11}, d_{22}, \rho_{1}) - F_{0} e^{C_{F}, q_{t}} N_{2}(d_{13}, d_{24}, \rho_{1}) \right] \\ - \left[A_{0} e^{C_{A}} N_{2}(d_{11}, d_{12}, \rho_{1}) - F_{0} e^{C_{F}, q_{t}} N_{2}(d_{13}, d_{14}, \rho_{1}) \right] \end{array} \right\}$$

where

$$d_{22} = \frac{\frac{\ln KD_0}{V_0} - \left(\rho_{VA,q_t}\sigma_{V,q_t}\sigma_A - 0.5\sigma_{V,q_t}^2 - \rho_{DA,q_t}\sigma_{D,q_t}\sigma_A + 0.5\sigma_{D,q_t}^2\right)}{\sqrt{\sigma_{V,q_t}^2 + \sigma_{D,q_t}^2 - 2\rho_{VD,q_t}\sigma_{V,q_t}\sigma_{D,q_t}}}$$
$$d_{24} = \frac{\ln \frac{KD_0}{V_0} - \left(\rho_{VF,q_t}\sigma_{V,q_t}\sigma_{F,q_t} - 0.5\sigma_{V,q_t}^2 - \rho_{DF,q_t}\sigma_{D,q_t}\sigma_{F,q_t} + 0.5\sigma_{D,q_t}^2\right)}{\sqrt{\sigma_{V,q_t}^2 + \sigma_{D,q_t}^2 - 2\rho_{VD,q_t}\sigma_{V,q_t}\sigma_{D,q_t}}}$$

Similarly, if we further consider the model with the stochastic interest rate, we have to change the measure to the forward measure for pricing the PBGC insurance. With the dynamics of pension benefits, accrued benefit, firm's assets under the forward measure, we can obtain the fair insurance prices with the intervention termination as follows,

$$P_{0} = E_{q}^{Q} \left\{ \begin{array}{c} \left[A_{0}e^{C_{A}}N_{2}(d_{11}', d_{22}', \rho_{1}') - F_{0}e^{C_{F,q_{t}}}N_{2}(d_{13}', d_{24}', \rho_{1}') \right] \\ - \left[A_{0}e^{C_{A}}N_{2}(d_{11}', d_{12}', \rho_{1}') - F_{0}e^{C_{F,q_{t}}}N_{2}(d_{13}', d_{14}', \rho_{1}') \right] \\ = \Sigma_{i=1}^{2}\gamma_{i} \left\{ \begin{array}{c} \left[A_{0}e^{C_{A}}N_{2}(d_{11}', d_{22}', \rho_{1}') - F_{0}e^{C_{F,q_{t}}}N_{2}(d_{13}', d_{24}', \rho_{1}') \right] \\ - \left[A_{0}e^{C_{A}}N_{2}(d_{11}', d_{12}', \rho_{1}') - F_{0}e^{C_{F,q_{t}}}N_{2}(d_{13}', d_{14}', \rho_{1}') \right] \end{array} \right\}$$

where

$$d'_{22} = \frac{\frac{\ln KD_0}{V_0} - \left(\alpha_{VA,q_t} - h_{V,q_t} + \rho_{AVB,q_t}\sigma_{AB,q_t}\sigma_{V,q_t} - 0.5\sigma_{V,q_t}^2 - \alpha_{D,q_t} + h_{D,q_t} - \rho_{ADB,q_t}\sigma_{AB,q_t}\sigma_{D,q_t} + 0.5\sigma_{D,q_t}^2\right)}{\sqrt{\sigma_{V,q_t}^2 + \sigma_{D,q_t}^2 - 2\rho_{VD,q_t}\sigma_{V,q_t}\sigma_{D,q_t}}}$$
$$d'_{24} = \frac{\frac{\ln KD_0}{V_0} - \left(\alpha_{VA,q_t} - h_{V,q_t} + \rho_{FVB,q_t}\sigma_{V,q_t}\sigma_{FB,q_t} - 0.5\sigma_{V,q_t}^2 - \alpha_{D,q_t} + h_{D,q_t} - \rho_{DFB,q_t}\sigma_{D,q_t}\sigma_{FB,q_t} + 0.5\sigma_{D,q_t}^2\right)}{\sqrt{\sigma_{V,q_t}^2 - \sigma_{VD,q_t}^2 - \alpha_{D,q_t} + h_{D,q_t} - \rho_{DFB,q_t}\sigma_{D,q_t}\sigma_{FB,q_t} + 0.5\sigma_{D,q_t}^2}}}$$

$$_{4} = \frac{\frac{1}{V_{0}} - \left(\alpha_{VA,q_{t}} - n_{V,q_{t}} + \rho_{FVB,q_{t}} \sigma_{V,q_{t}} \sigma_{FB,q_{t}} - 0.5 \sigma_{V,q_{t}} - \alpha_{D,q_{t}} + n_{D,q_{t}} - \rho_{DFB,q_{t}} \sigma_{D,q_{t}} \sigma_{V,q_{t}} + 0.5 \sigma_{D,q_{t}}}{\sqrt{\sigma_{V,q_{t}}^{2} + \sigma_{D,q_{t}}^{2} - 2\rho_{VD,q_{t}} \sigma_{V,q_{t}} \sigma_{D,q_{t}}}}$$

The first part of the pricing formula standards for the insurance value when the firm's liability reaches the intervention threshold at maturity, but the company does not induce the distress termination. Therefore, the second part is the value of the insurance price under the distress termination, and it has a negative sign which indicates a deduction term in the insurance prices.

5. Parameters Estimation and Empirical Analysis

In this study, we combine the PSO algorithm proposed by Kennedy and Eberhart (1995) and the EM-Gradient algorithm proposed by Lange (1995) to estimate the parameters of the Markov regime-switching model with hidden variables. Dempster et al. (1977) presented the EM algorithm, which continuously calculates expectation values and looks for the maximum likelihood for incomplete information. They deduced that the EM algorithm converges to the MLE and illustrated that the EM algorithm can be widely used in situations such as missing values, truncated data, and finite mixed models. Lange (1995) proposed the

EM-Gradient algorithm to solve the problem of slow convergence of the EM algorithm in various situations. He used the Newton–Raphson method to find the maximum likelihood value of the EM algorithm. The advantage of this method is that it can find the maximum at the secondary convergence speed, greatly increasing the calculation speed.

Kennedy and Eberhart (1995) proposed the particle swarm optimization (PSO) algorithm. The random variable in the PSO is called a 'particle', and the movement of each particle is influenced by its inertia, its own best experience, and the best experience of the group. It iterates to advance accuracy so as to derive parameter estimates; it does this by continuously updating the best experience values to obtain the parameters. The combination of PSO and the EM algorithm has been studied for model estimation in engineering and biomedical fields, such as Wen et al. (2015), Santos et al. (2016), Sauvageau and Kumral (2018), and Dai et al. (2021). We further apply the algorithm for the hidden Markov regime-switching model in the financial field.

5.1. The EM Algorithm

In this study, we use the EM-Gradient algorithm and the EM-PSO-Gradient algorithm to estimate model parameters. Dynamic processes of the model to be evaluated are converted into log returns. The log-returns for pension benefits, accrued benefits, firm's assets and liabilities, and zero-coupon bonds are represented by R_F , R_A , R_V , R_D , and R_B respectively, and their corresponding rates are deducted through Ito's lemma as follows:

$$R_F = \log\left(\frac{F(t)}{F(t-1)}\right) = \left(F_{F,q_t} + \alpha_{F,q_t} - 0.5\sigma_{F,q_t}^2\right) + \sigma_{F,q_t}Z_F,$$

$$R_A = \log\left(\frac{A(t)}{A(t-1)}\right) = \left(C_A + \alpha_A - 0.5\sigma_A^2\right) + \sigma_A Z_A,$$

$$R_V = \log\left(\frac{V(t)}{V(t-1)}\right) = \left(\alpha_{V,q_t} - 0.5\sigma_{V,q_t}^2\right) + \sigma_{V,q_t}Z_V,$$

$$R_D = \log\left(\frac{D(t)}{D(t-1)}\right) = \left(\alpha_{D,q_t} - 0.5\sigma_{D,q_t}^2\right) + \sigma_{D,q_t}Z_D,$$

$$R_B = \log\left(\frac{B(t,T')}{B(t-1,T')}\right) = \left(\mu_{B,q_t} - 0.5\sigma_{B,q_t}^2\right) + \sigma_{B,q_t}Z_B,$$

 Z_F , Z_A , Z_V , Z_D , and Z_B represent the standard normal distribution corresponding to the dynamics.

Assuming that the return of pension benefits is $\widetilde{R_F} = \{R_{F_1}, R_{F_2}, \dots, R_{F_T}\}$; the return of accrued benefits is $\widetilde{R_A} = \{R_{A_1}, R_{A_2}, \dots, R_{A_T}\}$; the return of the fair value of a firm's assets is $\widetilde{R_V} = \{R_{V_1}, R_{V_2}, \dots, R_{V_T}\}$; the return of company liabilities is $\widetilde{R_D} = \{R_{D_1}, R_{D_2}, \dots, R_{D_T}\}$; the return of ZCB is $\widetilde{R_B} = \{R_{B_1}, R_{B_2}, \dots, R_{B_T}\}$; and the market regime is $\widetilde{q} = \{q_1, q_2, \dots, q_T\}$ $q_i \in \{1, 2\}$.

When the total number of observed T is large, there may be 2^T number of possible combinations of the market regime and the calculation volume of the incomplete-data likelihood function will be huge. To make estimates more efficient, we adopt the EM algorithm to obtain the model parameters in this paper. The algorithm's characteristic uses the complete-data likelihood function to find the maximum estimation of the likelihood function of the model.

The EM algorithm consists of Step E and Step M. At the *k*th iteration, Step E gives observable data and the parameter values estimated in the (k - 1)th iteration. We derive the conditional expected value of the log-likelihood function for the data as $Q(\theta^{(k)}|\theta^{(k-1)})$ and we leave the details in Appendix B. Next, Step M aims to find the maximum conditional

expected value of the log-likelihood function in Step E in the process of *k*th iteration, expressed as

$$\theta^{(k)} = \underset{\theta}{\operatorname{argmax}} Q\Big(\theta^{(k)} | \theta^{(k-1)}\Big)$$

Then, we use the parameter estimation result of Step M back to Step E. After continuous iterations, the maximum likelihood estimate that maximizes the incomplete-data likelihood function can be obtained.

As can be seen from the conditional expected value of the log-likelihood function $Q(\theta^{(k)}|\theta^{(k-1)})$, this function consists of three parts: the first part is related to the initial probability of market regime. Since the hidden starting state is unknown, we assume that each state's initial probability is equal. The second part is associated with the switching probability of market regime; the third part is the Brownian motion item of returns.

We can calculate the maximum value of each of the three components to obtain the maximum value of the entire function $Q(\theta^{(k)}|\theta^{(k-1)})$. The characteristics of sufficient statistics are used to identify the maximum likelihood estimates for the first part π_i and the second part p_{ij} of the *k*th iteration, respectively shown as

$$\hat{\pi}_i^{(k)} = rac{P(q_1 = i | R, \theta^{(k-1)})}{\sum_{i=1}^2 P(q_1 = i | R, \theta^{(k-1)})},$$

and

$$\hat{p}_{ij}^{(k)} = \frac{\sum_{t=2}^{T} P(q_{t-1} = i, q_t = j | R, \theta^{(k-1)})}{\sum_{i=1}^{2} \sum_{t=2}^{T} P(q_{t-1} = i, q_t = j | R, \theta^{(k-1)})}$$

The third part is a nonlinear function, and the closed-form solution of the maximum likelihood estimate for parameters cannot be found, so EM-Gradient and EM-PSO-Gradient algorithms can be used to calculate the parameter of the third component.

Lange (1995) proposed the EM-Gradient algorithm, using the Newton–Raphson optimization algorithm to find the parameters of the *k*th iteration, and $\theta^{(k)}$ that can maximize $Q(\theta^{(k)}|\theta^{(k-1)})$:

$$\theta^{(k)} = \theta^{(k-1)} - \left[d^2 Q\left(\theta^{(k)}|\theta^{(k-1)}\right)\right]^{-1} d^1 Q\left(\theta|\theta^{(k-1)}\right),$$

where $d^1Q(\theta|\theta^{(k-1)})$ is the first-order differential matrix of $Q(\theta|\theta^{(k-1)})$, $d^2Q(\theta|\theta^{(k-1)})$ is the second-order differential matrix of $Q(\theta|\theta^{(k-1)})$. Lange (1995) also illustrated that since $d^2Q(\theta|\theta^{(k-1)})$ is a negative definite matrix, it can find a set of parameter estimates through successive iterations, and locally maximize the likelihood function of the observed values. The advantage of EM-Gradient is to find the optimizer through the gradient method, and its rate of convergence is quadratic convergence, greatly increasing the rate of convergence. However, if the initial point is not well selected, or the second-order derivative of the optimized function does not exist, then the optimizer may not converge. This problem can be solved through the EM-PSO-Gradient algorithm.

Through the PSO algorithm proposed by Kennedy and Eberhart (1995), given the range of search D, n number of particles $S = \{\theta_1, \theta_2, ..., \theta_n\} \subset D$ are randomly extracted. Then, given the boundary of velocity $v_i \in [v_{min}, v_{max}]$, the best value of initial particle experience $pbest_i$, and best value of group experience $pbest_i$ and accuracy. The PSO algorithm adjusts particle position as

$$v_i(t) = wv_i(t-1) + c_1 \cdot r_1 \cdot (pbest_i - \theta_i(t-1) + c_2 \cdot r_2(gbest - \theta_i(t-1)))$$
$$\theta_i(t) = \theta_i(t-1) + v_i(t)$$

where w is the inertia weight, c_1 and c_2 are cognitive learning factor and social learning factor respectively, and r_1 and r_2 are random variables following uniform [0,1]. We can substitute particles into the function as

If
$$Q(\theta_i(t)|\theta^{(k-1)}) > Q(pbest_i|\theta^{(k-1)})$$
, $pbest_i = \theta_i(t)$.
If $Q(\theta_i(t)|\theta^{(k-1)}) < Q(pbest_i|\theta^{(k-1)})$, $pbest_i = pbest_i$.

Then compare the best value of each particle experience with the best value of group experience,

If
$$\max_{i} Q(pbest_{i}|\theta^{(k-1)}) > Q(gbest|\theta^{(k-1)})$$
, $gbest = \arg_{i} Q(pbest_{i}|\theta^{(k-1)})$.
If $\max_{i} Q(pbest_{i}|\theta^{(k-1)}) < Q(gbest|\theta^{(k-1)})$, $gbest = gbest$.

Then, carry out successive iterations of the best values of particle experience and group experience until reaching the accuracy as set initially, and then stop and obtain the parameter value under PSO $\theta_t^{(k)}$.

This paper combines the PSO algorithm proposed by Kennedy and Eberhart (1995) and the EM-Gradient algorithm proposed by Lange (1995), uses $\theta_t^{(k)}$ as the initial value of the iteration $Q(\theta|\theta^{(k-1)})$ to obtain a more accurate parameter value of the k^{th} iteration

$$\theta^{(k)} = \theta^{(k-1)} - \left[d^2 Q \left(\theta_t^{(k)} | \theta^{(k-1)} \right) \right]^{-1} d^1 Q \left(\theta_t^{(k)} | \theta^{(k-1)} \right)$$

Based on the advantages of the PSO algorithm, we first find out the parameter estimation solutions through the PSO algorithm, then select the initial point for the Gradient algorithm, thereby solving the accuracy problem of the PSO algorithm through the Gradient algorithm to make the solutions more accurate.

5.2. Estimation Results

The Compustat on WRDS database contains the annual pension data of each company. However, in the database, the numbers of companies' data are small, and the data are not complete, so they cannot be used to estimate the model parameters of this paper. Therefore, we refer to the parameter setting method proposed by Marcus (1987) and Pennacchi and Lewis (1994).

Firstly, under different regimes, define the net contribution rate of pension benefits as C_{F,q_t} and the growth rate of accrued benefits due to demographic factors as C_A . Considering that the reference interest rate is 3% and the growth rate of PBGC's total insured liabilities has been 4.7% in recent years, C_A is set to 0.017, and C_{F,q_t} is set to be slightly higher than C_A —that is, 0.023 under Regime 1 and 0.02 under Regime 2—in order to coordinate with the enforcement of the Pension Protection Act in 1987.

Next, under different regimes, define the standard deviation of pension benefits as σ_{F,q_t} , the standard deviation of the fair value of a firm's assets as σ_{V,q_t} and the standard deviation of the firm's liabilities as σ_{D,q_t} . Set the standard deviation of pension benefits to be slightly lower than the standard deviation of S&P 500 returns, 0.13 under Regime 1 and 0.29 under Regime 2. With regard to the standard deviation of the fair value of a firm's assets σ_{V,q_t} , it is set to be the variation of stock market returns, 0.16 under Regime 1 and 0.32 under Regime 2. For the standard deviation of a firm's liabilities σ_{D,q_t} , it is set to be the variation of a firm's liabilities σ_{D,q_t} , it is set to be the standard deviation of a firm's liabilities σ_{D,q_t} , it is set to be the fluctuation of non-default bonds during the premium collection period, that is 0.02 under Regime 1 and 0.05 under Regime 2. Finally, the change of the mortality rate of pension participants σ_A and the correlation coefficients matrix refer to the configuration values of Pennacchi and Lewis (1994).

Table 1 lists the parameter estimation results of S&P 500 index returns and the one-year treasury bill returns that are evaluated through the Geometric Brownian Motion model (GBM) and the regime-switching model (RSM), respectively. The results show that the estimation values of the different methods are very similar regardless of the model, and the values of log-likelihood function estimated by the RSM are larger than those of the GBM, and its AIC and BIC are both smaller than those of the GMB, indicating that the RSM is a more appropriate model. Then, we further compare the values of the log-likelihood function estimated by the the values of the log-likelihood function estimated by the EM-PSO are smaller than those of the EM-PSO-Gradient. Then, we compare the results of EM-Gradient and EM-PSO-Gradient in the RSM model, and find that using the value given by PSO as the initial value for EM-Gradient and then estimating through EM-Gradient makes the estimates more accurate than only using EM-Gradient for estimation. However, in the GBM, the values of the log-likelihood function are almost identical, which is caused by the simple model structure. Therefore, using PSO can solve the initial value problem of Gradient, making estimates more accurate.

Table 1. Parameter estimation of S&P 500 returns and treasury bills under the GBM and the regimeswitching models.

		p_{i1}	p_{11}	<i>p</i> ₂₂	μ_{s1}	μ_{s2}	σ_{s1}	σ_{s2}	LogL	AIC	BIC
BM	PSO				0.0002		0.0129		10,935.15	-21,866.30	-21,853.85
0	PSOG				0.0002 (0.0002)		0.0129 (0.0002)		10,935.15	-21,866.30	-21,853.85
RSM	EM_G	0.9994 (0.0277)	0.8942 (0.0058)	0.5000 (0.0168)	0.0006 (0.0002)	-0.0018 (0.0008)	0.0081 (0.0001)	0.0238 (0.0006)	11,391.46	-22,768.91	-22,725.34
	EM_PSO	0.9918	0.8942	0.5000	0.0006	-0.0018	0.0081	0.0238	11,391.45	-22,768.91	-22,725.33
	EM_PSOG	0.9979 (0.0238)	0.8945 (0.0016)	0.5001 (0.0093)	0.0006 (0.0002)	-0.0018 (0.0009)	0.0081 (0.0001)	0.0238 (0.0007)	11,469.55	-22,925.10	-22,881.52

The number in parentheses is the standard error of the corresponding estimated parameter.

Extending to the correlated models, Table 2 lists the parameter estimation results on S&P 500 index returns and one-year treasury bill returns under the correlated GBM and correlated RSM model. That is, we further consider the correlation between the Brownian motions between stock index and treasury bills. In Table 2, we observe that the estimated values of different methods are very similar regardless of the model. The values of the log-likelihood function estimated by correlated RSM are all greater than those of correlated GBM. In comparison, its AIC and BIC values are smaller than those estimated by correlated GBM, indicating that the correlated RSM is a more appropriate model. We further find that the values of the log-likelihood function estimated by EM-PSO are smaller than those of EM-PSO-Gradient. Then, we further compare EM-PSO and EM-PSO-Gradient in the correlated RSM, and it can be found that using the value given by PSO as the initial value for EM-PSO and then estimating through EM-PSO makes the estimates more accurate than only using EM-PSO for estimation. However, in the correlated GBM, the values of the log-likelihood function are almost identical, which is caused by the simple model structure. Therefore, we find that using PSO to obtain the initial value for Gradient can improve the accuracy.

Σ		p_{i1}	p_{11}	<i>p</i> ₂₂	μ_{s1}	μ_{s2}	σ_{s1}	σ_{s2}	μ_{b1}	μ_{b2}	σ_{b1}	σ_{b2}	$ ho_1$	ρ_2	AIC	BIC
Corr. GB	PSO				0.0002		0.00001		0.0129		0.0004		-0.2497		-69,322	-69,290
	PSOG				0.0002 (0.0007)		0.00001 (0.0000)		0.0129 (0.0002)		0.0004 (0.0001)		-0.2504 (0.0113)		-69,322	-69,290
MS	EM	0.9888 (0.0972)	0.9429 (0.0040)	0.8530 (0.0109)	0.0006 (0.0002)	-0.0009 (0.0006)	0.0085 (0.0001)	0.0202 (0.0004)	$\begin{array}{c} -1\times 10^{-5} \\ 1\times 10^{-6} \end{array}$	$\begin{array}{c} 7\times10^{-5} \\ 2\times10^{-5} \end{array}$	0.0002 (0.0000)	0.0007 (0.0000)	-0.1118 (0.0224)	-0.2941 (0.0133)	-72,023	-71,942
н К	EM_PSO	0.6219	0.9440	0.8530	0.0006	-0.0008	0.0086	0.0200	$-1 imes 10^{-5}$	$6 imes 10^{-5}$	0.0002	0.0007	-0.1139	-0.2934	-72,023	-71,942
Cor	EM_PSOG	0.9985 (0.0604)	0.9429 (0.0055)	0.8530 (0.0165)	0.0006 (0.0002)	-0.0009 (0.0009)	0.0086 (0.0001)	0.0202 (0.0010)	$-1 imes 10^{-5} \ 1 imes 10^{-6}$	$\begin{array}{c} 7\times10^{-5}\\ 2\times10^{-5}\end{array}$	0.0002 (0.0000)	0.0007 (0.0000)	-0.1118 (0.0457)	-0.2941 (0.1397)	-72,023	-71,942

Table 2. Parameter estimation of S&P 500 returns and treasury bills under the correlated GBM and the regime-switching models.

The number in parentheses is the standard error of the corresponding estimated parameter.

5.3. Sensitivity Analysis

Based on the evaluation formulae developed in Section 4 of this paper, we adjust one of the parameters to observe the impact on the evaluation formulae. Assume that accrued benefits and company liabilities are both 100, the ratio of accrued benefits to pension benefits is 0.7, and the ratio of company liabilities to value is 0.85 in case of distress termination and 0.35 in case of intervention termination. Tables 3 and 4 list the sensitivity analyses for distress termination and intervention termination, respectively. Tables 5 and 6 list the sensitivity analysis for distress termination and intervention termination under the assumptions that interest rates are random, α_{F,q_t} , α_{V,q_t} , and α_{D,q_t} are 0.035 and 0.025 under Regime 1 and Regime 2 respectively, and α_A is 0.03.

From Panel 1 of Tables 3 and 5—the sensitivity analysis for distress termination—we find that as the ratio of the present value of accrued benefits to the present value of pension benefits increases, the insurance value increases. That is because pension benefits may become increasingly inadequate to cover accrued benefits in the future, as a result, in the event of a financial crisis of the company, PBGC liabilities will increase as the gap between pension benefits and accrued benefits widens. Then, as the ratio of the present value of company liabilities to the fair present value of the firm's assets rises, the insurance value goes up. This is because the company's financial situation may become worse in the future, resulting in higher probabilities for PBGC to bear liabilities. Panel 2 and Panel 3 show that when fluctuation goes up, the insurance value will increase because of higher uncertainties in the future, leaving PBGC to assume higher risks.

Panel 1		0.4	0.5	0.6	A_0 / F_0 0.7	0.8	0.9	1.0
$\frac{D_0}{V_0}$	0.7 0.8 0.9	0.00263 0.00305 0.00328	0.02235 0.02779 0.03115	0.08830 0.11722 0.13730	0.22483 0.32384 0.40238	0.44172 0.72149 0.99637	0.72553 1.36723 2.15385	1.03804 2.19686 3.86923
	1.0	0.00338	0.03300	0.14954	0.45343	1.19249	2.79935	5.42862
Panel 2		0.11	0.12	$\sigma_{F,1}$ 0.13	0.14	0.15		
$\sigma_{F,2}$	0.27 0.28 0.29	0.28315 0.31615 0.35058	0.28967 0.32266 0.35710	0.29908 0.33207 0.36650	0.31193 0.34493 0.37936	0.32870 0.36170 0.39613 0.42100		
	0.30	0.38636	0.39287 0.42989	0.40228	0.41514 0.45216	0.43190		
Panel 3		0.14	0.15	$\sigma_{V,1}$ 0.16	0.17	0.18		
	0.30 0.31	0.35621 0.36021	0.35757 0.36157	0.35874 0.36275	0.35977 0.36378	0.36067 0.36468		
$\sigma_{V,2}$	0.32 0.33 0.34	0.36397 0.36750 0.37083	0.36533 0.36886 0.37219	0.36650 0.37004 0.37337	0.36753 0.37106 0.37439	0.36843 0.37197 0.37529		
Par	Panel 4		0.015	$\sigma_{D,1}$ 0.020	0.025	0.030		
σ _{D,2}	0.040 0.045 0.050	0.36504 0.36534 0.36560	0.36550 0.36580 0.36605	0.36595 0.36625 0.36650	0.36640 0.36670 0.36695	0.36684 0.36714 0.36739		
	0.055 0.060	0.36581 0.36598	0.36627 0.36644	0.36672 0.36689	0.36717 0.36734	0.36761 0.36778		

Table 3. Sensitivity analysis for distress termination insurance.

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1.8 0.14562 0.14816 0.15103 0.15423 0.15774 2.0 0.21208 0.21662 0.22184 0.22772 0.23427	
2 0 0 21208 0 21662 0 22184 0 22772 0 23427	
$\sigma_{F,1}$	
Panel 3 0.11 0.12 0.13 0.14 0.15	
0.27 0.17753 0.17905 0.18117 0.18396 0.18746	
0.28 0.19750 0.19902 0.20114 0.20393 0.20743	
$\sigma_{F,2}$ 0.29 0.21819 0.21972 0.22184 0.22463 0.22812	
0.30 0.23955 0.24107 0.24319 0.24598 0.24948	
0.31 0.26150 0.26303 0.26515 0.26794 0.27144	
$\sigma_{V,1}$	
Panel 4 0.14 0.15 0.16 0.17 0.18	
0.30 0.20325 0.20454 0.20595 0.20742 0.20890	
0.31 0.21152 0.21281 0.21422 0.21569 0.21717	
$\sigma_{V,2}$ 0.32 0.21914 0.22043 0.22184 0.22331 0.22479	
0.33 0.22610 0.22739 0.22880 0.23027 0.23175	
0.34 0.23241 0.23370 0.23511 0.23658 0.23806	
$\sigma_{D,1}$	
Panel 5 0.010 0.015 0.020 0.025 0.030	
0.040 0.22114 0.22130 0.22148 0.22168 0.22189	
0.045 0.22132 0.22149 0.22167 0.22186 0.22208	
$\sigma_{D,2}$ 0.050 0.22149 0.22166 0.22184 0.22203 0.22225	
0.055 0.22165 0.22181 0.22199 0.22219 0.22240	
0.060 0.22179 0.22195 0.22213 0.22233 0.22254	

Table 4. Sensitivity analysis for intervention termination insurance.

 Table 5. Scenarios: distress termination insurance under stochastic interest rates.

Panel 1					A_0/F_0			
		0.4	0.5	0.6	0.7	0.8	0.9	1.0
	0.7	0.00263	0.02234	0.08826	0.22477	0.44173	0.72584	1.03886
D/W	0.8	0.00305	0.02779	0.11720	0.32382	0.72175	1.36856	2.20025
D_0 / V_0	0.9	0.00328	0.03115	0.13729	0.40239	0.99670	2.15565	3.87441
	1.0	0.00338	0.03300	0.14953	0.45345	1.19272	2.80071	5.43304
Panel 2				$\sigma_{F,1}$				
		0.11	0.12	0.13	0.14	0.15		
	0.27	0.28312	0.28964	0.29906	0.31194	0.32874		
	0.28	0.31612	0.32264	0.33207	0.34494	0.36174		
$\sigma_{F,2}$	0.29	0.35056	0.35709	0.36651	0.37939	0.39619		
	0.30	0.38635	0.39287	0.40229	0.41517	0.43197		
	0.31	0.42338	0.42990	0.43932	0.45220	0.46900		

Panel 3		0.14	0.15	$\sigma_{V,1}$ 0.16	0.17	0.18	
	0.30	0.35621	0.35757	0.35874	0.35977	0.36067	
	0.31	0.36022	0.36158	0.36275	0.36378	0.36468	
$\sigma_{V,2}$	0.32	0.36398	0.36533	0.36651	0.36753	0.36843	
	0.33	0.36751	0.36887	0.37004	0.37107	0.37197	
	0.34	0.37084	0.37220	0.37337	0.37440	0.37530	
D	1.4			$\sigma_{D,1}$			
Panel 4		0.010	0.015	0.020	0.025	0.030	
	0.040	0.36521	0.36556	0.36598	0.36641	0.36684	
	0.045	0.36549	0.36584	0.36626	0.36669	0.36713	
$\sigma_{D,2}$	0.050	0.36574	0.36609	0.36651	0.36694	0.36737	
	0.055	0.36595	0.36630	0.36672	0.36715	0.36758	
	0.060	0.36612	0.36647	0.36689	0.36732	0.36775	

Table 5. Cont.

 Table 6. Scenarios: intervention termination insurance under stochastic interest rates.

Par	nel 1	0.4	0.5	0.6	A_0 / F_0 0.7	0.8	0.9	1.0
	0.2	0.00034	0.00188	0.00521	0.00989	0.01503	0.01996	0.02435
	0.3	0.00191	0.01475	0.05350	0.12492	0.22256	0.33287	0.44312
D_0 / V_0	0.4	0.00271	0.02591	0.11199	0.31393	0.70672	1.34859	2.17590
	0.5	0.00231	0.02586	0.12656	0.40469	1.11232	2.68746	5.28879
				σA				
Par	nel 2	0.06	0.07	0.08	0.09	0.10		
	1.2	0.01141	0.01153	0.01166	0.01180	0.01195		
	1.4	0.03908	0.03956	0.04009	0.04068	0.04131		
k	1.6	0.08533	0.08656	0.08793	0.08946	0.09113		
	1.8	0.14562	0.14812	0.15095	0.15411	0.15759		
	2.0	0.21212	0.21662	0.22179	0.22763	0.23412		
	1.0			$\sigma_{F,1}$				
Panel 3		0.11	0.12	0.13	0.14	0.15		
	0.27	0.17746	0.17899	0.18112	0.18392	0.18743		
	0.28	0.19743	0.19896	0.20109	0.20389	0.20741		
$\sigma_{F,2}$	0.29	0.21813	0.21966	0.22179	0.22459	0.22810		
	0.30	0.23949	0.24102	0.24315	0.24595	0.24946		
	0.31	0.26145	0.26298	0.26511	0.26791	0.27143		
D	-1.4			$\sigma_{V,1}$				
Panel 4		0.14	0.15	0.16	0.17	0.18		
	0.30	0.20317	0.20447	0.20588	0.20735	0.20883		
	0.31	0.21145	0.21275	0.21416	0.21563	0.21711		
$\sigma_{V,2}$	0.32	0.21908	0.22038	0.22179	0.22326	0.22474		
	0.33	0.22606	0.22735	0.22876	0.23023	0.23172		
	0.34	0.23237	0.23367	0.23508	0.23655	0.23803		
D 1 F				$\sigma_{D,1}$				
Panel 5		0.010	0.015	0.020	0.025	0.030		
	0.040	0.22122	0.22133	0.22149	0.22167	0.22188		
	0.045	0.22138	0.22148	0.22164	0.22183	0.22203		
$\sigma_{D,2}$	0.050	0.22153	0.22163	0.22179	0.22198	0.22218		
	0.055	0.22167	0.22177	0.22193	0.22211	0.22232		
	0.060	0.22179	0.22190	0.22206	0.22224	0.22245		

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Panel 2 of Tables 5 and 6 is the sensitivity analysis for intervention termination. The insurance value increases as the limit of intervention rises, because as the limit of intervention rises, it is easier for PBGC to intervene and terminate the contract, take over pension benefits early, and bear losses, leading to a higher insurance value. With regard to Panel 1, the ratio of the present value of accrued benefits to the present value of pension benefits and other degrees of volatility in relation to the trend of insurance value, the explanations are the same as those for distress termination.

6. Conclusions

This paper considers two market states and proposes a regime-switching process model for pension benefits. In the theoretical part, this paper first converts pension benefits, accrued benefits, and the fair value of a firm's assets and liabilities into the risk-neutral probability measures and the forward probability measures. Then, we derive insurance pricing formulae for distress termination and intervention of contracts by a PBGC under constant interest rates. Next, considering stochastic interest rates, we derive insurance pricing formulae for distress termination and intervention under the forward measure.

In the empirical part, we take the S&P 500 index and one-year treasury bills, use the EM algorithm in different algorithms to maximize the likelihood function, and estimate the model's parameters under the regime-switching process. We use likelihood value, AIC, and BIC principle for model selection, and we find that the EM-PSO-Gradient can make the parameter estimates more accurate.

Finally, we conduct sensitivity analysis on the regime-switching model's evaluation of insurance value under a PBGC's distress termination and intervention termination of the contracts. The results show that when adjusting one parameter value and keeping other parameters unchanged, the insurance value rises under the following circumstances: the ratio of the present value of accrued benefits to the present value of pension benefits $\frac{A_0}{F_0}$ rises; the ratio of the present value of the firm's liabilities to the fair present value of the firm's assets $\frac{D_0}{V_0}$ rises; and the limit of intervention and the fluctuation degree of the model rise. Because pension benefits may become more and more inadequate to cover accrued benefits in the future, the company's financial situation will worsen. The early takeover of pension funds may occur due to a higher level of intervention, and there will be more significant uncertainties.

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Appendix A. Change of Measure

Under the assumption of no arbitrage, we can change the probability measure to an equivalent risk-neutral measure. First, we define β_t as the money market account (MMA), and the saving process is growth with a risk-free rate r,

$$\frac{d\beta_t}{\beta_t} = rdt$$

Under the risk-neutral measure, the dynamic process of any tradable asset with MMA as a numeraire will be a martingale. Therefore, at any time *t*, the dynamic of the pension benefit relative to MMA without considering net worth follows the martingale condition

$$\frac{F_t}{\beta_t} = E_t^Q \left(\frac{F_\tau}{\beta_\tau}\right), \ \tau \ge t$$

It can be further expressed as

$$d\frac{F_t}{\beta_t} = \frac{F_t}{\beta_t} \left(-r + \alpha_{F,q_t} \right) dt + \frac{F_t}{\beta_t} \sigma_{F,q_t} dW_F$$

According to the stochastic differential equation of pension asset and MMA, we can obtain the relation between the Brownian motions under the physical measure and the risk-neutral measure through the martingale condition. We use the property that the draft term of the relative process is zero.

$$dW_F^Q = dW_F + rac{-r + lpha_{F,q_1}}{\sigma_{F,q_t}} dt$$

Substituting the Brownian motions to the pension benefit under the regime-switching model, we can obtain the pension benefit dynamic process under the risk-neutral measure as

$$\frac{dF_t}{F_t} = (C_{F,q_t} + r)dt + \sigma_{F,q_1}dW_F^Q, q_t = 1, 2$$

By Ito's lemma, the stochastic differential equation of the pension benefit leads to the relation of the value between the times (t, T)

$$F(T) = F(t) \exp \left\{ \left(C_{F,q_t} + r - 0.5\sigma_{F,q_t}^2 \right) (T-t) + \sigma_{(F,q_t)} W_F^Q(T-t) \right\}.$$

Similarly, the risk-neutral measure, the accrued benefits, and the firm's assets and liabilities should be martingale process when taking MMA as a numeraire. Then, we can derive the martingale condition through the zero drafts to obtain the Brownian motions under the risk-neutral measure as

$$dW_A^Q = dW_A + \frac{-r + \alpha_A}{\sigma_A} dt, \ dW_V^Q = dW_V + \frac{-r + \alpha_{V,q_t}}{\sigma_{V,q_t}} dt, \text{ and } dW_D^Q = dW_D + \frac{-r + \alpha_{D,q_t}}{\sigma_{D,q_t}} dt.$$

Then we have the dynamics under the risk-neutral measure

$$\frac{dA_t}{A_t} = (C_A + r)dt + \sigma_A dW^Q_A,$$
$$\frac{dV_t}{V_t} = rdt + \sigma_{V,q_t} dW^Q_V,$$
$$\frac{dD_t}{D_t} = rdt + \sigma_{D,q_t} dW^Q_D,$$

and the value between the times (t, T)

$$\begin{split} A(T) &= A(t) \exp \left\{ \left(C_A + r - 0.5 \sigma_A^2 \right) (T - t) + \sigma_A W_A^Q (T - t) \right\}, \\ V(T) &= V(t) \exp \left\{ \left(r - 0.5 \sigma_{V,q_t}^2 \right) (T - t) + \sigma_{V,q_t} W_V^Q (T - t) \right\}, \\ D(T) &= D(t) \exp \left\{ \left(r - 0.5 \sigma_{D,q_t}^2 \right) (T - t) + \sigma_{D,q_t} W_D^Q (T - t) \right\}. \end{split}$$

When we extend the model to consider the interest rate risk, the stochastic interest rate brings uncertainty to the saving process, so the MMA is not an appropriate numeraire under the risk-neutral measure. To avoid the interest rate risk, we can use the ZCB as a numeraire to determine the value of total interest rates. The measure that any asset using ZCB as numeraire and that follows a martingale is called a 'forward measure', Q^T .

We build the dynamic process of zero-coupon bonds so that the dynamic of the pension benefit relative to ZCB under the physical measure will be

$$\frac{d\frac{F_t}{B_t}}{\frac{F_t}{B_t}} = \left(\alpha_{F,q_t} - \mu_{B,q_t} + \sigma_{B,q_t}^2 - \rho_{FB,q_t}\sigma_{F,q_t}\sigma_{B,q_t}\right)dt + \sigma_{FB,q_t}dW_{FB},$$

According to the martingale condition, the drift term of the dynamic should be zero, and we can obtain the change the measure from the physical measure to the forward measure through the relation of Brownian motions as

$$dW_{FB}^{Q^T} = dW_{FB} + rac{lpha_{F,q_t} - \mu_{B,q_t} + \sigma_{B,q_t}^2 -
ho_{FB,q_t}\sigma_{F,q_t}\sigma_{B,q_t}}{\sigma_{FB,q_t}}dt.$$

Given the Brownian motions under the forward measure, we are equivalent having the dynamics process of pension benefits, and we further apply the Ito's lemma to obtain the relation of the pension benefits between the times (t, T) under the measure Q^T .

$$F(T) = F(t) \exp\left\{ \left(C_F, q_t + \alpha_{F,q_t} - h_{F,q_t} - 0.5\sigma_{F,q_t}^2 \right) (T-t) + \sigma_{F,q_t} W_G^{Q^T}(T-t) \right\},\$$

where $h_{F,q_t} = \frac{\sigma_{F,q_t} \left(\alpha_{F,q_t} - \mu_{(B,q_t)} + \sigma_{B,q_t}^2 - \rho_{FB,q_t} \sigma_{F,q_t} \sigma_{B,q_t} \right)}{\sigma_{FB,q_t}}.$

Similarly, we can obtain the value of accrued benefits, firm's assets, and firm's liabilities in the time (t, T) under the forward measure

$$A(T) = A(t) \exp\left\{ \left(C_A + \alpha_A - h_{A,q_t} - 0.5\sigma_A^2 \right) (T-t) + \sigma_A W_A^{Q^T}(T-t) \right\},\$$

$$V(T) = V(t) \exp\left\{ \left(\alpha_{V,q_t} - h_{V,q_t} - 0.5\sigma_{V,q_t}^2 \right) (T-t) + \sigma_{V,q_t} W_V^{Q^T}(T-t) \right\},\$$

$$D(T) = D(t) \exp\left\{ \left(\alpha_{D,q_t} - h_{D,q_t} - 0.5\sigma_{D,q_t}^2 \right) (T-t) + \sigma_{D,q_t} W_D^{Q^T}(T-t) \right\},\$$
where $h_{A,q_t} = \frac{\sigma_A \left(\alpha_A - \mu_{B,q_t} + \sigma_{B,q_t}^2 - \rho_{AB,q_t} \sigma_A \sigma_{B,q_t} \right)}{\sigma_{AB,q_t}},\$

$$h_{V,q_t} = \frac{\sigma_{V,q_t} \left(\alpha_{V,q_t} - \mu_{B,q_t} + \sigma_{B,q_t}^2 - \rho_{VB,q_t} \sigma_{V,q_t} \sigma_{B,q_t} \right)}{\sigma_{VB,q_t}},$$

$$h_{D,q_t} = \frac{\sigma_{D,q_t} \left(\alpha_{D,q_t} - \mu_{B,q_t} + \sigma_{B,q_t}^2 - \rho_{VB,q_t} \sigma_{V,q_t} \sigma_{B,q_t} \right)}{\sigma_{DB,q_t}},$$

Appendix B. The Conditional Expected Value of the Log-Likelihood Function

Assuming that the return of pension benefits is $\widetilde{R_F} = \{R_{F_1}, R_{F_2}, \dots, R_{F_T}\}$; the return of accrued benefits is $\widetilde{R_A} = \{R_{A_1}, R_{A_2}, \dots, R_{A_T}\}$; the return of the fair value of a firm's assets is $\widetilde{R_V} = \{R_{V_1}, R_{V_2}, \dots, R_{V_T}\}$; the return of company liabilities is $\widetilde{R_D} = \{R_{D_1}, R_{D_2}, \dots, R_{D_T}\}$; the return of ZCB is $\widetilde{R_B} = \{R_{B_1}, R_{B_2}, \dots, R_{B_T}\}$; and the market regime is $\widetilde{q} = \{q_1, q_2, \dots, q_T\}$ $q_i \in \{1, 2\}$.

Among which, the probability value is $0 \le \pi_1$, p_{11} , $p_{22} \le 1$; the parameter of the growth rate of the dynamic process are $-\infty \le C_{\{F\},\{1,2\}}, C_A, \alpha_{\{F,V,D\},\{1,2\}}, \alpha_A \le \infty$; the fluctuation is positive, $\sigma_{\{F,V,D\},\{1,2\}}, \sigma_A > 0$; and the dynamic correlation coefficient is in $[-1, 1], -1 \le \rho_{\{AF,FV,DF,BF,AD,AV,AB,DV,BD,BV\},\{1,2\}} \le 1$.

Then, the complete-data likelihood function under regime-switching is represented as

$$L_{C}\left(\theta|\widetilde{R_{F}},\widetilde{R_{A}},\widetilde{R_{V}},\widetilde{R_{D}},\widetilde{R_{B}},\widetilde{q}\right) = \prod_{t=1}^{T} P\left(R_{F,t},R_{A,t},R_{V,t}R_{D,t},\widetilde{R_{B}},q_{t}|\theta\right)$$
$$= \pi_{q_{1}}\prod_{t=2}^{T} P_{q_{t-1},q_{t}}\left(\prod_{t=1}^{T} P(R_{F,t},R_{A,t},R_{V,t},R_{D,t},R_{B,t}|q_{t},\theta)\right)$$

where θ is the set of parameters.

In the market, the only observation is return data, while the market regime is a hidden variable, so we cannot directly use the complete-data likelihood function to estimate the parameter. We define the incomplete-data likelihood function for the market regime-switching model as $L_{IC}\left(\Theta | \widetilde{R_F}, \widetilde{R_A}, \widetilde{R_V}, \widetilde{R_D}, \widetilde{R_B}\right)$, which is represented as

$$L_{IC}\left(\theta|\widetilde{R_{F}},\widetilde{R_{A}},\widetilde{R_{V}},\widetilde{R_{D}},\widetilde{R_{B}}\right) = \prod_{t=1}^{T} \sum_{q_{t}=1}^{2} L\left(\theta|\widetilde{R_{F}},\widetilde{R_{A}},\widetilde{R_{V}},\widetilde{R_{D}},\widetilde{R_{B}},\widetilde{q}\right)$$
$$= \sum_{q_{1}}^{2} \pi_{q_{1}} \prod_{t=2}^{T} \Sigma_{q_{t}=1}^{2} P(R_{F,t},R_{A,t},R_{V,t},R_{D,t},R_{B,t}|q_{t},\theta)$$

The EM algorithm consists of Step E and Step M. At the k^{th} iteration. Step E gives observable data and the parameter values estimated in the $(k - 1)^{\text{th}}$ iteration. The conditional expected value of the log-likelihood function for calculating the data is:

$$\begin{aligned} &Q\left(\theta^{(k)}|\theta^{(k-1)}\right) \\ &= E\left(log L_{C}\left(\Theta|\widetilde{R_{F}},\widetilde{R_{A}},\widetilde{R_{V}},\widetilde{R_{D}},\widetilde{R_{B}},q\right)|\widetilde{R_{F}},\widetilde{R_{A}},\widetilde{R_{V}},\widetilde{R_{D}},\widetilde{R_{B}},\theta^{(k-1)}\right) \\ &= \Sigma_{i=1}^{2}log \pi_{i} P\left(q_{1}=i|\widetilde{R_{F}},\widetilde{R_{A}},\widetilde{R_{V}},\widetilde{R_{D}},\widetilde{R_{B}},\theta^{(k-1)}\right) \\ &+ \Sigma_{i=1}^{2}\Sigma_{j=1}^{2}\Sigma_{t=2}^{T}log p_{ij} P\left(q_{t-1}=i,q_{t}=j|\widetilde{R_{F}},\widetilde{R_{A}},\widetilde{R_{V}},\widetilde{R_{D}},\widetilde{R_{B}},\theta^{(k-1)}\right) \\ &+ \Sigma_{i=1}^{2}\Sigma_{t=1}^{T}\left[-\frac{1}{2}log(2\pi^{4}|\Sigma_{t}|) - \frac{1}{2}(r_{t}-\mu_{t})'\Sigma_{t}^{-1}(r_{t}-\mu_{t})\right] P\left(q_{t}=j|\widetilde{R_{F}},\widetilde{R_{A}},\widetilde{R_{V}},\widetilde{R_{D}},\widetilde{R_{B}},\theta^{(k-1)}\right) \end{aligned}$$

where

$$\mu_{t} = \begin{bmatrix} C_{F,q_{t}} + \alpha_{F,q_{t}} - 0.5\sigma_{F,q_{t}}^{2} \\ C_{A} + \alpha_{A} - 0.5\sigma_{A}^{2}, \alpha_{V,q_{t}} - 0.5\sigma_{V,q_{t}}^{2} \end{bmatrix}, \Sigma_{t} = \begin{bmatrix} \sigma_{F,q_{t}}^{2} & \sigma_{AF,q_{t}} & \sigma_{FV,q_{t}} & \sigma_{DF,q_{t}} & \sigma_{FB,q_{t}} \\ \sigma_{AF,q_{t}} & \sigma_{A}^{2} & \sigma_{AV,q_{t}} & \sigma_{AD,q_{t}} & \sigma_{AB,q_{t}} \\ \sigma_{FV,q_{t}} & \sigma_{AV,q_{t}} & \sigma_{V,q_{t}}^{2} & \sigma_{DV,q_{t}} & \sigma_{VB,q_{t}} \\ \sigma_{DF,q_{t}} & \sigma_{AD,q_{t}} & \sigma_{DV,q_{t}} & \sigma_{DB,q_{t}} \\ \sigma_{FB,q_{t}} & \sigma_{AB,q_{t}} & \sigma_{VB,q_{t}} & \sigma_{DB,q_{t}} & \sigma_{B,q_{t}} \end{bmatrix}$$

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