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**Pricing the Bund term structure
with linear regressions – without
an observable short rate**

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Non-technical summary

Research Question

Long-term interest rates on government bonds are just as relevant for investors' strategies as they are for analyzing macroeconomic developments or monetary policy decisions. For example, it is currently important for European monetary policymakers to know whether long-term interest rates have risen recently because market participants are expecting further policy rate hikes or whether an anticipated faster reduction in the Eurosystem's bond stock increasing the risk premium on long-term bonds.

Term structure models can decompose the data in this manner. Recent years have seen the development of statistical methods that use simple linear regressions to determine the structural parameters of a term structure model. They replace time-consuming, non-linear optimization. However, the existing linear regression approaches require a specific data structure, in particular an observable short-term interest rate. Although these requirements are met for US Treasury bonds, even German Bunds do not have an observable short-term interest rate.

Contribution

I develop two new linear regression approaches – called Difference Estimators – which determine the structural parameters of a term structure model *without* an observable short-term interest rate. The basic idea is to implicitly derive short-term interest rates from longer-term interest rates. I apply the new Difference Estimators to the term structure of German Bunds and determine interest rate expectations and risk premia for a history starting in 1967.

Results

A term structure model with three factors is sufficient to describe the dynamics of the German Bund yield curve since 1967. The expectation component benefits from additional constraints in the model that increase the persistence of interest rates. The implied short-term Bund rates derived from the Difference Estimators display plausible dynamics compared to central bank policy rates. Risk premia also have desirable properties: They move contrary to the economic cycle and are positively related to uncertainty measures. The long history allows the interest rate components to be classified in the macroeconomic environment. Currently, the comparison with the 1970s is particularly relevant. At that time, high supply-driven inflation, as a result of the oil crises, increased risk premia. In the 1980s, inflation was on the decline but the risk premium remained high initially. It took time to convince investors that the successful inflation reduction was permanent. At present, risk premia are not rising significantly, in spite of the high inflation. A clear stability-oriented monetary policy in the euro area can prevent higher (inflation) risk premia in the future and thus lay the basis for favorable financing conditions.

Nichttechnische Zusammenfassung

Fragestellung

Langfristige Zinsen von Staatsanleihen sind für Strategien von Investoren ebenso relevant wie für die Analyse gesamtwirtschaftlicher Entwicklungen oder geldpolitischer Entscheidungen. Beispielsweise ist es gegenwärtig für die Geldpolitik im Euroraum wichtig zu wissen, ob langfristige Zinsen zuletzt deutlich gestiegen sind, weil Marktteilnehmer weitere Leitzinserhöhungen erwarten oder ob ein antizipierter schnellerer Abbau des geldpolitischen Anleiheportfolios die Risikoprämie langlaufender Anleihen steigen lässt.

Zinsstrukturkurvenmodelle erlauben eine solche Zerlegung. In den letzten Jahren wurden statistische Methoden entwickelt, die mit Hilfe einfacher linearer Regressionen die strukturellen Parameter eines Zinsstrukturmodells bestimmen. Sie ersetzen damit eine aufwändige, nichtlineare Optimierung. Die bestehenden linearen Regressionsansätze stellen jedoch starke Anforderungen an die Datenstruktur, insbesondere muss ein Kurzfristzins vorhanden sein. Diese Anforderungen sind zwar für US Staatsanleihen erfüllt, aber selbst bei deutschen Bundeswertpapieren fehlt ein beobachtbarer Kurzfristzins.

Beitrag

Ich entwickle zwei neue lineare Regressionsansätze – genannt Differenzenschätzer – die *ohne* einen beobachtbaren Kurzfristzins die strukturellen Parameter eines Zinsstrukturmodells bestimmen. Die Grundidee besteht darin, die Kurzfristzinsen implizit aus längerfristigen Zinssätzen abzuleiten. Ich wende die neuen Differenzenschätzer auf die Zinsstruktur von Bundeswertpapieren an und bestimme Zinserwartungen und Risikoprämien für eine 1967 beginnende Historie.

Ergebnisse

Ein Zinsstrukturmodell mit drei Faktoren reicht aus, um die Dynamik der deutschen Zinsstrukturkurve seit 1967 zu beschreiben. Die Zinserwartungskomponente profitiert von zusätzlichen Beschränkungen im Modell, welche die Persistenz der Zinsen erhöhen. Gemessen an der Leitzinsentwicklung weisen die impliziten Kurzfristzinsen deutscher Bundeswertpapiere aus den Differenzenschätzern eine plausible Dynamik auf. Auch die hergeleiteten Risikoprämien weisen wünschenswerte Eigenschaften auf: Sie verlaufen dem Wirtschaftszyklus entgegengesetzt und sind hoch in Zeiten großer Unsicherheit. Die lange Historie erlaubt eine Einordnung der Zinskomponenten in das makroökonomische Umfeld. Gegenwärtig ist insbesondere der Vergleich mit den 1970er Jahren relevant. Damals steigerte die hohe angebotsgetriebene Inflation als Folge der Ölkrisen auch die Risikoprämien. Der Rückgang der Inflation in den 1980er Jahren konnte nur allmählich bei Investoren glaubwürdig verankert werden – die Risikoprämie blieb zunächst hoch. Aktuell steigen die Risikoprämien trotz hoher Inflation nicht deutlich an. Eine klare Stabilitätsorientierung der Geldpolitik im Euroraum kann einen Anstieg der (Inflations)-Risikoprämien auch künftig verhindern und damit die Grundlage für günstige Finanzierungsbedingungen legen.

Pricing the Bund term structure with linear regressions – without an observable short rate

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Abstract

Affine term structure models of bond yields are important tools for analyzing fixed income markets and monetary policy. Estimators of [Adrian, Crump, and Mönch \(2013\)](#) and [Diez de Los Rios \(2015\)](#) replace time-consuming nonlinear search procedures with a set of simple linear regressions. However, these estimators require an observable short rate, which is not available at a one-month maturity for many markets, including that for German Bunds. This paper introduces new regression-based Difference Estimators that require no observable short rate but are determined from long-term yields only. My new estimators replicate results of the traditional estimators for US Treasuries, although my approaches omit the available US short rate. For German Bund data since 1967, three factors are sufficient to represent the yield curve dynamics, and additional restrictions improve yield forecasts. Implicit short Bund rates are in line with policy interest rates. The term premium is counter-cyclical to business activity and positively related to uncertainty. Thus, the Bund yield decomposition into short-rate expectations and the term premium is suitable for a wide range of policy applications.

Keywords: Yield Curve, Affine Term Structure Model, Bund Term Premium

JEL classification: C13, E43, G12.

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1 Introduction

Affine term structure models are important tools for the analysis of fixed-income markets. They decompose long-term government bond yields into expected short-term yields and term premia and thus allow different channels of monetary policy to be identified. Conventional policy rate adjustments and forward guidance are supposed to drive rate expectations, while asset purchase programs are designed to affect the term premium.¹ Affine models have also been applied to inflation-indexed markets to derive inflation expectations or in a multi-country setting to investigate international linkages of monetary policy.

Affine term structure models take the form of a state space-model with nonlinear parameters. Standard filtering techniques and maximum likelihood estimation are therefore the natural choice for their estimation. However, those approaches used to be time-consuming and prone to local minima. For example, [Duffee \(2002, p. 418\)](#) states for the US Treasury market that “general three-factor affine models are already computationally difficult to estimate [with maximum likelihood] owing to the number of parameters. Adding another factor would make this investigation impractical.” Increasing computing power, alternative model formulations, in particular [Joslin, Singleton, and Zhu \(2011, JSZ\)](#), and novel estimation techniques like [Hamilton and Wu \(2012\)](#) allowed the computational burden of maximum likelihood estimation to be reduced.

More recently, new approaches avoid numerical optimization altogether. [Adrian et al. \(2013, ACM\)](#) and [Diez de Los Rios \(2015, DLR\)](#) estimate an affine Gaussian term structure model with observable factors using a set of linear regressions.² Without nonlinear optimizations, these estimators complete an estimation in milliseconds. This allows standard errors to be bootstrapped or extensive searches to be conducted for parameter restrictions, as in [Golinski and Spencer \(2018\)](#). Despite their elegance and numerical performance, the existing linear regression approaches of DLR and ACM require a specific data structure: Most importantly, they require an observable short rate whose maturity determines the data frequency.³ Thus, the estimation with existing linear regressions on a monthly frequency requires a one-month yield. By contrast, maximum likelihood approaches do *not* require an observable one-month rate but can be estimated from long-term yields only.

An observable one-month short rate is not available in many markets, including that for German Bunds. The German Bund yield curve published by the Bundesbank – Bbk in the remainder – is interpolated with a [Svensson \(1994\)](#) model from all traded German government bonds with a residual maturity of more than three months since 1972.⁴ There-

¹Policy applications of term structure models in the euro area include [De Backer, Schupp, and Vladu \(2021\)](#), [Deutsche Bundesbank \(2023\)](#) and [Horny, Sabes, and Sahuc \(2018\)](#).

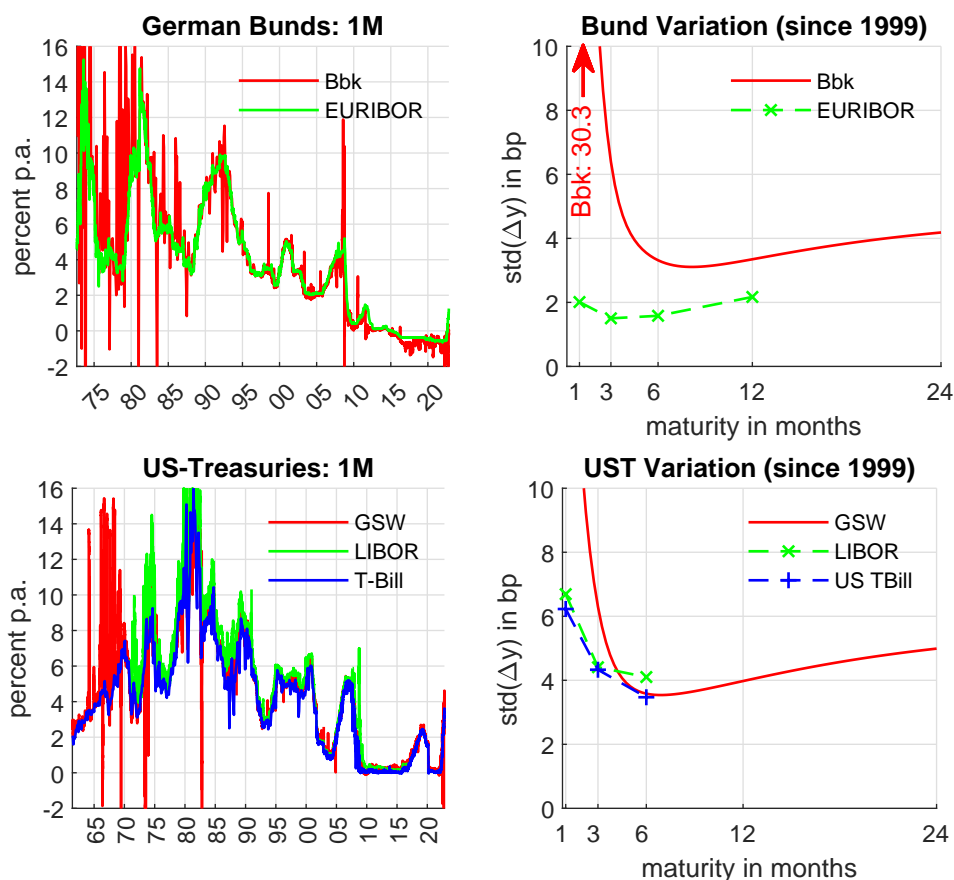
²Extensions to multiple curves are [Abrahams, Adrian, Crump, and Mönch \(2013\)](#) with inflation-protected US Treasuries and [Diez de Los Rios \(2017\)](#) for a joint term structure model with seven countries.

³A second requirement are consecutive maturities (i.e. for monthly data frequency 60- and 61-months yield maturities) which does not constitute an effective restriction in the interpolated government bond yield datasets studied in this paper.

⁴Money market papers (Bubills) are not included even if their residual maturity exceeds three months. See [Schich \(1997\)](#) for details of the estimation. The data are available for download on the Bundesbank webpage <https://www.bundesbank.de/en/statistics/money-and-capital-markets/interest-rates-and-yields/term-structure-of-interest-rates>.

Figure 1: Short-Term Yield Characteristics

Last observation October 2022. First observation September 1972 (Germany) and June 1961 (USA).



fore, the one-month rate is extrapolated and observes irregular movements with spikes of implausibly large negative or positive values, as illustrated in Figure 1. As a result, yields with a maturity below six months have a high volatility that reflects extrapolation noise rather than economic fluctuations. The yield curve for US Treasuries constructed by [Gürkaynak, Sack, and Wright \(2007\)](#) – GSW in the remainder – imposes a minimum maturity of three months to estimate the interpolation of [Svensson \(1994\)](#). The lower row in Figure 1 highlights that extrapolation noise is present in the GSW dataset as well.⁵ Since the implementation of a minimum maturity is standard in the field of government yield curve construction, see [BIS \(2005, Tab. 1\)](#), an extrapolated and noisy one-month government bond yield is a stylized fact of government bond yield curve datasets.

Researchers are well aware of the problems with the extrapolated one-month yield from GSW or Bbk. They try to circumvent them by adapting the data: For US Treasury curves, many authors use the one-month Treasury Bill rate (T-Bill), including [Adrian](#)

⁵GSW exclude on-the-run bonds and apply other data-cleaning procedures to get a more homogenous bond dataset that enters the interpolation. Furthermore, more frequent issues (and expirations) of US Treasuries compared to Bunds ensure a maturity of the shortest bond close to three months in GSW which reduces the extrapolation magnitude. The data are available for download on the Federal Reserve Board webpage <https://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html>.

et al. (2013). However, T-Bills experience jumps in Figure 1 as well.⁶ Amihud and Mendelson (1991) or Lenel, Piazzesi, and Schneider (2019) find a specialness premium of T-Bills over Treasury bonds. For Germany, the Bubill issuance policy does not allow a one-month rate to be constructed for a long history. Furthermore Brand, Ferrante, and Hubert (2019) document a specialness premia for German Bubills on the specific repo market. Lemke and Werner (2020) correct for outliers in the one-month Bbk yield by interpolating between the previous and the following months' the observation of the 1M Bbk yield. Golinski and Spencer (2018) use interbank rates to proxy the one-month Bund yield which creates heterogeneity in the German curve from interbank default risk and missing Bund specialness premia.⁷ Lemke and Vladu (2017), Horny et al. (2018), Geiger and Schupp (2018) or Berardi and Plazzi (2022) avoid the Bund curve and study the euro-area term structure with overnight index swaps.⁸ Interest rate swap data are available with a granular maturity spectrum for short maturities including an observable one-month rate. However, swaps with a long maturity are unreliable in the first years of the European monetary union. The Canadian yields used in Diez de Los Rios (2015) are available for maturities in three-month maturity steps only.⁹ As a consequence, he estimates his model at a quarterly but not a monthly frequency. Thus, the available maturity spectrum for Canadian data reduced the data frequency and increased the step size for the short-rate expectation path to three months. Overall, the data structure of the US Treasury market – with an observable one-month yield – consistent with the data requirements of ACM and DLR on a monthly frequency, seems to be the exception rather than the rule.

The major contribution of this paper is to develop new linear regression estimators that adapt to an imperfect structure of the dataset under investigation. Forming differences between two yields or returns of *long-term* bonds eliminates the dependence on the short rate: With the new estimators – called Yield Difference estimator (YD) and Return Difference estimator (RD) – one can estimate a term structure model from *long-term* yields at a monthly frequency without actually observing a one-month rate. This enables an estimation without extrapolation noise or a short rate that is inconsistent with the rest of the yield curve. Thus, the new regression approaches can be estimated without an observable short rate, and there is no reason to prefer a maximum likelihood estimation over Difference Estimators.

I benchmark the new Difference Estimators against the traditional approaches of DLR and ACM for the US Treasury curve which contain an observable one-month T-Bill rate. The goal of this exercise is to investigate whether the Difference Estimators achieve sim-

⁶The 1M data from the Federal Reserve Economic Database (FRED) used in this paper are the most recently issued 4-week T-Bill. Between the weekly issues of these papers, the maturity declines and the change in the underlying Bill may lead to rate jumps. The data are available for download on the FRED webpage <https://fred.stlouisfed.org/series/TB4WK>.

⁷The early paper of Cassola and Luis (2003) uses Bbk data from 1972 to 1998 with a maximum likelihood estimation. They assume non-zero measurement error for factors and use a minimum maturity of one year (which has a relatively large fitting error). A similar estimation approach is chosen by Fendel (2008).

⁸The bulletin article of Horny et al. (2018) is a policy application of Monfort, Pegoraro, Renne, and Roussellet (2017) to the euro area.

⁹See Bolder, Johnson, and Metzler (2004) for the data construction with exponential splines of Vasicek and Fong (1982). The data are available for download on the Bank of Canada webpage <http://www.bankofcanada.ca/rates/interest-rates/bond-yield-curves/>.

ilar results to the traditional estimators even if the available short rate is not used by the Difference Estimators. There are three major findings: (1) Although Difference Estimators omit the short rate, they are able to derive plausible implicit one-month short rates for the US and reproduce the short-rate expectation component published by the Federal Reserve Bank of New York based on ACM. (2) The short rate should be excluded from the factors driving the model dynamics if it is noisy or inconsistent with the rest of the curve. (3) Short-rate parameters should be determined implicitly from long-term yields but not from noisy observed short rates. Overall, there is no reason to prefer traditional approaches that include a short rate like ACM and DLR over the new Difference Estimators for US data. Moreover, traditional linear regression approaches yield robust results even for extrapolated and noisy short rates if factors and short-rate parameters are determined without the short rate, i.e. if results (2) and (3) are respected.

The empirical contribution of this paper is the longest available history of yield decompositions for German Bunds into expected short rates and term premia starting in 1967. Three factors are sufficient to represent the yield curve dynamics. In particular, restrictions that increase the time-series persistence of the yield dynamics improve yield forecasts. Model-implied Bund short rates are in line with Bundesbank and ECB policy rates. The term premium is counter-cyclical to business activity and positively related to uncertainty. With these desirable properties, the model decomposition of Bund yields into short-rate expectations and the term premium is suitable for a wide range of policy applications. Given the current record-high inflation caused by supply shocks, a comparison with similar periods in the 1970s and early 1980s from my long sample is particularly valuable. Currently, term premia remain low despite the high inflation. However, a stability-oriented monetary policy in the euro area is needed to reduce inflation and thus ensure low (inflation) risk premia and favorable financing conditions.

The remainder of the paper is structured as follows. Section 2 reviews the affine model structure and illustrates the role of the observable short rate in traditional linear regression estimators. Readers familiar with these models can skip this section and proceed directly to the derivation of the new Yield Difference estimator in section 3 that is independent of an observable short rate. The empirical application to German Bunds is contained in section 5 after I show for US Treasuries in section 4, that my new Difference Estimators replicate traditional estimates even if they omit an available short rate. Section 6 concludes.

2 Affine Term Structure Models

This section reviews the model structure of an affine term structure model with Gaussian innovations (section 2.1) and the estimation by maximum likelihood (section 2.2). Lastly, the linear regression estimators of DLR and ACM are introduced with a focus on the role of the short rate (section 2.3). This creates the starting point for the development of the new Difference Estimators in section 3.

2.1 Model Structure

An affine term structure model is characterized by a log price p_t^n of a zero-coupon bond with maturity n that is an affine function of K factors \mathcal{P}_t and measurement errors v_t^n :

$$p_t^n = \bar{A}_{\mathcal{P}}^n + \bar{B}_{\mathcal{P}}^n \mathcal{P}_t + \bar{v}_t^n \quad (1)$$

Duffie and Kan (1996) show that affine log prices require an affine process for the risk-neutral dynamics to ensure no-arbitrage:¹⁰

$$\mathcal{P}_{t+1} = \nu_{\mathbb{Q}\mathcal{P}} + \Phi_{\mathbb{Q}\mathcal{P}} \mathcal{P}_t + \varepsilon_{t+1}^{\mathbb{Q}\mathcal{P}} \quad (2)$$

$$\varepsilon_{t+1}^{\mathbb{Q}\mathcal{P}} \sim N(0, \Omega_{\mathcal{P}}) \quad (3)$$

The risk-neutral parameters $\Phi_{\mathbb{Q}\mathcal{P}}$ and $\nu_{\mathbb{Q}\mathcal{P}}$ as well as the error covariance $\Omega_{\mathcal{P}}$ are structural parameters that need to be estimated from the cross-section of log prices (1).

No-arbitrage requires a consistent pricing of long-term zero-coupon bonds with a maturity n and the short-term bond with a maturity that corresponds to the data frequency. These restrictions imply a recursive form of the parameters $\bar{A}_{\mathcal{P}}^n$ and $\bar{B}_{\mathcal{P}}^n$ from the log price equation (1):¹¹

$$\bar{B}_{\mathcal{P}}^{n+1} = \bar{B}_{\mathcal{P}}^1 + \bar{B}_{\mathcal{P}}^n \Phi_{\mathbb{Q}\mathcal{P}} \quad (4)$$

$$\bar{A}_{\mathcal{P}}^{n+1} = \bar{A}_{\mathcal{P}}^1 + \bar{A}_{\mathcal{P}}^n + \bar{B}_{\mathcal{P}}^n \nu_{\mathbb{Q}\mathcal{P}} + \frac{1}{2} \bar{B}_{\mathcal{P}}^n \Omega_{\mathcal{P}} (\bar{B}_{\mathcal{P}}^n)' \quad (5)$$

The boundary conditions $\bar{A}_{\mathcal{P}}^0 = 0$ and $\bar{B}_{\mathcal{P}}^0 = 0$ ensure full repayment at maturity ($p_t^0 = 0$) independent of the state of the economy represented by \mathcal{P}_t . For default-free (Treasury) bonds, the short-rate constant $\bar{A}_{\mathcal{P}}^1$ and loadings $\bar{B}_{\mathcal{P}}^1$ are parameters of the model that need to be determined.

I follow Dai and Singleton (2000) and Duffee (2002) who find Gaussian time-series \mathbb{P} -dynamics are best suited to close the model:

$$\mathcal{P}_{t+1} = \nu_{\mathbb{P}\mathcal{P}} + \Phi_{\mathbb{P}\mathcal{P}} \mathcal{P}_t + \varepsilon_{t+1}^{\mathbb{P}\mathcal{P}} \quad (6)$$

$$\varepsilon_{t+1}^{\mathbb{P}\mathcal{P}} \sim N(0, \Omega_{\mathcal{P}}) \quad (7)$$

The variation $\Omega_{\mathcal{P}}$ of the one-period innovations $\varepsilon_t^{\mathbb{P}\mathcal{P}}$ is identical to the variation $\Omega_{\mathcal{P}}$ in equation (3). The error covariance $\Omega_{\mathcal{P}}$ creates a link between the time-series dynamics and the constant $\bar{A}_{\mathcal{P}}^n$ of the log bond price.

The recursion (4) of the loadings $\bar{B}_{\mathcal{P}}^n$ of the log bond prices depends only on the risk-neutral $\Phi_{\mathbb{Q}\mathcal{P}}$ but not on the time-series dynamics $\Phi_{\mathbb{P}\mathcal{P}}$. This is a prerequisite to estimate $\Phi_{\mathbb{P}\mathcal{P}}$ and $\Phi_{\mathbb{Q}\mathcal{P}}$ in separate and independent steps in all linear regression approaches studied in this paper. However, the independence of the cross-section of yields from the time-series dynamics and the loadings' (B^n) independence from the yield level prevents the linear regression approach from being carried out for lower-bound models inspired by Black

¹⁰Here we concentrate on Gaussian innovations in line with most of the existing literature starting with Dai and Singleton (2000) and Duffee (2002).

¹¹Appendix A.1 contains a derivation. The first discrete-time derivation is contained in Backus, Foresi, and Telmer (1996).

(1995), like [Pribsch \(2013\)](#) and [Wu and Xia \(2016\)](#) for the US or [Lemke and Vladu \(2017\)](#), [Horny et al. \(2018\)](#) and [Geiger and Schupp \(2018\)](#) for the euro area.

For monetary policy analysis, log prices are less relevant than zero-coupon yields y_t^n :

$$y_t^n = A_{\mathcal{P}}^n + B_{\mathcal{P}}^n \mathcal{P}_t + v_t^n \quad (8)$$

Yield parameters and price parameters are related by the maturity in multiples n of the data frequency Δt which is denoted in years: $\bar{A}_{\mathcal{P}}^n = -n \cdot \Delta t \cdot A_{\mathcal{P}}^n$ and $\bar{B}_{\mathcal{P}}^n = -n \cdot \Delta t \cdot B_{\mathcal{P}}^n$. The decomposition of an n -period yield into its expectation and term premium components relies on the average short-rate expectation $E_t^n(y^1)$ contained in an n -period yield:

$$E_t^n(y^1) = \frac{1}{n} \sum_{h=0}^{n-1} (A_{\mathcal{P}}^1 + B_{\mathcal{P}}^1 \cdot E_t^{\mathbb{P}}(\mathcal{P}_{t+h})) \quad (9)$$

The factor expectations $E_t^{\mathbb{P}}(\mathcal{P}_{t+h})$ are determined from the time-series dynamics (6). The term premium TP_t^n including the constant convexity is the difference between the fitted yield and short-rate expectations $TP_t^n = y_t^n - E_t^n(y^1)$.

2.2 Maximum Likelihood Estimation

The affine term structure model takes a state-space form with the transition equation (6) and the measurement equation (1) or (8). The parameters to be determined are the time-series dynamics $(\Phi_{\mathbb{P}\mathcal{P}}, \nu_{\mathbb{P}\mathcal{P}}, \Omega_{\mathcal{P}})$, the risk-neutral dynamics $(\Phi_{\mathbb{Q}\mathcal{P}}, \nu_{\mathbb{Q}\mathcal{P}})$ and the short-rate parameters $(A_{\mathcal{P}}^1, B_{\mathcal{P}}^1)$.¹² The yield loadings $\bar{B}_{\mathcal{P}}^n$ or $B_{\mathcal{P}}^n$ of the measurement equation depend on $\Phi_{\mathbb{Q}\mathcal{P}}$ in a nonlinear fashion. Furthermore, the error covariance $\Omega_{\mathcal{P}}$ of the transition equation has a nonlinear impact on the constant $\bar{A}_{\mathcal{P}}^n$ or $A_{\mathcal{P}}^n$ of longer-term bonds in the measurement equation. Therefore, the natural way to estimate a term structure model is by maximum likelihood.

[Joslin et al. \(2011\)](#) introduce a new identification scheme and develop a canonical form that facilitates the estimation. They assume that factors are linear combinations of yields ($\mathcal{P}_t = W \cdot y_t$) measured without error ($0 = W \cdot v_t$). In that case, the parameters $\nu_{\mathbb{P}\mathcal{P}}$ and $\Phi_{\mathbb{P}\mathcal{P}}$ from the real-world dynamics (6) can be estimated by least squares with standard vector autoregression techniques for the observable factors \mathcal{P}_t .¹³ Furthermore, the risk-neutral constant $\nu_{\mathbb{Q}\mathcal{P}}$ can be concentrated out of the likelihood function, and the short-rate parameters are normalized. That reduces the nonlinear parameters in [Joslin et al. \(2011\)](#) to specific elements of $\Phi_{\mathbb{Q}}$ and Ω and speeds up maximum likelihood estimation considerably.

2.3 Linear Regression Estimator of DLR

DLR and ACM estimate the risk-neutral dynamics $\nu_{\mathbb{Q}\mathcal{P}}$ and $\Phi_{\mathbb{Q}\mathcal{P}}$ from two-step regressions instead of a nonlinear search in maximum likelihood estimation. To this end, both

¹²Not all elements of these parameters need to be estimated. Some are normalized to achieve unique factor identification.

¹³Technically, the assumption of no measurement error implies that the likelihood can be split into two independent parts: one governing the cross-sectional pricing and the other the transition of the observable factors. The latter can be estimated by least squares.

approaches first require an observable short rate. Second, they require consecutive maturities, i.e. for each bond with maturity n used in the estimation, an observable yield of maturity $n + 1$ is needed. The remainder of this section reviews their methods and highlights the use of the observable short rate.

ACM and DLR both use observable yield factors \mathcal{P}_t , as in [Joslin et al. \(2011\)](#). Therefore, they estimate the real-world dynamics $\nu_{\mathbb{P}\mathcal{P}}$, $\Phi_{\mathbb{P}\mathcal{P}}$ and $\Omega_{\mathcal{P}}$ by least squares. The short-rate constant $A_{\mathcal{P}}^1$ and loadings $B_{\mathcal{P}}^1$ are proxied by a regression of the observable short rate y_t^1 on the observed factors \mathcal{P}_t . What distinguishes DLR from ACM is the method used to determine the risk-neutral dynamics $\nu_{\mathbb{Q}\mathcal{P}}$ and $\Phi_{\mathbb{Q}\mathcal{P}}$. In the remainder I will concentrate on DLR, while ACM is covered in [Appendix A.2](#).

The idea of DLR is to determine, in a first step, observable proxies of the log-price loadings $\bar{B}_{\mathcal{P}}^n$, $\bar{B}_{\mathcal{P}}^{n+1}$, and $\bar{B}_{\mathcal{P}}^1$ that are contained in the recursions (4). DLR treats equation (1) as a least-squares regression of observable log prices on the observable factors.¹⁴ The coefficients from this first regression for maturity n are denoted \hat{a}_p^n and \hat{b}_p^n . Subscript p indicates the origin from a log price regression. For each maturity n , regressions for a matching maturity $n + 1$ are estimated. The regression parameters for N log prices with maturity n are collected in \hat{a}_p and \hat{b}_p , the N matching $n + 1$ maturities are collected in \hat{a}_p^+ and \hat{b}_p^+ . The coefficients \hat{b}_p , \hat{b}_p^+ , and \hat{b}_p^1 are regression-based proxies for $\bar{B}_{\mathcal{P}}^n$, $\bar{B}_{\mathcal{P}}^{n+1}$, and $\bar{B}_{\mathcal{P}}^1$. The short rate parameters $A_{\mathcal{P}}^1$ and $\bar{B}_{\mathcal{P}}^1$ are proxied by regression coefficients \hat{a}_p^1 and \hat{b}_p^1 of the observable short bond p_t^1 on the factors \mathcal{P}_t .

The second step of DLR inverts the pricing recursion (4) for the risk-neutral dynamics $\Phi_{\mathbb{Q}\mathcal{P}}$ using the proxies from the first regression step \hat{b}_p , \hat{b}_p^+ , and \hat{b}_p^1 :

$$\Phi_{\mathbb{Q}\mathcal{P}} = \left(\hat{b}_p' \hat{b}_p \right)^{-1} \hat{b}_p' \left(\hat{b}_p^+ - 1_{N \times 1} \cdot \hat{b}_p^1 \right) \quad (10)$$

Given the risk-neutral $\Phi_{\mathbb{Q}\mathcal{P}}$, the risk-neutral constant $\nu_{\mathbb{Q}\mathcal{P}}$ is determined from the recursions of the log price constant (5) with the first-step regression constants \hat{a}_p that proxy $\bar{A}_{\mathcal{P}}$:

$$\nu_{\mathbb{Q}\mathcal{P}} = \left((\bar{B}_{\mathcal{P}})' \bar{B}_{\mathcal{P}} \right)^{-1} (\bar{B}_{\mathcal{P}})' \left(\hat{a}_p^+ - \hat{a}_p^1 - \hat{a}_p - \frac{1}{2} \bar{B}_{\mathcal{P}} \Omega_{\mathcal{P}} (\bar{B}_{\mathcal{P}})' \right) \quad (11)$$

The convexity adjustment is based on the implicit $\Omega_{\mathcal{P}}$ from the time-series regression. $\bar{B}_{\mathcal{P}}$ are recursion-based with maturities n matched to \hat{a}_p .

Thus, the traditional DLR requires an observable short rate to determine the risk-neutral dynamics: It uses \hat{b}_p^1 in the inversion (10) for $\Phi_{\mathbb{Q}\mathcal{P}}$. Furthermore, the short rate is used to determine the short-rate parameters $\bar{A}_{\mathcal{P}}^1$ and $\bar{B}_{\mathcal{P}}^1$. If the short rate contains data errors from extrapolation or is inconsistent with the rest of the curve, the risk-neutral dynamics are estimated with error.

¹⁴In DLR's special case of constructing the yield portfolio factors (principal components) from all maturities, rescaled eigenvectors replace the first regression step.

3 Yield Difference Estimator

My approach to circumventing the dependence of the risk-neutral dynamics $\Phi_{\mathbb{Q}\mathcal{P}}$ on an observable short rate is to take differences between two bond prices of longer-term maturities. The next section derives the Yield Difference estimator (section 3.1) for the risk-neutral dynamics $\Phi_{\mathbb{Q}\mathcal{P}}$.¹⁵ The time-series parameters are determined from least squares of observable yield factors, as in JSZ or DLR. The implicit short-rate parameters and the risk-neutral constant $\nu_{\mathbb{Q}\mathcal{P}}$ – which determine the level of all yields – are derived from long-term yields in section 3.2.

3.1 The Yield Difference Estimator

Dependence on the observable short-rate parameters \hat{b}^1 in DLR can be overcome by forming a difference in log prices of long-term bonds with maturity $n + 1$ and $k + 1$. This approach will be called Yield Difference estimator (YD) in the remainder. Both maturities n and k that form the difference are longer-term maturities which are unaffected by data noise, for example between six months and ten years for Bunds from the Bbk dataset or US Treasuries from the GSW dataset.

From a theoretical point of view, forming differences and replacing the coefficients for maturity $n + 1$ and $k + 1$ with its elements from the no-arbitrage recursion (4) eliminates the short-rate parameters $\bar{B}_{\mathcal{P}}^1$:

$$\begin{aligned} p_t^{n+1} - p_t^{k+1} &= \bar{A}_{\mathcal{P}}^{n+1} - \bar{A}_{\mathcal{P}}^{k+1} - (\bar{B}_{\mathcal{P}}^{n+1} - \bar{B}_{\mathcal{P}}^{k+1})\mathcal{P}_t \\ &= \bar{A}_{\mathcal{P}}^{n+1} - \bar{A}_{\mathcal{P}}^{k+1} + (\bar{B}_{\mathcal{P}}^n - \bar{B}_{\mathcal{P}}^k)\Phi_{\mathbb{Q}\mathcal{P}}\mathcal{P}_t \end{aligned} \quad (12)$$

The first estimation step of YD is similar to DLR: Estimate (12) using a regression of log price differences on the factors \mathcal{P}_t . This gives the regression coefficients $\hat{a}_{YD,n,k} = \bar{A}_{\mathcal{P}}^{n+1} - \bar{A}_{\mathcal{P}}^{k+1}$ and $\hat{c}_{YD,n,k} = (\bar{B}_{\mathcal{P}}^n - \bar{B}_{\mathcal{P}}^k)\Phi_{\mathbb{Q}\mathcal{P}}$. $\bar{B}_{\mathcal{P}}^i$ are proxied by log price regression coefficients \hat{b}_p^i from DLR.¹⁶ The inversion for the parameters of the \mathbb{Q} -dynamics is then given by:¹⁷

$$\Phi_{\mathbb{Q}\mathcal{P}} = \left(\left(\hat{b}_p^n - \hat{b}_p^k \right)' \left(\hat{b}_p^n - \hat{b}_p^k \right) \right)^{-1} \left(\hat{b}_p^n - \hat{b}_p^k \right)' \hat{c}_{YD} \quad (13)$$

The calculation of the risk-neutral constant $\nu_{\mathbb{Q}\mathcal{P}}$ in Appendix A.3 is an extension of the DLR approach. The Yield Difference estimator is an extension of DLR without an observable one-period bond. If there is an observable short rate, DLR is a special case of YD for $k = 0 \forall k$.

¹⁵The Return Difference estimator is developed in Appendix A.2 as an extension of the ACM estimator.

¹⁶In my empirical implementation, n and k are maturities used to construct the principal components \mathcal{P}_t . In that special case, the maturity-scaled eigenvectors W exactly correspond to the regression coefficients $\hat{b}_p^n = -n \cdot \Delta t \cdot W^n$. This follows the implementation of Diez de Los Rios (2015).

¹⁷Here I use a slight abuse of notation, \hat{b}^n and \hat{b}^k is now a vector of all \hat{b}_p^i with maturities i matched to the maturities used to calculate the price differences and determine \hat{c}_{YD} .

3.2 Short-Rate Parameters and Risk-Neutral Constants

The risk-neutral factor dynamics $\Phi_{\mathbb{Q}\mathcal{P}}$ and $\nu_{\mathbb{Q}\mathcal{P}}$ are determined from the Yield Difference estimator or the traditional DLR and ACM approaches. They are accompanied by the real-world dynamics $\Phi_{\mathbb{P}\mathcal{P}}$, $\nu_{\mathbb{P}\mathcal{P}}$ and $\Omega_{\mathcal{P}}$ from a time-series VAR. Thus, we finally need to estimate the short-rate parameters $\bar{A}_{\mathcal{P}}^1$ and $\bar{B}_{\mathcal{P}}^1$ which determine the level of the yields.

If the short rate is noisy or not observed, its parameters can be backed out from the parameters of long-term bonds. The recursions of the loadings (4) can be expressed in a form that is linear in short-term loadings $\bar{B}_{\mathcal{P}}^1$:

$$\bar{B}_{\mathcal{P}}^n = \bar{B}_{\mathcal{P}}^1 (I - \Phi_{\mathbb{Q}\mathcal{P}}^n) (I - \Phi_{\mathbb{Q}\mathcal{P}})^{-1} \quad (14)$$

In the empirical implementation, I proxy $\bar{B}_{\mathcal{P}}^n$ on the left-hand side by maturity-scaled eigenvectors W for all estimators ($\bar{B}_{\mathcal{P}}^n \approx -n \cdot \Delta t \cdot W^n$).¹⁸ $\Phi_{\mathbb{Q}\mathcal{P}}$ is given by traditional ACM, DLR or the new Yield Difference estimator. Thus, short-rate parameters $\bar{B}_{\mathcal{P}}^1$ are determined from post-multiplying (14) by the inverse of the terms in brackets. In general, one is free to choose one or more arbitrary maturities n to determine the short-rate loadings. In the empirical estimation below, n are all yield maturities that are used to determine the observable factors.¹⁹

The constants of the short-rate $\bar{A}_{\mathcal{P}}^1$ affect long-term log price constants $\bar{A}_{\mathcal{P}}^n$ in a linear fashion.

$$\bar{A}_{\mathcal{P}}^n = n \cdot \bar{A}_{\mathcal{P}}^1 + \sum_{i=1}^{n-1} \bar{B}_{\mathcal{P}}^i \nu_{\mathbb{Q}\mathcal{P}} + \frac{1}{2} \sum_{i=1}^{n-1} \bar{B}_{\mathcal{P}}^i \Omega_{\mathcal{P}} (\bar{B}_{\mathcal{P}}^i)' \quad (15)$$

$\bar{B}_{\mathcal{P}}^n$ are model-consistent from recursions with $\bar{B}_{\mathcal{P}}^1$ and $\Phi_{\mathbb{Q}\mathcal{P}}$. The risk-neutral constant $\nu_{\mathbb{Q}\mathcal{P}}$ is from the linear regressions (see Appendix A.3). The convexity adjustment is determined with $\Omega_{\mathcal{P}}$ from the time-series regression.

An observed equivalent of $\bar{A}_{\mathcal{P}}^n$ is based on the average log price $\hat{a}_{\mathcal{P}}^n = p^n - \bar{B}_{\mathcal{P}}^n \cdot \mathcal{P}$. Thus, the estimate of the short-rate constant $\bar{A}_{\mathcal{P}}^1$ depends on the sample average of yields y^n and factors \mathcal{P} . In line with the determination of the short-rate loadings, n are all maturities used to construct the observed factors \mathcal{P}_t .

3.3 Transformation into Self-Consistent Estimators

All linear regression estimators of the risk-neutral dynamics $\Phi_{\mathbb{Q}\mathcal{P}}$ are overidentified – the traditional ACM or DLR and the new Yield Difference estimator. That means there are more free parameters than necessary: In a model with K factors, all regression-based risk-neutral dynamics have $K^2 + K$ parameters in $\Phi_{\mathbb{Q}\mathcal{P}}$ and $\bar{B}_{\mathcal{P}}^1$. The regression-based constants have $K + 1$ coefficients in $\nu_{\mathbb{Q}\mathcal{P}}$ and $\bar{A}_{\mathcal{P}}^1$. A self-consistent canonical form like

¹⁸We could as well proxy the long-term loadings on the left-hand side with first-step regression parameters $\hat{b}_{\mathcal{P}}^n$ for DLR and YD or \hat{b}_r^n for ACM.

¹⁹Again, DLR and ACM are a special case in my approach since they use the observable short rate ($n = 1$) to determine the short-rate loadings. In that case, the brackets on the right-hand side of equation (14) cancel. The short-rate loadings correspond to the regression coefficients of the observed factors on the log price of the short bond $\bar{B}_{\mathcal{P}}^n = \bar{B}_{\mathcal{P}}^1 \approx \hat{b}_{\mathcal{P}}^1$. The same applies to the determination of the short-rate constant in equation (15) below $\bar{A}_{\mathcal{P}}^n = \bar{A}_{\mathcal{P}}^1 \approx \hat{a}_{\mathcal{P}}^1$.

the one of [Joslin et al. \(2011\)](#) has risk-neutral dynamics that are driven by K parameters and one constant for each curve.²⁰ With more parameters than necessary in the linear regression estimators, they are prone to overfitting.

I transform the overidentified $\Phi_{\mathcal{QP}}$ from the linear regression estimator into a self-consistent JSZ form using the eigenvalue decomposition of [Golinski and Spencer \(2018\)](#).²¹ This approach allows me to prevent overfitting and preserves the numerical performance of all linear regression approaches.

3.4 Why use the Yield Difference Estimator?

The Yield Difference estimator needs no observable short rate. Therefore, extrapolation noise in short maturities does not affect the results. We can derive an implicit one-period short rate even if the minimum yield maturity in the data is larger. For example, for the interpolated Bund curve investigated in section 5, with long-term maturities n (and k) of one year or more, I use the Yield Difference estimator to determine $\Phi_{\mathcal{QP}}$. The latter avoids the noisy observable short rate with a maturity of one month and can always be transformed into the self-consistent JSZ canonical form with the original method of [Golinski and Spencer \(2018\)](#) outlined in section 3.3. This possibility is even more important for datasets that have a large minimum maturity by design.²²

In principal, by choosing the maturities n and k to form the differences in the Yield Difference estimator, one can assign specific weights to maturity segments that have the best data quality for the estimation of the risk-neutral dynamics and the implicit short-rate parameters. For interbank rate swap curves, a larger weight for (non-interpolated) short maturities may be desired. In that case, forming differences with respect to the shortest maturity ($k = \min(n)$) may offer little benefit over the existing DLR and ACM approaches. For inflation markets, seasonality or the indexation lag have a comparably large impact on short maturities such that the longest maturity may serve as a reference point for differencing ($k = \max(n)$). From a theoretical perspective, using neighboring maturities is close to the double differencing over time and maturity suggested by [Crump and Gospodinov \(2019\)](#). However, for the interpolated government bond yield datasets used in this paper, the choice of n and k has a negligible effect on the results. Therefore, I use $N \cdot (N + 1)/2$ differences between all N maturities used to construct the observable factors \mathcal{P}_t .

Overall, one can use the Yield Difference estimator for all datasets that are unsuited for the use of the traditional linear regression approaches of DLR and ACM and used to require maximum likelihood estimation.

²⁰The same number of parameters are counted in the alternative canonical form of [Hamilton and Wu \(2012\)](#).

²¹See Appendix A.4 for details. [Diez de Los Rios \(2015\)](#) uses a recursive approach to ensure self-consistency.

²²One important example are inflation-linked products. Inflation swaps usually have a minimum maturity of one year. [Gürkaynak, Sack, and Wright \(2010\)](#) recommend, for their real yield curve derived from US Treasury Inflation Protected Security (TIPS), using interpolated real yields only for maturities exceeding two years. With the Difference Estimators, a TIPS or inflation swap term structure model with a minimum maturity of two years ($n \geq 24$) can be estimated at a monthly frequency.

Table 1: Overidentification (OI) versus Self-Consistency (SC) for US Treasuries

Root-mean-square error (RMSE) for all yields from 1Y to 10Y with one-month spacing in basis points. Four observable factors are determined from yields with maturities from 6M to 10Y with 6M spacing. “DLR GSW” indicates that the 1M short rate is extrapolated from the GSW dataset, and “DLR TB” indicates that the 1M T-Bill rate is used as the short rate. “YD” is the new Yield Difference estimator.

	DLR GSW	DLR TB	YD
Traditional OI, 1M short rate	9.5	11.5	283.9
OI, long rates	5.1	4.5	3.9
SC, long rates	4.7	4.2	4.0

4 US Treasury Curve

The market for US Treasuries allows the new Yield Difference estimator (YD) to be compared with traditional DLR in a dataset that contains an observable one-month yield. The key question of this section is to investigate, whether important dynamics are lost by discarding the available short rate from the estimation. The short answer is: no.

All models are estimated from end-of-month observations of the GSW dataset from 1961 to 2019. Principal components serve as observable factors \mathcal{P}_t and are determined from yields implied by the GSW parameters with maturities from 6M to 10Y with 6M spacing. These 20 yields are used to derive the observable factors \mathcal{P}_t and represent n and k in the YD estimation, with the differences being taken between all 190 unique combinations. The benchmark model has four factors and is transformed into a self-consistent representation. The standard deviation of yield errors is one basis point from my four-factor model. It corresponds to the error variation of the five-factor excess return model of [Adrian et al. \(2013\)](#). Choosing three or five factors in my dataset leads to explosive risk-neutral yield dynamics (eigenvalues above one) for DLR models, which is implausible from an economic perspective.²³

4.1 Yield Difference versus Traditional DLR

The major contribution of this paper is to avoid the impact of a noisy or inconsistent short rate by using the Yield Difference estimator: first, in the risk-neutral dynamics by forming differences between longer-term yields (section 3.1). Second, in the short-rate parameters by estimating them from long-term yields (section 3.2). This section zooms in the implications of both parts.

Table 1 compares the yield fit in terms of RMSE of the traditional DLR approach to my new Yield Difference (YD) estimator. “DLR TB” uses the 1M US Treasury Bill as the short rate, which is the approach normally used in the literature to estimate short-rate parameters. “DLR GSW” uses the extrapolated and noisy 1M GSW yield, which proxies the imperfect data availability for most countries including Germany, where a

²³For the US Treasury curve, [Mönch \(2019\)](#) uses four factors in the ACM approach. Three factors are used by [Duffee \(2002\)](#), [Kim and Wright \(2005\)](#), [Joslin et al. \(2011\)](#), [Bauer \(2018\)](#) or [Golinski and Spencer \(2018\)](#). The latter authors use a more sparse maturity set for the yields to determine the factors. Using a maturity set that is closer to their choice (6M, 1Y, 2Y, 3Y, 5Y, 7Y, 10Y), I find overfitting in maximum likelihood approaches if more than three factors are used, i.e. the out-of-sample RMSE increase for four or more factors in the JSZ MLE approach. Thus, the yield selection underlying the factor construction has a crucial impact on the preferred factor dimension.

T-Bill rate is unavailable. The yields to measure the RMSE are at a monthly frequency, with monthly-spaced maturities from one year to ten years. Thus, the statistic in Table 1 is a mix of in-sample errors for every 6M maturity and out-of-sample errors for all other monthly maturities. I leave out the short-term yields below one-year maturity to construct the RMSE, since a good fit of those maturities might represent fitting the extrapolation noise of the 1M yield.

The first row, “Traditional OI, 1M short rate”, of Table 1 contains the traditional approach to estimating a linear regression model: The risk-neutral dynamics are overidentified (OI) and the short-rate parameters correspond to the regression parameters of the observed one-month short rate ($\bar{A}_{\mathcal{P}}^1 = \hat{a}_p^1$, $\bar{B}_{\mathcal{P}}^1 = \hat{b}_p^1$). The YD estimator fits extremely poorly, which implies that the risk-neutral dynamics derived from long-term yields are inconsistent with the short-rate loadings from a one-month yield. Interestingly, using the extrapolated one-month GSW yield in DLR results in a better fit than using the one-month T-Bill rate for DLR. Thus, potential inconsistencies between Treasury Bills and seasoned Treasury bonds seem more problematic in the traditional approaches than erratic noise from extrapolation in the one-month GSW yield.

The fit is considerably better in the second row, “OI, long rates”, in which the risk-neutral dynamics are still overidentified but the implicit short-rate parameters originate from yields with the longer maturities as outlined in section 3.2. For that approach, short rates in DLR only affect the results through \hat{b}_p^1 in the inversion for the risk-neutral dynamics (10). The RMSEs of both DLR approaches are only half the size of the first row. YD is completely independent of the short rate and reaches a RMSE of only 3.9 bps.

The last row, “SC, long rates”, of Table 1 contains results for the self-consistent model (SC) which may reduce overfitting (see section 3.3). The comparison of the last two rows in Table 1 provides evidence of minor overfitting in both DLR variants, while there is no overfitting in the YD model. Five-factor models that allow for more flexibility in the linear regression are more affected by overfitting (not displayed).

The results in Table 1 are in line with Golinski and Spencer (2018), who show that a self-consistent estimation (3rd row in Tab 1) improves the fit over the traditional estimation based on an observable short rate (1st row in Tab 1). However, Table 1 suggests the improvement they identify them for US Treasuries is caused to a large extent by the independence of short-rate parameters from the observable short rate but *not* from self-consistency. Overall, it is better to avoid the noisy short maturity spectrum altogether, which is only possible with the new Yield Difference estimator.

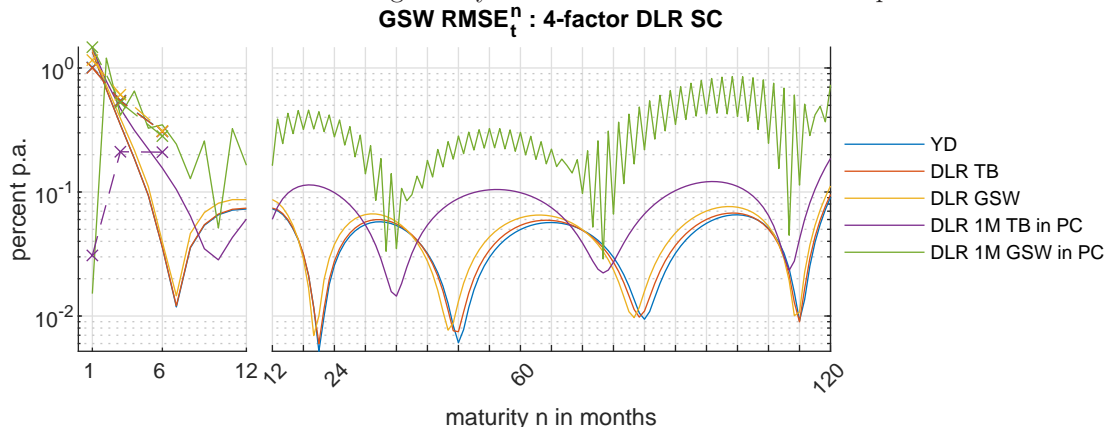
4.2 Factor Construction With(Out) a Short Rate

All self-consistent estimators in Table 1 have a broadly similar fit in terms of RMSE. This might be surprising at least for “DLR GSW”, which uses the extrapolated and noisy 1M GSW yield with implausible spikes in Figure 1.²⁴ Both DLR estimators in Table 1 perform well despite the use of noisy data because the short rate is *not* used to construct the observable factors \mathcal{P}_t . Hence, the noisy data in DLR only affect $\Phi_{\mathcal{Q}\mathcal{P}}$ through \hat{b}_p^1 in equation (10). Excluding the short rate from the factor dynamics is supposed to exclude

²⁴This finding is not driven by a specific maturity spectrum in Figure 2. Even for short maturities in the left part, YD performs just as well or poorly as the DLR variations that take into account an observable short rate (the blue line is hardly visible below the red line).

Figure 2: DLR versus Yield Difference for US Treasuries

Root mean squared error (RMSE) of the yield errors. The gray vertical grid lines indicate maturities used to construct the principal components that serve as observed factors. The errors of those maturities are in-sample. The other monthly-spaced maturities are out-of-sample errors. Dashed lines with \times marker are T-Bills with 1M, 3M and 6M maturity. “1M ... in PC” indicates that the 1M rate is contained in the principal components (PC) that serve as observable factors \mathcal{P}_t . All displayed models are from a self-consistent model that uses the long-term yields to determine the short-rate parameters.



potential data noise from the economic dynamics of \mathcal{P}_t . However, this potentially ignores economically relevant information about the short end of the yield curve.²⁵

In Figure 2, the violet and green lines are models that include a 1M yield in the factor construction in addition to the 6M-spaced yields from 6M to 10Y. This implies that the variation of the 1M rate – including the extrapolation noise – is considered to be economically relevant. Since the number of factors is unchanged at four, this comes at the cost of a worse fit of the longer maturities in the right part. If the 1M T-Bill rate (1M GSW yield) is included in the factor dynamics, the average RMSE between 1Y and 10Y is about eight basis points (above 30 basis points).²⁶

Using the one-month rate in the factor construction changes it from an out-of-sample to an in-sample measure. Therefore, the fitting error of that particular in-sample yield is lower for both GSW and TB. However, the improved fit of the 1M rate has no evident positive effect on the neighboring short-term maturities. If the 1M rates were dominated by economic signals but not by noise, we would expect to have a better fit of the short maturities because the short end has more weight in the factor construction. However, for the six-month GSW yield, the RMSE is only about 4 bps for models that exclude the 1M rate, compared with more than 10 bps for both models with a 1M rate in the factor construction. For the GSW dataset, data noise dominates economic information contained in the one-month horizon. For the violet model that includes the 1M T-Bill rate, the T-Bills with 3M and 6M maturity have a better fit compared to the linear regression estimators that exclude a one-month maturity. However, their RMSE is at an elevated level: the RMSE for 3M and 6M maturity T-Bills is 20bps which is a multiple

²⁵ Adrian et al. (2013) construct factors without the 1M US Treasury Bill rate, while Golinski and Spencer (2018) include it in their factor construction.

²⁶This is larger than the average (in-sample) RMSE of three to four basis points in Golinski and Spencer (2018), who include the one month T-Bill in the factor construction. The better fit might be due to the fact that they include only eight yields in their estimation as opposed to 20 yields in my estimation.

of the 1M T-Bill rate RMSE and at a comparable level to the worst fit of a GSW yield, the 10Y maturity. Thus, the 1M US T-Bill rate contains a considerable amount of data noise or a T-Bill premium as suggested by [Lenel et al. \(2019\)](#). An affine term-structure model of the US Treasury yield curve should exclude an observable short rate from the factor construction.

4.3 Expected Treasury Short Rates

A term structure model is estimated to deliver insights into the term premium and the expected short-rate path contained in long-term yields. [Figure 3](#) compares the expectation component of the 10-year US Treasury yield for a selection of my estimations to model-based benchmarks from the literature. Also shown are survey expectations of the average 3M T-Bill rate over 10 years from Consensus Economics.²⁷

The first benchmark is estimated by staff of the Federal Reserve Bank of New York (FRBNY), which updates the [Adrian et al. \(2013\)](#) estimation based on the GSW dataset.²⁸ It is a five-factor model with only yield data estimated using the overidentified method of ACM. The 1M T-Bill rate is not used to construct principal components that serve as observable pricing factors \mathcal{P}_t . It is only used to estimate short-rate parameters. Thus, the data sample and the methods of my estimates are similar to the FRBNY ACM. The same can be said for the expectation component's dynamics of the 10-year US Treasury yield in [Figure 3](#).²⁹ This holds not only for the linear regression estimators but also for the JSZ models estimated by maximum likelihood: hardly any other model's expectation component is visible below the five-factor YD. Thus, I confirm the major finding of [Li, Meldrum, and Rodriguez \(2017\)](#) that the expectation component is quite similar, no matter whether maximum likelihood or linear regressions are used for the estimation as long as the yield dataset and the model structure are similar.

The second benchmark is estimated by Federal Reserve Board (FRB) staff based on [Kim and Wright \(2005, KW\)](#). It is a three-factor model estimated with maximum likelihood that includes yields and interest rate surveys.³⁰ It is estimated with a history starting in 1990, shortly after the short-horizon surveys become available. From a technical perspective, the advantage of using surveys is that the small sample downward bias of the persistence in the time-series dynamics can be alleviated. Furthermore, the long-term expectation of the short-rate level is anchored by the level of the survey expectations. There is a consensus in the literature that the dynamics of the expectations are similar across model specifications.³¹ [Figure 3](#) illustrates a decline in short-rate expectations for all models since 1990 but the magnitude of the fluctuations and the level differ. The decline

²⁷There are no surveys available at Consensus Economics concerning the 1M T-Bill, so the 3M T-Bill is the closest survey-based proxy for the expected short rate. The survey for a horizon up to 10 years starts in 1998, and shorter forecast horizons are available since 1989.

²⁸The updated data is available for download at the FRBNY webpage https://www.newyorkfed.org/research/data_indicators/term_premia.html

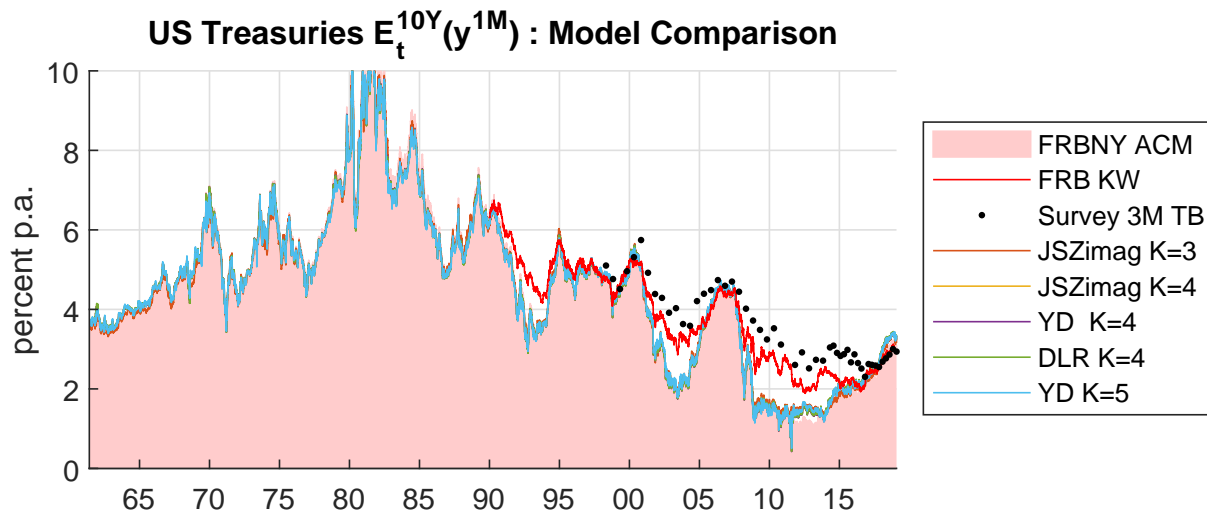
²⁹The remaining difference results from slightly different data specifications (maturities used to construct factors and excess returns). Furthermore, the FRBNY ACM includes a convexity adjustment in the constant, and their estimates are not transformed into the self-consistent canonical form of JSZ.

³⁰The updated data are available for download at the Federal Reserve Board webpage <https://www.federalreserve.gov/data/three-factor-nominal-term-structure-model.htm>

³¹See [Li et al. \(2017\)](#), [Cohen, Hördahl, and Xia \(2018\)](#) and [Adrian, Crump, Mills, and Mönch \(2014\)](#) for the US. [Malik and Meldrum \(2016\)](#) come to a similar conclusion for UK data.

Figure 3: Expectation Component of US Treasury Models

The expectation components are re-based to match the the average level between of the FRBNY ACM expectation component. Data is censored at 10 percent.



of the survey expectation data after the financial crisis was more moderate. Therefore, the model-implied expectations of the FRB KW data is at a higher level compared to the yields-only models like the FRBNY and my estimates. The long-term rate expectations of the latter reflect only the sample mean of the yield data.

The difference between the model estimates is less affected by the estimator than by the data choice – that means whether using surveys or not. Therefore, the main takeaway from this section on US Treasuries is that the Yield Difference estimator works well even in a data environment for which the data structure allows estimations of the traditional approaches of DLR (and ACM). The ability of the Yield Difference estimator to replicate well-established model dynamics in the US gives confidence to apply the Yield Difference estimator in the next section to German Bunds, which have a more challenging dataset structure.

5 German Bund Curve

The final part of this paper covers German Bund data. I decompose the Bund yield curve into short-rate expectations and term premia with a long history starting in January 1967. The yield maturities used for the construction of the observed factors (principal components \mathcal{P}_t) have maturities between one and ten years with six-month spacing. Appendix B contains details of the dataset. All models are estimated using the Yield Difference approach and transformed to the self-consistent version. Three principal components explain 99.99% of the yield variation. The in-sample standard deviation of the

cross-sectional error is less than two basis points.³²

While the estimation of the risk-neutral dynamics is determined by the canonical form of JSZ with the Yield Difference estimator, there are uncounted potential specifications of the time-series dynamics that determine the short-rate expectations. My model choice is based on three economic criteria. First, a yield decomposition should provide reasonable implicit short rates (section 5.2). They constitute the starting point for the path of short-rate expectations. Second, following the consumption asset pricing theory, the term premium should be a compensation for systematic risks and, therefore, be related to macroeconomic drivers and uncertainty (section 5.3). Third, and most importantly, only a model that delivers a reliable forecast of future yields provides a trustworthy expectation component from the yield decomposition for current investment or policy decisions (section 5.1).

5.1 Out-of-Sample Bund Yield Forecasts

The major criterion to narrow the set of model candidates is the out-of-sample forecasting ability: If a model was unsuited to forecast yields in the past, it is unlikely to produce reasonable current expected policy paths. The extremely fast estimation of the Yield Difference estimator allows for a recursive estimation to perform out-of-sample forecasts for many model specifications. The models displayed in the remainder represent only a subset of candidate models.³³

The imposition of No Arbitrage in an observed factor model alone does not improve the forecasting performance, see Duffee (2011a).³⁴ The reason is obvious from the canonical form of Joslin et al. (2011) used in the linear regression estimators: The risk-neutral parameters that ensure no-arbitrage are separated from the time-series dynamics. Therefore, we can use OLS to estimate the time-series dynamics of an unrestricted model. However, it is notoriously difficult for time-series models – and therefore unrestricted affine term structure models – to beat a simple random walk forecast for yields. An econometric reason for that result is emphasized by Bauer, Rudebusch, and Wu (2012): They highlight a small-sample bias in the OLS estimation of the time-series VAR. This bias leads to a lower estimated persistence in the time-series dynamics than in the true data generating process. Since the yield data themselves are very persistent, the naive random walk forecast of a yield may outperform the forecasts of a time-series VAR with

³²Using four factors reduces the in-sample error standard deviation to less than one basis point. The economic results of the four-factor model are comparable to those of the three-factor version. Therefore, I choose the parsimonious model with three factors. The conditional Sharpe Ratios of the all self-consistent models with three factors introduced below are between 0.4 and 0.55. This corroborates the evidence against overfitting.

³³Not shown here are models estimated with two to six factors. Furthermore, variations of the restricted models introduced below were considered. Variations include different risk ranks, variations of the econometric bias adjustment and a variation to the random walk. Lastly, different samples were estimated, most importantly a start of the data in 1983 and a dataset based on Schich (1997) with a limited maturity spectrum starting in 1972.

³⁴The key assumption here is that the observed factors are measured without error, i.e. $0 = W \cdot v_t$ with v_t from equation (8). If there is a measurement error, the No-Arbitrage restrictions help because they give structure to the loadings in the measurement equation. But in that case, principal components as observed factors used by Joslin et al. (2011) or linear regression approaches contain measurement errors and one has to resort to filtering techniques and maximum likelihood estimation.

a downward-biased persistence.

No-Arbitrage restrictions may help to forecast Bund yields if they are combined with auxiliary restrictions on the risk premia or the time-series dynamics. I consider three approaches from the literature that can be easily integrated into the linear-regression framework in a way that preserves its numerical performance.³⁵

Rank-restricted risk premium: A single rank restriction on the risk premium implies that the risk premium in all yields is driven by a single linear combination of the K factors, and not by K combinations, as in an unrestricted canonical model. By restricting the flexibility of the risk process, the time-series dynamics are tied to the risk-neutral dynamics. Since the latter are usually very persistent, the implicit real-world dynamics become more persistent. Rank restrictions on the risk process are motivated by the excess return regressions of [Cochrane and Piazzesi \(2005\)](#) and implemented in affine term structure models by [Duffee \(2011b\)](#) and [Joslin et al. \(2011\)](#) for US data. [Bauer \(2018\)](#), [Joslin, Priebsch, and Singleton \(2014\)](#) and [Cochrane and Piazzesi \(2008\)](#) use zero restrictions on the risk premium that implicitly reduce its rank.³⁶ Details of the implementation of a rank restriction in the linear regression framework can be found in [Appendix A.6](#).

Level factor with random walk: A slow-moving random walk is contained in shifting endpoint models as in [Bauer and Rudebusch \(2020\)](#), [van Dijk, Koopman, van der Wel, and Wright \(2014\)](#) or [Cieslak and Povala \(2015\)](#). A slow-moving random walk usually requires a filter to estimate a slow-moving trend. A simple alternative is chosen by [Duffee \(2011a\)](#), who imposes a random walk for the observed level factor. Compared to the shifting endpoint, the random walk of the observed level implies more noise in the implicit long-term rate. While the noise might be an undesirable property from a theoretical viewpoint, the simple estimation of the level random walk model is its key advantage: In the time-series regression, the level random walk translates into a VAR with simple linear parameter restrictions that can be estimated by least squares, see e.g. [Lütkepohl \(2005, section 5.2\)](#): The autoregressive parameter of the level factor is set to one and zero restrictions ensure that the level is neither predictable by other factors nor that the level factor (being a random walk) will affect the stationary slope and curvature factor. Only parameters that describe the joint movement of slope and curvature factor are estimated by GLS.

Econometric bias adjustment: The parametric approach of [Kilian \(1998\)](#) corrects for the bias in the VAR based on sample size, and error covariance. In the Bund dataset under investigation, the raw correction yields explosive real-world dynamics. The shrinkage suggested by [Kilian \(1998\)](#) ensures stable dynamics. For Bund data, the result is basically a unit root, since the largest time-series eigenvalue is above 0.9999. The bias adjustment

³⁵All restricted models create a link between the yield level A^n and the time-series dynamics by the covariance Ω . This requires estimating with GLS and iterating on the Ω as outlined in [Appendix A.6](#).

³⁶The Bayesian approach of [Bauer \(2018\)](#) prefers zero restrictions that imply a risk rank of one in a three-factor model. The model search in a five-factor macro-finance model of [Joslin et al. \(2014\)](#) results in a risk rank of two. The return prediction factor of [Cochrane and Piazzesi \(2008\)](#) implies a risk rank of one.

Table 2: Out-of-Sample Bund Yield Forecasts

The table shows Theil’s U for the model in the row header compared to a forecast from a yield random walk. A value below (above) one indicates that the model provides better (worse) forecasts than the yield random walk. The + + + / + + / + (- - - / - - / -) signs indicate an increase (decrease) in the model’s forecasting ability versus the yield random walk at the 1%/5%/10% significance level in the test of Diebold and Mariano (1995).

Panel A: 6-month yield								
	h=3M		h=12M		h=3Y		h=5Y	
Unrestricted	0.95	++	0.99		1.23	---	1.43	---
Risk Rank	0.93		0.89	++	1.31	---	1.26	---
Level Random Walk	0.92	+++	0.90	+++	0.79	+++	0.65	+++
Bias Adjustment	0.93	+++	0.95	++	1.01		0.85	+++
Forward	1.35	--	0.91		0.72	+++	0.95	

Panel B: 10-year par yield								
	h=3M		h=12M		h=3Y		h=5Y	
Unrestricted	1.03		1.18	---	1.63	---	1.89	---
Risk Rank	1.11	--	1.07		0.67	+++	0.48	+++
Level Random Walk	1.00		1.04	--	1.18	---	1.25	---
Bias Adjustment	0.99		0.97	+	0.92	++	0.93	++
Forward	1.03	---	1.04	--	1.11	---	1.10	--

is slightly more flexible compared to the level random walk restrictions of Duffee (2011a) because there are no additional zero restrictions imposed on the time-series dynamics.

Table 2 contains the out-of-sample forecast performance of the four models from January 1999 to October 2022. The forecast horizons h span from three months to five years. The estimation window is recursively extended each month from 1994 to 2022 for all models.³⁷ The upper panel contains forecasts of future short rates. It represents the path of future central bank policy rates.³⁸ The lower panel contains forecasts of the ten-year Bund par yield. From an investor perspective, the forecasts of future risk premia contained in the 10-year maturity are of particular importance. Displayed in Table 2 is Theil’s U. It measures the performance of the model relative to a simple yield random walk forecast. Values below (above) one indicate that the term structure model outperforms (underperforms) the simple yield random walk forecast. The plus (minus) signs that the model’s forecast performance is significantly better (worse) than the yield random walk based on the test of Diebold and Mariano (1995).

The unrestricted term structure model in the first row provides significantly better forecasts for short rates than the yield random walk only at short horizons. This is a remarkable result, because the six-month maturity is not contained in the maturity spectrum used for estimation (1Y-10Y). Consistent with downward-biased persistence in the unrestricted model, the forecast performance deteriorates for longer horizons. For long-term yields in the lower panel, the yield random walk forecast cannot be beaten by an unrestricted term structure model.

³⁷The start in 1994 is dictated by the five-year forecast horizon for the yields in 1999. The factors change over time because the eigenvectors W and principal components \mathcal{P}_t are re-determined for every monthly estimation.

³⁸Since only extrapolated data is available for the one-month short rate, the six-month maturity is the best proxy of the short rate without extrapolation noise.

The model with a rank-restricted risk premium in the second row of Table 2 is well-suited for long-horizon forecasts of long-term Bund yields, most likely due to its superior performance in predicting risk premia in the distant future. However, short-horizon forecasts of the long-term yield are significantly worse than the random-walk forecast of the yield. Short-rate forecast performance of the rank-restricted risk premium model is broadly in line with the unrestricted model.

The term structure model with a random walk for the level factor in the third row produces the best forecasts of the short rate. The forecast performance relative to the simple yield random walk benchmark improves as the horizon increases. For all horizons, the model forecasts are significantly better than those of a yield random walk. Thus, the slope and curvature factors have predictive power. The good forecast performance for short rates of the model with level random walk is in line with the results for US Treasuries in Duffee (2011a). However, long-term yield forecasts cannot be improved by imposing a random walk on the yield level.

The econometric bias adjustment model in the fourth row improves forecasts compared to the yield random walk for both maturities and for short as well as long forecast horizons. However, the magnitude of the improvement is limited. Its forecasting performance is inferior to the level random walk forecast for short rates. Thus, the additional zero restrictions in the level random walk compared to bias adjustment seem to be justified. Regarding long-term yields in the lower panel, the bias adjustment delivers inferior forecasts compared to the rank-restricted model for long horizons.

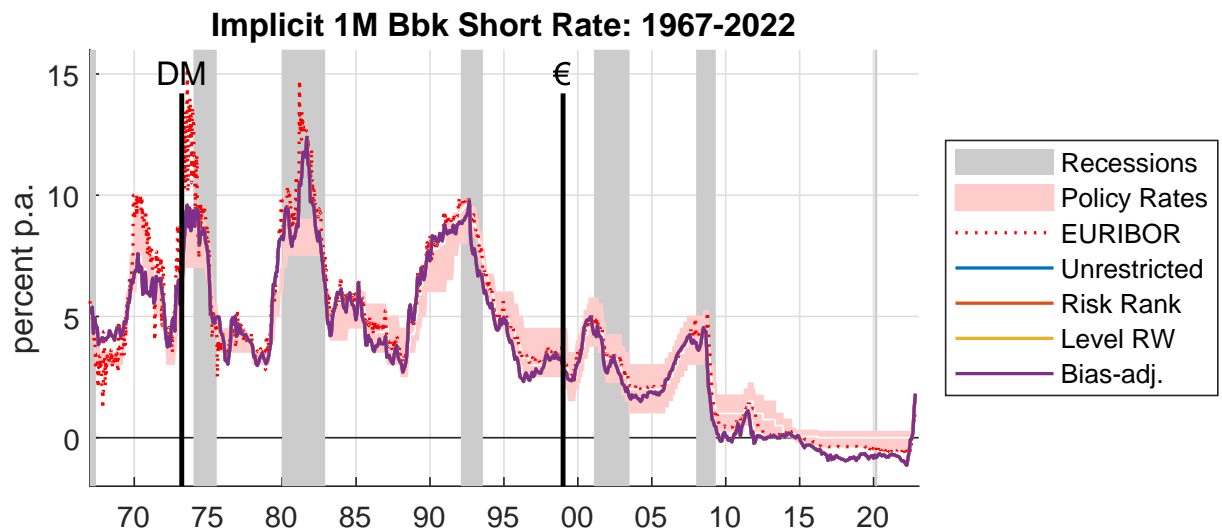
The implicit forwards of the Bund yield curve in the last row of Table 2 provide a simple alternative forecast that requires no estimation of a dynamic no-arbitrage term structure model. For the long-term yield in the lower panel, the forward rates provide significantly worse forecasts than the simple yield random walk benchmark. Thus, the decomposition into rate expectations and term premia of affine models seems to help forecast future term premia contained in the ten-year yield. Regarding the short-rate forecasts in the upper panel, the forward rates' performance relative to the yield random walk is U-shaped. At long horizons, the forward rate forecasts the short rate slightly (but insignificantly) better compared to the yield random walk. Moreover, it is forecasting better than an unrestricted term structure model or a model with a risk rank. However, the long-term forecasts by the forward rate are significantly worse than the model-implied forecasts of term structure models with a level random walk or an econometric bias adjustment. For intermediary horizons of twelve months and three years, the evidence is mixed. The poor performance at the three-month forecast horizon of the short rate is likely to represent extrapolation noise in the three-month yield. This highlights the advantage of affine term structure models that derive a reasonable implicit short rate from long-term yields and provide a reasonable starting point for the expected path of future short rates. This is the topic in the next section.

5.2 Implicit Bund Short Rates and their Expected Path

Figure 4 compares the model-implied one-month Bund short rates to the policy rates of the European Central Bank since 1999 and of the Bundesbank before that. The upper (lower) bound of the red-shaded area is the marginal lending facility rate (deposit facility rate) of the ECB and the Lombard rate (discount rate) of Bundesbank. This “discount

Figure 4: Implicit One-Month Short Rate

The implicit Bund short rate with a one-month maturity is determined from the full-sample estimates from January 1967 to October 2022. Month-end observations. Gray bars indicate German recessions.



“window” is seen as representing an upper and lower bound for short-term safe interest rates: otherwise banks would profit from lending to (borrowing from) third parties and refinancing (depositing) the funds with the central bank. Thus, we expect the implied one-month rates from the term structure models to be within the discount window. A white line within the discount window starting in 1985 represents the “Pensionsatz” (repo rate) of the Bundesbank and the minimum bid rate in the main refinancing operations (MRO) of the ECB. Before the unconventional monetary policy of the ECB started in 2008, the MRO rate or “Pensionsatz” was the implicit target of the central bank for short-term interbank rates and acted as an anchor for short-term rates in other market segments.

The implicit short rate for the different models has identical dynamics. In Figure 4, only the violet line from the econometric bias adjustment model is visible. The short rate reflects risk-neutral dynamics (i.e. eigenvalues) which are determined independently from the time-series dynamics that distinguish the models.³⁹ The white MRO rate, or “Pensionsatz”, is usually slightly higher than the model-implied Bund short rates until the financial crisis in 2008. During that time, the Bundesbank and later the ECB tried to keep the short-term interbank rates (EURIBOR) close to the “Pensionsatz” and MRO. When the ECB created excess liquidity during the financial crisis, the Bund short rate fell close to the deposit facility rate, i.e. the lower border of the discount window. Bunds still remained in line with interbank rates (EURIBOR) until the start of the Eurosystem’s public sector purchase program in 2015. The reduction of Bunds in the hands of private investors after 2015 created asset scarcity and additional excess liquidity pushed the implied Bund short rate below EURIBOR, the cost of short-term funding for the banking sector. A premium of that kind is also visible in the difference between Bund yields and

³⁹The difference in level A^1 across models is so tiny that it is invisible.

interest rate swaps.⁴⁰ Implicit Bund short rates can be used as an indicator of Bund scarcity and the effects of unconventional monetary policy – which is impossible for the noisy extrapolated yields at short maturities. Thus, unconventional monetary policy may cause the implicit Bund short rate to deviate from policy rates.⁴¹

Taking unconventional monetary policy into account, the implicit short Bund rates have reasonable dynamics. This is an important sanity check for the affine term structure models since all decompositions of yields start at the implicit short rate to derive short-rate expectations in Figure 5. All models share the declining trend in the expectation component despite some differences in the magnitude of the decline. The expectation component’s dynamics in the rank-restricted risk premium model shows the largest decline: It used to be the largest value among the models up to the 1970s and is now the smallest. Since 2015, the other three models’ expectation components diverge stronger compared to the earlier sample. In particular, the unrestricted model implies higher future short rates compared to the level random walk or bias adjustment. From the out-of-sample forecasts in the last section, we know that the unconditional model did in fact underestimate the persistence of the low yields during the period of unconventional monetary policy.⁴²

Figure 5 contains the expectation component of two yield decompositions from the literature and survey-based expectations. The first benchmark is the ACM model for Bunds of [Lemke and Werner \(2020\)](#). They use Bbk data only from 1999 to 2012 and estimate with conventional overidentified ACM. Their motivation for the rather early sample end is to exclude the effect of the Eurosystem’s asset purchase programs on model dynamics. The yield decomposition displayed here is based on their sample estimates applied to updated yield data. Their expectation component is highly correlated to my estimates but fluctuates less. This may be due to a lower persistence in their small sample.⁴³ This in turn implies a more stable expectation component at a higher level.

The second benchmark is an effective lower bound (ELB) model for overnight index swaps of [Geiger and Schupp \(2018\)](#). They include Consensus Economics surveys in their optimization. Therefore, their expectation component follows the survey observations, the black dots in Figure 5. From 2008 to 2022, the expectation component of [Geiger and Schupp \(2018\)](#) was at a higher level compared to the models of this paper and [Lemke and](#)

⁴⁰For the effect of the asset purchases on Bunds, see [Deutsche Bundesbank \(2018\)](#). The special premium of the benchmark bonds in a currency area is an international phenomenon, see [Diamond and Van Tassel \(2022\)](#). In the euro area, the Bund also has a premium over the bonds issued by the EU and EFSF, see [Bletzinger, Greif, and Schwaab \(2022\)](#) and [Carriero, Ricci, and Vangelista \(2022\)](#).

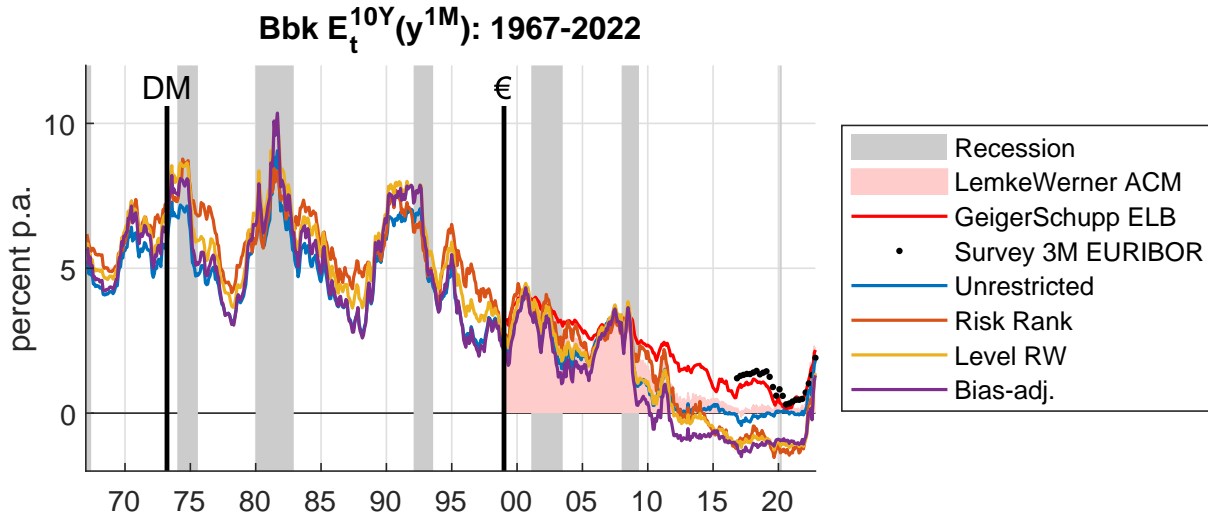
⁴¹Another example of unconventional monetary policy that pushes the Bund short rate out of the discount window is the higher “Sonderlombard” rate of the Bundesbank during the early 1980s. The “Sonderlombard” (1973/74 and 1981/82) is an interest rate above the discount window displayed in Figure 4, which banks had to pay to Bundesbank. Before the 1980s, non-price elements of monetary policy like quotas for discount or Lombard credit were more prominent in the conduct of monetary policy, in particular during the fixed-exchange rate regime of the Bretton Woods System until 1973 (“DM” in Fig. 4). Therefore, the link between the implicit Bund short rate and the policy rates is looser at the start of my sample.

⁴²The dis-advantage of the unrestricted model compared to the models with an econometric bias-adjusted or level random walk increases when the out-of-sample period contains only the period of unconventional monetary policy from 2008 onwards (and not 1999 as in Table 2).

⁴³[Lemke and Werner \(2020\)](#) adapt the bias correction of [Kilian \(1998\)](#) outlined above in a way that the maximum eigenvalue does not exceed 0.999. Thus, their persistence is below the de facto random walk of my econometric bias adjustment.

Figure 5: Expectation Component of ten-year German Bunds

The expectation components determined from the full-sample estimates from 1967 to 2022. Month-end observations. Long-term survey data from Consensus Economics are not available prior to 2016. Gray bars indicate German recessions.



Werner (2020). They declined less than the yields-only models’ expectations after 2008 when the unconventional monetary policy was implemented. After 2015, they contain more variation compared to the other models.

Including surveys leads to a breakdown of the separation of risk-neutral and time-series parameters, which is the key to the simplified estimation of all linear regression approaches and the simplified maximum likelihood estimation of JSZ. Since surveys have a relatively short history, a long history of short-rate expectations and term premia has to rely on yields-only models, which can be estimated with the Yield Difference estimator.

Overall, all models share a decline in the expectation component since the 1990s. Furthermore, they agree on an increase in 2022, when high inflation prompted the Eurosystem to increase policy rates sharply. The different models only disagree on the magnitude of the change in the expectation component. These results are in line with the findings for the US. Whether surveys should be included in a term structure model remains an open question that is beyond the scope of this paper. Starting with Kim and Orphanides (2012), many authors include surveys to address the small-sample bias in the estimation of the time-series dynamics and to fix the level of long-horizon expectations. Surveys may contain (unspanned) information beyond the term structure of yields to predict bond returns, see Duffee (2018) and Bauer and Rudebusch (2017). Yield forecasting ability of surveys is mixed, in particular for long-term yields, see for example Chun (2012). Piazzi, Salomao, and Schneider (2015) find that expectations from time series models and surveys do not align. It is unclear whether one can learn about yields from surveys and which measure is “correct”.⁴⁴ Lastly, the usage of surveys reduces the desired cyclicity of the US term premium based compared with term premia estimates based on ACM, see Li et al. (2017). The cyclicity of the German term premium concludes this paper in the

⁴⁴To make the issue even more complicated, even ex post “wrong” investor expectations collected in surveys may be relevant for the pricing of the securities.

Table 3: Cyclicity of Bund Term Premia: Correlation

Correlation of term premium of the models in the row header with the indicators. The year in which the sample starts is indicated in brackets.

	<i>ifo</i> (1967)	π (1967)	<i>Bund – Vola</i> (2001)	<i>Disp</i> (y^{3M}) (1989)	<i>Disp</i> (par^{10Y}) (1989)
Unrestricted	-0.432	0.114	0.195	0.597	0.263
Rank-Restricted	-0.277	0.011	0.144	0.544	0.165
Level Random Walk	-0.270	-0.283	0.264	0.221	0.153
Bias Adjustment	-0.292	-0.256	0.249	0.284	0.162

next section.

5.3 Cyclicity of the Bund Term Premium

The term premium of long-term bonds reflects the compensation for a long-term investment relative to a sequential short-term investment. Following the consumption-based asset pricing model, an asset’s risk premium in general reflects the asset price’s co-movement with the business cycle. Table 3 contains correlation coefficients of term premia from the different term structure models and indicators of macroeconomic developments and uncertainty. Figure 6 displays the time series of the indicators and the term premium of the model with a random walk for the level factor (Level RW) which provided the best forecasts of future short-term yields in section 5.1.

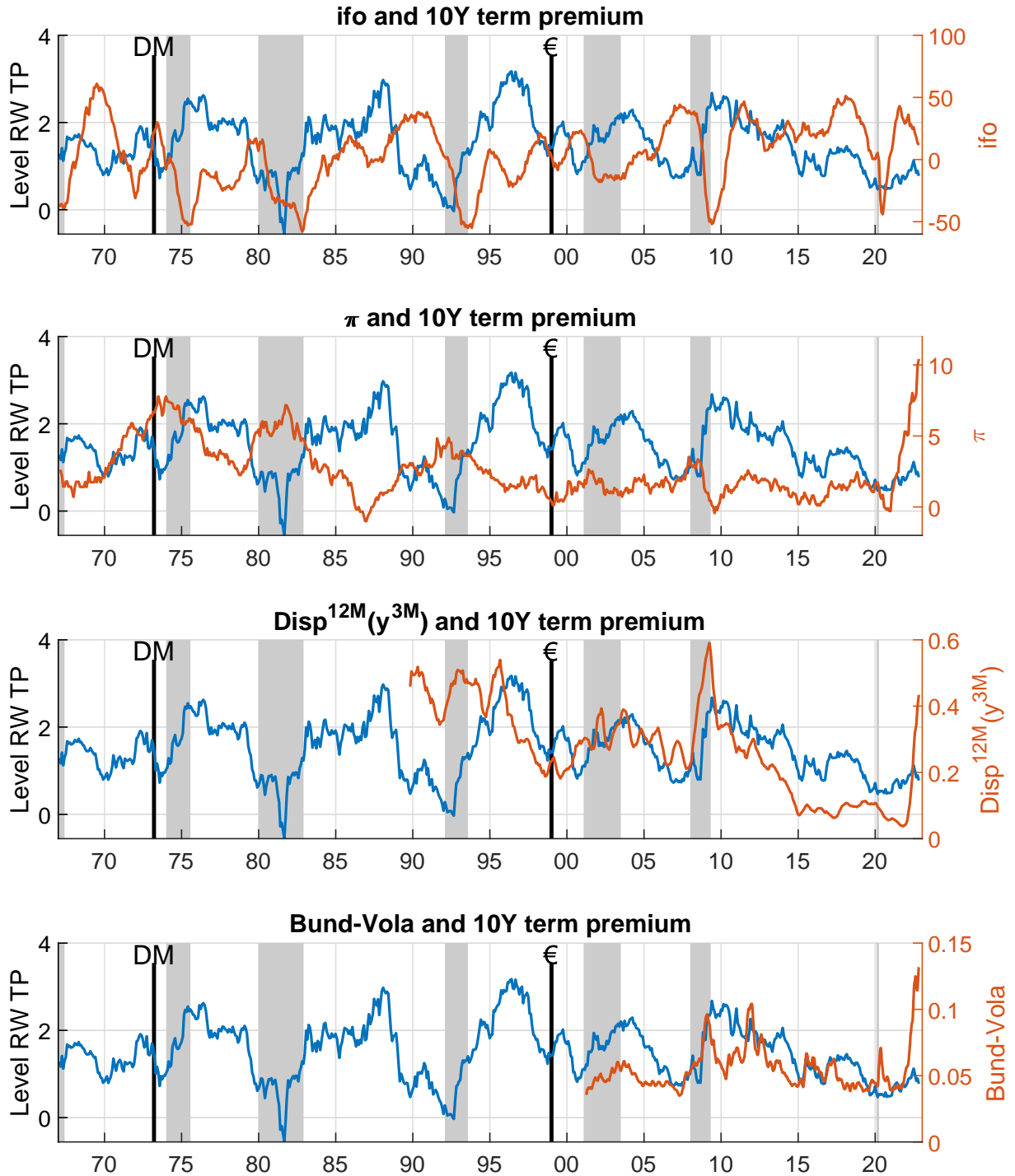
Risk premia in general should be countercyclical, which means they are high in bad states. Long-term Bunds are usually regarded to be free of default risk so their payments are fixed. However, the market value of long-term Bunds changes when yields move over the business cycle. The first panel of Figure 6 shows that the term premium in fact increases in German recessions (gray areas) and is high when business confidence (ifo index) is low. Table 3 summarizes, in the first column, that all models have a countercyclical term premium, similar to the US. Thus, the cyclical behavior of all models’ Bund term premium is in line with economic theory.

Inflation (π) in the second panel of Figure 6 determines the Bund’s future payments’ purchasing power. The sign of the relation of inflation to the term premium crucially depends on the type of economic shocks that investors are most concerned about. The correlation in Table 3 mixes different periods and the models weight them differently, resulting in different signs. Until the great moderation, high inflation was the major concern of policymakers and investors. Driven by the supply shocks of the oil crisis, inflation used to be particularly high in the recessions of the 1970s and 1980s.⁴⁵ The Bundesbank tried to control inflation by increasing policy rates (Fig. 4), such that long-term Bunds lost value (in Deutsche Mark) during recessions. The purchasing power of the lower long/term Bund value was further deteriorated by higher inflation. Moreover, high inflation in the 1970s and early 1980s induced high uncertainty about future inflation and, therefore, about the future purchasing power of long-term Bunds. As a consequence, investors required a higher compensation (i.e. term premium) for holding a long-term Bund relative to a sequential short-term investment in the earlier part of my sample.

⁴⁵Eickmeier and Hofmann (2022) provide a decomposition of inflation and growth into demand and supply factors.

Figure 6: Cyclicity of the Bund Term Premia

Month-end observations for the term premium. Monthly data for inflation π and business climate index *ifo*. The survey dispersions at the twelve-month horizon $Disp^{12M}(\cdot)$ are six-month averages of monthly data. Bund-Vola is the 50 trading-day average measured at month-end. *ifo* index in points, all other measures in percent p.a.. Gray bars indicate German recessions.



After successful disinflation in the 1980s, inflation became more stable. But although inflation decreased at that time, the term premium remained at a high level. Such a negative relation of inflation and the term premium is in line with a learning mechanism.⁴⁶ After the losses on long-term suffered due to high inflation in the 1970s, investors could only gradually be convinced about the low inflation environment in the 1980s despite the low levels of realized inflation. This motivates the slow-moving and backward-looking inflation process in the term structure model of [Cieslak and Povala \(2015\)](#). The results of [Blanco, Ottonello, and Ranosova \(2022\)](#) are in line with such a learning mechanism: Following a strong inflationary shock, long-term inflation expectations from survey participants remain high even after a successful disinflation.

Demand shocks played a relatively larger role in the economy from the 1990s onwards compared to the 1970s and early 1980s. A demand-driven recession is accompanied by low inflation and low policy rates. In such an environment, long-term Bunds do in fact serve as insurance, lowering the term premium. After the global financial crisis, inflation was well below 2%, the Eurosystem’s price stability objective. A deflationary scenario became more likely in which nominal long-term bonds have an even stronger insurance character for purchasing power. The reduction of term premia was accelerated further by the Eurosystem’s asset purchases, which sought to directly lower the term premium as a way of pushing up inflation.⁴⁷ Term premia for all models reached record-low levels in late 2019 and early 2020. It was only after the supply chain disruptions following Covid in 2021 and the energy price surge after Russia’s invasion of Ukraine, supply shocks with high inflation reversed the decline in term premia.

Uncertainty measures are positively related to term premia in [Table 3](#) and [Figure 6](#).⁴⁸ The implicit volatility in Bund Future options (Bund-Vola) peaks in the financial crisis in 2008 and the European sovereign debt crisis in 2012. While the spikes in Bund volatility are rather short-lived, the term premium increases at that time were more persistent. At the end of the sample in October 2022, Bund-Vola was at its highest level, reflecting huge uncertainty about the future path of monetary policy. The dispersion of analysts in surveys about future yields is another measure of uncertainty. Interestingly, the term premium of 10-year Bunds is more closely related to the disagreement about future short rates ($Disp(y^{3M})$) than to the disagreement about the long-term Bund yield itself ($Disp(par^{10Y})$). This emphasizes the crucial role of monetary policy reflected in short rates for long-term bond pricing.

Comparing the different models in [Table 3](#), macro cyclicalities is higher for the unrestricted model than for the restricted models that have superior forecasting performance.⁴⁹ There seems to be a slight trade-off between restricted models that deliver good forecasts and the unrestricted model that has more pronounced cyclical properties. Given that all models deliver plausible term premia characteristics and forecast better than naive yield random walk forecasts at least for specific segments, no model should be discarded.

⁴⁶The correlation between inflation and the term premium is negative for all models in the 1980s.

⁴⁷See [Eser, Lemke, Nyholm, Radde, and Vladu \(2019\)](#) for the effect of Eurosystem purchases on the term premium, and [Li and Wei \(2013\)](#) for quantitative easing in the US. Both implement the theoretical model of [Vayanos and Vila \(2009\)](#).

⁴⁸See [Abrahams et al. \(2013\)](#) for the US and the international comparison in [Cohen et al. \(2018\)](#).

⁴⁹A technical explanation for this finding is, that all restricted models increase time-series persistence. Hence, they reduce the role of the expectation component in the yield decomposition at the expense of the term premium.

6 Conclusion

The new Difference Estimators can be flexibly adapted to the dataset structure under investigation. Without needing an observable one-month short rate, my new Difference Estimators bring the models' data requirement closer to the actual structure of existing bond datasets. This is important for the Bund dataset, and other markets like inflation-linked products that lack a one-month rate by construction. For developed bond markets like US Treasuries, the Yield Difference estimator performs better than the classic DLR estimator if the observable short rate is noisy or inconsistent with longer maturities. Extensions to unspanned factors as in [Adrian et al. \(2013, section 4.1\)](#) or auxiliary restrictions on the variation of expectations in [Halberstadt \(2021\)](#) can be easily integrated. Thus, Difference Estimators can be used in basically all instances where a yield-factor model of JSZ can be estimated with maximum likelihood.

Difference Estimators deliver instantaneous results. This allows error bands to be bootstrapped for estimated objects of interest like the expectation component or the term premium. Furthermore, Difference Estimators are not subject to the curse of dimensionality inherent in existing maximum likelihood approaches which is especially relevant for multicurve models with many factors.⁵⁰ Even in datasets that are not balanced and/or contain too much interpolation noise, using Difference Estimators as a first step with an imperfect dataset may be useful: First, they deliver an endogenous structure of the real or complex risk-neutral eigenvalues, which needs to be specified in maximum likelihood approaches. Furthermore, the results of a Difference Estimator may also serve as starting values in nonlinear search procedures.⁵¹

Given that the current record-high inflation in the euro area is caused by supply shocks, it is useful to examine evidence from the 1970s. The new long history of yield decomposition data for German Bunds since 1967 enables a comparison of current yield developments with those of that time. Existing decompositions, in contrast, are usually based on data from the late 1990s onwards that were dominated by a decline in the yield level, stable inflation and demand shocks. Since the end of 2021, yields have increased both due to higher expected rates but also on account of higher term premia. In the light of the experience from the 1970s, the joint increase of term premia and inflation represents a warning sign, even if term premia are still at a low level by historical standards. However, if investors consider that European monetary policy lacks either the determination or ability to reduce inflation to the 2% objective, inflation risk may increase term premia further. After a failure to ensure price stability, it may take some time to restore monetary policy credibility, like in the 1980s, when the term premium was still high despite decreasing inflation. Therefore, a stability-oriented monetary policy currently seems to be the best way to keep the inflation component in the term premium low and, thus, ensure favorable financing conditions.

⁵⁰For example, estimating a ten-factor yield curve model with seven currencies using an adapted DLR in [Diez de Los Rios \(2017\)](#) avoids estimating 213 parameters in a maximum likelihood estimation.

⁵¹In a simulation exercise, [Golinski and Spencer \(2018\)](#) show that starting values based on DLR and ACM are better suited to achieving the global optimum than with random numbers usually used as starting values in JSZ maximum likelihood. Examples of coupon-based estimation are [Christensen, Lopez, and Rudebusch \(2010\)](#), [Andreasen, Christensen, and Riddell \(2020\)](#) or [Andreasen, Christensen, and Rudebusch \(2019\)](#).

A Term Structure Modelling and Estimation

A.1 Pricing Recursions

No-arbitrage implies investing in an $n+1$ period bond has the same price as an investment strategy that shifts from a short-term bond into an n -period bond when the short-term bond expires after one period. With the risk-neutral dynamics (2) we can equate the log price of both strategies:

$$\begin{aligned}
 p_t^{n+1} &= p_t^1 + E_t^{\mathbb{Q}}(p_{t+1}^n) + \frac{1}{2} \text{var}_t(p_{t+1}^n) \\
 \bar{A}_{\mathcal{P}}^{n+1} + \bar{B}_{\mathcal{P}}^{n+1} \mathcal{P}_t &= \bar{A}_{\mathcal{P}}^1 + \bar{B}_{\mathcal{P}}^1 \mathcal{P}_t + E_t^{\mathbb{Q}}(\bar{A}_{\mathcal{P}}^n + \bar{B}_{\mathcal{P}}^n \mathcal{P}_{t+1}) + \frac{1}{2} \text{var}_t(\bar{A}_{\mathcal{P}}^n + \bar{B}_{\mathcal{P}}^n \mathcal{P}_{t+1}) \\
 \bar{A}_{\mathcal{P}}^{n+1} + \bar{B}_{\mathcal{P}}^{n+1} \mathcal{P}_t &= \bar{A}_{\mathcal{P}}^1 + \bar{B}_{\mathcal{P}}^1 \mathcal{P}_t + \bar{A}_{\mathcal{P}}^n + \bar{B}_{\mathcal{P}}^n \cdot E_t^{\mathbb{Q}}(\mathcal{P}_{t+1}) + \frac{1}{2} (\bar{B}_{\mathcal{P}}^n) \text{var}_t(\mathcal{P}_{t+1}) (\bar{B}_{\mathcal{P}}^n)' \\
 \bar{A}_{\mathcal{P}}^{n+1} + \bar{B}_{\mathcal{P}}^{n+1} \mathcal{P}_t &= \bar{A}_{\mathcal{P}}^n + \bar{A}_{\mathcal{P}}^1 + \bar{B}_{\mathcal{P}}^n \nu_{\mathbb{Q}\mathcal{P}} + \frac{1}{2} \bar{B}_{\mathcal{P}}^n \Omega_{\mathcal{P}} (\bar{B}_{\mathcal{P}}^n)' + (\bar{B}_{\mathcal{P}}^1 + \bar{B}_{\mathcal{P}}^n \Phi_{\mathbb{Q}\mathcal{P}}) \mathcal{P}_t
 \end{aligned}$$

The constant and loadings of the factors \mathcal{P}_t on both sides have to be equal to ensure no-arbitrage and result in the recursions (4) and (5) in the main text:

$$\begin{aligned}
 \bar{B}_{\mathcal{P}}^{n+1} &= \bar{B}_{\mathcal{P}}^1 + \bar{B}_{\mathcal{P}}^n \Phi_{\mathbb{Q}\mathcal{P}} \\
 \bar{A}_{\mathcal{P}}^{n+1} &= \bar{A}_{\mathcal{P}}^n + \bar{A}_{\mathcal{P}}^1 + \bar{B}_{\mathcal{P}}^n \nu_{\mathbb{Q}\mathcal{P}} + \frac{1}{2} \bar{B}_{\mathcal{P}}^n \Omega_{\mathcal{P}} (\bar{B}_{\mathcal{P}}^n)'
 \end{aligned}$$

The boundary conditions are $\bar{A}_{\mathcal{P}}^0 = 0$ and $\bar{B}_{\mathcal{P}}^0 = 0$. For default-free nominal bonds, the short-rate constant $\bar{A}_{\mathcal{P}}^1$ and loadings $\bar{B}_{\mathcal{P}}^1$ are parameters of the model that are estimated.

A.2 The Return-Difference Estimator

This appendix introduces the Return-Difference Estimator as an extension to the traditional ACM estimator. It follows the structure of sections 2.3 and 3: First, I introduce the approach of ACM before turning to the new Return-Difference Estimator. Lastly, I illustrate why the Yield-Difference Estimator is the preferred approach for German Bunds.

Traditional Estimator of ACM: ACM begins by representing the bond return with maturity $n+1$ in excess of the 1-period short rate return in terms of priced factors. They replace the loadings $\bar{B}_{\mathcal{P}}^{n+1}$ with their components from the No-Arbitrage recursion (4) such that the short-rate loadings $\bar{B}_{\mathcal{P}}^1$ drop out of the excess return:⁵²

$$\begin{aligned}
 rx_{t+1}^{n+1} &= p_{t+1}^n - p_t^{n+1} + p_t^1 \\
 &= \bar{A}_{\mathcal{P}}^n - \bar{A}_{\mathcal{P}}^{n+1} + \bar{A}_{\mathcal{P}}^1 + \bar{B}_{\mathcal{P}}^n \mathcal{P}_{t+1} - \bar{B}_{\mathcal{P}}^n \Phi_{\mathbb{Q}\mathcal{P}} \mathcal{P}_t
 \end{aligned} \tag{16}$$

⁵²This formulation is not contained in Adrian et al. (2013) but was used in the working paper version Abrahams et al. (2013) of Abrahams, Adrian, Crump, Mönch, and Yu (2016) to write the model depending on $\Phi_{\mathbb{Q}}$ instead of risk premia parameters. Here I use the version with $\Phi_{\mathbb{Q}}$ to remain in line with DLR and to have a common starting point for the derivation of the self-consistent JSZ form with the method of Golinski and Spencer (2018).

The first step of the empirical implementation of ACM is to treat the identity (16) as a regression of the future realized excess returns on future factors (with coefficient $\hat{b}_r^n = \bar{B}_P^n$) and current factors (with coefficient $\hat{c}_r^n = \bar{B}_P^n \cdot \Phi_{\mathbb{Q}\mathcal{P}}$). The second step of ACM is to use \hat{b}_r for all bonds in order to invert \hat{c}_r for $\Phi_{\mathbb{Q}\mathcal{P}}$:

$$\Phi_{\mathbb{Q}\mathcal{P}} = \left(\hat{b}'_r \hat{b}_r \right)^{-1} \hat{b}'_r \hat{c}_r \quad (17)$$

The risk-neutral constant $\nu_{\mathbb{Q}\mathcal{P}}$ can be determined by formula (11) using the relation between excess return regression constant and yield regression constants $-\hat{a}_r = \hat{a}_p^+ - \hat{a}_p^1 - \hat{a}_p$ from equation (16).

The Return-Difference Estimator The observable short rate in ACM is needed to construct the excess return. Forming differences in returns of long-term bonds with maturity $n + 1$ and $k + 1$ removes the short rate. The difference between two holding period returns rh_{t+1}^{n+1} (not in excess of the one-period bond) is defined by:

$$\begin{aligned} rh_{t+1}^{n+1} - rh_{t+1}^{k+1} &= (p_{t+1}^n - p_t^{n+1}) - (p_{t+1}^k - p_t^{k+1}) \\ &= \bar{A}_P^n - \bar{A}_P^{n+1} - (\bar{A}_P^k - \bar{A}_P^{k+1}) \\ &\quad + (\bar{B}_P^n - \bar{B}_P^k) \mathcal{P}_{t+1} - (\bar{B}_P^n - \bar{B}_P^k) \Phi_{\mathbb{Q}\mathcal{P}} \mathcal{P}_t \end{aligned} \quad (18)$$

The first estimation step of RD is to estimate (18) by a regression of return differences on future and current factors that results in the regression coefficients $\hat{a}_{RD,n,k} = \bar{A}_P^n - \bar{A}_P^{n+1} - (\bar{A}_P^k - \bar{A}_P^{k+1})$, $\hat{b}_{RD,n,k} = (\bar{B}_P^n - \bar{B}_P^k)$ and $\hat{c}_{RD,n,k} = (\bar{B}_P^n - \bar{B}_P^k) \Phi_{\mathbb{Q}\mathcal{P}}$. With all maturities' coefficients stacked together, the inversion for the \mathbb{Q} -dynamics parameters is given by:

$$\Phi_{\mathbb{Q}\mathcal{P}} = \left(\hat{b}'_{RD} \hat{b}_{RD} \right)^{-1} \hat{b}'_{RD} \hat{c}_{RD} \quad (19)$$

The Return-Difference Estimator is an extension of ACM without an observable one-period bond. If there is an observable short rate, ACM is a special case of RD for $k = 0 \forall k$, i.e. the return difference in ACM is always made with to the short-term bond.

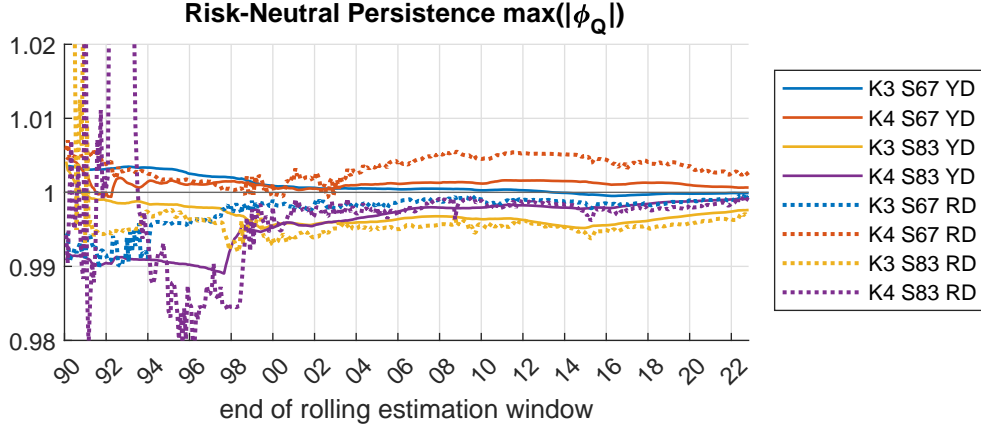
Empirical Results For the Bbk dataset, Figure 7 compares the persistence under the risk-neutral dynamics for expanding window estimates. The sample starts in 1967 (“S67”) or 1983 (“S83”).⁵³

The risk-neutral dynamics of the RD model become less stable as the window expands, in particular during the 1990s and for a short sample that starts in 1983 (“S83”). In contrast, the risk-neutral process of YD is relatively stable over time. YD seems more robust and is the preferred estimator.⁵⁴

⁵³The specification S83 contains a six-month yield in addition to the maturities in S67. It is restricted to the sample where the Bbk data have little interpolation problems in the six-month yield, see Appendix B.

⁵⁴The choice between RD and YD can be based on \mathbb{Q} -eigenvalues alone: The time-series persistence is identical for all unrestricted models of YD and RD since the principal components in the time-series VAR are identical and the VAR can be estimated by OLS without information about the \mathbb{Q} -dynamics.

Figure 7: Rolling Estimation of Return Difference Estimator



A.3 Overidentified Risk-Neutral Constants of Difference Estimators

For the Difference Estimators, the constant $\nu_{\mathbb{Q}\mathcal{P}}$ of the risk-neutral dynamics is determined from the first-step regression constants $\hat{a}_{YD,n,k}$ of the Yield-Difference estimator or $\hat{a}_{RD,n,k}$ of the Return-Difference estimator.

For Yield Difference, use $\hat{a}_p^n = \bar{A}_p^n$ and replace \bar{A}_p^{n+1} with its components of the No-Arbitrage recursion (5) such that the short-rate constant \bar{A}_p^m canceled out:

$$\begin{aligned} \hat{a}_{YD,n,k} &= \bar{A}_p^{n+1} - \bar{A}_p^{k+1} \\ \hat{a}_{YD,n,k} - \hat{a}_p^n + \hat{a}_p^k &= (\bar{B}_p^n - \bar{B}_p^k) \nu_{\mathbb{Q}\mathcal{P}} + \frac{1}{2} \bar{B}_p^n \Omega_{\mathcal{P}} (\bar{B}_p^n)' - \frac{1}{2} \bar{B}_p^k \Omega_{\mathcal{P}} (\bar{B}_p^k)' \end{aligned} \quad (20)$$

For Return Difference, use the relation between the regression constants $-\hat{a}_{RD,n,k} = \hat{a}_{YD,n,k} - \hat{a}_p^n + \hat{a}_p^k$.

For both Difference Estimators, the loadings \bar{B}_p^n are model-implied by \bar{B}_p^1 and $\Phi_{\mathbb{Q}\mathcal{P}}$ and the covariance matrix $\Omega_{\mathcal{P}}$ is taken from the time-series VAR. Then, equation (20) can be inverted for $\nu_{\mathbb{Q}\mathcal{P}}$ as in equation (11) for DLR and ACM in the main text. Finally, $\nu_{\mathbb{Q}\mathcal{P}}$ will serve as an input to determine the short-rate constant \bar{A}_p^1 in equation (15) of section 3.2.

A.4 Self-Consistent Risk-Neutral Dynamics

The latent factors of the self-consistent canonical JSZ form are denoted X_t in the remainder to distinguish them from the observed factors \mathcal{P}_t . The key feature of the JSZ canonical form is that $\Phi_{\mathbb{Q}X}$ contains K eigenvalues $\phi_{\mathbb{Q}X}$ in an ordered Jordan form.⁵⁵ For only real distinct eigenvalues, the Jordan form is diagonal. The canonical JSZ specification must satisfy the relation between the observable yield principal components \mathcal{P}_t and the latent structural factors X_t that underlay an identified model:

$$\mathcal{P}_t = W \cdot y_t = W \cdot A_X + W \cdot B_X X_t \quad (21)$$

⁵⁵See Lütkepohl (2005, App. A.9) and Joslin et al. (2011, App. C) for technical details of the Jordan decomposition.

Basically, the transformation from the JSZ canonical form into the observable factor model is an invariant transformation with the matrices $W \cdot A_X$ and $W \cdot B_X$. However, the matrix B_X depends on $\Phi_{\mathbb{Q}X}$ which is supposed to be determined, and a simple inversion from $\Phi_{\mathbb{Q}\mathcal{P}}$ to $\Phi_{\mathbb{Q}X}$ is not possible. Therefore, the maximum likelihood estimation of JSZ treats the eigenvalues $\phi_{\mathbb{Q}X}$ as nonlinear parameters.

Golinski and Spencer (2018) show that eigenvalues $\phi_{\mathbb{Q}\mathcal{P}}$ of the regression-based, but overidentified risk-neutral factor dynamics $\Phi_{\mathbb{Q}\mathcal{P}}$ correspond to eigenvalues $\phi_{\mathbb{Q}X}$ of the self-consistent but latent JSZ process X_t . Thus, a simple eigenvalue decomposition of the regression-based $\Phi_{\mathbb{Q}\mathcal{P}}$ yields the eigenvalues $\phi_{\mathbb{Q}X}$ in Jordan form and replaces a nonlinear search over the eigenvalues in the maximum likelihood estimation of JSZ. In a single-curve setting, $\Phi_{\mathbb{Q}X}$ alone determines B_X since short-rate loadings B_X^m are normalized to a vector of ones in JSZ. The \mathbb{P} -dynamics for X_t including Ω_X can be determined with A_X and B_X at hand from a simple invariant transformation of the VAR-based parameters of the observed factor \mathcal{P}_t dynamics.

A.5 Self-Consistent Risk-Neutral Constants

The yield constants A_X^n of the self-consistent JSZ form are linear in the short-rate constant A_X^1 and the constant terms in the risk neutral dynamics $\nu_{\mathbb{Q}X}$. The structure is identical to the structure of the recursions of the constant for the observable factors \mathcal{P}_t in equation (15). In the self-consistent model of JSZ, there is only one free parameter. Thus, K arbitrary elements of A_X^1 and $\nu_{\mathbb{Q}X}$ can be set equal to zero. In line with JSZ, I set the short rate constant to zero ($A_X^1 = 0$) and choose the first element in $\nu_{\mathbb{Q}X}$ as the free parameter which will be called $\nu_{\mathbb{Q}X}^\infty$. Then, the yield loading of a n -period yield can be represented as:

$$A_X^n = A_{\nu_X}^n \nu_{\mathbb{Q}X}^\infty + A_{\Omega_X}^n$$

The scalar $A_{\nu_X}^n$ is the effect of the first element of the drift in the \mathbb{Q} -dynamics of the yields. Technically, $A_{\nu_X}^n$ corresponds to the yield constant of an n -period bond with $A_X^1 = 0$, $\Omega_X = 0$, and $\nu_{\mathbb{Q}X} = e_1$. It depends only on B_X^n , i.e. the eigenvalues $\phi_{\mathbb{Q}X}$. The scalar $A_{\Omega_X}^n$ describes the impact of convexity on the yields, which is independent of the constants being estimated. Technically, it is the constant from the recursions with $\nu_{\mathbb{Q}X} = 0$ and $A_X^1 = 0$. It depends on B_X^n and Ω_X (derived from $\Omega_{\mathcal{P}}$ using an invariant transformation).

A_X collects all constants of yields that form the principal components. It is not observable, but the yield constant $A_{\mathcal{P}}$ of the observable factors from section 3.2 is. Thus, the maturities selected to determine short-rate parameters are identical for the overidentified models and the self-consistent models. With the invariant transformation of the yield constant $A_{\mathcal{P}}$ of the observable factors into the unobservable and structural yield constant A_X , we can determine the risk-neutral constant $\nu_{\mathbb{Q}X}^\infty$:

$$\begin{aligned} A_{\mathcal{P}} &= \underbrace{(I - B_X(W \cdot B_X)^{-1}W)}_H (A_{\nu_X}^n \nu_{\mathbb{Q}X}^\infty + A_{\Omega_X}^n) \\ \bar{y} - B_{\mathcal{P}}\bar{\mathcal{P}} - HA_{\Omega_X}^n &= HA_{\nu_X}^n \nu_{\mathbb{Q}X}^\infty \\ \nu_{\mathbb{Q}X}^\infty &= ((HA_{\nu_X}^n)'(HA_{\nu_X}^n))^{-1}(HA_{\nu_X}^n)'(\bar{y} - B_{\mathcal{P}}\bar{\mathcal{P}} - HA_{\Omega_X}^n) \quad (22) \end{aligned}$$

The constant $\nu_{\mathbb{Q}X} = (\nu_{\mathbb{Q}X}^\infty \ 0_{1 \times K-1})'$ is the last missing parameter for a self-consistent model. Risk-neutral eigenvalues that determine the Jordan form are determined in section A.4 and the short-rate parameters of the canonical model are fixed $A_X^1 = 0$ and $B_X^1 = 1_{1 \times K}$.

A.6 Risk Premium Restrictions

In a canonical model without any further restrictions, the real-world dynamics can be estimated based on OLS in a VAR of the observed yield principal components \mathcal{P}_t as in JSZ. [Cochrane and Piazzesi \(2005\)](#), [Duffee \(2011b\)](#) and others suggest that risk premia of US Treasuries are driven by a single factor. Technically speaking, this implies that the risk premium parameters are driven by one linear combination of the factors and the risk premium parameters $\Lambda_1 = \Phi_{\mathbb{Q}} - \Phi_{\mathbb{P}}$ have a rank of one.

More generally, a model with auxiliary restrictions on the market price of risk or the real-world dynamics is estimated by GLS conditional on the risk-neutral dynamics with the weighting matrix $\Omega_{\mathcal{P}}$:

$$\begin{aligned} \mathcal{P}_{t+1} &= \nu_{\mathbb{P}\mathcal{P}} + \Phi_{\mathbb{P}\mathcal{P}}\mathcal{P}_t + \varepsilon_t^{\mathcal{P}} \\ \mathcal{P}_{t+1} &= (\nu_{\mathbb{Q}\mathcal{P}} + \Lambda_{0\mathcal{P}}) + (\Phi_{\mathbb{Q}\mathcal{P}} + \Lambda_{1\mathcal{P}})\mathcal{P}_t + \varepsilon_t^{\mathcal{P}} \\ \mathcal{P}_{t+1} - \nu_{\mathbb{Q}\mathcal{P}} - \Phi_{\mathbb{Q}\mathcal{P}}\mathcal{P}_t &= \Lambda_{0\mathcal{P}} + \Lambda_{1\mathcal{P}}\mathcal{P}_t + \varepsilon_t^{\mathcal{P}} \end{aligned} \quad (23)$$

Thus, putting restrictions on Λ_1 or $(\Lambda_0 \ \Lambda_1)$ creates a link through $\Omega_{\mathcal{P}}$ between the physical dynamics and the short-rate constant. However, the yield loadings B^n are not affected since they only depend on the risk-neutral dynamics $\Phi_{\mathbb{Q}\mathcal{P}}$.

I use a rank restriction on Λ_1 such that the K -dimensional matrix Λ_1 is only driven by $rk < K$ linear combinations. Technically, this corresponds to an eigenvalue decomposition of Λ_1 into eigenvectors and eigenvalues. In the ordered Jordan form, the $K - rk$ smallest eigenvalues are set equal to zero before Λ_1 is reconstructed based on the eigenvectors and the rk largest eigenvalues in Jordan form. I use the code published by [Joslin et al. \(2011\)](#) for the implementation.

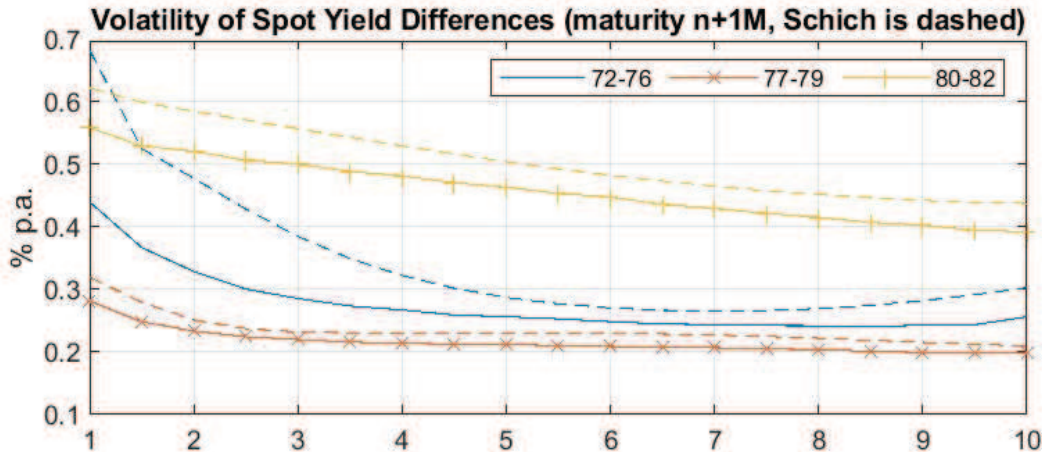
In the maximum likelihood estimation of JSZ, the elements of the weighting matrix Ω are optimized in a nonlinear way and will be considered in the GLS estimation of equation (23). For the linear regressors, I follow the iterative approach for $\Omega_{\mathcal{P}}$ introduced by [Golinski and Spencer \(2018\)](#):

1. Take the risk-neutral eigenvalues and the normalized short-rate loadings as given to determine implicit $B_{\mathcal{P}}$.
2. Initiate the variance-covariance matrix with the OLS estimate $\Omega_{\mathcal{P}}^{[0]}$
3. Determine the implicit constant parameters $\nu_{\mathbb{Q}\mathcal{P}}$ conditional on $\Omega_{\mathcal{P}}^{[i]}$.
4. Estimate the restricted \mathbb{P} -dynamics (23) conditional on $\Omega_{\mathcal{P}}^{[i]}$. Determine $\Omega_{\mathcal{P}}^{[i+1]}$ from estimated \mathbb{P} -dynamics.
5. Repeat iteration steps 3. and 4. until convergence when the change in the norm of $\Omega_{\mathcal{P}}^{[i]}$ is smaller than 10^{-6} . This corresponds to the choice of [Golinski and Spencer \(2018\)](#).

B Bund Term Structure Data since 1967

The Bundesbank yield curve data interpolated using the method of [Svensson \(1994\)](#) – outlined in [Schich \(1997\)](#) – contains month-end observations since September 1972. During the first years, only a few Bunds are outstanding and the maturity spectrum is limited, see [Schich \(1997, Abb. 5\)](#). Because bonds expire only infrequently, even one-year yields from the Bbk dataset are often extrapolated before 1983. Similar problems – albeit to a lesser extent – exist for the interpolated 10-year Bbk yield from the Schich dataset due to infrequent issues of 10-year Bunds and missing longer maturities. [Figure 8](#) (dashed blue line) illustrates the increasing variation of extrapolated Schich yields between 1972 and 1976 for maturities above eight years. The problem declines in the later samples when bonds are available across the whole maturity spectrum. Thus, the mathematical optimizer of the cross-sectional interpolation seems to place more weight on intermediary maturities at the cost of a worse fit at the borders of the maturity spectrum. For a fairly large spectrum of long and short maturities, however, the volatility before 1977 are elevated, which is likely to reflect extrapolation noise. Using the extrapolated short maturities to construct the observed factors ($\mathcal{P}_t = W \cdot y_t$) would introduce data noise in the economic dynamics.

Figure 8: Spot Yield Volatility of Schich and Bootstrap



[Cassola and Luis \(2003\)](#) estimate a Bund model from 1972 to 1998 with Schich data. They impose a minimum maturity of one year to reduce extrapolation noise. Furthermore, their traditional Kalman Filter approach does not assume perfectly measured observed factors and allows for a rather large measurement error for the one-year maturity. [Fendel \(2008\)](#) takes a similar approach for the Schich data starting in 1979 and adds money market rates maturities below one year.

For the early sample, I use a different dataset to estimate the yield curve. The sample starting in 1967 is based on a par yield curve (Renditestruktur) constructed by [Deutsche Bundesbank \(1978\)](#).⁵⁶ The month-end data are available for maturities from one year to ten years with six-month spacing. From the par yield curve, I bootstrap zero-coupon yield

⁵⁶The data were published on a regular basis until December 1982 in Statistisches Beiheft, Reihe 2, Tabelle 8d.

curves under the assumption of piecewise constant forward yields following the method of [Fama and Bliss \(1987\)](#). The par yield data is smoothed, so additional smoothing in the bootstrap is not necessary. The bootstrapped yields and the Schich yields are highly correlated for intermediate maturities where data noise from extrapolation is not a problem. The time series of medium-term to long-term spot yields have been identical within a few basis points since the late 1970s.

The interpolation underlying the par yield curve is less flexible than the [Svensson \(1994\)](#) approach and is therefore less prone to extrapolation noise. In all periods, the volatility of the Schich data is higher than that of the bootstrap (see [Fig. 8](#)). That might reflect a lack of flexibility in the par yields or excessive flexibility in the Schich data, or both. Visual inspections of the shape of the yield curve data in 1982 suggests minor extrapolation noise in several months of the Schich data. Therefore, I use Schich data from 1983 to 2022 extended by bootstrapped par yields from 1967 to 1982.

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