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Research Report

Error estimation of the integral tau method for fourth order overdetermined ODES

ERROR ESTIMATION OF THE INTEGRAL TAU METHOD FOR FOURTH ORDER OVERDETERMINED ODES

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Abstract

In this paper, the error estimate for the integrated formulation of the tau method for over determined ordinary differential equations is presented. In the earlier works of these authors, the generalized error estimation of tau approximates for the ODES for the integrated formulation for non-overdetermined ODES were reported. In this work emphasis is laid on the error and error estimate, we established the accuracy of the results using the principle of mathematical induction and through some numerical experiments.

Keywords: Ordinary Differential equation, Variance, Formulation and Error Estimation

Introduction

The Tau method , is a method initially formulated as a tool for the approximation of special functions of Mathematical Physics which could be expressed in terms of simple differential equations. Adeniyi[1]. Error estimation is the study of kind and quantity of error, or uncertainty that may be present in the solution to a problem. Its a method used to document the error that appear in learner language, determine whether those errors are systematic and possibly explain what caused them.[2&5] The first attempt on error estimation of the Tau method was developed by Lanczos[8] as a simple algebraic approach using the relation of chebysev polynomial to trigonometric functions limited to first order problem. Fox[6] later developed an approach which could handle higher order problems in ordinary and partial differential equations. In this vein , Yisa and Adeniyi [12] developed a generalization of the error and the error estimation process of the Ortiz recursive formulation of the Tau method. The error estimation for ODE's has been reported by many researchers including [4,12,2&9] to mention a view. Also Ojo and Adeniyi[10 and 11] discussed the two variants namely the differential and the integrated formulations. Namasivayam[9] reported an error analysis of the Tau method. Moreover [5] discussed a prior Integration of Numerical Integration Scheme for certain IBVP in Partial Differential Equations with Error Estimation .However, a polynomial error approximation of the error function en(x) of the

Lanczos Tau method for ODE's based on the error of Lanczos economization process was constructed by Adeniyi[3] dependence of the approximation error on the choice of the perturbation term. where he modified the approximant by perturbing some of the homogeneous conditions of en(x).]. This paper is aimed at obtaining a better approximation by the integrated formulation of the Tau method with its error estimation for fourth order overdetermined differential equations, for the selected problems due to the higher order of the perturbation term in order to improve on the work of Erebholo[6].

Let us consider the m th order linear differential system.

$$Ly(x) = \sum_{r=0}^{m} xr\,(x)y^{(r)}(x) = \sum_{r}^{f} frx^{r}, a \le x \le b \qquad (1a)$$

$$L^{\mathrm{x}}\mathrm{yOGk}) := \sum_{r=0}^{m-1} \mathrm{Grky}^{(\mathrm{r})}(\mathrm{xn}) = \infty k1k = 0(1)(m-1) \qquad (1b)$$

With smooth solution y(x), a ≤ x ≤ b, |a| < + ∞ |b| < + ∞

1. Error Estimation of the Integrated Formulation of the Tau Method

The integrated error estimation is considered here with the aim of improving the accuracy of the estimate of differential form of tau method. To this end, let

$$\iint\cdots\int^{m} L(\cdot)dx \qquad (1c)$$

The integral form of the error equation

$$Le_n(x) = -H_n(x) \qquad (1ds)$$

is therefore

$$I_L(e_n(x)) = -\iint\cdots\int^{m} H_n(x)dx + C_{m-1}(x) \qquad (2)$$

We consider the perturbed form of (2) i.e, perturbed error equation

$$I_L(e_n(x))_{n+1} = -\iint\cdots\int^{m} H_n(x)dx + C_{m-1}(x)\hat{H}_{n+m+1}(x)$$

which is equivalent to

$$= -\iint\cdots\int^{m}\left(\sum_{r=0}^{m+s-1}\tau_{m+s-r}T_{n-m+r+1}(x)\right)dx + C_{m-1}(x)$$

$$+\sum_{r=0}^{m+s-1}\hat{\tau}_{m+s-r}T_{n-m+r+s}(x) \tag{3}$$

and which is satisfied by

$$(e_n(x))_{n+1}=\frac{\hat{\phi}_n x^m T_{n+m+1}(x)}{C_{n-m+1}^{n-m+1}} \tag{4}$$

and where

$$\hat{H}_{n+m+1}(x)=\hat{\tau}_1 T_{n+m+1}(x)+\hat{\tau}_2 T_{n+m+s}(x)+\cdots+\hat{\tau}_{m+s}T_{n+s} \tag{5}$$

We insert (5) into (6) and equate the coefficients of powers of $x^{n+m+s+1}, x^{n+m+s}, \cdots x^{n+m}$ for the determination of the parameter $\hat{\phi}_n$ of $(e_n(x))_{n+1}$. We then have

$$\varepsilon^* = \frac{|\hat{\phi}_n|}{2^{2n-2m+1}} \tag{6}$$

as an estimate of ε.

We shall carry out these steps for obtaining $\hat{\phi}_n$ with various values of s and m remains fourth order and then generalise the results to obtain a recursive formula for $\hat{\phi}_n$.

Derivation of Error Estimate for the Fourth Order Integrated Formulation of the Tau Method

In this section, we consider the error estimation of the integrated formation of Lanczos Tau method for varying degrees for the class of problems:

$$Ly(x):=\sum_{r=0}^{m}\alpha_r(x)y^{(r)}(x)=\sum_{r}^{F}f_r x^r, a\le x\le b \tag{7}$$

$$L^* y(x_{rk}):=\sum_{r=0}^{m-1}a_{rk}y^{(r)}(x_{rk})=\alpha_k, k=0(1)(m-1) \tag{8}$$

We shall consider case $m=4, s=3$. From (7) and (8) above we have the problem

$$\begin{aligned}Ly(x):=&\left(\alpha_{40}+\alpha_{41}x+\alpha_{42}x^2+\alpha_{43}x^3+\alpha_{44}x^4\alpha_{45}x^5+\alpha_{46}x^6+\alpha_{47}x^7\right)y^{iv}(x)\\&+\left(\alpha_{30}+\alpha_{31}x+\alpha_{32}x^2+\alpha_{33}x^3+\alpha_{34}x^4+\alpha_{35}x^5+\alpha_{36}x^6\right)y'''(x)\\&+\left(\alpha_{20}+\alpha_{21}x+\alpha_{22}x^2+\alpha_{23}x^3+\alpha_{24}x^4+\alpha_{25}x^5\right)y''(x)\\&+\left(\alpha_{10}+\alpha_{11}x+\alpha_{12}x^2+\alpha_{13}x^3+\alpha_{14}x^4\right)y'(x)\\&+\left(\alpha_{00}+\alpha_{01}x+\alpha_{02}x^2+\alpha_{03}x^3\right)y(x)=\sum_{r}^{F}f_r x^r \ ,0\le x\le 1\end{aligned} \tag{10}$$

$$y(0)=\rho_0, y'(0)=\rho_1, y''(0)=\rho_2, y'''(0)=\rho_3 \tag{11}$$

Where ∞_{rk}, ρ_0, ρ_1, ρ_2, ρ_3 are known real numbers

Which leads to the perturbed integrated error equation

$$\int_0^x\int_0^u\int_0^t\int_0^s\Big[\big(\alpha_{40}+\alpha_{41}v+\alpha_{42}v^2+\alpha_{43}v^3+\alpha_{44}v^4\alpha_{45}v^5$$
$$+\alpha_{46}v^6+\alpha_{47}v^7\big)\big(e_n^{iv}(v)\big)_{n+1}+\big(\alpha_{30}+\alpha_{31}v+\alpha_{32}v^2+$$
$$\alpha_{33}v^3+\alpha_{34}v^4+\alpha_{35}v^5+\alpha_{36}v^6\big)\big(e_{n'''}(v)\big)_{n+1}+\big(\alpha_{20}+\alpha_{21}v$$
$$+\alpha_{22}v^2+\alpha_{23}v^3+\alpha_{24}v^4+\alpha_{25}v^5\big)\big(e_{n''}(v)\big)_{n+1}+\big(\alpha_{10}+\alpha_{11}v$$
$$\alpha_{12}v^2+\alpha_{13}v^3+\alpha_{14}v^4\big)\big(e_{n'}(v)\big)_{n+1}$$
$$+\big(\alpha_{00}+\alpha_{01}v+\alpha_{02}v^2+\alpha_{03}v^3\big)\big(e_n(v)\big)_{n+1}\Big]dvdsdtdu$$
$$=-\int_0^x\int_0^u\int_0^t\int_0^s H_n(v)dvdsdtdu+\hat{H}_{n+m+1}(v) \tag{12}$$

and where

$$\big(e_n(x)\big)_{n+1}=\frac{\hat{\phi}_n x^4 T_{n-3}}{C_{n-3}^{(n-3)}} \tag{13}$$

$$H_n(x)=\tau_7 T_{n-3}(x)+\tau_6 T_{n-2}(x)+\tau_5 T_{n-1}(x)+\tau_4 T_n(x)$$
$$+\tau_3 T_{n+1}(x)+\tau_2 T_{n+2}(x)+\tau_1 T_{n+3}(x) \tag{14}$$

$$\hat{H}_{n+m+1}(x)=\hat{\tau}_1 T_{n+8}(x)+\hat{\tau}_2 T_{n+7}(x)+\hat{\tau}_3 T_{n+6}(x)$$
$$+\hat{\tau}_4 T_{n+5}(x)+\hat{\tau}_5 T_{n+4}(x)+\hat{\tau}_6 T_{n+3}(x)+\hat{\tau}_7 T_{n+2}(x) \tag{15}$$

Now,

$$\big(e_n(x)\big)_{n+1}=\frac{\hat{\phi}_n x^4 T_{n-3}}{C_{n-3}^{(n-3)}} \tag{16}$$

$$\big(e_n(x)\big)_{n+1}=\frac{\hat{\phi}_n x^4 T_{n-3}}{2^{2n-7}}(x) \tag{17}$$

$$\big(e_n(x)\big)_{n+1}=\frac{\hat{\phi}_n}{p_1}\left[p_1 x^{n+1}+p_2 x^n+p_3 x^{n-1}+p_4 x^n+\cdots\right] \tag{18}$$

where

$$p_1 = C_{n-3}^{n-3}, p_2 = C_{n-4}^{n-3}, p_3 = C_{n-5}^{n-3}, p_4 = C_{n-6}^{n-3} \ ,$$

$$p_5 = C_{n-7}^{n-3}, p_6 = C_{n-8}^{n-3}, p_7 = C_{n-9}^{n-3} \ ,\cdots \tag{19}$$

Thus from(12) we have

$$\int_0^x\int_0^u\int_0^t\int_0^s \left(e_n(v)\right)_{n+1} dvdsdtdu = \frac{\phi_n}{p_1}\left[\frac{p_1 x^{n+5}}{(n+2)(n+3)(n+4)(n+5)}\right.$$

$$+\frac{p_2 x^{n+4}}{(n+1)(n+2)(n+3)(n+4)}+\frac{p_3 x^{n+3}}{(n)(n+1)(n+2)(n+3)}$$

$$+\frac{p_4 x^{n+2}}{(n-1)(n)(n+1(n+2)}+\frac{p_5 x^{n+1}}{(n-2)(n-1)(n)(n+1)}$$

$$\left.+\frac{p_6 x^{n}}{(n-3)(n-2)(n-1)(n)}\right] \tag{20}$$

$$\int_0^x\int_0^u\int_0^t\int_0^s \left(e_{n'}(v)\right)_{n+1} dvdsdtdu = \frac{\hat{\phi}_n}{p_1}\left[\frac{p_1 x^{n+4}}{(n+2)(n+3)(n+4)}\right.$$

$$+\frac{p_2 x^{n+3}}{(n+1)(n+2)(n+3)}+\frac{p_3 x^{n+2}}{(n)(n+1)(n+2)}+\frac{p_4 x^{n+1}}{(n-1)(n)(n+1)}$$

$$\left.+\frac{p_5 x^{n}}{(n-2)(n-1)(n)}+\frac{p_6 x^{n-1}}{(n-3)(n-2)(n-1)}\right] \tag{21}$$

$$\int_0^x\int_0^u\int_0^t\int_0^s \left(e_{n''}(v)\right)_{n+1} dvdsdtdu = \frac{\hat{\phi}_n}{p_1}\left[\frac{p_1 x^{n+3}}{(n+2)(n+3)}\right.$$

$$+\frac{p_2 x^{n+2}}{(n+1)(n+2)}+\frac{p_3 x^{n+1}}{(n)(n+1)}+\frac{p_4 x^{n}}{(n-1)(n)}$$

$$\left.+\frac{p_5 x^{n-1}}{(n-2)(n-1)}+\frac{p_6 x^{n-2}}{(n-3)(n-2)}\right] \tag{22}$$

$$\int_0^x\int_0^u\int_0^t\int_0^s \left(e_{n'''}(v)\right)_{n+1} dvdsdtdu = \frac{\hat{\phi}_n}{p_1}\left[\frac{p_1 x^{n+2}}{(n+2)}\right.$$

$$\left. + \frac{p_2 x^{n+1}}{(n+1)} + \frac{p_3 x^n}{(n)} + \frac{p_4 x^{n-1}}{(n-1)} + \frac{p_5 x^{n-2}}{(n-2)} + \frac{p_6 x^{n-3}}{(n-3)}\right] \tag{23}$$

$$\int_0^x\int_0^u\int_0^t\int_0^s \left(e_n^{iv}(v)\right)_{n+1} dvdsdtdu = \frac{\hat{\phi}_n}{p_1}\left[p_1 x^{n+1}\right.$$

$$\left. + p_2 x^n + p_3 x^{n-1} + p_4 x^{n-2} + p_5 x^{n-3} + p_6 x^{n-4}\right] \tag{24}$$

Substituting (20), (21), (22), (23) and (24) into (12), collecting the like terms and equating the corresponding coefficients of $x^{n+12}, x^{n+11}, x^{n+10} \cdots x^n$, the following system of equations obtained below were solved to get $\hat{\phi}_n$:

$$\hat{\tau}_1 C_{n+8}^{(n+8)} = \frac{\hat{\phi}_n}{C_{n-3}^{(n-3)}}\left[\frac{3!}{7!}\alpha_{03}\frac{C_{n-7}^{(n-3)}}{\substack{(n-2)(n-1)\\(n)(n+1)}} + \frac{2!}{6!}\alpha_{02}\frac{C_{n-6}^{(n-3)}}{\substack{(n-1)(n)\\(n+1)(n+2)}}\right.$$

$$+ \frac{1!}{5!}\alpha_{01}\frac{C_{n-5}^{(n-3)}}{\substack{(n)(n+1)\\(n+2)(n+3)}} + \frac{1!}{4!}\alpha_{00}\frac{C_{n-4}^{(n-3)}k6}{\substack{(n+1)(n+2)\\(n+3)(n+4)}} + \frac{4!}{8!}\alpha_{14}\frac{C_{n-7}^{(n-3)}}{\substack{(n-2)(n-1)\\(n)}}$$

$$+ \frac{3!}{7!}\alpha_{13}\frac{C_{n-6}^{(n-3)}}{\substack{(n-1)(n)\\(n+1)}} + \frac{2!}{6!}\alpha_{12}\frac{C_{n-5}^{(n-3)}}{\substack{(n)(n+1)\\(n+2)}} + \frac{1!}{5!}\alpha_{11}\frac{C_{n-4}^{(n-3)}}{\substack{(n+1)(n+2)\\(n+3)}}$$

$$+ \frac{1!}{4!}\alpha_{10}\frac{C_{n-3}^{(n-3)}}{\substack{(n+2)(n+3)\\(n+4)}} + \frac{5!}{9!}\alpha_{25}\frac{C_{n-7}^{(n-3)}}{(n-2)(n-1)} + \frac{4!}{8!}\alpha_{24}\frac{C_{n-6}^{(n-3)}}{(n-1)(n)}$$

$$+ \frac{3!}{7!}\alpha_{23}\frac{C_{n-5}^{(n-3)}}{(n)(n+1)} + \frac{2!}{6!}\alpha_{22}\frac{C_{n-4}^{(n-3)}}{(n+1)(n+2)} + \frac{1!}{5!}\alpha_{21}\frac{C_{n-3}^{(n-3)}}{(n+2)(n+3)}$$

$$+ \frac{5!}{9!}\alpha_{35}\frac{C_{n-6}^{(n-3)}}{(n-1)} + \frac{4!}{8!}\alpha_{34}\frac{C_{n-5}^{(n-3)}}{(n)} + \frac{3!}{7!}\alpha_{33}\frac{C_{n-4}^{(n-3)}}{(n+1)}$$

$$+ \frac{2!}{6!}\alpha_{32}\frac{C_{n-3}^{(n-3)}}{(n+2)} + \frac{7!}{11!}\alpha_{47}C_{n-7}^{(n-3)} + \frac{6!}{10!}\alpha_{46}C_{n-6}^{(n-3)}$$

$$\left. + \frac{5!}{9!}\alpha_{45}\frac{C_{n-5}^{(n-3)}}{(n)} + \frac{4!}{8!}\alpha_{44}C_{n-4}^{(n-3)} + \frac{3!}{7!}\alpha_{43}C_{n-3}^{(n-3)}\right] \tag{25}$$

Solving the above equations by forward elimination method to get $\hat{\varphi}_n$ using the relation

$$C_n^n = 2^{2n-1}, C_{n-1}^{(n)} = \frac{1}{2} n C_n^{(n)} = -n 2^{2n-2}$$

Thus; solving the resulting equations above for $\hat{\phi}_n$ gives

$$\hat{\varphi}_n = \left[\frac{m_1}{\prod_{r=1}^{m} (n+s+r)} - \frac{m_2}{\prod_{r=1}^{m} (n+s+r)} + \frac{m_3}{\prod_{r=1}^{m} (n+s+r)} \right.$$

$$- \frac{m_4}{\prod_{r=1}^{m} (n+s+r)} + \frac{m_5}{\prod_{r=1}^{m} (n+s+r)} - \frac{m_6}{\prod_{r=1}^{m} (n+s+r)} - - - - - - - -$$

$$\left. + \frac{m_{123}}{\prod_{r=1}^{m} (n+s+r-6)} \right] \frac{P_1 \tau_5}{R_8} - \left[\frac{m_{124}}{\prod_{r=1}^{m} (n+s+r-5)} \right.$$

$$\left. + \frac{m_{125}}{\prod_{r=1}^{m} (n+s+r-6)} \right] \frac{P_1 \tau_6}{R_8} - \frac{P_1^2 \tau_{m+s}}{\prod_{r=1}^{m} (n+s+r-6) R_{m+s+1}} \qquad \textbf{(26)}$$

where

$$m_1 = \frac{C_n^{(n)} C_{n+2}^{(n+7)} C_{n+7}^{(n+8)}}{C_{n+2}^{(n+2)} C_{n+7}^{(n+7)} C_{n+8}^{(n+8)}}, \; m_2 = \frac{C_{n+2}^{(n+8)} C_n^{(n)}}{C_{n+8}^{(n+8)} C_{n+2}^{(n+2)}}$$

$$m_3 = \frac{C_{n+2}^{(n+6)} C_{n+6}^{(n+8)} C_n^{(n)}}{C_{n+6}^{(n+6)} C_{n+8}^{(n+8)} C_{n+2}^{(n+2)}}, \; m_4 = \frac{C_{n+2}^{(n+6)} C_{n+6}^{(n+7)} C_{n+7}^{(n+8)} C_n^{(n)}}{C_{n+6}^{(n+6)} C_{n+2}^{(n+7)} C_{n+8}^{(n+8)} C_{n+2}^{(n+2)}} \quad - - - - - -$$

$$m_{59} = \frac{C_{n+5}^{(n+6)} C_{n+6}^{(n+8)} C_n^{(n+3)}}{C_{n+5}^{(n+5)} C_{n+6}^{(n+6)} C_{n+8}^{(n+8)}} \; \cdots$$

Where expression for R_s is obtained recursively as follows

$$R_1 = \gamma_1, \; R_2 = \gamma_2 - \frac{C_{n+7}^{(n+8)}}{C_{n+8}^{(n+8)}} R_1, \; R_3 = \gamma_3 - \frac{C_{n+6}^{(n+7)}}{C_{n+7}^{(n+7)}} R_2 - \frac{C_{n+7}^{(n+8)}}{C_{n+8}^{(n+8)}} \gamma_1$$

$$R_4 = \gamma_4 - \frac{C_{n+5}^{(n+6)}}{C_{n+6}^{(n+6)}} R_3 - \frac{C_{n+5}^{(n+7)}}{C_{n+7}^{(n+7)}} R_2 - \frac{C_{n+5}^{(n+8)}}{C_{n+8}^{(n+8)}} \gamma_1, \text{ - -- - - -}$$

$$R_9 = \gamma_9 - \frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}} R_8 - \frac{C_n^{(n+2)}}{C_{n+2}^{(n+2)}} R_7 - \frac{C_n^{(n+3)}}{C_{n+3}^{(n+3)}} R_6 - \frac{C_n^{(n+1)}}{C_{n+4}^{(n+1)}} R_5$$

$$- \frac{C_n^{(n+5)}}{C_{n+6}^{(n+5)}} R_4 - \frac{C_n^{(n+6)}}{C_{n+6}^{(n+6)}} R_3 - \frac{C_n^{(n+7)}}{C_{n+7}^{(n+7)}} R_2 - \frac{C_n^{(n+8)}}{C_{n+8}^{(n+8)}} \gamma_1 \quad (27)$$

and

$$\gamma_1 = \left[\frac{\left(\frac{3!}{7!}\alpha_{03} + (n+1)\frac{4!}{8!}\alpha_{14} + (n)(n+1)\frac{5!}{9!}\alpha_{25} + \alpha_{3,6}\frac{6!}{10!}(n+1)(n)(n-1) + {}^{(n-2)(n-1)}_{(n)(n+1)} \frac{7!}{11!}\alpha_{47} \right) P_5}{(n-2)(n-1)n(n+1)} \right.$$

$$+ \frac{\left(\frac{2!}{6!}\alpha_{02} + (n+2)\frac{3!}{7!}\alpha_{13} + {}^{(n+2)}_{(n+1)} \frac{4!}{8!}\alpha_{24} + {}^{(n)(n+1)}_{(n+2)} \frac{5!}{9!}\alpha_{35} + {}^{(n-1)(n)}_{(n+1)(n+2)} \frac{6!}{10!}\alpha_{46} \right) P_4}{(n-1)n(n+1)(n+2)}$$

$$+ \frac{\left(\frac{1!}{5!}\alpha_{01} + (n+3)\frac{2!}{6!}\alpha_{12} + {}^{(n+2)}_{(n+3)} \frac{3!}{7!}\alpha_{23} + {}^{(n+1)(n+2)}_{(n+3)} \frac{45!}{8!}\alpha_{34} + {}^{(n)(n+1)}_{(n+2)(n+3)} \frac{5!}{9!}\alpha_{45} \right) P_3}{(n)(n+1)(n+2)(n+3)}$$

$$+ \frac{\left(\frac{1!}{4!}\alpha_{00} + (n+4)\frac{1!}{5!}\alpha_{11} + {}^{(n+3)}_{(n+4)} \frac{2!}{6!}\alpha_{22} + {}^{(n+2)(n+2)}_{(n+4)} \frac{3!}{7!}\alpha_{33} + {}^{(n+1)(n+2)}_{(n+3)(n+4)} \frac{4!}{8!}\alpha_{44} \right) P_2}{(n+1)(n+2)(n+3)(n+4)}$$

$$\left. + \frac{\left(\frac{1!}{4!}\alpha_{10} + (n+4)\frac{1!}{5!}\alpha_{21} + {}^{(n+3)}_{(n+4)} \frac{2!}{6!}\alpha_{32} + {}^{(n+2)(n+3)}_{(n+4)} \frac{3!}{7!}\alpha_{43} \right) P_1}{(n+2)(n+3)(n+4)} \right] \quad (28)...$$

.

$$\gamma_9 = \left(\frac{1!}{4!}\alpha_{40} \right) P_6 \quad (29)$$

Thus, from (6) we have the following expressions for ε^* :

$$\varepsilon^{*}=\left[\frac{m_1}{\prod_{r=1}^{m}(n+s+r)}-\frac{m_2}{\prod_{r=1}^{m}(n+s+r)}+\frac{m_3}{\prod_{r=1}^{m}(n+s+r)}\right.$$

$$-\frac{m_4}{\prod_{r=1}^{m}(n+s+r)}+\frac{m_5}{\prod_{r=1}^{m}(n+s+r)}-\frac{m_6}{\prod_{r=1}^{m}(n+s+r)}\quad\text{-------}$$

$$\left.+\frac{m_{125}}{\prod_{r=1}^{m}(n+s+r-6)}\right]\frac{\tau_6}{R_8}-\frac{P_1T_{m+s}}{\prod_{r=1}^{m}(n+s+r-6)R_{m+s+1}}\qquad(30)$$

Where:

$$m_1=\frac{C_{n+m+s-5}^{(n+m+s)}C_{n+m+s}^{(n+m+s+1)}C_{n}^{(n)}}{C_{n+m+s}^{(n+m+s)}C_{n+m+s+1}^{(n+m+s+1)}C_{n+m+s-5}^{(n+m+s-5)}},\ m_2=\frac{C_{n+m+s-5}^{(n+m+s+1)}C_{n}^{(n)}}{C_{n+m+s+1}^{(n+m+s+1)}C_{n+m+s-5}^{(n+m+s-5)}}$$

$$m_3=\frac{C_{n+m+s-5}^{(n+m+s-1)}C_{n+m+s-1}^{(n+m+s+1)}}{C_{n+m+s-1}^{(n+m+s-1)}C_{n+m+s+1}^{(n+m+s+1)}C_{n+m+s-5}^{(n+m+s-5)}}\quad\text{- - - - -}\qquad(31)$$

$$m_{59}=\frac{C_{n+m+s-2}^{(n+m+s+1)}C_{n+m+s-1}^{(n+m+s+1)}C_{n}^{(n+m+s-4)}}{C_{n+m+s-2}^{(n+m+s-2)}C_{n+m+s-1}^{(n+m+s-1)}C_{n+m+s+1}^{(n+m+s+1)}}\text{ etc.}$$

Summarily, the above results can be expressed in the general form:

Studying the above very well, we arrived at the general expression for RV as:

$$R_v=\gamma_v-\sum_{i=1}^{v-1}\frac{C_{n+m+s+2-v}^{(n+m+s+2)}}{C_{n+m+s+2}^{(n+m+s+2-i)}}R_i,\ v=1,2,\cdots m+s\qquad(32)$$

where γ_v is given as

$$\gamma_v=\frac{\sum_{i=1}^{v}\{\sum_{j=0}^{m}\left(\alpha_{j,s-v+i+j}\right)\binom{s-v+i+j!}{m+s+j!}\binom{n+2-i}{j}\}}{\prod_{r=1}^{m+i-v}(n-s-m+3+v+r)}P_{m+s-r},\qquad(33)$$

In order to establish the validity of (32) and (33) above, we shall state the following theorem

Theorem: Let D_k, $l \le k \le e$ ($l = 62$ and $e = 125$) for the class $m = 4$ of fourth order differential equations with overdeterminanion $s = 3$ be the quotients of some products of the coefficients $C_p^{(q)}$, $n \le p$, $q < n + m + s + 1$, of Chebyshev polynomials. The factor R_{m+s+1} of the error estimate ε^* in equation (21) is given recursively by

$$R_v = \gamma_v - \sum_{i=1}^{v-1} \frac{C_{n+m+s+2-v}^{(n+m+s+2)}}{C_{n+m+s+2}^{(n+m+s+2-i)}} R_i, \; v = 1,2,\cdots m+s \tag{34}$$

where γ_v is given as

$$\gamma_v = \frac{\sum_{i=1}^{v} \{ \sum_{j=0}^{m} \left(\alpha_{j,s-v+i+j} \right) \binom{s-v+i+j!}{m+s+j!} \binom{n+2-i}{j} \}}{\prod_{r=1}^{m+i-v} (n-s-m+3+v+r)} P_{m+s-r},$$

$$r = 2,3\cdots(m+s-1), \quad r = 1,2,3\cdots m+s+1 \tag{35}$$

Proof:

We shall employ the principle of mathematical induction over the summation variables m and v to establish the validity of (35) above. This is done by varying one of the variables at a time while the other is fixed.

$$\gamma_v = \frac{\sum_{l=1}^{v} \{ \sum_{j=0}^{m} \left(\alpha_{j,s-v+i+j} \right) \binom{s-v+i+j!}{m+s+j!} \binom{n+2-i}{j} \}}{\prod_{r=1}^{m+i-v} (n-s-m+3+v+r)} P_{m+s-r} \tag{36}$$

Let $v = 1$ be fixed at one in (34), that is, we assume that (34) is true for $v = 1$, so that

$$\gamma_1 = \frac{\{ \sum_{j=0}^{m} \left(\alpha_{j,s+j} \right) \binom{s+j!}{m+s+j!} \binom{n+2-i}{j} \}}{\prod_{r=1}^{m} (n-s-m+4+r)} P_{m+s-r}, r = 2,3\cdots(m+s-1), i = v = 1 \tag{37}$$

Now using induction on m for fixed $v = 1$, we shall show that formula (34) holds for $m = 4$. That is, for $m = 4$, we have

$$\gamma_1 = \left(\frac{\alpha_{0,s} \frac{s!}{(m+s)!} + (n+1)\alpha_{1,(s+1)} \frac{(s+1)!}{(m+s+1)!} + \binom{(n)}{(n+1)} \alpha_{2,(s+2)} \frac{(s+2)!}{(m+s+2)}! + \cdots + \binom{(n-1)(n-2)}{(n+1)} \alpha_{4,(s+4)} \frac{(s+4)!}{(m+s+4)!}}{\prod_{r=1}^{4} (n-s+r)} \right) P_5$$

$$+\left(\frac{\alpha_{0,(s-1)}\dfrac{(s-1)!}{(m+s-1)!}+(n+2)\alpha_{1,s}\dfrac{s!}{(m+s)!}+{}^{(n+1)}_{(n+2)}\alpha_{2,(s+1)}\dfrac{(s+1)!}{(m+s+1)}!+\cdots+{}^{(n-1)(n)}_{(n+1)(n+2)}\alpha_{4,(s+3)}\dfrac{(s+3)!}{(m+s+3)!}}{\prod_{r=1}^{4}(n-(s-1)+r)}\right)P_4$$

$$+\left(\frac{\alpha_{0,(s-2)}\dfrac{(s-2)!}{(m+s-2)!}+(n+3)\alpha_{1,(s-1)}\dfrac{(s-1)!}{(m+s-1)!}+{}^{(n+2)}_{(n+3)}\alpha_{2,(s)}\dfrac{(s)!}{(m+s)}!+\cdots+{}^{(n)(n+1)}_{(n+2)(n+3)}\alpha_{4,(s+2)}\dfrac{(s+2)!}{(m+s+2)!}}{\prod_{r=1}^{4}(n-(s-1)+r)}\right)P_3$$

$$+\left(\alpha_{0,(s-3)}\frac{(s-3)!}{(m+s-3)!}+(n+s+1)\frac{\alpha_{1,(s-2)}\dfrac{(s-2)!}{(m+s-2)!}+{}^{(n+s)}_{(n+s+1)}\alpha_{2,(s-1)}\dfrac{(s-1)!}{(m+s-1)!}+\cdots+{}^{(n+s-2)(n+s-1)}_{(n+s)(n+s+1)}\alpha_{4,(s+1)}\dfrac{(s+1)!}{(m+s+1)!}}{\prod_{r=1}^{4}(n-(s-3)+r)}\right)P_2$$

$$+\left(\alpha_{1,(s-3)}\frac{(s-3)!}{(m+s-3)!}+(n+s+1)\frac{\alpha_{2,(s-2)}\dfrac{(s-2)!}{(m+s-2)!}+{}^{(n+s)}_{(n+s+1)}\alpha_{3,(s-1)}\dfrac{(s-1)!}{(m+s-1)!}+{}^{(n+s-1)}_{(n+s)(n+s+1)}\alpha_{4,s}\dfrac{s!}{(m+s)!}}{\prod_{r=1}^{4}(n-(s-4)+r)}\right)P_1$$

(38)

which is the same as γ_1 in equations (28).

Assuming that (37) is true for $m=l$, thus, (37) gives

$$\gamma_1=\left\{\frac{\sum_{j=0}^{l}(\alpha_{j,s+j})\binom{s+j}{m+s+j}\binom{n+2-j}{j}}{\prod_{r=1}^{l}(n-s+4+r)}\right\}P_{m+s} \tag{39}$$

To show that the formula (37) holds for $m=l$.

From the construction of γ_1 in the cases considered for $m=4$ up to $m=l$, we have

$$\gamma_1=\frac{\left\{\sum_{j=0}^{l}\left(\alpha_{j,s+j}\right)\binom{s+j!}{m+s+j!}\binom{n+2-i}{j}\right\}}{\prod_{r=1}^{l}(n-s-l+4+r)}P_{m+s-r},r=2,3\cdots(m+s-1)+$$

$$\frac{\{(\alpha_{l+1,s+l+1})[\binom{s+l+1!}{m+s+l+1!}\binom{n+2-i}{l+1}]\}}{\prod_{r=1}^{l+1}(n-s-l+4+r)}P_{m+s-r}(2,3\cdots(m+s-1)$$

$$=\left\{\frac{\sum_{j=0}^{l}(\alpha_{j,s+j})\binom{s+j!}{m+s+j!}\binom{n+2-i}{j}}{(n-s-l+4+r)}+\frac{\alpha_{l+1,s+l+1}[\binom{s+l+1!}{m+s+l+1!}]\binom{n+2-i}{l+1}}{\prod_{r=1}^{l+1}(n-s-l+4+r)}\right\}$$

$$=\frac{\{\sum_{j=0}^{l+1}(\alpha_{j,s+j})\binom{s+j!}{m+s+j!}\binom{n+2-i}{j}\}}{\prod_{r=1}^{l+1}(n-s-l+4+r)}P_{m+s-r},r=2,3\cdots(m+s-1) \quad (50)$$

Thus, (32) holds for $m=l+1$, and hence it holds positive values of m.

Also, we assume that (32) holds for $v=l$, i.e

$$\gamma_l=\frac{\sum_{l=1}^{l}\{\sum_{j=0}^{m}(\alpha_{j,s-l+i+j})\binom{s-l+i+j!}{m+s+j!}\binom{n+2-i}{j}\}}{\prod_{r=1}^{m}(n-s-m+3+l+r)}P_{m+s-r},r=2,3\cdots(m+s-1) \quad (51)$$

Then, we show that it holds for $v=l+1$, i.e

$$\gamma_{l+1}=\frac{\sum_{l=1}^{l+1}\{\sum_{j=0}^{m}(\alpha_{j,s-l+i+j})\binom{s-l+i+j!}{m+s+j!}\binom{n+2-i}{j}\}}{\prod_{r=1}^{m+\sigma-x-1}(n-s-m+3+l+r)}P_{m+s-r},r=2,3\cdots(m+s-1) \quad (52)$$

By the construction of γ_{l+1}, we have

$$\gamma_{l+1}=\frac{\sum_{i=1}^{l}\{\sum_{j=0}^{m}(\alpha_{j,s-l+i+j})\binom{s-l+i+j!}{m+s+j!}\binom{n+2-i}{j}\}}{\prod_{r=1}^{m+i-1}(n-s-m+3+l+r)}P_{m+s-q},q=2,3\cdots(m+s-1)$$

$$\frac{\{\sum_{j=0}^{m}(\alpha_{j,s-j+1})\binom{s-l+i+j!}{m+s+j!}\binom{n+2-i}{j}\}}{\prod_{r=1}^{m+i}(n-s-m+3+l+r)}P_{l+i} \quad (53)$$

which leads to:

$$\gamma_{l+1} = \frac{\sum_{l=1}^{l+1}\{\sum_{j=0}^{m}\left(\alpha_{j,s-l+i+j}\right)\binom{s-l+i+j!}{m+s+j!}\binom{n+2-i}{j}\}}{\prod_{r=1}^{m+i-1}(n-s-m+3+l+r)} P_{m+s-q}, q = 2,3\cdots(m+s-1) \quad (54)$$

Thus (32) holds for all m, v and we follow the same procedure for the proof of γ_v in (36)

3 Numerical Experiments

Having validated the results above using mathematical induction principle, some numerical examples were presented to further establish the accuracy of the work reported in this paper. We tagged the program written to automated integrated formulation of the Tau method as mapple-tau-program (MTP) which was used to solve all the numerical examples.

In this section we present some numerical results to support the work of the preceding section. The exact error is defined as :

$$\varepsilon = max\{|\, y(x_k) - y_n(x_k)\,| : a \le x_k \le b\}$$

where

$$\{x_k\} = \{0.01k\}, \text{for } k = 0(1)100.$$

3.0.1 Example 4.1:

$$Ly(x) \equiv y^{iv} + \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)y''(x) = -1 + 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5$$

$$y(0) = -1, y'(0) = 0, y''(0) = 2, y'''(0) = 0.$$

with analytical solution

$$y(x) = 1 - 2\cos x. \quad 0 \le x \le 1.$$

we assume a Tau approximant $y_n(x), n = 7,8,9$

The numerical results for this example are presented in Table 4.1 below for the error estimate and actual error.

Example 4.2

$$Ly(x) := y^{iv}(x) + \frac{1}{4}(16 + x^3)y''(x) + x^3 y(x) = x^3$$

with analytical solution

$$y(x) = 1 + \sin 2x$$

$$y(0) = 1,\ y'(0) = 2,\ y''(0) = 0,\ y'''(0) = -8$$

See Table 4.2 for the numerical results for this example.

Example 4.3

$$Ly(x) := y^{iv}(x) + x^3 y(x) = 0$$

$$y(0) = 1, y'(0) = 1, y''(0) = 0, y'''(0) = 3$$

See Table 4.3 for the numerical results for this example.

Table 4.1: Error and Error Estimate for Example 4.1

n / Error	7	8	9
Exact	2.58×10^{-4}	9.1×10^{-6}	8.67×10^{8}
Estimate	2.43×10^{-5}	2.43×10^{-6}	2.43×10^{-8}

Table 4.2: Error and Error Estimate for Example 4.2

n / Error	7	8	9
Exact	3.54×10^{-6}	1.81×10^{-7}	9.21×10^{-9}
Estimate	2.13×10^{-4}	1.43×10^{-5}	4.36×10^{-6}

Table 4.3: Error and Error Estimate for Example 4.3

n / Error	8	9
Exact	7.09×10^{-8}	4.12×10^{-9}

Estimate	1.55×10^{-11}	4.15×10^{-13}

Remarks

1. In Table 4.1, both the exact error and error estimate get improved as the degree of approximation increases. The error estimate closely captures the order of the tau approximant.
2. In Table 4.2, as the degree of approximation increases, the exact error and error estimate also improved, and the error estimate gradually closes up with the exact error; actually the order of the error is exactly captured by the estimate.
3. In Table 4.1, the error estimate closely captures the order of the exact error and serves as upper bound as n increases.
4. In Table 4.3, we note the effectiveness of the error estimate in providing a close upper bound to the exact error.

5 GENERAL CONCLUSION

1 Summary

The study focused on the integral form of the Lanczos tau method for the numerical solution of ordinary differential equations. The class of problems considered is the class of fourth order ordinary differential equations whose solution satisfies some given initial conditions and which are of the second and third degree of over determination. The process of the determination of the tau approximation to the desired unknown solution yielded some recurrence relations for the class.

This rendered the method suitable for development of a computer programme written to implement the technique. Furthermore, an error estimation of the integral tau method for the class under consideration was carried out. The estimation obtained was theoretically validated by the Mathematical induction principle. Both the methods and the error estimates were tested on some problems which were formulated.

The resulting numerical results showed the closeness of the tau approximation to the analytic solution as well as the accuracy of the error estimates.

2 Discussion of Results

We also note the effectiveness of the error estimation technique reported in this thesis judging by the numerical evidences from Tables 4.1, 4.2 and 4.3s. The error estimate improved as the degree of tau approximation also increased.

This estimate, in fact, captures the order of the exact error as the degree of tau approximation increases. This conforms with the features of most numerical algorithms where the performance is expected to improve, as the step length, h, decreases and/or the degree of approximation increases. The only exception to this rule is in the case of numerical differentiation where an optimal value of the step length h may have to be determined for an improved accuracy of results.

3 Conclusion

A method for the solution of the class of fourth order differential equations with second and third degree over determination by the integral tau method has been presented.

A process for estimating the associated or corresponding error has also been reported. While the integral tau method closely approximates the analytic solution, the error estimate closely captures the order of the exact error. Numerical evidences obtained from some selected problems show that the method is accurate and effective for use for the class of problems within the scope of the study.

4 Contribution to Knowledge

The following contributions were made to the body of knowledge:

- the derivation of recurrence relations for the integral tau method;
- the derivation of the error estimate for the integral tau method;
- the validation of the error estimate by Mathematical induction; and
- the proposed error estimate closely captures the order of the tau approximation.

5 Recommendation

The integral tau method is known to perform most accurately among the three variants of the tau method. It is therefore recommended that this variant be adopted for use in solving fourth order problems with second and third degree over determination.

By virtue of the fact that the error polynomial leading to the error estimate involves a minimum number of unknown parameters (one, in this case), the error estimation readily recommends itself to users for accuracy, efficiency and effectiveness.

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