

Dammann, Felix; Ferrari, Giorgio

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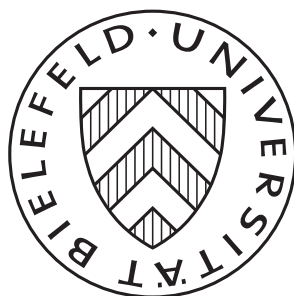
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# Optimal Execution with Multiplicative Price Impact and Incomplete Information on the Return

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Felix Dammann and Giorgio Ferrari



# OPTIMAL EXECUTION WITH MULTIPLICATIVE PRICE IMPACT AND INCOMPLETE INFORMATION ON THE RETURN

FELIX DAMMANN AND GIORGIO FERRARI

ABSTRACT. We study an optimal liquidation problem with multiplicative price impact in which the trend of the asset's price is an unobservable Bernoulli random variable. The investor aims at selling over an infinite time-horizon a fixed amount of assets in order to maximize a net expected profit functional, and lump-sum as well as singularly continuous actions are allowed. Our mathematical modelling leads to a singular stochastic control problem featuring a finite-fuel constraint and partial observation. We provide the complete analysis of an equivalent three-dimensional degenerate problem under full information, whose state process is composed of the asset's price dynamics, the amount of available assets in the portfolio, and the investor's belief about the true value of the asset's trend. The optimal execution rule and the problem's value function are expressed in terms of the solution to a truly two-dimensional optimal stopping problem, whose associated belief-dependent free boundary  $b$  triggers the investor's optimal selling rule. The curve  $b$  is uniquely determined through a nonlinear integral equation, for which we derive a numerical solution allowing to understand the sensitivity of the problem's solution with respect to the relevant model's parameters.

**Keywords:** optimal execution problem, multiplicative price impact, singular stochastic control, partial observation, optimal stopping.

**MSC2020 subject classification:** 93E20, 93C41, 49L20, 91G80

**JEL classification:** G11, C61

## 1. INTRODUCTION

In this paper, we consider an investor who possesses a fixed amount of assets and aims at selling them on the market. We assume that the investor faces the issue of causing an adverse price reaction, so that fast selling depresses the stock price, while splitting the order over time may take too long. This problem – also known as the *optimal execution problem* in algorithmic trading – thus deals with the question of how to trade optimally in order to maximize a given profit, and therefore of how to determine the time as well as the size of the order.

Dating back to the early works of Bertsimas and Lo [9], Almgren and Chriss [1] and Almgren [2], the study of optimal execution strategies has received much attention and resulted in a series of important contributions in various settings, which, amongst other modeling features, can be distinguished with respect to the considered type of price impact: Additive or multiplicative. A comprehensive discussion on the latter class of models can be found in Guo and Zervos [41], who also point out that models with multiplicative price impact seem to be more natural since they ensure prices to remain positive. Amongst those works dealing with multiplicative price impact, let us mention Bertsimas et al. [10] for a discrete-time framework, Forsyth et al. [35] for a continuous-time model à la Black-Scholes, Guo and Zervos [41] and Becherer et al. [5] for settings involving singular stochastic controls.

A common feature in the literature is the assumption that the investor has full information on the trend of the asset. This, however, can be a strong requirement. As pointed out by Ekström and Lu [27], a statistical estimation of the drift is not an efficient procedure, and obtaining a reasonable precision would need data of decades or even centuries under the same market conditions – which is simply not feasible in reality (see also the discussion in Rogers [54], Section 4.2). In some cases, such as initial public offerings, this price history does not even exist.

To account for this fact, we propose a model of optimal execution with multiplicative price impact in which the drift of the stock price dynamics is a random variable, which is not directly observable by the investor. Through monitoring the evolution of the price on the market, the investor is able to update her belief regarding the drift value. However, such observation is noisy as the investor cannot perfectly distinguish whether price variations are caused by the drift or the stochastic driver of the underlying dynamics. From a mathematical point of view, our model leads to a finite-fuel singular stochastic control problem under partial observation, and we investigate how the presence of incomplete information influences the selling strategy of the investor. In particular, we show that the flow of incoming information – through the observation of the asset’s market price – has a direct effect on the optimal execution rule. Indeed, differently to the case of full information treated in Guo and Zervos [41], the decision to sell is no longer triggered by a constant critical price, but the execution threshold changes dynamically depending on the investor’s current belief on the future trend of the asset. Our results show that the optimal execution strategy is in fact determined by a boundary that is increasing in the belief towards the larger drift value, underlying the intuition that the decision maker chooses to delay selling assets if future prices are expected to increase.

In this regard, our work relates to the bunch of economic and financial literature where questions of optimal decision-making under partial observation have been considered; amongst a large number of contributions, we refer to the seminal papers on portfolio selection by Detemple [25] and Gennotte [40]; to Veronesi [59] for an equilibrium model with uncertain dividend drift; to Sass and Haussmann [55] for a terminal-wealth portfolio optimization problem, and to the more recent Colaneri et al. [15] for an optimal liquidation problem with rate strategies and partial observation. Notably, the recent Drissi [26] and Bismuth et al. [11] incorporate Bayesian learning in a model of multi-asset optimal execution, although restricting the agent to absolutely continuous (regular) controls.

Furthermore, we contribute to those models dealing with problems of optimal stopping and singular stochastic control. To name just a few recent works, Callegaro et al. [13] for public debt control, De Angelis [21] and Décamps and Villeneuve [23] for dividend payments, Décamps et al. [22] for investment timing, Ekström and Lu [27] as well as Ekström and Vaicenavicius [28] for asset liquidation, Federico et al. [30] for inventory management, Johnson and Peskir [43] for quickest detection, Gapeev [37] for the pricing problem of perpetual commodity equities, and Gapeev and Rodosthenous [38] for a zero-sum optimal stopping games associated with perpetual convertible bonds.

**Our model, approach and overview of the mathematical analysis.** We now discuss the mathematical modeling and analysis. We consider an investor holding a fixed amount  $y$  of assets in her portfolio. In absence of the investor’s actions, the stock price evolves according to a geometric Brownian motion  $dS_t = \beta S_t dt + \sigma S_t dW_t$ , where  $W$  is a standard Brownian motion and  $\sigma > 0$  a constant volatility parameter. Furthermore, the price process exhibits a random future trend  $\beta$ , which is however unknown to the decision maker, and is assumed to be a random variable, independent of the Brownian noise, taking two values  $\beta_0 < \beta_1$ , for some  $\beta_0, \beta_1 \in \mathbb{R}$  and  $\beta_0 < 0$ .

The decision maker is able to sell the assets on the market over an infinite time horizon, and we denote by  $\xi_t$  the cumulative amount of assets liquidated up to time  $t$ . Consequently, the remaining assets in the portfolio follow the deterministic dynamics  $Y_t^\xi = y - \xi_t$ . Clearly, it has to be  $\xi_t \leq y$  at any time  $t \geq 0$  (finite-fuel constraint), since no more than the initial amount of assets can be sold. As anticipated, we assume that the investor causes an adverse price reaction upon selling, which, following Guo and Zervos [41], we assume to be of multiplicative type. Hence, the controlled asset’s price evolves as

$$dS_t^\xi = \beta S_t^\xi dt + \sigma S_t^\xi dW_t - \alpha S_t^\xi \circ d\xi_t, \quad S_{0-}^\xi = s > 0,$$

where  $\alpha > 0$  denotes the parameter of price impact, and the operator  $\circ$  is defined as in (2.3) below so to take care of the continuous and jump components of any admissible selling strategy  $\xi$ . Notice that the multiplicative price impact structure allows to express the asset’s price process as  $S^\xi = \exp(X^\xi)$ .

Here,  $X^\xi$  is then a linearly controlled drifted Brownian motion with volatility  $\sigma > 0$  and drift value  $\mu = \beta - \frac{1}{2}\sigma^2$ .

The investor aims at maximizing the total expected discounted reward upon selling, net of transaction costs; that is,

$$\sup_{\xi} \mathbb{E} \left[ \int_0^\infty e^{-rt} (e^{X_t^\xi} - \kappa) \circ d\xi_t \right],$$

where the optimization is taken over a suitable admissible class of selling strategies. The latter is a finite-fuel singular stochastic control problem under partial observation.

By relying on classical filtering techniques (cf. Shiryaev [57], Section 4.2), we begin by determining an equivalent Markovian problem – the so-called *separated problem* – under full information (see Fleming and Pardoux [32] as a classical reference on the separated problem). To this end, we introduce the process  $\Pi$ , according to which the investor can update her belief regarding the true value of the drift. This is done by observing the evolution of the process  $X^0$  (denoting the uncontrolled version of the process  $X^\xi$ ), whose natural filtration  $\mathcal{F}_t^{X^0}$  models the overall information available up to time  $t$ . More precisely, after forming a prior  $\pi := \mathbb{P}(\mu = \mu_1) \in (0, 1)$ , the investor dynamically updates her belief upon the arrival of new information through observing the process  $X^0$ , so that the belief process is given by  $\Pi_t = \mathbb{P}(\mu = \mu_1 \mid \mathcal{F}_t^{X^0})$ . Notice that a value of  $\Pi$  close to 1 indicates a strong belief towards the larger value of the drift, while  $\Pi$  close to 0 displays a strong belief in the lower value. Hence, we expect the investor to change the liquidation strategy dynamically and not solely base it on the current price on the market, but also on the present belief at that time.

The separated problem turns out to be a *three-dimensional degenerate finite-fuel singular stochastic control problem*, so that obtaining explicit solutions through a traditional “guess-and-verify approach” is in general not feasible.<sup>1</sup>

In order to tame the multidimensional nature of the resulting optimal execution problem under full information, we then follow a direct approach which hinges on the study of a suitable optimal stopping problem with value  $v$ , that we expect to be associated to the singular stochastic control problem. This method was studied and refined by many authors such as Beneš et al. [6], El Karoui and Karatzas [29], and Karatzas and Shreve [45], or De Angelis [21], De Angelis et al. [19] and [20], and Guo and Tomecek [42] for more recent contributions. The optimal stopping problem, which involves the underlying two-dimensional diffusion  $(X^0, \Pi)$  taking values in  $\mathbb{R} \times (0, 1)$ , can be interpreted as an optimal selling problem and exhibits a structure similar to that of the problem treated by Décamps et al. [22] (see also Ekström and Lu [27] for a parabolic version). However, the specific choice of possible drift values assumed in Décamps et al. [22] allows for the explicit construction of a solution to the related variational inequality, which is instead not possible in our context. We then solve the optimal stopping problem by relying on techniques from free-boundary theory (as illustrated in the monography by Peskir and Shiryaev [53]) and first show that the optimal stopping rule is characterized through a belief-dependent free boundary  $a(\pi)$  for  $\pi \in (0, 1)$ .

However, the coupled dynamics of the underlying processes  $X^0$  and  $\Pi$ , as well as the fact that they are driven by the same Brownian motion, makes a further study of the free boundary and the value function  $v$  not feasible. It is for that reason we proceed by deriving two equivalent representations of the optimal stopping problem, which allow for a thorough analysis. First, via a change of measure, the state process  $(X^0, \Pi)$  is transformed into  $(X^0, \Phi)$  taking values in  $\mathbb{R} \times (0, \infty)$  and with decoupled dynamics. Here, the process  $\Phi$  is the so-called “likelihood ratio”. Again, we can express the optimal stopping strategy in terms of a free boundary  $\varphi \mapsto b(\varphi)$ , which results from a simple transformation of the boundary  $\pi \mapsto a(\pi)$ . Second, we pass yet to another formulation by deriving the intrinsic parabolic formulation of the stopping problem in coordinates  $(X^0, Z)$ , in which the process  $Z$  now follows purely deterministic dynamics and takes values in  $\mathbb{R}$ . Even though the monotonicity result

<sup>1</sup>A “guess-and-verify approach” is applicable if we take  $\beta_0 = -\beta_1$ , which indeed allows for a dimension reduction; see, e.g., Décamps and Villeneuve [23]. In this paper, however, we do not consider any relation amongst  $\beta_0$  and  $\beta_1$  other than  $\beta_0 < \beta_1$ .

of the associated free boundary  $z \mapsto c(z)$  is certainly not trivial to derive and calls for a rigorous technical analysis, it is in this formulation that we are able to provide further regularity results of  $c$  and of the transformed optimal stopping value function  $\widehat{v}$ . In fact, borrowing arguments from De Angelis [21], suitably adapted to the present setting, we achieve a global regularity of  $\widehat{v}$ , namely  $\widehat{v} \in C^1(\mathbb{R}^2)$ . The latter result also allows proving  $\widehat{v}_{xx} \in L_{\text{loc}}^\infty(\mathbb{R}^2)$ , and finally obtaining a nonlinear integral equation uniquely solved by the optimal stopping boundary  $c$ . It is worth mentioning that such a characterization can be traced back to both optimal stopping boundaries  $b$  and  $a$  and is thus tantamount to a complete specification of the optimal stopping rule in the original  $(x, \pi)$ -coordinates.

The thorough analysis developed for the optimal stopping problem is then exploited in order to identify an optimal execution strategy. In fact, the derived regularity results for  $\widehat{v}$  permits us to prove a verification theorem, that identifies an optimal execution rule and shows that the optimal stopping value function  $v$  indeed coincides with a directional derivative of the separated problem's value function  $V$ . Namely, we show that

$$V(x, y, \pi) := \frac{1}{\alpha} \int_{x-\alpha y}^x v(x', \pi) dx', \quad (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1).$$

Notice, that if  $\alpha \downarrow 0$ , one finds  $V(x, y, \pi) = yv(x, \pi)$ , which is the value of the problem in which the investor has no market impact.

The optimal execution rule can be thought of as a “myopic one”. Indeed, it prescribes to sell assets as if the size of the investor's portfolio were infinite, and to stop selling once the asset's inventory is depleted (see also Karatzas [46] and El Karoui and Karatzas [29]). The optimal selling rule involves lump-sum executions (whenever the asset's price is sufficiently large), that could eventually result into an immediate depletion of the portfolio (if the initial portfolio size is sufficiently small). However, for relatively large portfolios, an initial lump-sum selling is followed by a policy of oblique reflection type. This is triggered by the belief-dependent boundary  $\varphi \mapsto b(\varphi)$  (equivalently,  $\pi \mapsto a(\pi)$ ). Notably, given that all the transformations developed for the resolution of the optimal stopping problem are one-to-one and onto, the integral equation for the boundary  $z \mapsto c(z)$  yields an integral equation for  $\varphi \mapsto b(\varphi)$ , and therefore a complete characterization of the optimal execution rule. In order to provide insights about the sensitivity of the optimal decision mechanism of the investor with respect to the model's parameters, we develop a recursive numerical scheme, which relies on an application of the Monte-Carlo method.

**Our contributions.** Overall, we believe that the contributions of this paper are the following. Even though the literature on optimal execution problems is extensive (see, to name just a few, Almgren and Chriss [1], Almgren [2], Becherer et al. [5], Bertsimas and Lo [9], Bertsimas et al. [10], Colaneri et al. [15], Gatheral and Schied [39], Guo and Zervos [41], Moreau et al. [50], Schied and Schöneborn [56]), the combination of incomplete information on the future price trend while allowing for lump-sum as well as singularly continuous executions constitutes a novelty. Furthermore, the present study on the optimal execution strategy complements as well as extends the literature on problems with a similar structure under full information. As a matter of fact, the derived optimal execution rule exhibits a broader structure and prescribes to take actions depending on the current belief on the future trend of the asset.

From a mathematical point of view, to the best of our knowledge, ours is the first work providing a complete characterization of the value function and of the optimal control rule in a finite-fuel singular stochastic control problem under partial observation (which, in the present setting, is equivalent to a three-dimensional degenerate singular stochastic control problem). Furthermore, we believe that the optimal stopping (selling) problem, studied as a device to characterize the optimal solution of the optimal execution problem, is of interest of its own. By performing a thorough analysis on the regularity of (a transformed version of) its value function and free boundary, we are able to provide a complete characterization of the optimal selling rule through a nonlinear integral equation, thus extending the results of the related model studied by Décamps et al. [22]. Notice, that an

integral equation for the free boundary has been obtained also in Ekström and Lu [27] and Ekström and Vaicenavicius [28], though in settings where the parabolic nature of the problem is arising because of an explicit time-dependency. Finally, the probabilistic numerical approach developed for the resolution of the free boundary's integral equation allows to understand the dependency of the investor's optimal execution strategy on relevant model's parameters such as volatility and trend.

**Organization of the paper.** The rest of the paper is organized as follows. In Section 2 we present our setting and first preliminary results. In Section 3 we investigate the benchmark problem under full information, before we consider a corresponding optimal stopping problem and its optimal boundary in Section 4. In Section 5 and 6 we derive two equivalent formulations of this problem, which allow for a more thorough study. Eventually, in Section 7, we return to the optimal control problem and characterize the optimal selling rule of the investor. A numerical comparative statics analysis is then carried out in Section 8.

## 2. SETTING AND PROBLEM FORMULATION

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space, rich enough to accommodate a standard one-dimensional Brownian motion  $(W_t)_{t \geq 0}$  and an independent random variable  $\beta$  taking two values  $\beta_0$  and  $\beta_1$ . We denote by  $\mathbb{F}^W := (\mathcal{F}_t^W)_{t \geq 0}$  the filtration generated by  $(W_t)_{t \geq 0}$  augmented by  $\mathbb{P}$ -null sets of  $\mathcal{F}_0$ . We assume that, in absence of any actions of the investor, the asset's price on the stock market evolves stochastically according to a geometric Brownian motion

$$(2.1) \quad dS_t^0 = \beta S_t^0 dt + \sigma S_t^0 dW_t, \quad S_0^0 = s > 0,$$

where  $\sigma > 0$  is a constant volatility. The investor holds a finite amount  $y \geq 0$  of assets, which she is able to sell. We identify the cumulative amount of assets sold up to time  $t \geq 0$ , which we denote by  $\xi_t$ , as the investor's control variable. We denote the natural filtration of any process  $Z$  by  $\mathbb{F}^Z := (\mathcal{F}_t^Z)_{t \geq 0}$  and hence, the set of admissible execution strategies in this context is given by

$$\mathcal{A}(y) := \left\{ \xi : \Omega \times [0, \infty) : (\xi_t)_{t \geq 0} \text{ } \mathbb{F}^{S^0}\text{-adapted, increasing, càdlàg, and } \xi_{0-} = 0, \xi_t \leq y \text{ a.s.} \right\},$$

where the last condition naturally arises from the fact that the investor cannot sell more than the initial amount of assets. Moreover, the remaining assets in the portfolio evolve according to the deterministic dynamics

$$Y_t^\xi = y - \xi_t, \quad Y_{0-}^\xi = y \geq 0,$$

where we stress the dependency on the selling strategy  $\xi$ . Following Guo and Zervos [41], in our model we assume that the investor's transactions on the market have a proportional impact on the asset's price. More precisely, when selling a small amount  $\epsilon > 0$  of assets at time  $t$ , the price exhibits a jump of size

$$\Delta S_t = S_t - S_{t-} = -\alpha \epsilon S_t,$$

for  $\alpha > 0$  denoting the parameter of permanent price impact (see Almgren and Chriss [1], Almgren [2] for early works and Becherer et al. [4], Ferrari and Koch [31], Guo and Zervos [41] for more recent contributions). Hence, a small transaction is such that  $S_t = (1 - \alpha \epsilon) S_{t-} \simeq e^{-\alpha \epsilon} S_{t-}$  and, by interpreting a lump-sum sale of  $\Delta \xi_t$  shares as a sequence of  $N$  individual sales of size  $\epsilon = \Delta \xi_t / N$ , we have

$$S_t = e^{-\alpha N \epsilon} S_{t-} = e^{-\alpha \Delta \xi_t} S_{t-},$$

for  $N$  large enough. It follows that, for any  $\xi \in \mathcal{A}(y)$ , we can model the controlled asset's price process by

$$(2.2) \quad dS_t^\xi = \beta S_t^\xi dt + \sigma S_t^\xi dW_t - \alpha S_t^\xi \circ d\xi_t, \quad S_{0-}^\xi = s,$$

where

$$(2.3) \quad \int_0^\cdot S_t^\xi \circ d\xi_t := \int_0^\cdot S_t^\xi d\xi_t^c + \sum_{t \leq \cdot : \Delta\xi_t \neq 0} \frac{1}{\alpha} S_{t-}^\xi (1 - e^{-\alpha \Delta\xi_t}) = \int_0^\cdot S_t^\xi d\xi_t^c + \sum_{t \leq \cdot : \Delta\xi_t \neq 0} S_{t-}^\xi \int_0^{\Delta\xi_t} e^{-\alpha u} du,$$

$\xi^c$  denotes the continuous part of the process  $\xi$ , and  $\Delta\xi_t := \xi_t - \xi_{t-}$ . The solution to (2.2) can be explicitly determined via Itô's formula and it is given by

$$(2.4) \quad S_t^\xi = s \exp\left(\left(\beta - \frac{1}{2}\sigma^2\right)t + \sigma W_t - \alpha \xi_t\right) = S_t^0 \exp(-\alpha \xi_t),$$

where  $S^0$  is the solution to (2.1) and we observe that the price impact of selling is additive to the logarithm of the asset's price.

We assume that the investor aims at maximizing the total expected (discounted) profits, net of the total cost of selling, and thus seeks to solve

$$(2.5) \quad \begin{aligned} & \sup_{\xi \in \mathcal{A}(y)} \mathbb{E} \left[ \int_0^\infty e^{-rt} (S_t^\xi - \kappa) \circ d\xi_t \right] \\ &= \sup_{\xi \in \mathcal{A}(y)} \mathbb{E} \left[ \int_0^\infty e^{-rt} (S_t^\xi - \kappa) d\xi_t^c + \sum_{t: \Delta\xi_t \neq 0} e^{-rt} \int_0^{\Delta\xi_t} (S_{t-}^\xi e^{-\alpha u} - \kappa) du \right]. \end{aligned}$$

Here,  $\kappa > 0$  is a proportional transaction cost, which, thinking of  $S_t^\xi$  as the mid-price of the stock at time  $t$ , can also be interpreted as a constant bid spread. Notice that the structure of the expected net-profit functional in (2.5) can also be justified through stability results in the Skorokhod  $M_1$ -topology in probability (see Becherer et al. [5]). Moreover, problem (2.5) has finite value due to  $\xi_t \leq y$  a.s. Thanks to (2.4) we have  $S_t^\xi = \exp(X_t^\xi)$ , where

$$(2.6) \quad dX_t^\xi = \mu dt + \sigma dW_t - \alpha d\xi_t, \quad X_0^\xi = x,$$

with  $x := \ln(s)$  and  $\mu := \beta - \frac{1}{2}\sigma^2$ . In particular, the drift can take two values  $\mu_i = \beta_i - \frac{1}{2}\sigma^2$ ,  $i = 0, 1$ . In the following, when needed, we let  $X^0$  denote the solution to (2.6) with  $\xi \equiv 0$ , which is then an arithmetic Brownian motion. Furthermore, we state the following assumption.

**Assumption 2.1.** *We have  $\beta_1 > \beta_0$  and  $\beta_0 < 0$ , which implies  $\mu_0 < 0$ .*

The maximization problem (2.5) thus can be rewritten in terms of (2.6) as

$$(2.7) \quad \sup_{\xi \in \mathcal{A}(y)} \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( e^{X_t^\xi} - \kappa \right) \circ d\xi_t \right].$$

Notice that for a constant non-random drift coefficient, a close variant of this problem was considered and solved by Guo and Zervos [41], who also incorporate the option of buying shares of assets and the constraint that the whole inventory has to be depleted at terminal time. However - due to the presence of incomplete information on the drift of the asset - Problem (2.7) is not of Markovian nature and thus requires a thoroughly different analysis. In order to obtain an equivalent Markovian formulation of (2.7), we rely on classical results from filtering theory, dating back to the contribution of Shiryaev in the context of quickest detection models (see Shiryaev [58] for a survey). To this end, we introduce the *belief* process

$$\Pi_t := \mathbb{P} \left( \mu = \mu_1 \mid \mathcal{F}_t^{X^0} \right), \quad t \geq 0,$$

which reflects the probability at time  $t$  that  $\mu = \mu_1$ , conditional on the observations of the price process up to that time (indeed,  $\mathbb{F}^{X^0} = \mathbb{F}^{S^0}$ ). According to this process, the investor is able to update the belief regarding the true value of the drift, based on the arrival of new information by observing the asset's price evolution on the market. Notice that a large value of  $\Pi$  close to 1 implies a



strong belief towards the larger drift value  $\mu_1$ , while a low value of  $\Pi$  implies the contrary. It follows (see, e.g., Shiryaev [57], Section 4.2) that the dynamics of  $X^\xi, \Pi$  and  $Y^\xi$  can be written as

$$(2.8) \quad \begin{cases} dX_t^\xi = (\mu_1 \Pi_t + \mu_0(1 - \Pi_t))dt + \sigma d\bar{W}_t - \alpha d\xi_t, & X_{0-}^\xi = x \in \mathbb{R}, \\ d\Pi_t = \gamma \Pi_t(1 - \Pi_t)d\bar{W}_t, & \Pi_0 = \pi \in (0, 1), \\ Y_t^\xi = y - \xi_t, & Y_{0-}^\xi = y \geq 0, \end{cases}$$

where  $\gamma = (\mu_1 - \mu_0)/\sigma$  is the *signal-to-noise ratio* and

$$d\bar{W}_t = \frac{dX_t^0}{\sigma} - \left( \frac{\mu_0}{\sigma} + \gamma \Pi_t \right) dt$$

denotes the *innovation process*, which is an  $\mathbb{F}^{X^0}$ -Brownian motion on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Moreover,  $\pi := \mathbb{P}(\mu = \mu_1)$  reflects the initial subjective belief of the investor regarding the true value of the drift. We do not question the origin of this initial belief, this can either be an instinctive decision or even the result of a constructive approach, for instance by observing the trends of similar assets over the past years. In the new formulation, the process  $(X^\xi, Y^\xi, \Pi)$  is an  $\mathbb{F}^{X^0}$ -adapted and time-homogeneous Markov process, as it is the unique and strong solution to the system of stochastic differential equations in (2.8). Furthermore, we observe that the drift  $\mu$  is replaced by its conditional estimate and the process  $\Pi$  is a bounded martingale on  $[0, 1]$  with  $\Pi_\infty \in \{0, 1\}$ , as all information will eventually get revealed. Denoting  $\mathbb{E}_{(x,y,\pi)}[\cdot] = \mathbb{E}[\cdot | X_{0-}^\xi = x, Y_{0-}^\xi = y, \Pi_0 = \pi]$ , we can thus reformulate the problem of incomplete information as a so-called *separated problem* (cf. Bensoussan [8], Chapter 7.1 and Fleming and Pardoux [32])

$$(2.9) \quad V(x, y, \pi) := \sup_{\xi \in \mathcal{A}(y)} J(x, y, \pi, \xi),$$

with

$$(2.10) \quad J(x, y, \pi, \xi) := \mathbb{E}_{(x,y,\pi)} \left[ \int_0^\infty e^{-rt} (e^{X_t^\xi} - \kappa) d\xi_t^c + \sum_{t: \Delta \xi_t \neq 0} e^{-rt} \int_0^{\Delta \xi_t} (e^{X_{t-}^\xi - \alpha u} - \kappa) du \right],$$

for any  $(x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1)$ . Notice indeed that  $\Pi_t \in (0, 1)$  for all  $t \geq 0$  a.s. if  $\pi \in (0, 1)$ , while  $\Pi_t \equiv \pi_0$  for all  $t \geq 0$  a.s. if  $\pi_0 \in \{0, 1\}$ . Problem (2.9) is equivalent to (2.5): They share the same value and, because of the uniqueness of the strong solution to (2.8), a control is optimal for (2.5) if and only if it is optimal for (2.9).

**The Hamilton-Jacobi-Bellman equation.** Problem (2.9) takes the form of a *three-dimensional singular stochastic control problem with finite-fuel constraint* (cf. Baldursson [3], Beneš et al. [6], El Karoui and Karatzas [29], Karatzas [44] and Karatzas et al. [48] for early contributions). We start our analysis by providing a heuristic derivation of the dynamic programming equation, that we expect the value function  $V$  to satisfy. To this end, we notice that the investor is faced with two possible actions at initial time. On the one hand, the investor could choose to wait for a short period of time  $\Delta t$ , not sell any fraction of the assets and then continue with an optimal execution strategy (supposing that one exists). Since this strategy is not necessarily optimal, we obtain

$$V(x, y, \pi) \geq \mathbb{E}_{(x,y,\pi)} \left[ e^{-r\Delta t} V(X_{\Delta t}, y, \Pi_{\Delta t}) \right], \quad (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1).$$

If we assume that the value function  $V$  has enough regularity, we can apply Itô's formula, divide by  $\Delta t$  and invoke the mean value theorem in order to let  $t \rightarrow 0$ , so to obtain

$$(\mathcal{L}_{X,\Pi} - r)V \leq 0.$$

Here,  $\mathcal{L}_{X,\Pi}$  denotes the second-order differential operator, acting on twice-continuously differentiable functions,

$$(2.11) \quad \mathcal{L}_{X,\Pi} := \frac{1}{2} \gamma^2 \pi^2 (1 - \pi)^2 \partial_{\pi\pi} + \frac{1}{2} \sigma^2 \partial_{xx} + (\pi \mu_1 + (1 - \pi) \mu_0) \partial_x + \sigma \gamma \pi (1 - \pi) \partial_{x\pi}.$$

On the other hand, the investor can instantaneously sell an amount  $\epsilon > 0$  of the assets and then proceed by following an optimal execution strategy. Again, this strategy is a priori suboptimal and, since this action is associated with the inequality

$$V(x, y, \pi) \geq V(x - \alpha\epsilon, y - \epsilon, \pi) + \frac{1}{\alpha}e^x(1 - e^{-\alpha\epsilon}) - \kappa\epsilon,$$

adding and subtracting  $V(x - \alpha\epsilon, y, \pi)$ , and dividing by  $\epsilon$ , yields

$$\frac{V(x, y, \pi) - V(x - \alpha\epsilon, y, \pi)}{\epsilon} \geq \frac{V(x - \alpha\epsilon, y - \epsilon, \pi) - V(x - \alpha\epsilon, y, \pi)}{\epsilon} + \frac{1}{\alpha}e^x \frac{(1 - e^{-\alpha\epsilon})}{\epsilon} - \kappa.$$

Hence, by letting  $\epsilon \downarrow 0$ , we obtain

$$\alpha V_x(x, y, \pi) \geq -V_y(x, y, \pi) + e^x - \kappa.$$

Since only one of these actions should be optimal, and given the Markovian setting of problem (2.9), we thus expect that the value function  $V$  should identify with an appropriate solution to the Hamilton-Jacobi-Bellman equation

$$(2.12) \quad \max\{(\mathcal{L}_{X,\Pi} - r)u, -\alpha u_x - u_y + e^x - \kappa\} = 0, \quad (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1),$$

with boundary condition  $u(x, 0, \pi) = 0$ , since  $y = 0$  implies  $\mathcal{A}(0) = \{\xi \equiv 0\}$  and  $J(x, 0, \pi, 0) = 0$ . It is worth noticing that the variable  $y$  plays the role of a parameter in (2.12), which is then a two-dimensional elliptic partial differential equation with a state-dependent directional derivative constraint, parametrized by  $y > 0$ . With reference to (2.12) and the reasoning above, we can introduce the *waiting region*

$$(2.13) \quad \mathbb{W}_1 := \{(x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1) : (\mathcal{L}_{X,\Pi} - r)V = 0, -\alpha V_x - V_y + e^x - \kappa < 0\},$$

in which it is expected to be suboptimal to sell any assets, and the *selling/execution region*, where it should be profitable for the investor to sell a fraction of the assets:

$$(2.14) \quad \mathbb{S}_1 := \{(x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1) : (\mathcal{L}_{X,\Pi} - r)V \leq 0, -\alpha V_x - V_y + e^x - \kappa = 0\}.$$

Due to the multi-dimensional structure of the problem, a traditional *guess-and-verify approach*, as seen for instance in Guo and Zervos [41] and Ferrari and Koch [31], is not effective. In fact, this would require the construction of an explicit solution to the second-order PDE with state dependent gradient constraint seen in (2.12) above, which is not feasible in general. Instead, we use a different approach and construct an optimal stopping problem connected to the stochastic control problem (2.9), which is then of a simpler structure. Before we do so, and in order to get insights from a benchmark problem, we briefly discuss the problem under *full information*, i.e. where the drift coefficient is constant and equal to either  $\mu_0$  or  $\mu_1$ .

### 3. BENCHMARK PROBLEM UNDER FULL INFORMATION

Suppose that the initial subjective belief  $\pi = \mathbb{P}(\mu = \mu_1)$  is such that  $\pi \in \{0, 1\}$ . Observe that there exists no uncertainty in the model other than the Brownian one and the belief process  $\Pi$  will remain constant, as the investor is already certain at initial time regarding the true value of the drift. Hence - in this formulation - we are in the case of *full information*. The problem we address in this section has a similar structure to the ones studied by Guo and Zervos [41] as well as Koch [49], Chapter 2, and we therefore do not provide full details. Let us assume  $\pi = 0$ , we thus obtain  $\Pi_t = 0$  for all  $t \geq 0$  and the dynamics of  $X^\xi$  and  $Y^\xi$  then write as

$$(3.1) \quad \underline{X}_t^\xi = x + \mu_0 t + \sigma W_t - \alpha \xi_t, \quad Y_t^\xi = y - \xi_t.$$

We denote the corresponding value function as

$$(3.2) \quad V_0(x, y) := \sup_{\xi \in \mathcal{A}(y)} \mathbb{E}_{(x,y)} \left[ \int_0^\infty e^{-rt} (e^{X_t^\xi} - \kappa) \circ d\xi_t \right], \quad (x, y) \in \mathbb{R} \times (0, \infty),$$

where  $\mathbb{E}_{(x,y)}[\cdot] = \mathbb{E}[\cdot | X_{0-}^{\xi} = x, Y_{0-}^{\xi} = y]$ . By employing similar arguments as in the case of incomplete information, we can expect that  $V_0$  should identify with an appropriate solution to the HJB equation

$$(3.3) \quad \max\{(\mathcal{L}_X - r)w, -\alpha w_x - w_y + e^x - \kappa\} = 0, \quad \text{with } \mathcal{L}_X = \frac{1}{2}\sigma^2\partial_{xx} + \mu_0\partial_x,$$

and  $w(x, 0) = 0$ . Defining the associated waiting and selling regions as

$$(3.4) \quad \mathbb{W}^{\mu_0} := \{(x, y) \in \mathbb{R} \times [0, \infty) : (\mathcal{L}_X - r)w(x, y) = 0, -\alpha w_x - w_y + e^x - \kappa < 0\},$$

$$(3.5) \quad \mathbb{S}^{\mu_0} := \{(x, y) \in \mathbb{R} \times [0, \infty) : (\mathcal{L}_X - r)w(x, y) \leq 0, -\alpha w_x - w_y + e^x - \kappa = 0\},$$

we suppose that the investor is only willing to sell a share of assets when its price is sufficiently large. Hence, we guess that for every  $y \geq 0$  there exists a critical price  $G(y)$  such that (3.4)-(3.5) rewrite as

$$\mathbb{W}^{\mu_0} = \{(x, y) \in \mathbb{R} \times [0, \infty) : y > 0 \text{ and } x < G(y)\} \cup (\mathbb{R} \times \{0\}),$$

$$\text{and } \mathbb{S}^{\mu_0} = \{(x, y) \in \mathbb{R} \times [0, \infty) : y > 0 \text{ and } x \geq G(y)\}.$$

Notice that the candidate value function should then satisfy  $(\mathcal{L}_X - r)w(x, y) = 0$  for all  $(x, y) \in \mathbb{W}^{\mu_0}$ . It is well-known that the latter equation admits two fundamental strictly positive solutions; the only solution that remains bounded as  $x \downarrow -\infty$  is then given by

$$w(x, y) = A(y)e^{nx},$$

for some functions  $A : [0, \infty) \rightarrow \mathbb{R}$  and where  $n$  is the positive solution to  $(\sigma^2/2)n^2 + \mu_0n - r = 0$ . On the other hand, for  $(x, y) \in \mathbb{S}^{\mu_0}$ , we expect that the value function  $V_0$  should instead satisfy

$$-\alpha w_x - w_y + e^x - \kappa = 0 \quad \text{and thus} \quad -\alpha w_{xx} - w_{yx} + e^x = 0.$$

In order to derive the solutions for  $A(y)$  and  $G(y)$ , we evaluate the two previous formulas at  $x = G(y)$ , require that  $A(0) = 0$  and obtain

$$(3.6) \quad G(y) = \ln\left(\frac{\kappa n}{n-1}\right) =: x_0^* \quad \text{and} \quad A(y) = \frac{\kappa}{\alpha n(n-1)}\left(\frac{\kappa n}{n-1}\right)^{-n}\left(1 - e^{-\alpha n y}\right).$$

Notice that the optimal execution threshold - determining the price at which the investor should sell - is independent of the current amount of assets in the portfolio. Moreover, the selling region is partitioned into

$$\mathbb{S}_1^{\mu_0} := \{(x, y) \in \mathbb{R} \times (0, \infty) : x \geq x_0^*, y \leq \frac{x - x_0^*}{\alpha}\}$$

$$\text{and } \mathbb{S}_2^{\mu_0} := \{(x, y) \in \mathbb{R} \times (0, \infty) : x \geq x_0^*, y > \frac{x - x_0^*}{\alpha}\},$$

and we suppose that for  $(x, y) \in \mathbb{S}_1^{\mu_0}$  it should be optimal to sell the complete amount of assets instantaneously, while for  $(x, y) \in \mathbb{S}_2^{\mu_0}$  the investor is expected to make a lump-sum execution and then follow the strategy that keeps the process  $(X, Y)$  inside  $\overline{\mathbb{W}}^{\mu_0}$  until all assets are sold. The candidate value function, according to our previous considerations, then takes the shape

$$(3.7) \quad w(x, y) = \begin{cases} A(y)e^{nx} & \text{for } (x, y) \in \mathbb{W}^{\mu_0}, \\ A(y - \frac{x-x_0^*}{\alpha})e^{nx_0^*} + \frac{1}{\alpha}(e^x - e^{x_0^*}) - \frac{\kappa}{\alpha}(x - x_0^*) & \text{for } (x, y) \in \mathbb{S}_2^{\mu_0}, \\ \frac{1}{\alpha}e^x(1 - e^{-\alpha y}) - \kappa y & \text{for } (x, y) \in \mathbb{S}_1^{\mu_0}, \end{cases}$$

and via a verification theorem (cf. Guo and Zervos [41], Prop. 5.1, Koch [49], Prop. 2.4.1), one can indeed show that  $w$  is a  $C^{2,1}$  solution to the HJB equation (3.3) and coincides with the value function  $V_0$  of (3.2). Moreover, the process

$$(3.8) \quad \xi_t^{\mu_0} := y \wedge \sup_{0 \leq s \leq t} \frac{1}{\alpha} \left[ x - x_0^* + \mu_0 s + \sigma W_s \right]^+, \quad t \geq 0, \quad \xi_0^{\mu_0} = 0,$$

belongs to  $\mathcal{A}(y)$  and provides an optimal execution strategy for problem (3.2) (cf. Guo and Zervos [41], Prop. 5.1; recall that here we are not assuming  $\lim_{T \uparrow \infty} Y_T^{\xi} = 0$  as admissibility condition). In light of

our subsequent analysis, it is interesting to notice that the directional derivative  $v_0 := \alpha \partial_x V_0 + \partial_y V_0$  can be checked from (3.7) to identify with the value function of an optimal stopping problem. More precisely, for any  $x \in \mathbb{R}$  one has

$$(3.9) \quad \alpha \partial_x V_0 + \partial_y V_0 =: v_0(x) = \sup_{\tau \geq 0} \mathbb{E}_x [e^{-r\tau} (e^{\underline{X}_\tau^0} - \kappa)],$$

where  $\underline{X}^0$  denotes the solution to (3.1) with  $\xi_t \equiv 0$ , the optimization is performed over all stopping times of the Brownian filtration and  $\mathbb{E}_x$  is the expectation under  $\mathbb{P}_x(\cdot) = \mathbb{P}(\cdot \mid \underline{X}_0^0 = x)$ . Moreover, the stopping time

$$(3.10) \quad \tau_0^*(x) := \inf\{t \geq 0 : \underline{X}_t^0 \geq x_0^*\}, \quad \mathbb{P}_x\text{-a.s.}, \quad x \in \mathbb{R},$$

is optimal for (3.9). We can interpret (3.10) as the optimal time at which the investor should sell another unit of shares, and notice that it in fact characterizes the time at which the marginal expected profit  $\alpha \partial_x V_0 + \partial_y V_0$  coincides with the marginal instantaneous net profit  $e^x - \kappa$  from selling.

**Remark 3.1.** *It is easily checked that the results we obtained for the case  $\mu \equiv \mu_0$  can be replicated for the case  $\mu \equiv \mu_1$ . More precisely, considering the dynamics*

$$(3.11) \quad \overline{X}_t^\xi = x + \mu_1 t + \sigma W_t - \alpha \xi_t, \quad t \geq 0,$$

and the value function

$$(3.12) \quad V_1(x, y) := \sup_{\xi \in \mathcal{A}(y)} \left[ \int_0^\infty e^{-rt} (e^{\overline{X}_t^\xi} - \kappa) \circ d\xi_t \right], \quad (x, y) \in \mathbb{R} \times (0, \infty),$$

we can verify the existence of an optimal execution threshold  $x_1^*$ , which triggers the selling strategy of the investor through the optimal control  $\xi^{\mu_1}$ , which is of similar structure as (3.8), with  $\mu_0$  replaced by  $\mu_1$ . Furthermore, we have

$$(3.13) \quad \alpha \partial_x V_1 + \partial_y V_1 =: v_1(x) = \sup_{t \geq 0} \mathbb{E}_x [e^{-rt} (e^{\overline{X}_t^0} - \kappa)],$$

where  $\overline{X}^0$  denotes the solution to (3.11) with  $\xi_t \equiv 0$ , and the stopping time  $\tau_1^*(x) := \inf\{t \geq 0 : \overline{X}_t^0 \geq x_1^*\}$ ,  $\mathbb{P}_x$ -a.s., is optimal for problem (3.13).

#### 4. A RELATED OPTIMAL STOPPING PROBLEM

Motivated by the connection to optimal stopping observed in the benchmark problem of Section 3, we here introduce and study an optimal stopping problem, that we expect to be associated to the singular stochastic control problem (2.9). We recall that  $(X_t^0, \Pi_t)_{t \geq 0}$  is the two-dimensional strong Markov process solving

$$(4.1) \quad \begin{cases} dX_t^0 = (\mu_1 \Pi_t + \mu_0 (1 - \Pi_t)) dt + \sigma d\overline{W}_t, & X_0^0 = x, \\ d\Pi_t = \gamma \Pi_t (1 - \Pi_t) d\overline{W}_t, & \Pi_0 = \pi, \end{cases}$$

and in the following - in order to simplify notation - we write  $X$  instead of  $X^0$ . For a stopping time  $\tau$  of the filtration  $\mathbb{F}^X$ , we then define

$$(4.2) \quad \Psi(x, \pi, \tau) := \mathbb{E}_{(x, \pi)} [e^{-r\tau} (e^{X_\tau} - \kappa)], \quad (x, \pi) \in \mathbb{R} \times (0, 1),$$

and consider the optimal stopping problem

$$(4.3) \quad v(x, \pi) := \sup_{\tau} \Psi(x, \pi, \tau).$$

Above, and in the following,  $\mathbb{E}_{(x, \pi)}[\cdot] = \mathbb{E}[\cdot \mid X_0 = x, \Pi_0 = \pi]$ . Also, denoting  $(X_t^{x, \pi}, \Pi_t^\pi)_{t \geq 0}$  the unique strong solution to (4.1) we will often employ the following equivalent notation  $\mathbb{E}[f(X_t^{x, \pi}, \Pi_t^\pi)] = \mathbb{E}_{x, \pi}[f(X_t, \Pi_t)]$ , for any integrable measurable function  $f : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ .

In order to guarantee well-posedness of problem (4.3) we make the next natural **standing assumption**.

**Assumption 4.1.** *We assume that  $r > \mu_1 + \frac{1}{2}\sigma^2$ ; that is  $r > \beta_1$ .*

In the following, we derive some preliminary results of the optimal stopping problem (4.3) and its associated free boundary. Noticing that  $(x, \pi) \mapsto X_t^{x, \pi}$  as well as  $\pi \mapsto \Pi_t^\pi$  are continuous and nondecreasing, due to classical comparison theorems for strong solutions to stochastic differential equations, the proof of the following lemma follows from standard arguments and it is therefore skipped.

**Lemma 4.2.** *The value function  $v$  of (4.3) is such that*

- i)  $x \mapsto v(x, \pi)$  is nondecreasing;
- ii)  $\pi \mapsto v(x, \pi)$  is nondecreasing;
- iii)  $x \mapsto v(x, \pi)$  is continuous.

Furthermore, using that  $(x, \pi) \mapsto (X_t^{x, \pi}, \Pi_t^\pi)$  is continuous  $\mathbb{P}$ -a.s., it follows that

$$(x, \pi) \mapsto \mathbb{E}[e^{-r\tau}(e^{X_\tau^{x, \pi}} - \kappa)],$$

is continuous and hence,  $(x, \pi) \mapsto v(x, \pi)$  is lower-semicontinuous. As it is customary in optimal stopping theory, we introduce the continuation and stopping regions associated to  $v$  as

$$(4.4) \quad \mathcal{C}_1 := \{(x, \pi) \in \mathbb{R} \times (0, 1) : v(x, \pi) > (e^x - \kappa)\},$$

$$(4.5) \quad \mathcal{S}_1 := \{(x, \pi) \in \mathbb{R} \times (0, 1) : v(x, \pi) = (e^x - \kappa)\}.$$

Then, the continuation region  $\mathcal{C}_1$  is an open set, while the stopping region  $\mathcal{S}_1$  in (4.5) is closed, and by Peskir and Shiryaev [53], Chapter 1, Section 2, Corollary 2.9, the stopping time

$$(4.6) \quad \tau^* = \tau^*(x, \pi) := \inf\{t \geq 0 : (X_t^{x, \pi}, \Pi_t^\pi) \in \mathcal{S}_1\},$$

is optimal whenever it is  $\mathbb{P}$ -a.s. finite, otherwise it is an optimal Markov time. We set

$$(4.7) \quad a(\pi) := \inf\{x \in \mathbb{R} : v(x, \pi) \leq (e^x - \kappa)\},$$

with the convention  $\inf \emptyset = +\infty$ , and state the following lemma.

**Lemma 4.3.** *It holds*

$$\mathcal{C}_1 = \{(x, \pi) \in \mathbb{R} \times (0, 1) : x < a(\pi)\} \quad \text{and} \quad \mathcal{S}_1 = \{(x, \pi) \in \mathbb{R} \times (0, 1) : x \geq a(\pi)\}.$$

*Proof.* Recalling (2.11), an application of Dynkin's formula reveals

$$u(x, \pi) := v(x, \pi) - (e^x - \kappa) = \sup_{\tau} \mathbb{E}_{(x, \pi)} \left[ \int_0^{\tau} e^{-rt} \left( e^{X_t} (\mu_1 \Pi_t + (1 - \Pi_t) \mu_0 + \frac{1}{2} \sigma^2 - r) + r\kappa \right) dt \right].$$

For  $x_1 < x_2$  and  $\tau^*$  optimal for  $v(x_2, \pi)$  we have

$$\begin{aligned} u(x_1, \pi) - u(x_2, \pi) &\geq \mathbb{E} \left[ \int_0^{\tau^*} e^{-rt} \left( e^{X_t^{x_1, \pi}} - e^{X_t^{x_2, \pi}} \right) (\mu_1 \Pi_t^\pi + (1 - \Pi_t^\pi) \mu_0 + \frac{1}{2} \sigma^2 - r) dt \right] \\ &\geq \mathbb{E} \left[ \int_0^{\tau^*} e^{-rt} \left( e^{X_t^{x_2, \pi}} - e^{X_t^{x_1, \pi}} \right) (r - \mu_1 - \frac{1}{2} \sigma^2) dt \right] \geq 0, \end{aligned}$$

due to Assumption 4.1. For  $(x_1, \pi) \in \mathcal{S}_1$  and  $x_2 > x_1$ , we thus obtain  $0 \leq u(x_2, \pi) \leq u(x_1, \pi) = 0$ , so that  $(x_2, \pi) \in \mathcal{S}_1$ .  $\square$

The free boundary  $a(\pi)$  thus splits  $\mathbb{R} \times (0, 1)$  into the continuation and stopping region. In the following lemma we derive some preliminary properties.

**Lemma 4.4.** *One has:*

- i)  $\pi \mapsto a(\pi)$  is nondecreasing on  $(0, 1)$ ;
- ii)  $\pi \mapsto a(\pi)$  is left-continuous on  $(0, 1)$ ;
- iii) *There exist constants such that  $x_0^* \leq a(\pi) \leq x_1^*$  for all  $\pi \in (0, 1)$ .*

*Proof.* We prove the claims separately.

i) Let  $\pi_2 > \pi_1$  and  $(x, \pi_2) \in \mathcal{S}_1$ . We thus have  $x \geq a(\pi_2)$  and  $v(x, \pi_2) = e^x - \kappa$ . Since  $\pi \mapsto v(x, \pi)$  is nondecreasing,  $v(x, \pi_1) \leq v(x, \pi_2) = e^x - \kappa$ , which, together with  $v(x, \pi_1) \geq (e^x - \kappa)$ , gives  $(x, \pi_1) \in \mathcal{S}_1$ . Therefore,  $a(\pi_2) \geq a(\pi_1)$ .

ii) Let  $(\pi_n)_n$  be a sequence such that  $\pi_n \uparrow \pi$ . Due to i), the sequence  $a(\pi_n)$  is increasing as  $n \rightarrow \infty$  and  $a(\pi_n) \leq a(\pi)$ . Consequently, there exists  $\lim_n a(\pi_n) =: a(\pi^-)$  and  $a(\pi^-) \leq a(\pi)$ . Because  $v(a(\pi_n), \pi_n) = e^{a(\pi_n)} - \kappa$  for all  $n \in \mathbb{N}$ , by lower-semicontinuity of  $(x, \pi) \mapsto v(x, \pi)$  we find  $v(a(\pi^-), \pi) = e^{a(\pi^-)} - \kappa$ . Hence,  $a(\pi) \leq a(\pi^-)$  and thus  $\lim_n a(\pi_n) = a(\pi)$ .

iii) Recall  $v_0$  and  $v_1$  of (3.9) and (3.13), the value functions in the optimal stopping problems with full information when either  $\mu \equiv \mu_0$  or  $\mu \equiv \mu_1$ . The associated continuation regions are given by

$$\{x \in \mathbb{R} : x \geq x_1^*\} = \{x \in \mathbb{R} : v_1(x) \leq e^x - \kappa\} \quad \text{and} \quad \{x \in \mathbb{R} : x \geq x_0^*\} = \{x \in \mathbb{R} : v_0(x) \leq e^x - \kappa\},$$

where  $x_0^*$  and  $x_1^*$  are the optimal execution thresholds (cf. (3.6) and Remark 3.1). Recalling  $\mu_0 < \mu_1$  and  $\Pi_t \in (0, 1)$  for  $\pi \in (0, 1)$ , we have  $\underline{X}_t^0 \leq X_t \leq \bar{X}_t^0$   $\mathbb{P}$ -a.s. for any  $t \geq 0$ , due to classical comparison arguments and where  $\underline{X}^0$  and  $\bar{X}^0$  denote the solutions to (3.1) and (3.11) with  $\xi_t \equiv 0$ . Thus,  $v_0(x) \leq v(\pi, x) \leq v_1(x)$ , which implies

$$\{x \in \mathbb{R} : v_1(x) \leq e^x - \kappa\} \subset \{(x, \pi) \in \mathbb{R} \times (0, 1) : v(x, \pi) \leq e^x - \kappa\} \subset \{x \in \mathbb{R} : v_0(x) \leq e^x - \kappa\},$$

and the latter, combined with (4.7), allows to conclude that  $x_0^* \leq a(\pi) \leq x_1^*$ .  $\square$

## 5. DECOUPLING CHANGE OF MEASURE AND A NEW OPTIMAL SELLING PROBLEM

We notice that the underlying dynamics in (4.1) are coupled. In order to derive further results about the properties of the optimal stopping problem (4.3) and its associated free boundary, it is useful to address the problem under a different probability measure. With reference to related contributions (cf. De Angelis [21], Ekström and Lu [27], Johnson and Peskir [43] and Shiryaev [58] and references therein), we introduce the so-called *likelihood ratio process* via

$$\Phi_t := \frac{\Pi_t}{1 - \Pi_t}, \quad t \geq 0.$$

Through an application of Itô's formula we can derive its associated dynamics, given by

$$(5.1) \quad d\Phi_t = \gamma\Phi_t(\gamma\Pi_t dt + d\bar{W}_t), \quad \Phi_0 = \varphi := \frac{\pi}{1 - \pi},$$

and we aim to remove its dependency on the process  $\Pi$  through a change of measure. For a fixed  $T > 0$ , we define the measure  $\mathbb{Q}_T \sim \mathbb{P}$  on  $(\Omega, \mathcal{F}_T)$  via the Radon-Nikodym derivative

$$(5.2) \quad \eta_T := \frac{d\mathbb{Q}_T}{d\mathbb{P}} := \exp\left(-\int_0^T \gamma\Pi_s d\bar{W}_s - \frac{1}{2}\int_0^T \gamma^2\Pi_s^2 ds\right),$$

and notice that the process

$$(5.3) \quad dB_t = d\bar{W}_t + \gamma\Pi_t dt,$$

is a Brownian motion under  $\mathbb{Q}_T$  on  $[0, T]$ . Rewriting the state process  $(X, \Phi)$  under  $\mathbb{Q}_T$  then yields

$$(5.4) \quad \begin{cases} dX_t = \mu_0 dt + \sigma dB_t, & t \in (0, T], & X_0 = x, \\ d\Phi_t = \gamma\Phi_t dB_t, & t \in (0, T], & \Phi_0 = \varphi, \end{cases}$$

and we notice that the processes decouple under this formulation. In the following, when needed, we will write  $\mathbb{E}_{(x, \varphi)}^{\mathbb{Q}_T}$  to denote the expectation under  $\mathbb{Q}_T$ , conditioned on  $X_0 = x, \Phi_0 = \varphi$ . In order to rewrite problem (4.3) in terms of the new variables  $(X, \Phi)$ , we introduce

$$\Theta_t := \frac{1 + \Phi_t}{1 + \varphi}, \quad t \in [0, T],$$

and by an application of Itô's formula, it can be verified that  $\Theta$  admits the representation

$$(5.5) \quad \Theta_t = \exp \left( \int_0^t \gamma \Pi_s d\bar{W}_s + \frac{1}{2} \int_0^t \gamma^2 \Pi_s^2 ds \right) = \frac{1}{\eta_t}, \quad t \in [0, T].$$

Upon using (5.2) and (5.5), we find

$$(5.6) \quad \begin{aligned} \mathbb{E}_{(x,\pi)}[e^{-r(\tau \wedge T)}(e^{X_{\tau \wedge T}} - \kappa)] &= \mathbb{E}_{(x,\pi)}[e^{-r(\tau \wedge T)}(e^{X_{\tau \wedge T}} - \kappa)\eta_{\tau \wedge T}\Theta_{\tau \wedge T}] \\ &= \mathbb{E}_{(x,\varphi)}^{\mathbb{Q}_T} \left[ e^{-r(\tau \wedge T)}(e^{X_{\tau \wedge T}} - \kappa) \frac{1 + \Phi_{\tau \wedge T}}{1 + \varphi} \right] \\ &= (1 + \varphi)^{-1} \mathbb{E}_{(x,\varphi)}^{\mathbb{Q}_T} [e^{-r(\tau \wedge T)}(e^{X_{\tau \wedge T}} - \kappa)(1 + \Phi_{\tau \wedge T})], \end{aligned}$$

for any stopping time  $\tau$  and  $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ . With regard to (5.6) we introduce the stopping problems

$$\begin{aligned} v(x, \pi; T) &:= \sup_{\tau} \mathbb{E}_{(x,\pi)}[e^{-r(\tau \wedge T)}(e^{X_{\tau \wedge T}} - \kappa)], \\ \text{and} \quad v^{\mathbb{Q}_T}(x, \varphi; T) &:= \sup_{\tau} \mathbb{E}_{(x,\varphi)}^{\mathbb{Q}_T} [e^{-r(\tau \wedge T)}(e^{X_{\tau \wedge T}} - \kappa)(1 + \Phi_{\tau \wedge T})], \end{aligned}$$

and notice that (5.6) implies  $v^{\mathbb{Q}_T}(x, \varphi; T) = (1 + \varphi)v(x, \varphi/(1 + \varphi); T)$  for fixed  $T > 0$ . However, since the measure  $\mathbb{Q}_T$  changes with  $T$ , passing to the limit  $T \rightarrow \infty$  in the latter expression (5.6) requires a bit of care. To this end, we define a probability space  $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{Q}})$  with a Brownian motion  $\tilde{B}$  and a filtration  $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \geq 0}$ . Moreover, we let  $(\tilde{X}, \tilde{\Phi})$  be the strong solution to the stochastic differential equation (5.4) driven by the Brownian motion  $\tilde{B}$  instead of  $B$ . Let  $\tilde{\mathbb{E}}_{(x,\varphi)}[\cdot]$  denote the expectation under  $\tilde{\mathbb{Q}}$  and define the stopping problems

$$\bar{v}(x, \varphi; T) = \sup_{\tau} \tilde{\mathbb{E}}_{(x,\varphi)}[e^{-r(\tau \wedge T)}(e^{\tilde{X}_{\tau \wedge T}} - \kappa)(1 + \tilde{\Phi}_{\tau \wedge T})], \quad \bar{v}(x, \varphi) = \sup_{\tau} \tilde{\mathbb{E}}_{(x,\varphi)}[e^{-r\tau}(e^{\tilde{X}_{\tau}} - \kappa)(1 + \tilde{\Phi}_{\tau})].$$

Due to the equivalence in laws of the process  $(\tilde{X}_t, \tilde{\Phi}_t, \tilde{B}_t)_{t \geq 0}$  under  $\tilde{\mathbb{Q}}$  and the process  $(X_t, \Phi_t, B_t)_{t \geq 0}$  under  $\mathbb{Q}_T$  on  $[0, T]$ , we have  $v^{\mathbb{Q}_T}(x, \varphi; T) = \bar{v}(x, \varphi; T)$ . Moreover, upon using Fatou's lemma and simple comparison arguments, one can show that

$$\lim_{T \rightarrow \infty} v(x, \pi; T) = v(x, \pi) \quad \text{as well as} \quad \lim_{T \rightarrow \infty} \bar{v}(x, \varphi; T) = \bar{v}(x, \varphi).$$

Hence, we finally obtain

$$(5.7) \quad \begin{aligned} \bar{v}(x, \varphi) &= \lim_{T \rightarrow \infty} \bar{v}(x, \varphi; T) = \lim_{T \rightarrow \infty} v^{\mathbb{Q}_T}(x, \varphi; T) \\ &= (1 + \varphi) \lim_{T \rightarrow \infty} v(x, \varphi/(1 + \varphi); T) = (1 + \varphi)v(x, \varphi/(1 + \varphi)). \end{aligned}$$

For the sake of clarity - and with a slight abuse of notation - from now on we simply write  $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q}, \mathbb{E}^{\mathbb{Q}}, X, \Phi, B)$  instead of  $(\tilde{\Omega}, \tilde{\mathbb{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{Q}}, \tilde{\mathbb{E}}, \tilde{X}, \tilde{\Phi}, \tilde{B})$ . Henceforth, we thus study the optimal stopping problem

$$(5.8) \quad \bar{v}(x, \varphi) = \sup_{\tau} \mathbb{E}_{(x,\varphi)}^{\mathbb{Q}} [e^{-r\tau}(e^{X_{\tau}} - \kappa)(1 + \Phi_{\tau})].$$

In the sequel, we will often write  $\mathbb{E}_{(x,\varphi)}^{\mathbb{Q}}[f(X_t, \Phi_t)] = \mathbb{E}^{\mathbb{Q}}[f(X_t^x, \Phi_t^{\varphi})]$ , where  $(X_t^x, \Phi_t^{\varphi})_{t \geq 0}$  is the unique strong solution to (5.4). The continuation and stopping region associated to this problem are then given by

$$(5.9) \quad \mathcal{C}_2 := \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : \bar{v}(x, \varphi) > (e^x - \kappa)(1 + \varphi)\},$$

$$(5.10) \quad \mathcal{S}_2 := \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : \bar{v}(x, \varphi) = (e^x - \kappa)(1 + \varphi)\}.$$

With regard to the lower-semicontinuity of  $v$  and (5.7), we find that  $(x, \varphi) \mapsto \bar{v}(x, \varphi)$  is lower-semicontinuous as well. Hence, the stopping region  $\mathcal{S}_2$  of (5.10) is a closed set, while the continuation

region  $\mathcal{C}_2$  of (5.9) is open. Also,  $\tau^* := \tau^*(x, \varphi) := \inf\{t \geq 0 : (X_t^x, \Phi_t^\varphi) \in \mathcal{S}_2\}$  is optimal by Peskir and Shiryaev [53], whenever  $\mathbb{Q}$ -a.s. finite. Furthermore, we define

$$(5.11) \quad b(\varphi) := \inf\{x \in \mathbb{R} : \bar{v}(x, \varphi) \leq (e^x - \kappa)(1 + \varphi)\},$$

with  $\inf \emptyset = \infty$ . In the following lemma, we derive some preliminary properties of the value function (5.8). In light of the relation (5.7) we notice that some of the following results are a direct consequence of Lemma 4.2.

**Lemma 5.1.** *The value function  $\bar{v}$  of (5.8) is such that*

- i)  $0 \leq \bar{v}(x, \varphi) \leq K_1 e^x (1 + \varphi)$  for all  $(x, \varphi) \in \mathbb{R} \times (0, \infty)$  and some  $K_1 > 0$ ;
- ii)  $x \mapsto \bar{v}(x, \varphi)$  is nondecreasing;
- iii)  $\varphi \mapsto \bar{v}(x, \varphi)$  is nondecreasing;
- iv)  $(x, \varphi) \mapsto \bar{v}(x, \varphi)$  is locally Lipschitz over  $\mathbb{R} \times (0, \infty)$ ;
- v)  $\varphi \mapsto \bar{v}(x, \varphi)$  and  $x \mapsto \bar{v}(x, \varphi)$  are convex.

*Proof.* Property ii) follows from Lemma 4.2 i), upon using equality (5.7). We prove the remaining claims separately.

i) For the lower bound, we notice that  $\{(x, \varphi) \in \mathbb{R} \times (0, \infty) : (e^x - \kappa) < 0\} \subset \mathcal{C}_2$ . Hence, since  $\Phi^\varphi \geq 0$  a.s., we have  $\bar{v}(x, \varphi) \geq 0$  for all  $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ . For the upper bound, we observe that for any stopping time  $\tau$

$$\begin{aligned} \mathbb{E}_{(x, \varphi)}^{\mathbb{Q}} [e^{-r\tau} (e^{X_\tau} - \kappa)(1 + \Phi_\tau)] &= (1 + \varphi) \mathbb{E}_{(x, \pi)} [e^{-r\tau} (e^{X_\tau} - \kappa)] \\ &\leq (1 + \varphi) \mathbb{E} [e^{-r\tau} e^{x + \mu_1 \tau + \sigma W_\tau}] \leq K_1 e^x (1 + \varphi), \end{aligned}$$

for  $\pi = \varphi / (1 + \varphi)$  and the last inequality follows from standard estimates upon using Assumption 4.1.

iii) Let  $\varphi, \varphi' \in (0, \infty)$  with  $\varphi' > \varphi$  and notice that  $\Phi_t^\varphi = \varphi e^{-\frac{1}{2}\gamma^2 t + \gamma B_t}$ . For  $x \in \mathbb{R}$  and  $\tau^* := \tau^*(x, \varphi)$  optimal for  $\bar{v}(x, \varphi)$  we have

$$\begin{aligned} \bar{v}(x, \varphi') - \bar{v}(x, \varphi) &\geq \mathbb{E}_{(x, \varphi')}^{\mathbb{Q}} [e^{-r\tau^*} (e^{X_{\tau^*}} - \kappa)(1 + \Phi_{\tau^*})] - \mathbb{E}_{(x, \varphi)}^{\mathbb{Q}} [e^{-r\tau^*} (e^{X_{\tau^*}} - \kappa)(1 + \Phi_{\tau^*})] \\ &= \mathbb{E}^{\mathbb{Q}} [e^{-r\tau^*} (e^{X_{\tau^*}} - \kappa)(\varphi' - \varphi) e^{-\frac{1}{2}\gamma^2 \tau^* + \gamma B_{\tau^*}}] \geq 0, \end{aligned}$$

where the last inequality exploits that  $\{(x, \varphi) \in \mathbb{R} \times (0, \infty) : (e^x - \kappa) < 0\} \subset \mathcal{C}_2$ , and the claim follows.

iv) Let  $x, x' \in \mathbb{R}$ ,  $\pi \in (0, 1)$  and  $\varphi, \varphi' \in (0, \infty)$ . Recall  $v$  of (4.3). Again, standard estimates yield

$$|v(x, \pi) - v(x', \pi)| \leq K_1 |e^x - e^{x'}|, \quad \text{as well as} \quad |\bar{v}(x, \varphi) - \bar{v}(x, \varphi')| \leq K_2 e^x |\varphi - \varphi'|,$$

for some  $K_1, K_2 > 0$ . Hence, using (5.7), we obtain

$$(5.12) \quad \begin{aligned} |\bar{v}(x, \varphi) - \bar{v}(x', \varphi')| &\leq |\bar{v}(x, \varphi) - \bar{v}(x', \varphi)| + |\bar{v}(x', \varphi) - \bar{v}(x', \varphi')| \\ &\leq K_1 (1 + \varphi) |e^x - e^{x'}| + K_2 e^{x'} |\varphi - \varphi'|, \end{aligned}$$

and thus the locally-Lipschitz property follows.

iv) We first prove convexity regarding  $\varphi \in (0, \infty)$ . For  $\varphi_1, \varphi_2 \in (0, \infty)$ ,  $x \in \mathbb{R}$  and  $\lambda \in (0, 1)$  we set  $\bar{\varphi} := \lambda \varphi_1 + (1 - \lambda) \varphi_2$  and obtain

$$\begin{aligned} \bar{v}(x, \bar{\varphi}) &= \sup_{\tau} \mathbb{E}_{(x, \bar{\varphi})}^{\mathbb{Q}} [e^{-r\tau} (e^{X_\tau} - \kappa)(1 + \bar{\varphi} e^{-\frac{1}{2}\gamma^2 \tau + \gamma B_\tau})] \\ &\leq \sup_{\tau} \mathbb{E}^{\mathbb{Q}} [e^{-r\tau} (e^{X_\tau} - \kappa) \lambda (1 + \varphi_1 e^{-\frac{1}{2}\gamma^2 \tau + \gamma B_\tau})] \\ &\quad + \sup_{\tau} \mathbb{E}^{\mathbb{Q}} [e^{-r\tau} (e^{X_\tau} - \kappa) (1 - \lambda) (1 + \varphi_2 e^{-\frac{1}{2}\gamma^2 \tau + \gamma B_\tau})] \\ &= \lambda \bar{v}(x, \varphi_1) + (1 - \lambda) \bar{v}(x, \varphi_2), \end{aligned}$$



and the claim follows. Analogously, upon exploiting the convexity of  $x \mapsto e^x$ , one can prove the convexity of  $x \mapsto \bar{v}(x, \varphi)$ .  $\square$

**Lemma 5.2.** *The continuation and stopping region regions as in (5.9)-(5.10) are such that*

$$\mathcal{C}_2 = \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : x < b(\varphi)\}, \quad \mathcal{S}_2 = \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : x \geq b(\varphi)\}.$$

*Proof.* We proceed similarly to Lemma 4.3. We first notice that the second-order differential operator associated with the two-dimensional process  $(X, \Phi)$  is such that

$$(5.13) \quad \mathcal{L}_{X, \Phi} f = \mu_0 \partial_x f + \frac{1}{2} \sigma^2 \partial_{xx} f + \frac{1}{2} \gamma^2 \varphi^2 \partial_{\varphi\varphi} f + \gamma \varphi \sigma \partial_{x\varphi} f, \quad \forall f \in C^2(\mathbb{R} \times (0, \infty)),$$

and apply Dynkin's formula to obtain

$$(5.14) \quad \begin{aligned} \bar{u}(x, \varphi) &:= \bar{v}(x, \varphi) - (e^x - \kappa)(1 + \varphi) \\ &= \sup_{\tau} \mathbb{E}_{(x, \varphi)}^{\mathbb{Q}} \left[ \int_0^{\tau} e^{-rt} \left( e^{X_t} (\mu_0 + \frac{1}{2} \sigma^2 - r) + r\kappa + \Phi_t \left( e^{X_t} (\mu_1 + \frac{1}{2} \sigma^2 - r) + r\kappa \right) \right) dt \right]. \end{aligned}$$

For  $x_2 > x_1$  and  $\tau^* := \tau^*(x_2, \varphi)$  optimal for  $\bar{v}(x_2, \varphi)$  we have

$$\begin{aligned} \bar{u}(x_1, \varphi) - \bar{u}(x_2, \varphi) &\geq \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tau^*} e^{-rt} \left( (e^{X_t^{x_2}} - e^{X_t^{x_1}}) (r - \mu_0 - \frac{1}{2} \sigma^2) + \Phi_t (e^{X_t^{x_2}} - e^{X_t^{x_1}}) (r - \mu_1 + \frac{1}{2} \sigma^2) \right) dt \right] \\ &\geq 0, \end{aligned}$$

where the last inequality follows from  $X^{x_2} \geq X^{x_1}$   $\mathbb{Q}$ -a.s. and Assumption 4.1. Hence, for  $(x_1, \varphi) \in \mathcal{S}_2$  and  $x_2 > x_1$ , we obtain  $0 \leq \bar{u}(x_2, \varphi) \leq \bar{u}(x_1, \varphi) = 0$  and the claim follows.  $\square$

It is interesting to notice that there exists a one-to-one correspondence between the continuation regions  $\mathcal{C}_1$  and  $\mathcal{C}_2$  of (4.4) and (5.9) as well as the stopping regions  $\mathcal{S}_1$  and  $\mathcal{S}_2$  of (4.5) and (5.10). Indeed, introducing the diffeomorphism

$$(5.15) \quad T := (T_1, T_2) : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R} \times (0, \infty), \quad (T_1(x, \pi), T_2(x, \pi)) := \left( x, \frac{\pi}{1 - \pi} \right),$$

with inverse

$$T^{-1}(x, \varphi) := \left( x, \frac{\varphi}{1 + \varphi} \right), \quad (x, \varphi) \in \mathbb{R} \times (0, \infty),$$

one has

$$\mathcal{C}_2 = T(\mathcal{C}_1) \quad \text{as well as} \quad \mathcal{S}_2 = T(\mathcal{S}_1).$$

Furthermore, upon using Lemma 4.3 and Lemma 5.2, we find that

$$(5.16) \quad b(\varphi) = a\left(\frac{\varphi}{1 + \varphi}\right).$$

Due to this explicit relationship between the optimal stopping boundaries, we obtain some first results on  $b$  thanks to Lemma 4.4.

**Lemma 5.3.** *The boundary  $b(\varphi)$  of (5.11) is such that*

- i)  $\varphi \mapsto b(\varphi)$  is nondecreasing on  $(0, \infty)$ ;
- ii)  $\varphi \mapsto b(\varphi)$  is left-continuous;
- iii)  $b$  is bounded by  $x_0^* \leq b(\varphi) \leq x_1^*$  for all  $\varphi \in (0, \infty)$ , with  $x_0^*$  and  $x_1^*$  as in Lemma 4.4.

The relationship (5.16) and the transformation (5.15) allow us to trace back our results from this section - as well as from the following section - to the initial optimal stopping problem (4.3). Moreover, (5.16) turns out to be valuable in the proof of Lemma 5.3, since proving the monotonicity result i) as well as the boundedness iii) is not straightforward without exploiting the relation between  $b$  and  $a$  and the results of Lemma 4.4.

## 6. A PARABOLIC FORMULATION

Observe that the dynamics of the processes  $X$  and  $\Phi$  in (5.4) are driven by the same Brownian motion. In order to account for this degeneracy, we pass yet to another formulation of the optimal stopping problem. To this end, we rely on a transformation that reveals the true parabolic nature of the generator  $\mathcal{L}_{X,\Phi}$  as in (5.13); i.e. that poses it in its canonical form (cf. Folland [34]). Define

$$(6.1) \quad \bar{T} := (\bar{T}_1, \bar{T}_2) : \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}^2, \quad (\bar{T}_1(x, \varphi), \bar{T}_2(x, \varphi)) := \left( x, \frac{\sigma}{\gamma} \ln(\varphi) - x \right),$$

for any  $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ , which is a diffeomorphism with inverse given by

$$(6.2) \quad \bar{T}^{-1}(x, z) := \left( x, e^{\frac{\gamma}{\sigma}(x+z)} \right), \quad (x, z) \in \mathbb{R}^2.$$

With regard to the transformation (6.1) we can introduce the process

$$(6.3) \quad Z_t = \frac{\sigma}{\gamma} \ln(\Phi_t) - X_t, \quad t \geq 0,$$

and an application of Itô's formula reveals that its dynamics are given by

$$(6.4) \quad dZ_t = -\frac{1}{2}(\mu_1 + \mu_0)dt, \quad Z_0 = z := \frac{\sigma}{\gamma} \ln(\varphi) - x.$$

Furthermore, we can define the transformed version of the value function  $\bar{v}$  of (5.8) via

$$(6.5) \quad \hat{v}(x, z) := \bar{v}(x, e^{\frac{\gamma}{\sigma}(x+z)}) = \sup_{\tau} \mathbb{E}_{(x,z)}^{\mathbb{Q}} [e^{-r\tau} (e^{X_\tau} - \kappa)(1 + e^{\frac{\gamma}{\sigma}(X_\tau + Z_\tau)})],$$

for  $(x, z) \in \mathbb{R}^2$  and where now  $\mathbb{E}_{(x,z)}^{\mathbb{Q}}[\cdot] = \mathbb{E}^{\mathbb{Q}}[\cdot | X_0 = x, Z_0 = z]$ . In light of this explicit relationship between the value functions  $\bar{v}$  and  $\hat{v}$ , we can conclude the following result from Lemma 5.1.

**Lemma 6.1.** *The value function  $\hat{v}(x, z)$  of (6.5) is locally Lipschitz continuous over  $\mathbb{R}^2$ .*

The associated continuation and stopping region are given by

$$(6.6) \quad \mathcal{C}_3 := \{(x, z) \in \mathbb{R}^2 : \hat{v}(x, z) > (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)})\},$$

$$(6.7) \quad \mathcal{S}_3 := \{(x, z) \in \mathbb{R}^2 : \hat{v}(x, z) = (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)})\},$$

where  $\mathcal{C}_3$  is open and  $\mathcal{S}_3$  is closed. Furthermore, the global diffeomorphism (6.1) implies that  $\mathcal{C}_3 = \bar{T}(\mathcal{C}_2)$  as well as  $\mathcal{S}_3 = \bar{T}(\mathcal{S}_2)$ , with  $\mathcal{C}_2$  and  $\mathcal{S}_2$  as in (5.9)-(5.10). Notice that the second-order infinitesimal generator associated to the process  $(X, Z)$  is now such that

$$(6.8) \quad \mathcal{L}_{X,Z}f = \mu_0 \partial_x f + \frac{1}{2} \sigma^2 \partial_{xx} f - \frac{1}{2} (\mu_1 + \mu_0) \partial_z f, \quad \forall f \in C^{2,1}(\mathbb{R}^2).$$

We can rely on standard arguments from classical PDE theory as well as optimal stopping theory (see, e.g., Karatzas and Shreve [47], Section 2.7, Th. 7.7) and obtain the following lemma.

**Lemma 6.2.** *The value function  $\hat{v}$  of (5.7) is the unique classical  $C^{2,1}$ -solution to the boundary value problem*

$$(6.9) \quad (\mathcal{L}_{X,Z} - r)w = 0 \quad \text{in } \mathcal{R} \quad \text{and} \quad w|_{\partial \mathcal{R}} = \hat{v}|_{\partial \mathcal{R}},$$

for  $\mathcal{L}_{X,Z}$  as in (6.8) and any open set  $\mathcal{R}$  such that its closure is contained in the continuation region  $\mathcal{C}_3$  of (6.6). In particular,  $\hat{v} \in C^{2,1}(\mathcal{C}_3)$ .

In the following, we aim at investigating the geometry of the state space in the coordinates  $(X, Z)$ . To this end, we define the generalised inverse of the nondecreasing boundary  $b$  by

$$(6.10) \quad b^{-1}(x) := \inf\{\varphi \in (0, \infty) : b(\varphi) > x\},$$

such that the continuation region  $\mathcal{C}_2$  of (5.9) rewrites as

$$(6.11) \quad \mathcal{C}_2 = \{(x, \varphi) \in \mathbb{R} \times (0, \infty) : b^{-1}(x) < \varphi\}.$$

Since  $\varphi \mapsto b(\varphi)$  is nondecreasing by Lemma 5.3, we observe that

$$(x, z) \in \mathcal{C}_3 \iff (x, e^{\frac{\gamma}{\sigma}(x+z)}) \in \mathcal{C}_2 \iff e^{\frac{\gamma}{\sigma}(x+z)} > b^{-1}(x) \iff z > \frac{\sigma}{\gamma} \log(b^{-1}(x)) - x,$$

and by setting

$$(6.12) \quad c^{-1}(x) := \frac{\sigma}{\gamma} \log(b^{-1}(x)) - x,$$

we can rewrite (6.6) and (6.7) as

$$(6.13) \quad \mathcal{C}_3 = \{(x, z) \in \mathbb{R}^2 : z > c^{-1}(x)\}, \quad \mathcal{S}_3 = \{(x, z) \in \mathbb{R}^2 : z \leq c^{-1}(x)\}.$$

In contrast to the optimal stopping problems in the formulations (4.3) and (5.8), deriving the monotonicity of the boundary  $x \mapsto c^{-1}(x)$  is not straightforward. Moreover - and differently to related contributions such as Federico et al. [30] - we cannot trace it back to the monotonicity of the boundary  $b$  of (5.11), since its generalised inverse  $b^{-1}$  is nondecreasing as well, and this does not imply monotonicity of  $x \mapsto c^{-1}(x)$ . To this end, we follow and adapt arguments presented in Section 4.4 of De Angelis [21], which studies separately the two cases in which the deterministic process  $Z$  as in (6.4) is either increasing ( $\mu_0 + \mu_1 \geq 0$ ) or decreasing ( $\mu_0 + \mu_1 < 0$ ). To that end, we state the following assumption, which will be **standing from now on**.

**Assumption 6.3.** *We assume  $r > \frac{1}{2}\sigma^2 + \mu_1 + \frac{(2\mu_1 + \sigma^2)(\mu_1 - \mu_0)}{\sigma^2}$ .*

**Remark 6.4.** *Notice that Assumption 6.3 requires the discount factor  $r$  to be larger than the lower bound  $\frac{1}{2}\sigma^2 + \mu_1$ , required in Assumption 4.1 and needed for the well-posedness of the problem. Furthermore, recall that  $x_0^*$  denotes the optimal execution threshold in the case of full information, when the drift is constant and equal to  $\mu_0$  (see Section 3), define*

$$(6.14) \quad \tilde{x} := \log\left(\frac{r\kappa}{r - \frac{1}{2}\sigma^2 - \mu_1}\right),$$

and notice that  $(\mathcal{L}_{\bar{X}} - r)(e^x - \kappa) \geq 0$ , for all  $x \geq \tilde{x}$  and  $\mathcal{L}_{\bar{X}} = \frac{1}{2}\sigma^2\partial_{xx} + \mu_1\partial_x$ . Using the explicit expression of  $x_0^*$  given by (3.6), it can be verified that Assumption 6.3 precisely guarantees  $x_0^* > \tilde{x}$ .

For the following analysis, it is useful to define

$$(6.15) \quad \hat{u}(x, z) := \hat{v}(x, z) - (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}),$$

as well as

$$(6.16) \quad \begin{aligned} g(x, z) &:= (\mathcal{L}_{X,Z} - r)((e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)})) \\ &= e^x\left(\frac{1}{2}\sigma^2 + \mu_0 - r\right) + r\kappa + e^{\frac{\gamma}{\sigma}(x+z)}\left(e^x\left(\frac{1}{2}\sigma^2 + \mu_1 - r\right) + r\kappa\right), \end{aligned}$$

and we observe that an application of Dynkin's formula implies

$$(6.17) \quad \hat{u}(x, z) = \sup_{\tau} \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ \int_0^{\tau} e^{-rt} g(X_t, Z_t) dt \right], \quad (x, z) \in \mathbb{R}^2.$$

**Proposition 6.5.** *Let  $\mu_0 + \mu_1 \geq 0$ . Then there exists a nondecreasing function  $c : \mathbb{R} \rightarrow \mathbb{R}$  such that the continuation region  $\mathcal{C}_3$  of (6.6) rewrites as*

$$(6.18) \quad \mathcal{C}_3 = \{(x, z) \in \mathbb{R}^2 : x < c(z)\}.$$

*Proof.* Let  $(x_0, z_0) \in \mathcal{S}_3$ ,  $x_1 > x_0$  and notice that (6.13) implies  $(-\infty, z_0] \times \{x_0\} \in \mathcal{S}_3$ . Furthermore, Assumption 6.3 guarantees  $x_0 > x_0^*$  and since the process  $Z$  is decreasing, we observe that the process  $(X^{x_1}, Z^{z_0})$  crosses the half-line  $(-\infty, z_0] \times \{x_0\}$  before reaching the level  $x_0^*$ . Hence, we have  $\mathbb{Q}_{x_1, z_0}(\tau^* < \tau_{x_0^*}) = 1$ , where  $\tau_{x_0^*} := \inf\{t \geq 0 : X_t^{x_1} = x_0^*\}$  and  $\mathbb{Q}_{x_1, z_0}(\cdot) = \mathbb{Q}(\cdot | X_0 = x_1, Z_0 = z_0)$ , and

Assumption 6.3 implies  $\exp(X_s^{x_1})(r - \frac{1}{2}\sigma^2 - \mu_1) > r\kappa$  for all  $s \in [0, \tau^*)$ . Consequently, (6.16)-(6.17) imply  $\hat{u}(x_1, z_0) \leq 0$  for all  $x_1 > x_0$ , and therefore  $\{z_0\} \times [x_0, \infty) \in \mathcal{S}_3$ . We can thus define

$$(6.19) \quad c(z) := \inf\{x \in \mathbb{R} : (x, z) \in \mathcal{S}_3\}.$$

and observe that (6.13) implies that  $z \mapsto c(z)$  is nondecreasing.  $\square$

In order to establish the same result in the case when  $\mu_0 + \mu_1 < 0$ , we first state the following lemma.

**Lemma 6.6.** *We have*

$$(6.20) \quad \hat{v}_z(x, z) = \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ \frac{\gamma}{\sigma} e^{-r\tau^*} (e^{X_{\tau^*}} - \kappa) e^{\frac{\gamma}{\sigma}(X_{\tau^*} + Z_{\tau^*})} \mathbb{1}_{\{\tau^* < \infty\}} \right],$$

for all  $(x, z) \in \mathbb{R}^2 \setminus \partial\mathcal{C}_3$  and  $\tau^* := \tau^*(x, z)$ .

*Proof.* For  $(x, z) \in \mathcal{S}_3$  the claim follows immediately, since  $\mathbb{Q}_{(x,z)}(\tau^* = 0) = 1$ . Hence, we let  $(x, z) \in \mathcal{C}_3$  and for  $\epsilon > 0$  we obtain

$$(6.21) \quad \begin{aligned} \hat{v}(x, z + \epsilon) - \hat{v}(x, z) &\geq \mathbb{E}^{\mathbb{Q}} \left[ e^{-r(\tau^* \wedge t)} (\hat{v}(X_{\tau^* \wedge t}^x, Z_{\tau^* \wedge t}^{z+\epsilon}) - \hat{v}(X_{\tau^* \wedge t}^x, Z_{\tau^* \wedge t}^z)) \right] \\ &\geq \mathbb{E}^{\mathbb{Q}} \left[ e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa) e^{\frac{\gamma}{\sigma} X_{\tau^*}^x} (e^{\frac{\gamma}{\sigma} Z_{\tau^*}^{z+\epsilon}} - e^{\frac{\gamma}{\sigma} Z_{\tau^*}^z}) \mathbb{1}_{\{\tau^* < t\}} \right] \\ &\quad + \mathbb{E}^{\mathbb{Q}} \left[ e^{-rt} (\hat{v}(X_t^x, Z_t^{z+\epsilon}) - \hat{v}(X_t^x, Z_t^z)) \mathbb{1}_{\{\tau^* > t\}} \right], \end{aligned}$$

where the first inequality follows from the supermartingale property of  $(e^{-r(\tau \wedge t)} \hat{v}(X_{\tau \wedge t}^x, Z_{\tau \wedge t}^{z+\epsilon}))_t$  and the martingale property of  $(e^{-r(\tau \wedge t)} \hat{v}(X_{\tau \wedge t}^x, Z_{\tau \wedge t}^z))_t$  for  $\tau^* := \tau^*(x, z)$ . Upon employing a change of measure as in Section 5, we find

$$\begin{aligned} \mathbb{E}_{(x,z)}^{\mathbb{Q}} [e^{-rt} |\hat{v}(X_t, Z_t)|] &\leq \mathbb{E}_{(x,z)}^{\mathbb{Q}} [e^{-rt} |\bar{v}(x, e^{\frac{\gamma}{\sigma}(X_t + Z_t)})|] \leq K_1 \mathbb{E}_{(x, \exp(\frac{\gamma}{\sigma}(x+z)))}^{\mathbb{Q}} [e^{-rt} e^{X_t} (1 + \Phi_t)] \\ &= K_1 (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \mathbb{E}_{(x, \pi)} [e^{-rt} e^{X_t}], \end{aligned}$$

where  $\pi = e^{\frac{\gamma}{\sigma}(x+z)} / (1 + e^{\frac{\gamma}{\sigma}(x+z)})$ . It is then easy to verify that Assumption 4.1 implies

$$\lim_{t \uparrow \infty} \mathbb{E}_{(x,z)}^{\mathbb{Q}} [e^{-rt} \hat{v}(X_t, Z_t)] = 0,$$

and hence, applying dominated convergence in (6.21) as  $t \uparrow \infty$  yields

$$(6.22) \quad \hat{v}(x, z + \epsilon) - \hat{v}(x, z) \geq \mathbb{E}^{\mathbb{Q}} \left[ e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa) e^{\frac{\gamma}{\sigma} X_{\tau^*}^x} (e^{\frac{\gamma}{\sigma} Z_{\tau^*}^{z+\epsilon}} - e^{\frac{\gamma}{\sigma} Z_{\tau^*}^z}) \mathbb{1}_{\{\tau^* < \infty\}} \right].$$

Similar arguments show

$$(6.23) \quad \hat{v}(x, z) - \hat{v}(x, z - \epsilon) \leq \mathbb{E}^{\mathbb{Q}} \left[ e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa) e^{\frac{\gamma}{\sigma} X_{\tau^*}^x} (e^{\frac{\gamma}{\sigma} Z_{\tau^*}^{z+\epsilon}} - e^{\frac{\gamma}{\sigma} Z_{\tau^*}^z}) \mathbb{1}_{\{\tau^* < \infty\}} \right],$$

and since  $\hat{v} \in C^{2,1}(\mathcal{C}_3)$  (cf. Lemma 6.2), dividing (6.22) and (6.23) by  $\epsilon$  and letting  $\epsilon \downarrow 0$ , we obtain the desired result.  $\square$

**Proposition 6.7.** *Let  $\mu_0 + \mu_1 < 0$ . Under the additional assumption that  $r > \frac{\gamma}{2\sigma} |\mu_0 + \mu_1|$ , there exists a nondecreasing function  $c : \mathbb{R} \rightarrow \mathbb{R}$  such that the continuation region of (6.6) can be written as*

$$(6.24) \quad \mathcal{C}_3 = \{(x, z) \in \mathbb{R}^2 : x < c(z)\}.$$

*Proof.* Let  $(x, z) \in \mathbb{R}^2$ . Notice that  $x < x_0^*$  implies  $(x', z) \in \mathcal{C}_3$  for all  $x' < x$  and  $z \in \mathbb{R}$ , because of Lemma 4.4 and since the transformations  $T_1$  and  $\bar{T}_1$  of (5.15) and (6.1), respectively, are the identity; hence,  $\{(x, z) : x < x_0^*\} \subset \mathcal{C}_3$ . We can thus focus on the case that  $x \geq x_0^*$  and distinguish two possibilities:

- i)  $\hat{u}_x(x, z) \leq 0 \quad \forall x \in (x_0^*, \infty)$  such that  $(x, z) \in \mathcal{C}_3$ ;
- ii)  $\exists x_0 \in \mathbb{R}, x_0 > x_0^*$  such that  $(x_0, z) \in \mathcal{C}_3$  and  $\hat{u}_x(x_0, z) > 0$ .

In case i), the map  $x \mapsto \widehat{u}(x, z)$  is decreasing for  $x \in (x_0^*, \infty)$  and  $(x, z) \in \mathcal{C}_3$ . Hence, for any  $(x, z)$  in the latter region we obtain  $(-\infty, x] \times \{z\} \in \mathcal{C}_3$  and the claim follows in the same spirit as in Proposition 6.5. In case ii), we establish a contradiction scheme. As a first step, we show that ii) implies  $[x_0, \infty) \times \{z\} \in \mathcal{C}_3$ , which will then lead to a contradiction. We start by noticing that Lemma 6.2 and (6.17) imply

$$(6.25) \quad (\mathcal{L}_{X,Z} - r)\widehat{u}(x_0, z) = -g(x_0, z),$$

for  $(x_0, z)$  as given in ii) above. Since  $\mu_0 < 0$  and  $\widehat{u}_x(x_0, z) > 0$  we have  $\mu_0 \widehat{u}(x_0, z) < 0$ , and thus

$$(6.26) \quad \begin{aligned} \frac{1}{2}\sigma^2 \widehat{u}_{xx}(x_0, z) &= r\widehat{u}(x_0, z) - \mu_0 \widehat{u}_x(x_0, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x_0, z) - g(x_0, z) \\ &> r\widehat{u}(x_0, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x_0, z) - g(x_0, z). \end{aligned}$$

Next, we notice that we can rewrite (6.20) as

$$(6.27) \quad \widehat{v}_z(x, z) = \frac{\gamma}{\sigma} \left( \widehat{v}(x, z) - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa)] \right),$$

and since

$$\widehat{v}_z(x, z) = \widehat{u}_z(x, z) + \frac{\gamma}{\sigma} (e^x - \kappa) e^{\frac{\gamma}{\sigma}(x+z)} \quad \text{and} \quad \widehat{v}(x, z) = \widehat{u}(x, z) + (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}),$$

(6.27) gives

$$\widehat{u}_z(x, z) + \frac{\gamma}{\sigma} (e^x - \kappa) e^{\frac{\gamma}{\sigma}(x+z)} = \frac{\gamma}{\sigma} \left( \widehat{u}(x, z) + (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}) - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa)] \right),$$

which is equivalent to

$$\widehat{u}_z(x, z) = \frac{\gamma}{\sigma} \widehat{u}(x, z) + \frac{\gamma}{\sigma} \left( e^x - \kappa - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}^x} - \kappa)] \right).$$

We can thus plug this last equality into (6.26) and obtain

$$\begin{aligned} &\frac{1}{2}\sigma^2 \widehat{u}_{xx}(x_0, z) \\ &> r\widehat{u}(x_0, z) + \frac{1}{2}(\mu_0 + \mu_1) \left( \frac{\gamma}{\sigma} \widehat{u}(x_0, z) + \frac{\gamma}{\sigma} \left( e^{x_0} - \kappa - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}^{x_0}} - \kappa)] \right) \right) - g(x_0, z) \\ &= \left( r + \frac{1}{2}(\mu_0 + \mu_1) \frac{\gamma}{\sigma} \right) (\widehat{u}(x_0, z) + e^{x_0} - \kappa) - \frac{1}{2}(\mu_0 + \mu_1) \frac{\gamma}{\sigma} \mathbb{E}^{\mathbb{Q}}[e^{-r\tau^*} (e^{X_{\tau^*}^{x_0}} - \kappa)] - g(x_0, z) \\ &> 0, \end{aligned}$$

where the last inequality follows from  $r > \frac{\gamma}{2\sigma} |\mu_0 + \mu_1|$  and Assumption 6.3, upon noticing that  $x_0 > x_0^*$ . We deduce that  $\widehat{u}_x(\cdot, z)$  increases in a right-neighbourhood of  $x_0$  and repeating arguments for every  $x > x_0$  yields  $\widehat{u}_x(\cdot, z) > 0$  on  $[x_0, \infty)$ . It follows that  $\widehat{u}(\cdot, z)$  is increasing on  $[x_0, \infty)$  such that  $[x_0, \infty) \times \{z\} \in \mathcal{C}_3$  and (combining the latter with (6.13)) we have  $\mathcal{A} := [x_0, \infty) \times [z_0, \infty) \subset \mathcal{C}_3$ . However, this leads to a contradiction. To see this, let  $(x, z) \in \mathcal{A}$  and define  $\tau_{x_0} := \inf\{t > 0 : X_t^x \leq x_0\}$ . Since  $t \mapsto Z_t^z$  is increasing, the only possibility for the process  $(X^x, Z^z)$  to exit  $\mathcal{A}$  and thus eventually the continuation region, is by passing through the horizontal line  $[x_0, \infty) \times \{z_0\}$ . We thus have  $\tau_{x_0} \leq \tau^*$   $\mathbb{Q}_{(x,z)}$ -a.s. and moreover, since  $\mu_0 < 0$ , the stopping time  $\tau_{x_0}$  is finite a.s. Upon using Lemma 5.1 i) and (6.5), it follows that

$$\begin{aligned} (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}) &< \widehat{v}(x, z) = \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-r\tau_{x_0}} \widehat{v}(X_{\tau_{x_0}}, Z_{\tau_{x_0}}) \right] \\ &= \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-r\tau_{x_0}} \widehat{v}(x_0, z - \frac{1}{2}(\mu_0 + \mu_1)\tau_{x_0}) \right] \\ &\leq K_1 e^{x_0} \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-r\tau_{x_0}} \right] + K_1 e^{\frac{\gamma}{\sigma}(x_0+z)} e^{x_0} \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-(r - \frac{1}{2}\frac{\gamma}{\sigma} |\mu_0 + \mu_1|)\tau_{x_0}} \right]. \end{aligned}$$

Let now  $\widehat{r} := r - \frac{\gamma}{2\sigma}|\mu_0 + \mu_1| > 0$  and denote  $\phi_r$  (resp.  $\phi_{\widehat{r}}$ ) the strictly decreasing solution to  $\frac{1}{2}\sigma^2 f_{xx} + \mu_0 f_x - qf = 0$ , for  $q \in \{r, \widehat{r}\}$ . Then, by results on hitting times for one-dimensional diffusions (see, e.g., Borodin and Salminen [12], Ch. II), the above inequality is equivalent to

$$(6.28) \quad (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}) \leq K_1 e^{x_0} \frac{\phi_r(x)}{\phi_r(x_0)} + K_1 e^{x_0} e^{\frac{\gamma}{\sigma}(x_0+z)} \frac{\phi_{\widehat{r}}(x)}{\phi_{\widehat{r}}(x_0)},$$

which thus holds true for all  $(x, z) \in \mathcal{A}$ . Since  $\mathcal{A}$  is right-connected, we can let  $x \rightarrow \infty$  and notice that  $(e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}) \rightarrow \infty$ , while the right hand side of (6.28) decreases to 0 due to the decreasing property of  $x \rightarrow \phi_q(x)$  for  $q$  positive. We thus obtain a contradiction, which concludes our proof.  $\square$

In order to guarantee the existence of the nondecreasing boundary  $z \mapsto c(z)$  in the rest of the paper, we state the following **standing assumption**.

**Assumption 6.8.** *We let Assumption 6.3 hold and that either i)  $\mu_0 + \mu_1 \geq 0$  or ii)  $\mu_0 + \mu_1 < 0$  and  $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$ .*

**Remark 6.9.** *Notice that Propositions 6.5 and 6.7 imply that the function  $x \mapsto c^{-1}(x)$  of (6.12) is nondecreasing as well. Moreover, we notice that*

$$(6.29) \quad z > c^{-1}(x) \iff c(z) > x,$$

and hence, the function  $c^{-1}$  is the right-continuous inverse of  $c$  and thus admits the representation

$$(6.30) \quad c^{-1}(x) = \inf\{z \in \mathbb{R} : c(z) > x\}.$$

In light of the connection (6.12) between  $c^{-1}$  and  $b^{-1}$  (the generalised inverse of the boundary  $b$ ), equation (6.30) allows us to trace back our results to the formulation of Section 5 and then - through the representation (5.16) - to the original setting of Section 4.

**6.1. Regularity of the value function and of the optimal stopping boundary.** Under the Assumption 6.8 we established the existence of a nondecreasing boundary  $z \mapsto c(z)$ , such that  $\mathbb{R}^2$  is split into the continuation region  $\mathcal{C}_3$  of (6.6) and the stopping region  $\mathcal{S}_3$  of (6.7). In the following, we derive some further properties of the optimal stopping boundary and of the value function  $\widehat{v}$  of (6.5). We first state the following result, which will be helpful in the forthcoming analysis.

**Lemma 6.10.** *We have  $\widehat{u}_z(x, z) \geq 0$  for  $(x, z) \in \mathcal{C}_3$ .*

*Proof.* Because of (6.5) and (6.1), we have that  $\bar{v}(x, \varphi)$  as in (5.8) is such that  $\bar{v}(x, \varphi) = \widehat{v}(x, \frac{\sigma}{\gamma} \ln(\varphi) - x)$ ,  $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ . Since  $\widehat{v}_z \in C^0(\mathcal{C}_3)$  by Lemma 6.2, we then also have  $\bar{v}_\varphi \in C^0(\mathcal{C}_2)$ . Furthermore,  $\varphi \mapsto \bar{v}(x, \varphi)$  is convex on  $(0, \infty)$  by Lemma 5.1 iv) and thus also  $\varphi \mapsto \bar{u}(x, \varphi)$  of (5.14). Then, for  $(x, \varphi) \in \mathcal{C}_2$  and  $\varphi' = b^{-1}(x)$  such that  $(x, \varphi') \in \partial\mathcal{C}_2$ , we obtain (as  $\bar{u}_\varphi \in C^0(\mathcal{C}_2)$  as well)

$$0 \leq \bar{u}(x, \varphi) = \bar{u}(x, \varphi) - \bar{u}(x, b^{-1}(x)) \leq \bar{u}_\varphi(x, \varphi)(\varphi - b^{-1}(x)),$$

and  $\varphi > b^{-1}(x)$  implies  $\bar{u}_\varphi(x, \varphi) \geq 0$  for  $(x, \varphi) \in \mathcal{C}_2$ . In light of the relation (6.5) we then obtain  $\widehat{u}_z(x, z) \geq 0$  on  $\mathcal{C}_3$ .  $\square$

**Proposition 6.11.** *The optimal stopping boundary  $c(z)$  is such that  $x_0^* \leq c(z) \leq x_1^*$  for all  $z \in \mathbb{R}$  and with  $x_0^*$  and  $x_1^*$  as in Lemma 4.4. Furthermore, we have  $c \in C(\mathbb{R})$ .*

*Proof.* The first part of the claim follows from Lemma 5.3 iii) and by noticing that the transformation  $\bar{T}_1$  of (6.1) is the identity. We derive the continuity of  $z \mapsto c(z)$  in two steps.

1) *Left-Continuity:* Let  $z_0 \in \mathbb{R}$  and  $z_n \uparrow z_0$  as  $n \rightarrow \infty$ . Since  $z \mapsto c(z)$  is nondecreasing and  $\mathcal{S}_3$  is closed, we obtain  $\lim_{n \rightarrow \infty} (c(z_n), z_n) = (c(z_0-), z_0) \in \mathcal{S}_3$ , where  $c(z_0-)$  denotes the left limit of  $c$  at  $z_0$ . The definition of  $c$  in (6.19) implies  $c(z_0-) \geq c(z_0)$ , but since  $c$  is nondecreasing, we must have  $c(z_0-) = c(z_0)$  and the claim follows.

2) *Right-Continuity:* We argue by contradiction and assume there exists  $z_0 \in \mathbb{R}$  s.t.  $c(z_0) < c(z_0+)$ .

Using techniques developed in De Angelis [18], we take  $c(z_0) < x_1 < x_2 < c(z_0+)$  and a nonnegative function  $\phi \in C_c^\infty(x_1, x_2)$  such that  $\int_{x_1}^{x_2} \phi(x) dx = 1$ . Recalling (6.25), we have

$$(6.31) \quad \mathcal{L}_{X,Z} \widehat{u}(x, z) - r\widehat{u}(x, z) = -g(x, z),$$

for  $(x, z) \in (x_1, x_2) \times (z_0, \infty)$ . In the following, it is helpful to treat the cases i)  $\mu_0 + \mu_1 \geq 0$  and ii)  $\mu_0 + \mu_1 < 0$  separately. Let us start with i) and recall that  $\widehat{u}_z(x, z) \geq 0$  for  $x$  and  $z$  as above, due to Lemma 6.10. Integration by parts reveals

$$\begin{aligned} 0 &\geq -\frac{1}{2}(\mu_0 + \mu_1) \int_{x_1}^{x_2} \widehat{u}_z(x, z) \phi(x) dx \\ &= \int_{x_1}^{x_2} \left( r\widehat{u}(x, z) - \mu_0 \widehat{u}_x(x, z) - \frac{1}{2} \sigma^2 \widehat{u}_{xx}(x, z) - g(x, z) \right) \phi(x) dx \\ &= \int_{x_1}^{x_2} \left( r\widehat{u}(x, z) \phi(x) + \mu_0 \widehat{u}(x, z) \phi'(x) - \frac{1}{2} \sigma^2 \widehat{u}(x, z) \phi''(x) - g(x, z) \phi(x) \right) dx. \end{aligned}$$

Hence, employing dominated convergence as  $z \downarrow z_0$  and using  $\widehat{u}(x, z_0) = 0$ , yields

$$(6.32) \quad 0 \geq - \int_{x_1}^{x_2} g(x, z) \phi(x) dx > 0,$$

where the latter inequality follows from  $x_1, x_2 \geq x_0^*$  and Assumption 6.3, which implies  $x > \tilde{x}$  for all  $x \in [x_1, x_2]$  and  $\tilde{x}$  as in (6.14). We thus obtain a contradiction and  $c(z_0) = c(z_0+)$ .

In case ii), we rely on classical results of internal regularity of PDEs (cf. Th. 10 in Chapter 3 of Friedman [36]), which allow to take derivatives in (6.31) with respect to  $x$  and have  $\widehat{u}_x \in C^{2,1}(\mathcal{C}_3)$  solving

$$(\mathcal{L}_{X,Z} - r)\widehat{u}_x(x, z) = -g_x(x, z), \quad (x, z) \in (x_1, x_2) \times (z_0, \infty).$$

Then, for  $z > z_0$  we obtain

$$(6.33) \quad \int_{x_1}^{x_2} ((\mathcal{L}_{X,Z} - r)\widehat{u}_x(x, z) + g_x(x, z)) \phi(x) dx = 0.$$

Let  $F_\phi(z) := \int_{x_1}^{x_2} \widehat{u}_{xz}(x, z) \phi(x) dx$ . Integration by parts allows to rewrite (6.33) as

$$\begin{aligned} \frac{1}{2} |\mu_0 + \mu_1| F_\phi(z) &= \int_{x_1}^{x_2} \left( r\widehat{u}_x(x, z) - \frac{1}{2} \sigma^2 \widehat{u}_{xxx}(x, z) - \mu_0 \widehat{u}_{xx}(x, z) - g_x(x, z) \right) \phi(x) dx \\ &= \int_{x_1}^{x_2} \left( -r\widehat{u}(x, z) \phi'(x) + \frac{1}{2} \sigma^2 \widehat{u}(x, z) \phi'''(x) - \mu_0 \widehat{u}(x, z) \phi''(x) - g_x(x, z) \phi(x) \right) dx, \end{aligned}$$

and using dominated convergence as  $z \downarrow z_0$  as well as  $\widehat{u}(x, z_0) = 0$  results in

$$F_\phi(z_0+) = \frac{2}{|\mu_0 + \mu_1|} \int_{x_1}^{x_2} -g_x(x, z_0) \phi(x) dx \geq p_0 > 0,$$

for some  $p_0$ , where the second to last inequality again follows from Assumption 6.3. Thus, there exists  $\epsilon > 0$  such that  $F_\phi(z) \geq p_0/2$  for all  $z \in (z_0, z_0 + \epsilon)$  and we finally obtain

$$\begin{aligned} \frac{1}{2} p_0 \epsilon &\leq \int_{z_0}^{z_0 + \epsilon} F_\phi(z) dz = \int_{z_0}^{z_0 + \epsilon} \int_{x_1}^{x_2} \widehat{u}_{xz}(x, z) \phi(x) dx dz = - \int_{x_1}^{x_2} \int_{z_0}^{z_0 + \epsilon} \widehat{u}_z(x, z) \phi'(x) dz dx \\ &= - \int_{x_1}^{x_2} (\widehat{u}(x, z_0 + \epsilon) - \widehat{u}(x, z_0)) \phi'(x) dx = \int_{x_1}^{x_2} \widehat{u}_x(x, z_0 + \epsilon) \phi(x) dx \leq 0, \end{aligned}$$

where we used  $\widehat{u}(x, z_0) = 0$  as well as  $\widehat{u}_x(x, z) \leq 0$  for  $x \in [x_1, x_2]$  and  $z > z_0$  (cf. Proposition 6.7). Hence,  $c(z) = c(z+)$  for all  $z \in \mathbb{R}$  and together with 1) we conclude that  $z \mapsto c(z)$  is continuous.  $\square$

In the next step, we derive the regularity of the value function. Its proof can be found in Appendix A.

**Proposition 6.12.** *The value function  $\hat{v}$  of (6.5) satisfies  $\hat{v} \in C^1(\mathbb{R}^2)$  and  $\hat{v}_{xx} \in L_{loc}^\infty(\mathbb{R}^2)$ .*

In light of Proposition 6.12, we are able to derive an integral equation for the free boundary  $c$ . Let us first recall that by standard arguments, based on the strong Markov property and Proposition 6.12, the value function  $\hat{v}$  and the free boundary  $c$  solve the free-boundary problem

$$(6.34) \quad \begin{cases} (\mathcal{L}_{X,Z} - r)\hat{v}(x, z) \leq 0, & (x, z) \in \mathbb{R}^2, \\ (\mathcal{L}_{X,Z} - r)\hat{v}(x, z) = 0, & x < c(z), z \in \mathbb{R}, \\ \hat{v}(x, z) \geq (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}), & (x, z) \in \mathbb{R}^2, \\ \hat{v}(x, z) = (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)}), & x \geq c(z), z \in \mathbb{R}, \\ \hat{v}_x(x, z) = e^x(1 + e^{\frac{\gamma}{\sigma}(x+z)}) + \frac{\gamma}{\sigma}(e^x - \kappa)e^{\frac{\gamma}{\sigma}(x+z)}, & x = c(z), z \in \mathbb{R}, \\ \hat{v}_z(x, z) = \frac{\gamma}{\sigma}(e^x - \kappa)e^{\frac{\gamma}{\sigma}(x+z)}, & x = c(z), z \in \mathbb{R}. \end{cases}$$

In the next Proposition, upon using a suitable application of Itô's Lemma, we derive a probabilistic representation of the value function  $\hat{v}$ . Its proof is postponed to Appendix B.

**Proposition 6.13.** *Recall the free boundary  $c$  of (6.19) and the function  $g$  of (6.16). For any  $(x, z) \in \mathbb{R}^2$ , the value function  $\hat{v}$  can be written as*

$$(6.35) \quad \hat{v}(x, z) = \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ - \int_0^\infty e^{-rs} g(X_s, Z_s) \mathbb{1}_{\{X_s \geq c(Z_s)\}} ds \right].$$

Denote now by

$$(6.36) \quad G(w; m, v) := \frac{1}{\sqrt{2\pi v^2}} e^{-\frac{(w-m)^2}{2v^2}}, \quad w \in \mathbb{R}, m \in \mathbb{R}, v > 0,$$

the density function of a Gaussian random variable with mean  $m$  and variance  $v^2$ . Then, from Proposition 6.13 we obtain the following result.

**Proposition 6.14.** *Let*

$$\mathcal{M} := \{f : \mathbb{R} \mapsto \mathbb{R} : f \text{ is nondecreasing, continuous and s.t. } x_0^* \leq f(z) \leq x_1^*\}.$$

*Then, the free boundary  $c$  of (6.19) is the unique solution in  $\mathcal{M}$  to the integral equation*

$$(6.37) \quad (e^{c(z)} - \kappa)(1 + e^{\frac{\gamma}{\sigma}(c(z)+z)}) = \int_0^\infty e^{-rs} \left( \int_{\mathbb{R}} -g(w, Z_s) G(w; c(z) + \mu_0 s, \sigma \sqrt{s}) \mathbb{1}_{\{w \geq c(z)\}} dw \right) ds,$$

*with  $g$  as in (6.16) and  $G$  as in (6.36).*

*Proof.* We take  $x = c(z)$  in Proposition 6.13. Employing the continuity of the value function we find

$$(6.38) \quad (e^{c(z)} - \kappa)(1 + e^{\frac{\gamma}{\sigma}(c(z)+z)}) = \mathbb{E}^{\mathbb{Q}} \left[ - \int_0^\infty e^{-rs} g(X_s^{c(z)}, Z_s^z) \mathbb{1}_{\{X_s^{c(z)} \geq c(Z_s^z)\}} ds \right], \quad z \in \mathbb{R}.$$

By noticing that  $Z^z$  is deterministic and  $X_s^{c(z)}$  is Gaussian under  $\mathbb{Q}$  with mean  $c(z) + \mu_0 s$  and variance  $\sigma^2 s$ , we can reformulate (6.38) as (6.37), upon using (6.36). To show uniqueness one can employ a four-step-approach exploiting the superharmonic characterization of  $\hat{v}$ , as originally developed in Th. 3.1 of Peskir [51]. Since the present setting does not exhibit additional challenges, we omit details for the sake of brevity.  $\square$

**Remark 6.15.** *As it turns out, the integral equation (6.37) allows to derive an integral equation for the boundary  $b^{-1}$  of (6.10) as well. Indeed, taking  $z = c^{-1}(x)$  in (6.37) and using (6.12) yields*

$$(e^x - \kappa)(1 + b^{-1}(x)) = \mathbb{E}^{\mathbb{Q}} \left[ - \int_0^\infty e^{-rs} g(X_s^x, \frac{\sigma}{\gamma} \ln(\Phi_s^{b^{-1}(x)}) - X_s^x) \mathbb{1}_{\{\Phi_s^{b^{-1}(x)} \leq b^{-1}(X_s^x)\}} ds \right], \quad x \in \mathbb{R}.$$



In particular, it follows from the latter

(6.39)

$$b^{-1}(x) = \frac{1}{e^x - \kappa} \mathbb{E}^{\mathbb{Q}} \left[ - \int_0^\infty e^{-rs} g(X_s^x, \frac{\sigma}{\gamma} \ln(\Phi_s^{b^{-1}(x)}) - X_s^x) \mathbb{1}_{\{\Phi_s^{b^{-1}(x)} \leq b^{-1}(X_s^x)\}} ds \right] - 1, \quad x \in \mathbb{R}.$$

Notice that the domain of  $b^{-1}$  is given by the interval  $[x_0^*, x_1^*]$  (cf. Lemma 5.3) and hence, we do not encounter any problems when dividing by  $e^x - \kappa$  since Assumption 6.3 guarantees  $e^x - \kappa > 0$  for  $x \geq x_0^*$ .

## 7. SOLUTION OF THE OPTIMAL EXECUTION PROBLEM

In this section, we finally return to the optimal execution problem of Section 4 and provide its solution. Before we do so, it is helpful to transform the singular stochastic control problem (2.9) by arguing as for the optimal stopping problem in Sections 5 and 6, respectively. Since the arguments are in the same spirit of those developed in Section 5, details are omitted (see also Section 4 in Federico et al. [30]). First, we make a change of measure as in Section 5, and for  $\mathbb{Q}$  as introduced therein, we let

$$(7.1) \quad dX_t^\xi = \mu_0 dt + \sigma dB_t - \alpha d\xi_t, \quad X_{0-}^\xi = x,$$

denote the dynamics of the controlled process  $X^\xi$  under  $\mathbb{Q}$ . Hence, conditionally to  $X_{0-}^\xi = x$ ,  $Y_{0-}^\xi = y$  and  $\Phi_0 = \varphi$ , we introduce the transformed optimal control problem

(7.2)

$$\bar{V}(x, y, \varphi) := \sup_{\xi \in \mathcal{A}(y)} \mathbb{E}^{\mathbb{Q}}_{(x, y, \varphi)} \left[ \int_0^\infty e^{-rt} (e^{X_t^\xi} - \kappa) (1 + \Phi_t) \circ d\xi_t \right], \quad (x, y, \varphi) \in \mathbb{R} \times (0, \infty) \times (0, \infty),$$

and observe that  $\bar{V}(x, y, \varphi) = (1 + \varphi)V(x, y, \frac{\varphi}{1+\varphi})$ . Furthermore, we set

$$(7.3) \quad Z_t^\xi := \frac{\sigma}{\gamma} \log(\Phi_t) - X_t^\xi, \quad z := \frac{\sigma}{\gamma} \log(\varphi) - x,$$

for any  $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ , which, through an application of Itô-Meyer's formula, is easily shown to have dynamics

$$(7.4) \quad dZ_t^\xi = -\frac{1}{2}(\mu_0 + \mu_1)dt + \alpha d\xi_t, \quad Z_{0-}^\xi = z.$$

Finally, analogously to (6.5), we define

$$(7.5) \quad \widehat{V}(x, y, z) := \bar{V}(x, y, e^{\frac{\gamma}{\sigma}(x+z)}) = \sup_{\xi \in \mathcal{A}(y)} \mathbb{E}^{\mathbb{Q}}_{(x, y, z)} \left[ \int_0^\infty e^{-rt} (e^{X_t^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_t^\xi + Z_t^\xi)}) \circ d\xi_t \right],$$

for  $(x, y, z) \in \mathcal{O} := \mathbb{R} \times (0, \infty) \times \mathbb{R}$ , where  $\mathbb{E}^{\mathbb{Q}}_{(x, y, z)}$  denotes the expectation conditional on  $X_{0-}^\xi = x$ ,  $Y_{0-}^\xi = y$  and  $Z_{0-}^\xi = z$ .

In the following, we introduce a candidate for the value function  $V$  of (2.9) and - through the explicit relationships between the value functions  $v, \bar{v}$  and  $\widehat{v}$  - also for the value functions  $\bar{V}$  and  $\widehat{V}$  of (7.2) and (7.5). To this end, we set

$$(7.6) \quad U(x, y, \pi) := \frac{1}{\alpha} \int_{x-\alpha y}^x v(x', \pi) dx',$$

where  $v$  denotes the value function of (4.3). Upon using the explicit relationship (5.7) of  $v$  and  $\bar{v}$  it follows that

$$(7.7) \quad \bar{U}(x, y, \varphi) := (1 + \varphi)U(x, y, \frac{\varphi}{1+\varphi}) = (1 + \varphi) \frac{1}{\alpha} \int_{x-\alpha y}^x v(x', \frac{\varphi}{1+\varphi}) dx' = \frac{1}{\alpha} \int_{x-\alpha y}^x \bar{v}(x', \varphi) dx',$$

as the candidate for the value function  $\bar{V}$  of (7.2). Furthermore, by defining  $\widehat{U}(x, y, z) := \bar{U}(x, y, e^{\frac{\gamma}{\sigma}(x+z)})$  and exploiting the relationship (6.5) we can derive

$$(7.8) \quad \widehat{U}(x, y, z) = \frac{1}{\alpha} \int_{x-\alpha y}^x \widehat{v}(x', x+z-x') dx' = \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}(x+z-q, q) dq,$$

where the last equality above follows from a simple change of variables. With regard to Proposition 6.12 we can state the following result, whose proof is based on direct computations.

**Lemma 7.1.** *The function  $\widehat{U}$  of (7.8) is such that  $\widehat{U} \in C^1(\mathcal{O})$ . Moreover,  $\widehat{U}_{xy}, \widehat{U}_{yz} \in C(\mathcal{O})$  and  $\widehat{U}_{xx}, \widehat{U}_{xz} \in L_{loc}^\infty(\mathcal{O})$ .*

*Proof.* Notice that (7.8) gives

$$(7.9) \quad \widehat{U}_x(x, y, z) = \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_x(x+z-q, q) dq, \quad \widehat{U}_y(x, y, z) = \widehat{v}(x-\alpha y, z+\alpha y),$$

$$(7.10) \quad \begin{aligned} \widehat{U}_z(x, y, z) &= \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_z(x+z-q, q) dq + \frac{1}{\alpha} [\widehat{v}(x-\alpha y, z+\alpha y) - \widehat{v}(x, z)] \\ &= \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_z(x+z-q, q) dq, \end{aligned}$$

so that  $\widehat{U}_x, \widehat{U}_y$  and  $\widehat{U}_z$  are continuous due to Proposition 6.12. Moreover, Proposition 6.12 also implies that

$$(7.11) \quad \widehat{U}_{xx}(x, y, z) = \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_{xx}(x+z-q, q) dq,$$

is locally bounded, and the mixed derivatives

$$\widehat{U}_{xy}(x, y, z) = \widehat{v}_x(x-\alpha y, z+\alpha y), \quad \widehat{U}_{yz}(x, y, z) = \widehat{v}_z(x-\alpha y, z+\alpha y),$$

are continuous, while

$$\widehat{U}_{xz}(x, y, z) = \frac{1}{\alpha} \int_z^{z+\alpha y} \widehat{v}_{xz}(x+z-q, q) dq + \frac{1}{\alpha} [\widehat{v}_x(x-\alpha y, z+\alpha y) - \widehat{v}_x(x, z)],$$

is locally bounded. Furthermore, it is easy to see that  $\widehat{U}_{yx} = \widehat{U}_{xy}$ ,  $\widehat{U}_{xz} = \widehat{U}_{zx}$  and  $\widehat{U}_{yz} = \widehat{U}_{zy}$ .  $\square$

The proof of the next corollary follows from (7.9)-(7.10) and direct computations.

**Corollary 7.2.** *One has*

$$(7.12) \quad \alpha \widehat{U}_x(x, y, z) - \alpha \widehat{U}_z(x, y, z) + \widehat{U}_y(x, y, z) = \widehat{v}(x, z) \geq (e^x - \kappa) (1 + e^{\frac{\gamma}{\sigma}(x+z)}),$$

so

$$(7.13) \quad \mathbb{W}_3 := \{(x, y, z) \in \mathcal{O} : \alpha \widehat{U}_x(x, y, z) - \alpha \widehat{U}_z(x, y, z) + \widehat{U}_y(x, y, z) > (e^x - \kappa) (1 + e^{\frac{\gamma}{\sigma}(x+z)})\} = \mathcal{C}_3,$$

with  $\mathcal{C}_3$  as in (6.6). Furthermore,  $\widehat{U}_x - \widehat{U}_z = \bar{U}_x$  as well as  $\widehat{U}_y = \bar{U}_y$ , and we have

$$(7.14) \quad \mathbb{W}_2 := \{(x, y, \varphi) \in \mathbb{R} \times (0, \infty) \times (0, \infty) : \alpha \bar{U}_x(x, y, \varphi) + \bar{U}_y(x, y, \varphi) > (e^x - \kappa) (1 + \varphi)\} = \mathcal{C}_2,$$

with  $\mathcal{C}_2$  of (5.9).

**7.1. Construction of the optimal control for the state space process  $(X, Y, \Phi)$ .** Recall  $b$  as in (5.11), which is nondecreasing and left-continuous by Lemma 5.3. Then, for any  $(x, y, \varphi) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$ , define the admissible control strategy

$$(7.15) \quad \widehat{\xi}_t := y \wedge \sup_{0 \leq s \leq t} \frac{1}{\alpha} \left[ x - b(\Phi_s^\varphi) + \mu_0 s + \sigma B_s \right]^+, \quad t \geq 0, \quad \widehat{\xi}_{0-} = 0,$$

according to which the investor should only execute a lump-sum amount of shares whenever the process  $X_{t-}$  is strictly inside the selling region and hence strictly above the boundary  $b(\Phi_t)$ . More precisely, if  $y \leq \frac{1}{\alpha}(x - b(\varphi))$  it is optimal to sell the complete amount of shares instantaneously, while for  $y > \frac{1}{\alpha}(x - b(\varphi))$  the system is brought immediately to the level  $(X_0, Y_0, \Phi_0) = (b(\varphi), y - \frac{1}{\alpha}(x - b(\varphi)), \varphi)$ . Afterwards, the strategy (7.15) prescribes to take action whenever the process  $X_t$  approaches the boundary  $b(\Phi_t)$  from below and the process  $(X_t, Y_t)$  is obliquely reflected at the belief-dependent boundary  $b(\Phi_t)$  in the direction  $(-\alpha, -1)$ . Hence, the process  $X_t$  is kept inside the interval  $(-\infty, b(\Phi_t)]$  with “minimal effort”. These actions are the so-called *Skorokhod reflection-type policies* and caused by the continuous part  $\widehat{\xi}^c$  of the control  $\widehat{\xi}$ . Notice that the nondecreasing process  $\widehat{\xi}$ , and the induced random measure  $d\widehat{\xi}$  on  $[0, \infty)$ , are such that (recall (7.14))

$$(7.16) \quad \begin{cases} (X_t^{\widehat{\xi}}, Y_t^{\widehat{\xi}}, \Phi_t) \in \overline{\mathbb{W}_2}, & \mathbb{Q} \otimes dt\text{-a.s.}; \\ d\widehat{\xi}_t \text{ has support on } \{t \geq 0 : (X_{t-}^{\widehat{\xi}}, Y_{t-}^{\widehat{\xi}}, \Phi_t) \notin \mathbb{W}_2\}; \\ \widehat{\xi}_t \leq y, & t \geq 0. \end{cases}$$

Furthermore, due to (7.4)-(7.5) and Corollary 7.2, we can express the control  $\widehat{\xi}$  equivalently in terms of the state-process  $(X^{\widehat{\xi}}, Y^{\widehat{\xi}}, Z^{\widehat{\xi}})$  by (cf. (7.13))

$$(7.17) \quad \begin{cases} (X_t^{\widehat{\xi}}, Y_t^{\widehat{\xi}}, Z_t^{\widehat{\xi}}) \in \overline{\mathbb{W}_3}, & \mathbb{Q} \otimes dt\text{-a.s.}; \\ d\widehat{\xi}_t \text{ has support on } \{t \geq 0 : (X_{t-}^{\widehat{\xi}}, Y_{t-}^{\widehat{\xi}}, Z_{t-}^{\widehat{\xi}}) \notin \mathbb{W}_3\}; \\ \widehat{\xi}_t \leq y, & t \geq 0. \end{cases}$$

In the following, we prove that in fact  $\widehat{\xi}$  is an optimal control for problem (7.5) and  $\widehat{U} = \widehat{V}$ . As an immediate consequence we have that  $\overline{U} = \overline{V}$  and  $U = V$ .

**Theorem 7.3** (Verification Theorem). *Let  $(x, y, z) \in \mathbb{R} \times [0, \infty) \times \mathbb{R}$  and  $\widehat{U}(x, y, z)$  as in (7.8). Then, one has  $\widehat{U}(x, y, z) = \widehat{V}(x, y, z)$  and  $\widehat{\xi}$  as in (7.15) is optimal for the singular control problem (7.5).*

*Proof.* First of all, for  $y = 0$  we have  $\widehat{U}(x, 0, z) = 0 = \widehat{V}(x, 0, z)$ . Hence, in the following we assume  $(x, y, z) \in \mathcal{O}$ .

**1. We prove  $\widehat{U} \geq \widehat{V}$ .** Take an arbitrary control  $\xi \in \mathcal{A}(y)$  and for  $R > 0$  and  $N \in \mathbb{N}$  we set  $\tau_{R,N} := \inf\{s \geq 0 : |(X_s^\xi, Z_s^\xi)| > R\} \wedge N$ . Due to Lemma 7.1 we can proceed as in Fleming and Soner [33], Chapter 8, Th. 4.1 to obtain (after performing an approximation of  $\widehat{U}$  via mollifiers and

taking limits)

$$\begin{aligned}
& \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ e^{-r\tau_{R,N}} \widehat{U}(X_{\tau_{R,N}}^{\xi}, Y_{\tau_{R,N}}^{\xi}, Z_{\tau_{R,N}}^{\xi}) - \widehat{U}(x, y, z) \right] \\
&= \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ \int_0^{\tau_{R,N}} e^{-rs} (\mathcal{L}_{X,Z} - r) \widehat{U}(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) ds + \underbrace{\sigma \int_0^{\tau_{R,N}} e^{-rs} \widehat{U}_x(X_{\tau_{R,N}}^{\xi}, Y_{\tau_{R,N}}^{\xi}, Z_{\tau_{R,N}}^{\xi}) dB_s}_{=: M_{R,N}} \right. \\
&\quad + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \left( \widehat{U}(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) - \widehat{U}(X_{s-}^{\xi}, Y_{s-}^{\xi}, Z_{s-}^{\xi}) \right) \\
(7.18) \quad &\left. + \int_0^{\tau_{R,N}} e^{-rs} \left( -\alpha \widehat{U}_x(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) - \widehat{U}_y(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) + \alpha \widehat{U}_z(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) \right) d\xi_s^c \right].
\end{aligned}$$

Notice that

$$\begin{aligned}
& \widehat{U}(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) - \widehat{U}(X_{s-}^{\xi}, Y_{s-}^{\xi}, Z_{s-}^{\xi}) = \widehat{U}(X_{s-}^{\xi} - \alpha \Delta \xi_s, Y_{s-}^{\xi} - \Delta \xi_s, Z_{s-}^{\xi} + \alpha \Delta \xi_s) - \widehat{U}(X_{s-}^{\xi}, Y_{s-}^{\xi}, Z_{s-}^{\xi}) \\
&= \int_0^{\Delta \xi_s} \frac{\partial \widehat{U}(X_{s-}^{\xi} - \alpha u, Y_{s-}^{\xi} - u, Z_{s-}^{\xi} + \alpha u)}{\partial u} du \\
(7.19) \quad &= \int_0^{\Delta \xi_s} (-\alpha \widehat{U}_x - \widehat{U}_y + \alpha \widehat{U}_z)(X_{s-}^{\xi} - \alpha u, Y_{s-}^{\xi} - u, Z_{s-}^{\xi} + \alpha u) du.
\end{aligned}$$

Hence, combining (7.18) and (7.19), upon adding the term

$$\int_0^{\tau_{R,N}} e^{-rs} (e^{X_s^{\xi}} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^{\xi} + Z_s^{\xi})}) d\xi_s^c + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \int_0^{\Delta \xi_s} (e^{X_{s-}^{\xi} - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^{\xi} + Z_{s-}^{\xi})}) du,$$

on both sides, yields

$$\begin{aligned}
& \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ \int_0^{\tau_{R,N}} e^{-rs} (e^{X_s^{\xi}} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^{\xi} + Z_s^{\xi})}) d\xi_s^c \right. \\
&\quad \left. + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \int_0^{\Delta \xi_s} (e^{X_{s-}^{\xi} - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^{\xi} + Z_{s-}^{\xi})}) du - \widehat{U}(x, y, z) \right] \\
&= \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ \int_0^{\tau_{R,N}} e^{-rs} (\mathcal{L}_{X,Z} - r) \widehat{U}(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) ds + M_{R,N} - e^{-r\tau_{R,N}} \widehat{U}(X_{\tau_{R,N}}^{\xi}, Y_{\tau_{R,N}}^{\xi}, Z_{\tau_{R,N}}^{\xi}) \right. \\
&\quad + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \int_0^{\Delta \xi_s} (-\alpha \widehat{U}_x - \widehat{U}_y + \alpha \widehat{U}_z)(X_{s-}^{\xi} - \alpha u, Y_{s-}^{\xi} - u, Z_{s-}^{\xi} + \alpha u) \\
&\quad \quad \quad \left. + (e^{X_{s-}^{\xi} - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^{\xi} + Z_{s-}^{\xi})}) du \right. \\
(7.20) \quad &\left. + \int_0^{\tau_{R,N}} e^{-rs} \left( (-\alpha \widehat{U}_x - \widehat{U}_y + \alpha \widehat{U}_z)(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) + (e^{X_s^{\xi}} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^{\xi} + Z_s^{\xi})}) \right) d\xi_s^c \right].
\end{aligned}$$

We observe that (7.9)-(7.11) imply

$$(7.21) \quad (\mathcal{L}_{X,Z} - r) \widehat{U}(x, y, z) = \frac{1}{\alpha} \int_{x-\alpha y}^x (\mathcal{L}_{X,Z} - r) \widehat{v}(x', x + z - x') dx' \leq 0,$$

where the last inequality follows from the supermartingale property of  $(e^{-rt} \widehat{v}(X_t, Z_t))_t$  combined with the regularity obtained in Proposition 6.12. Hence, due to (7.12) and since  $\widehat{U} \geq 0$  and

$\mathbb{E}_{(x,y,z)}^{\mathbb{Q}}[M_{R,N}] = 0$ , (7.20) writes as

$$(7.22) \quad \widehat{U}(x, y, z) \geq \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ \int_0^{\tau_{R,N}} e^{-rs} (e^{X_s^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^\xi + Z_s^\xi)}) d\xi_s^c \right. \\ \left. + \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs} \int_0^{\Delta \xi_s} (e^{X_{s-}^\xi - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^\xi + Z_{s-}^\xi)}) du \right].$$

Taking limits as  $R \uparrow \infty$  as well as  $N \uparrow \infty$ , invoking the dominated convergence theorem due to Assumption 4.1, we obtain

$$(7.23) \quad \widehat{U}(x, y, z) \geq \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ \int_0^\infty e^{-rs} (e^{X_s^\xi} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^\xi + Z_s^\xi)}) d\xi_s^c \right. \\ \left. + \sum_{s: \Delta \xi_s \neq 0} e^{-rs} \int_0^{\Delta \xi_s} (e^{X_{s-}^\xi - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^\xi + Z_{s-}^\xi)}) du \right] \\ = J(x, y, z, \xi).$$

Since  $\xi$  was arbitrary, we have

$$(7.24) \quad \widehat{U}(x, y, z) \geq \widehat{V}(x, y, z),$$

for all  $(x, y, z) \in \mathcal{O}$ . That is,  $\widehat{U} \geq \widehat{V}$  on  $\mathcal{O}$ .

**2. We prove that  $\widehat{U} \leq \widehat{V}$ .** In order to accomplish that, let  $\widehat{\xi}$  satisfy the conditions in (7.17) and define  $\widehat{\tau}_{R,N} = \inf\{t \geq 0 : |(X_t^{\widehat{\xi}}, Z_t^{\widehat{\xi}})| > R\} \wedge N$ , again for  $R > 0$  and  $N \in \mathbb{N}$ . Notice that the properties of  $\widehat{\xi}$  imply equalities in (7.12) and (7.21), where the equality in (7.21) follows from the monotonicity of  $c$  and we can deduce that  $(x', x + z - x') \in \mathbb{W}_3$  for  $(x, y, z) \in \mathbb{W}_3$  and  $x' \leq x$ . Employing the same arguments as in the first part of the proof yields

$$(7.25) \quad \widehat{U}(x, y, z) = \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Y_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Z_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}) \right] \\ + \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ \int_0^{\widehat{\tau}_{R,N}} e^{-rs} (e^{X_s^{\widehat{\xi}}} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_s^{\widehat{\xi}} + Z_s^{\widehat{\xi}})}) d\widehat{\xi}_s^c \right. \\ \left. + \sum_{0 \leq s \leq \widehat{\tau}_{R,N}} e^{-rs} \int_0^{\Delta \widehat{\xi}_s} (e^{X_{s-}^{\widehat{\xi}} - \alpha u} - \kappa) (1 + e^{\frac{\gamma}{\sigma}(X_{s-}^{\widehat{\xi}} + Z_{s-}^{\widehat{\xi}})}) du \right].$$

It is thus left to prove that

$$(7.26) \quad \lim_{N \uparrow \infty} \lim_{R \uparrow \infty} \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Y_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Z_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}) \right] = 0,$$

since taking limits as  $R \uparrow \infty$  and  $N \uparrow \infty$  together with (7.22) implies  $J(x, y, z, \widehat{\xi}) = \widehat{U}(x, y, z)$  and hence  $\widehat{V}(x, y, z) \geq \widehat{U}(x, y, z)$  for all  $(x, y, z) \in \mathcal{O}$ . Combining the latter with (7.24) yields  $\widehat{U} = \widehat{V}$  on  $\mathcal{O}$ .

In order to prove (7.26) we notice that Lemma 5.1 i), (6.5) and (7.8) imply

$$(7.27) \quad \widehat{U}(x, y, z) \leq \frac{1}{\alpha} \int_z^{z+\alpha y} K_1 e^{x+z-q} (1 + e^{\frac{\gamma}{\sigma}(x+z-q+q)}) dq = \frac{1}{\alpha} K_1 e^x (1 + e^{\frac{\gamma}{\sigma}(x+z)}) (1 - e^{-\alpha y}),$$

and, since  $y \mapsto \widehat{U}(x, y, z)$  is increasing, we obtain

$$0 \leq e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Y_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, Z_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}) \leq e^{-r\widehat{\tau}_{R,N}} \widehat{U}(X_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}, y, Z_{\widehat{\tau}_{R,N}}^{\widehat{\xi}}) \\ \leq \frac{K_1}{\alpha} (1 - e^{-\alpha y}) e^{-r\widehat{\tau}_{R,N}} e^{X_{\widehat{\tau}_{R,N}}^0} (1 + e^{\frac{\gamma}{\sigma}(X_{\widehat{\tau}_{R,N}}^0 + Z_{\widehat{\tau}_{R,N}}^0)}),$$

where we used that  $X_t^\xi \leq X_t^0$  as well as  $X_t^\xi + Z_t^\xi = X_t^0 + Z_t^0$  a.s. Hence, taking expectations yields

$$\begin{aligned}
0 &\leq \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ e^{-r\hat{\tau}_{R,N}} \widehat{U}(X_{\hat{\tau}_{R,N}}^{\hat{\xi}}, Y_{\hat{\tau}_{R,N}}^{\hat{\xi}}, Z_{\hat{\tau}_{R,N}}^{\hat{\xi}}) \right] \\
&\leq \frac{K_1}{\alpha} (1 - e^{-\alpha y}) \mathbb{E}_{(x,y,z)}^{\mathbb{Q}} \left[ e^{-r\hat{\tau}_{R,N}} e^{X_{\hat{\tau}_{R,N}}^0} (1 + e^{\frac{\gamma}{\sigma}(X_{\hat{\tau}_{R,N}}^0 + Z_{\hat{\tau}_{R,N}}^0)}) \right] \\
(7.28) \quad &= \frac{K_1}{\alpha} (1 - e^{-\alpha y}) (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \mathbb{E}_{(x,y,\pi)} \left[ e^{-r\hat{\tau}_{R,N}} e^{X_{\hat{\tau}_{R,N}}^0} \right],
\end{aligned}$$

with  $\pi := e^{\frac{\gamma}{\sigma}(x+z)} / (1 + e^{\frac{\gamma}{\sigma}(x+z)})$  and the last equality follows from a change of measure as in Section 5. Upon using Assumption 4.1, it is easy to check that (7.26) holds true, thus completing the proof.  $\square$

**Remark 7.4.** We can use the transformation (6.1) from  $(x, z)$ - to  $(x, \varphi)$ -coordinates in order to show that  $\hat{\xi}$  is an optimal control for problem (7.2) as well. Indeed, recall (6.3) and  $\widehat{V}(x, y, z) = \widehat{V}(x, y, \frac{\sigma}{\gamma} \ln(\varphi) - x) = \bar{V}(x, y, \varphi)$  to conclude

$$\bar{V}(x, y, \varphi) = \mathbb{E}_{(x,y,\varphi)}^{\mathbb{Q}} \left[ \int_0^\infty e^{-rs} (e^{X_s^{\hat{\xi}}} - \kappa) (1 + \Phi_s) d\hat{\zeta}_s^c + \sum_{s: \Delta\hat{\xi}_s \neq 0} e^{-rs} \int_0^{\Delta\hat{\xi}_s} (e^{X_s^{\hat{\xi}} - \alpha u} - \kappa) (1 + \Phi_s) du \right].$$

Furthermore, the latter equation and (7.7) imply  $U(x, y, \pi) = V(x, y, \pi)$  for all  $(x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1)$ .

**Remark 7.5.** Letting  $\tilde{\tau}(x, y, \varphi) := \inf\{t \geq 0 : x + \mu_0 t + \sigma B_t \geq b(\Phi_s^\varphi)\}$ , the optimal execution strategy  $\hat{\xi}$  as in (7.15) converges as  $\alpha \downarrow 0$  to the execution strategy

$$\tilde{\xi}_t = \begin{cases} 0 & t < \tilde{\tau}(x, y, \varphi), \\ y & t \geq \tilde{\tau}(x, y, \varphi), \end{cases}$$

which prescribes to sell the total amount of shares instantaneously when the process  $X$  reaches the optimal execution boundary  $b(\Phi)$ . It is interesting to notice that the optimal solution and the value function are robust w.r.t. the parameter  $\alpha$ . Indeed, by L'Hôpital's rule, we see from (7.7) that  $\lim_{\alpha \downarrow 0} = y\bar{v}(x, \varphi)$ ,  $(x, y, \varphi) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$ . It is in fact easy to show via a verification theorem that  $y\bar{v}(x, \varphi)$  and  $\tilde{\xi}$  are the value function and the optimal execution rule in the problem with no market impact.

**Remark 7.6.** Let  $\hat{\sigma} := \inf\{t \geq 0 : Y_t^\xi = 0\}$  denote the time at which the portfolio is fully depleted. Imposing the constraint that the investor has to sell all assets until terminal time (cf. Guo and Zervos [41]), we notice that for  $y < \frac{1}{\alpha}(x - b(\varphi))$  the control strategy  $\hat{\xi}$  of (7.15) still defines an optimal control, as the complete amount of shares is sold immediately at time  $t = 0$ . However, for  $y > \frac{1}{\alpha}(x - b(\varphi))$ , simple calculations yield

$$\lim_{T \uparrow \infty} \mathbb{Q}(\hat{\sigma} > T) \geq 1 - \exp\left(\frac{2\mu_0}{\sigma^2}(\alpha y + x_0^* - x)\right),$$

and we notice that for increasing  $y$  and decreasing  $x$ , the probability increases that the investor does not sell the entire amount of shares until terminal time. Hence, if we restrict the admissible strategies to all  $\xi \in \mathcal{A}(y)$  such that  $\lim_{T \rightarrow \infty} Y_T^\xi = 0$ , the control strategy  $\hat{\xi}$  of (7.15) does not provide an admissible execution strategy. In this case, arguing as in Guo and Zervos [41], Proposition 5.1, we can use  $\hat{\xi}$  to construct a sequence of  $\epsilon$ -optimal strategies.

## 8. COMPARATIVE STATICS ANALYSIS

In this section we perform a brief comparative statics analysis on the optimal execution boundaries  $a$  and  $b$  of (4.7) and (5.11), respectively. Based on the integral equation (6.37) we implement a recursive numerical scheme, which relies on an application of the Monte-Carlo method. To this end, we let  $\zeta$  denote an auxiliary exponentially distributed random variable with parameter  $r$ , that is

independent of the Brownian motion  $B$ . Recalling that (6.37) can be reformulated as (6.39), we notice that the latter takes the shape of a fixed point problem

$$(8.1) \quad b^{-1}(x) = \Gamma(b^{-1}(x), x; b^{-1}),$$

for  $x \in \mathbb{R}$  and  $b^{-1}$  being the generalized inverse of  $b$  as in (6.10). Here, the operator  $\Gamma$  is defined via

$$(8.2) \quad \Gamma(\varphi, x; f) := \frac{1}{e^x - \kappa r} \frac{1}{\mathbb{E}^{\mathbb{Q}}} \left[ -g(X_{\zeta}^x, \frac{\sigma}{\gamma} \ln(\Phi_{\zeta}^{\varphi}) - X_{\zeta}^x) \mathbb{1}_{\{\Phi_{\zeta}^{\varphi} \leq f(X_{\zeta}^x)\}} \right] - 1,$$

for  $(x, \varphi) \in \mathbb{R} \times (0, \infty)$  and a function  $f : \mathbb{R} \rightarrow (0, \infty)$ . By employing techniques seen in Christensen and Salminen [14], Dammann and Ferrari [17] and Detemple and Kitapbayev [24], we aim to solve (8.1) via an iterative scheme. To this end, we let

$$(8.3) \quad (b^{-1})^{[n]}(x) = \Gamma((b^{-1})^{[n-1]}(x), x; (b^{-1})^{[n-1]}), \quad x \in \mathbb{R}, n \geq 1,$$

define a sequence of boundaries and - for a given boundary  $(b^{-1})^{[k]}$  - we estimate the expectation in (8.2) by

$$-\frac{1}{N} \sum_{i=1}^N g\left(X_{\zeta_i}^{i,x}, \frac{\sigma}{\gamma} \ln(\Phi_{\zeta_i}^{i,(b^{-1})^{[k]}(x)}) - X_{\zeta_i}^{i,x}\right) \mathbb{1}_{\{\Phi_{\zeta_i}^{i,(b^{-1})^{[k]}(x)} \leq (b^{-1})^{[k]}(X_{\zeta_i}^{i,x})\}},$$

where  $N$  denotes the total amount of realizations of the exponential random variable. We can choose the initial boundary  $(b^{-1})^{[0]}$  as a simple exponential function with  $(b^{-1})^{[0]}(x_0^*) = 0$  and  $(b^{-1})^{[0]}(x) \rightarrow \infty$  for  $x \uparrow x_1^*$  with  $x_0^*$  and  $x_1^*$  as in (3.6) and Remark 3.1, respectively. The numerical scheme (8.3) is then iterated until the variation between steps drops below a predetermined level. Finally, we calculate  $b$  from its generalized inverse  $b^{-1}$  and can transform the resulting boundary according to the explicit relationship (5.16). We can thus study the sensitivity of  $b(\varphi)$  as well as  $a(\pi)$  with respect to some of the model's parameters.

Furthermore, we can compare the belief-dependent boundaries to the strategy of a pre-committed agent, who - after forming an initial belief  $\pi = \mathbb{P}(\mu = \mu_1 | \mathcal{F}_t^X)$  - refrains from updating her belief and thus acts as if the drift value was constant and equal to  $\mu_1 \pi + \mu_0(1 - \pi)$ . The resulting strategy is then triggered by a constant execution threshold, which is of similar structure as the one derived in Section 3. Consequently, we observe that such an agent cannot react to any price movements on the market and is thus not able to decrease or increase the target price at which she would like to sell the asset.

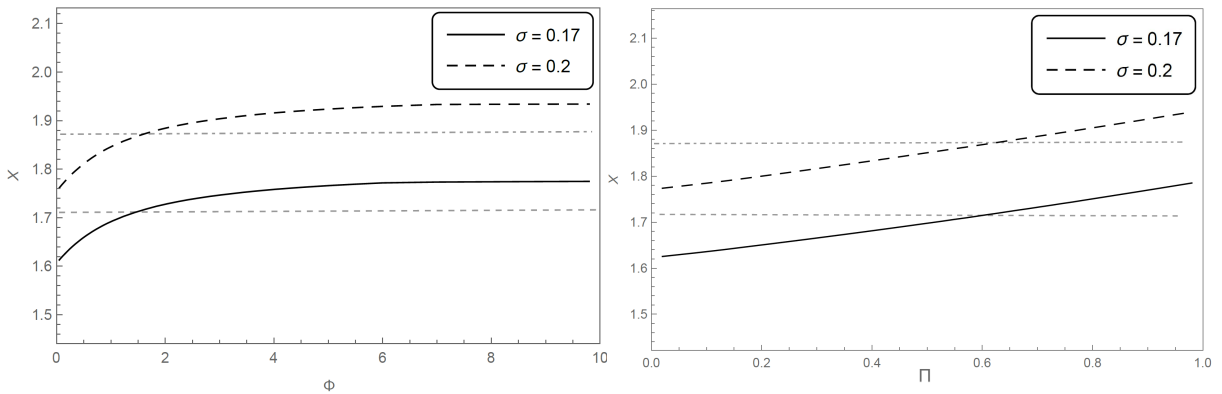


FIGURE 1. The optimal execution boundaries  $b(\varphi)$  and  $a(\pi)$  as well as the pre-committed strategies for different values of  $\sigma$  and following parameters:  $r = 0.07$ ,  $\mu_0 = -0.01$ ,  $\mu_1 = 0.007$ ,  $\kappa = 3$ ,  $\pi = 0.6$ .

Figure 1 shows the dependency of the optimal execution boundaries  $a$  and  $b$  with respect to the volatility  $\sigma$ . Clearly, the boundaries increase for a larger value of  $\sigma$  and hence, the investor aims at

selling the shares at a higher price. We can interpret a larger volatility coefficient as a higher level of uncertainty and thus larger price fluctuations on the market. The investor exploits the latter fact and delays selling the shares by waiting for higher prices to evolve.

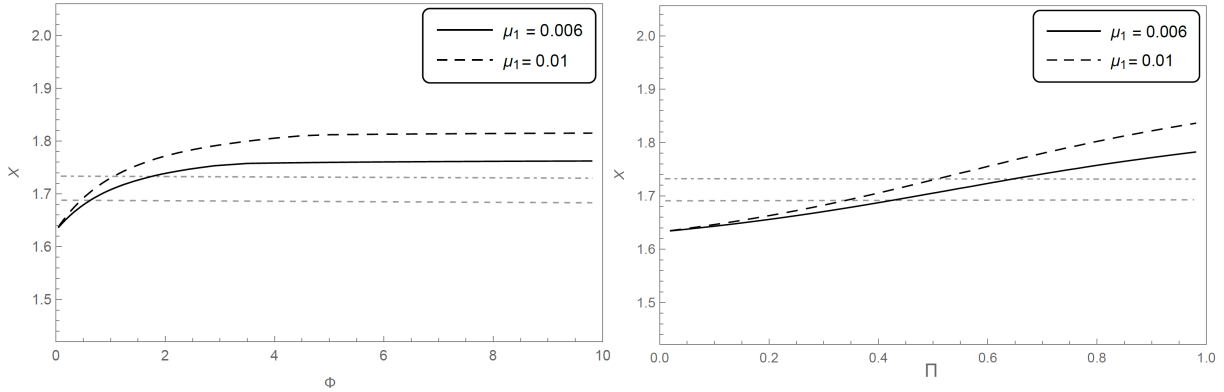


FIGURE 2. The optimal execution boundaries  $b(\varphi)$  and  $a(\pi)$  as well as the pre-committed strategies for different values of  $\mu_1$  and following parameters:  $r = 0.07$ ,  $\mu_0 = -0.01$ ,  $\sigma = 0.17$ ,  $\kappa = 3$ ,  $\pi = 0.6$ .

In Figure 2 we can observe the sensitivity of the optimal execution boundaries with respect to one of the possible drift values. Since an increase in  $\mu_1$  implies higher expected prices on the market, the investor again delays selling shares and waits for larger prices to evolve. Differently to a change in  $\sigma$ , as seen in Figure 1, the lower bound  $x_0^*$  remains untouched by a change in  $\mu_1$ , since it results from the case of full information when  $\mu = \mu_0$ . Consequently, for a strong belief towards the drift value  $\mu_0$ , the investor does not significantly changes her execution strategy.

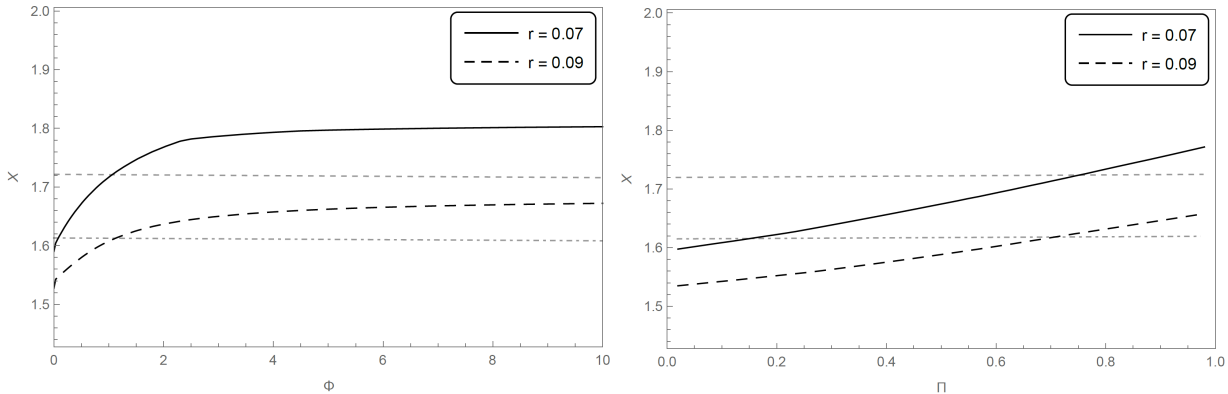


FIGURE 3. The optimal execution boundaries  $b(\varphi)$  and  $a(\pi)$  as well as the pre-committed strategies for different values of  $r$  and following parameters:  $\mu_0 = -0.01$ ,  $\mu_1 = 0.007$ ,  $\sigma = 0.17$ ,  $\kappa = 3$ ,  $\pi = 0.6$ .

Finally, Figure 3 shows the effect on the boundaries  $a$  and  $b$  for a change in  $r$ , the latter can be interpreted as the impatience of the investor. Consequently, we observe that the boundary decreases for larger values of  $r$  and hence, the investor is willing to accept a lower execution price when selling the shares for every belief.



## APPENDIX A. PROOF OF PROPOSITION 6.12

The proof follows the lines of Section 4 in [21], suitably adapted to the present setting, and it is obtained through a series of intermediate results. Let  $(x, z) \in \mathbb{R}^2$  be given and fixed and set

$$(A.1) \quad \sigma_* := \sigma_*(x, z) := \inf\{t \geq 0 : (X_t^x, Z_t^z) \in \mathcal{S}_3\}, \quad \hat{\sigma}_* := \hat{\sigma}_*(x, z) := \inf\{t \geq 0 : (X_t^x, Z_t^z) \in \text{int}(\mathcal{S}_3)\},$$

and observe that  $\sigma_* = \tau_*$   $\mathbb{Q}$ -a.s. on  $\mathbb{R}^2 \setminus \partial\mathcal{C}_3$  due to the continuity of paths. It is crucial to show that this equality also holds for the boundary points  $(x_0, z_0) \in \partial\mathcal{C}_3$ . As it turns out, the cases i)  $\mu_0 + \mu_1 \geq 0$  and ii)  $\mu_0 + \mu_1 < 0$  should be treated in different fashions and the latter case exhibits some more technical difficulties than the first case. Let us start with case i), in which the needed result follows upon using the law of iterated logarithm.

**Proposition A.1.** *Assume that  $\mu_0 + \mu_1 \geq 0$ . Let  $(x_n, z_n) \in \mathcal{C}_3$  be a sequence with  $(x_n, z_n) \rightarrow (x_0, z_0) \in \partial\mathcal{C}_3$ , such that  $x_0 = c(z_0)$ . We then have  $\tau^*(x_n, z_n) \downarrow 0$  as well as  $\hat{\sigma}_*(x_n, z_n) \downarrow 0$   $\mathbb{Q}$ -a.s.*

*Proof.* Fix  $\omega \in \Omega$  and assume that  $\limsup_{n \rightarrow \infty} \tau^*(x_n, z_n)(\omega) =: \delta > 0$ . Hence, there exists a subsequence (still labelled by  $(x_n, z_n)$ ) such that

$$(A.2) \quad X_t^{x_n}(\omega) < c(Z_t^{z_n}) \quad \forall n \in \mathbb{N}, \quad \forall t \in [0, \delta/2],$$

which is equivalent to

$$x_n + \mu_0 t + \sigma B_t(\omega) < c(z_n - \frac{1}{2}(\mu_0 + \mu_1)t) \quad \forall n \in \mathbb{N}, \quad \forall t \in [0, \delta/2].$$

Upon using that  $z \mapsto c(z)$  is continuous, we let  $n \rightarrow \infty$  and obtain

$$(A.3) \quad \sigma B_t(\omega) \leq c(z_0 - \frac{1}{2}(\mu_0 + \mu_1)t) - x_0 - \mu_0 t \leq c(z_0) - x_0 - \mu_0 t = -\mu_0 t \quad \forall t \in [0, \delta/2],$$

where the last inequality follows from  $\mu_0 + \mu_1 \geq 0$  and Proposition 6.5. On the other hand, by the law of iterated logarithm, there exists a sequence  $(t_n) \downarrow 0$  for all  $\epsilon > 0$  such that

$$(A.4) \quad B_{t_n} \geq (1 - \epsilon) \sqrt{2t_n \log(\log(\frac{1}{t_n}))} \quad \forall n \in \mathbb{N}.$$

Combining (A.3) and (A.4) implies

$$\frac{1}{t} \sigma (1 - \epsilon) \sqrt{2t \log(\log(\frac{1}{t}))} \leq -\mu_0,$$

but since  $\sqrt{2t \log(\log(1/t))}/t \rightarrow \infty$  for  $t \downarrow 0$ , (A.2) can only happen on a  $\mathbb{Q}$ -null set. Thus  $\tau^*(x_n, z_n) \downarrow 0$  and by replacing the strict inequality in (A.2) by " $\leq$ ", we obtain that  $\hat{\sigma}_*(x_n, z_n) \downarrow 0$  as well.  $\square$

Notice that the proof of Proposition A.1 cannot be replicated for the case ii), in which  $\mu_0 + \mu_1 < 0$ , since the last inequality in (A.3) does not longer apply. As it turns out, in order to prove the same result for case ii), we have to take a longer route. The reason for this lies in the fact that the process  $(X, Z)$  is moving towards the right in the state space and hence - keeping in mind that the continuation region  $\mathcal{C}_3$  of (6.18) lies below the increasing boundary  $c$  - could possibly evade from the stopping set. In the following, we show that this is not the case by adapting the procedure in of Section 4 in De Angelis [21]. As a first step, we state the following Lemma, whose proof follows the lines of Cox and Peskir [16], Corollary 8, and is thus omitted for the sake of brevity.

**Lemma A.2.** *Assume that  $\mu_0 + \mu_1 < 0$  and  $r > \frac{\gamma}{2\sigma} |\mu_0 + \mu_1|$ . We have  $\mathbb{Q}(\sigma_* = \hat{\sigma}_*) = 1$ .*

In the next step, we aim at proving *regularity* of the boundary points for the stopping set  $\mathcal{S}_3$  in the sense of diffusions, that is, for  $(x, z) \in \partial\mathcal{C}_3$  we have

$$(A.5) \quad \mathbb{Q}_{x,z}(\sigma_* > 0) = 0.$$

It is clear from Blumenthal's 0-1 law that if (A.5) does not hold, we have  $\mathbb{Q}_{x,z}(\sigma_* > 0) = 1$ . Due to the mentioned structure of the problem this is not a straightforward task, since we cannot apply an argument similar to the one on Proposition A.1. Instead, we establish the result in two steps and begin by showing that the classical smooth-fit property holds at the free-boundary, i.e. continuity of  $\widehat{v}_x(\cdot, z)$ .

**Lemma A.3.** *Assume that  $\mu_0 + \mu_1 < 0$  and  $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$ . For  $\widehat{v}$  of (6.5) we have  $\widehat{v}_x(\cdot, z) \in C(\mathbb{R})$ , or, equivalently,  $\widehat{u}_x(\cdot, z) \in C(\mathbb{R})$  for  $\widehat{u}$  of (6.15).*

*Proof.* From (6.25) we obtain

$$\frac{1}{2}\sigma^2\widehat{u}_{xx}(x, z) = r\widehat{u}(x, z) - \mu_0\widehat{u}_x(x, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x, z) - g(x, z),$$

for  $(x, z) \in \mathcal{C}_3$ , and due to (5.12) (which implies an analogous result for  $\widehat{v}$ ) we deduce that for a bounded set  $B$ , we must have that  $\widehat{u}_{xx}$  is bounded on the closure of  $B \cap \mathcal{C}_3$ . Moreover, we recall that  $\widehat{u}_x \leq 0$  in  $\mathcal{C}_3$ , as verified in the proof of Proposition 6.7. Aiming for a contradiction we now assume that for  $(x_0, z_0) \in \partial\mathcal{C}_3$ , such that  $x_0 = c(z_0)$ , we have

$$(A.6) \quad \widehat{u}_x(x_0-, z_0) < -\delta_0,$$

for some  $\delta_0 > 0$ . We now take a bounded rectangular neighbourhood of  $(x_0, z_0)$  and let  $\tau_B := \inf\{t > 0 : (X_t, Z_t) \notin B\}$ . Notice that

$$(A.7) \quad \widehat{u}(x_0, z_0) \geq \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} \left[ e^{-r(\tau_B \wedge t)} \widehat{u}(X_{\tau_B \wedge t}, Z_{\tau_B \wedge t}) + \int_0^{\tau_B \wedge t} e^{-rs} g(X_s, Z_s) ds \right],$$

from the supermartingale property of  $(e^{-rt}\widehat{v}(X_t, Z_t))_t$ . Recall Lemma 6.10 and since  $t \mapsto Z_{\tau_B \wedge t}$  is increasing, we have  $\widehat{u}(X_{\tau_B \wedge t}, Z_{\tau_B \wedge t}) \geq \widehat{u}(X_{\tau_B \wedge t}^{x_0}, Z_{\tau_B \wedge t}^{z_0})$   $\mathbb{Q}$ -a.s. Moreover, since the integrand on the right-hand side of (A.7) is bounded on  $B$ , we obtain

$$(A.8) \quad \widehat{u}(x_0, z_0) \geq \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} \left[ e^{-r(\tau_B \wedge t)} \widehat{u}(X_{\tau_B \wedge t}, z_0) - c_B(\tau_B \wedge t) \right],$$

where  $c_B$  is a constant depending on  $B$ . Due to the previously discussed local boundedness of  $\widehat{u}_{xx}$ , we can apply Itô-Tanaka's formula to the first term in the expectation of (A.8). Let  $\mathcal{L}_X := \frac{1}{2}\sigma^2\partial_{xx} + \mu_0\partial_x$  and denote the local time of  $X$  at  $x_0$  by  $L^{x_0}$ . Moreover, noticing that  $\widehat{u}_{xx}(\cdot, z_0) = 0$  for  $x > x_0$ , we obtain

$$\begin{aligned} \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} \left[ e^{-r(\tau_B \wedge t)} \widehat{u}(X_{\tau_B \wedge t}, z_0) \right] &= \widehat{u}(x_0, z_0) + \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} \left[ \int_0^{\tau_B \wedge t} e^{-rs} (\mathcal{L}_X - r) \widehat{u}(X_s, z_0) \mathbb{1}_{\{X_s \neq x_0\}} ds \right] \\ &\quad - \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} \left[ \int_0^{\tau_B \wedge t} e^{-rs} \widehat{u}_x(x_0-, z_0) dL_s^{x_0} \right], \end{aligned}$$

and, combining this with (A.8), as well as noticing that  $(\mathcal{L}_X - r)\widehat{u}(X_s, Z_s)$  is bounded on  $B$ , we find

$$\begin{aligned} 0 &\geq \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} \left[ \int_0^{\tau_B \wedge t} e^{-rs} (\mathcal{L}_X - r) \widehat{u}(X_s, z_0) \mathbb{1}_{\{X_s \neq x_0\}} ds - c_B(\tau_B \wedge t) \right] \\ &\quad - \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} \left[ \int_0^{\tau_B \wedge t} e^{-rs} \widehat{u}_x(x_0-, z_0) dL_s^{x_0} \right] \\ &\geq \delta_0 e^{-rt} \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} [L_{\tau_B \wedge t}^{x_0}] - c_B \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} [\tau_B \wedge t], \end{aligned}$$

where we used our assumption (A.6) in the last inequality. Since this is equivalent to  $c_B \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} [\tau_B \wedge t] \geq \delta_0 e^{-rt} \mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} [L_{\tau_B \wedge t}^{x_0}]$ , and  $\mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} [\tau_B \wedge t] \approx t$  while  $\mathbb{E}_{(x_0, z_0)}^{\mathbb{Q}} [L_{\tau_B \wedge t}^{x_0}] \approx \sqrt{t}$  (see, e.g., Peskir [52], Lemma 15), we obtain the desired contradiction. Hence,  $\widehat{u}_x(\cdot, z) \in C(\mathbb{R})$ .  $\square$

We can now state the regularity of the boundary points.

**Proposition A.4.** *Assume that  $\mu_0 + \mu_1 < 0$  and  $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$ . All points  $(x, z) \in \partial\mathcal{C}_3$  are regular, i.e. we have  $\mathbb{Q}_{x,z}(\sigma_* > 0) = 0$ .*

*Proof.* We argue by contradiction and show that if  $\mathbb{Q}_{(x_0, z_0)}(\sigma_* > 0) = 1$  for some boundary point  $(x_0, z_0) \in \partial\mathcal{C}_3$  it follows that  $\hat{u}_x(x_0-, z_0) < 0$ , which contradicts Lemma A.3. As a first step, we establish an upper bound for  $\hat{u}_x$ . Fix  $(x, z) \in \mathcal{C}_3$  such that  $x > \tilde{x}$ , with the latter given by (6.14). Define  $\tau_\epsilon := \tau_\epsilon(x) := \inf\{t \geq 0 : X_t^x = \tilde{x} + \epsilon\}$  and observe that - by strong Markov property - we have

$$(A.9) \quad \hat{u}(x, z) = \sup_{\tau} \mathbb{E}_{(x, z)}^{\mathbb{Q}} \left[ e^{-r\tau_\epsilon} \hat{u}(\tilde{x} + \epsilon, Z_{\tau_\epsilon}^z) \mathbb{1}_{\{\tau > \tau_\epsilon\}} + \int_0^{\tau_\epsilon \wedge \tau} e^{-rt} g(X_t, Z_t) dt \right].$$

Moreover, we let  $\tilde{\tau} := \tilde{\tau}(x) := \inf\{t > 0 : X_t^x = \tilde{x}\}$ , and for  $\tau' := \tau^*(x, z)$  we obtain

$$(A.10) \quad \hat{u}(x - \epsilon, z) = \mathbb{E}_{(x - \epsilon, z)}^{\mathbb{Q}} \left[ e^{-r\tilde{\tau}(x - \epsilon)} \hat{u}(\tilde{x}, Z_{\tilde{\tau}(x - \epsilon)}^z) \mathbb{1}_{\{\tau' > \tilde{\tau}(x - \epsilon)\}} + \int_0^{\tau' \wedge \tilde{\tau}(x - \epsilon)} e^{-rt} g(X_t, Z_t) dt \right].$$

Notice that  $\tau_\epsilon(x) = \tilde{\tau}(x - \epsilon)$ . Hence, subtracting (A.10) from (A.9) yields

$$\begin{aligned} \hat{u}(x, z) - \hat{u}(x - \epsilon, z) &= \mathbb{E}^{\mathbb{Q}} \left[ e^{-r\tau_\epsilon} (\hat{u}(\tilde{x} + \epsilon, Z_{\tau_\epsilon}^z) - \hat{u}(\tilde{x}, Z_{\tau_\epsilon}^z)) \mathbb{1}_{\{\tau' > \tau_\epsilon\}} \right] \\ &\quad + \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tau_\epsilon \wedge \tau'} e^{-rt} (g(X_t^x, Z_t^z) - g(X_t^{x - \epsilon}, Z_t^z)) dt \right]. \end{aligned}$$

Since  $(\tilde{x} + \epsilon, Z_{\tau_\epsilon}^z) \in \mathcal{C}_3$  on  $\{\tau' > \tau_\epsilon\}$  and  $\hat{u}_x \leq 0$  in  $\mathcal{C}_3$  (see Proposition 6.7), we must have

$$\hat{u}(\tilde{x}, Z_{\tau_\epsilon}^z) \geq \hat{u}(\tilde{x} + \epsilon, Z_{\tau_\epsilon}^z),$$

and we obtain

$$\hat{u}(x, z) - \hat{u}(x - \epsilon, z) \leq \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tau_\epsilon \wedge \tau'} e^{-rt} (g(X_t^x, Z_t^z) - g(X_t^{x - \epsilon}, Z_t^z)) dt \right].$$

If we now divide by  $\epsilon > 0$  and let  $\epsilon \downarrow 0$ , we obtain (since  $\tau_\epsilon \downarrow \tilde{\tau}$  and  $\tau' = \tau^*(x, z)$ )

$$\hat{u}_x(x, z) \leq \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tilde{\tau} \wedge \tau'} e^{-rt} g_x(X_t^x, Z_t) dt \right].$$

In the next step, we assume by contradiction that there exists  $(x_0, z_0) \in \partial\mathcal{C}_3$  with  $\mathbb{Q}_{x_0, z_0}(\sigma_* > 0) = 1$  and take an increasing sequence  $x_n \uparrow x_0$  such that  $x_n > \tilde{x}$  for all  $n \in \mathbb{N}$ , which is possible due to Assumption 6.3. Let  $\tau_n := \tau^*(x_n, z_n)$  and notice that  $\tau_n = \sigma_n := \sigma_*(x_n, z_n)$  for all  $n \in \mathbb{N}$  due to continuity of paths. Furthermore,  $\sigma_n$  decreases in  $n$  and  $\sigma_n \geq \sigma_* := \sigma_*(x_0, z_0)$ , since  $x \mapsto X_t^x$  is increasing. Set  $\tilde{\tau}^n := \tilde{\tau}(x_n)$  and notice that  $\tilde{\tau}^n \uparrow \tilde{\tau}$ . Moreover, we let  $\sigma^\infty := \lim_{n \rightarrow \infty} \sigma_n$  and have

$$\sigma^\infty \wedge \tilde{\tau} = \lim_{n \rightarrow \infty} (\sigma_n \wedge \tilde{\tau}^n) \geq \sigma_* \wedge \tilde{\tau} \quad \mathbb{Q}\text{-a.s.}$$

We then obtain

$$\begin{aligned} \hat{u}_x(x_0-, z_0) &= \lim_{n \rightarrow \infty} \hat{u}_x(x_n, z_0) \leq \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tilde{\tau} \wedge \sigma_n} e^{-rt} g_x(X_t^{x_n}, Z_t^{z_0}) dt \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \int_0^{\tilde{\tau} \wedge \sigma^\infty} e^{-rt} g_x(X_t^{x_0}, Z_t^{z_0}) dt \right] < 0, \end{aligned}$$

where we used  $x_0 > \tilde{x}$  as well as  $\tilde{\tau} \wedge \sigma^\infty > 0$  due to our assumption  $\mathbb{Q}_{x_0, z_0}(\sigma^\infty \geq \sigma_* > 0) = 1$ . But this contradicts Lemma A.3 and the claim follows.  $\square$

As a corollary of Lemma A.2 and Proposition A.4 we obtain

**Corollary A.5.** *Assume that  $\mu_0 + \mu_1 < 0$  and  $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$ . Then, for all  $(x, z) \in \mathbb{R}^2$  we have*

$$\mathbb{Q}_{x, z}(\tau^* = \sigma_* = \hat{\sigma}_*) = 1.$$

This result allows us to state the continuity result of the optimal stopping time with respect to the initial data.

**Lemma A.6.** *Assume that  $\mu_0 + \mu_1 < 0$  and  $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$ . We have  $\lim_{n \rightarrow \infty} \tau^*(x_n, z_n) = \tau^*(x, z)$  for any  $(x, z) \in \mathbb{R}^2$  and any sequence  $(x_n, z_n) \rightarrow (x, z)$ . In particular, if  $(x, z) \in \partial\mathcal{C}_3$ , the limit is zero.*

*Proof.* Let  $(x, z) \in \mathbb{R}^2$  and denote  $\tau_n := \tau^*(x_n, z_n)$  as well as  $\tau := \tau^*(x, z)$  for simplicity. In order to show lower-semicontinuity, we fix  $\omega \in \Omega$  outside of a null-set. For  $\tau(\omega) = 0$  we are finished and thus assume  $\tau(\omega) > \delta > 0$ . Due to Proposition 6.11 there exists  $k_{\delta, \omega} > 0$  such that

$$c(Z_t(\omega)) - X_t(\omega) > k_{\delta, \omega},$$

for all  $t \in [0, \delta]$ . The map  $(t, x, z) \mapsto c(Z_t^z(\omega)) - X_t^x(\omega)$  is uniformly continuous on any compact  $[0, \delta] \times K$ , hence we can find  $N_\omega \geq 1$  such that for all  $n \geq N_\omega$  and  $t \in [0, \delta]$

$$c(Z_t^{z_n}(\omega)) - X_t^{x_n}(\omega) > k_{\delta, \omega},$$

and therefore  $\liminf_n \tau_n(\omega) \geq \delta$ . Since  $\omega$  and  $\delta$  were arbitrary, we obtain  $\liminf_n \tau_n \geq \tau$   $\mathbb{Q}$ -a.s. and thus lower-semicontinuity. By employing similar arguments we can show  $\limsup_n \hat{\tau}_n \leq \hat{\tau}$   $\mathbb{Q}$ -a.s. and the claim thus follows together with Corollary A.5.  $\square$

Before we finally state the proof of Proposition 6.12, we can derive a probabilistic representation of  $v_x$  by employing arguments similar to those employed in the proof of Lemma 6.6.

**Lemma A.7.** *For all  $(x, z) \in \mathbb{R}^2 \setminus \partial\mathcal{C}_3$ , we have*

$$\hat{v}_x(x, z) = \mathbb{E}_{(x, z)}^{\mathbb{Q}} \left[ e^{-r\tau^*} \left( e^{X_{\tau^*}} (1 + e^{\frac{\gamma}{\sigma}(X_{\tau^*} + Z_{\tau^*})}) + \frac{\gamma}{\sigma} (e^{X_{\tau^*}} - \kappa) e^{\frac{\gamma}{\sigma}(X_{\tau^*} + Z_{\tau^*})} \right) \mathbb{1}_{\{\tau^* < \infty\}} \right].$$

We are therefore ready to prove Proposition 6.12.

**Proof of Proposition 6.12.** The first statement trivially holds true for  $(x, z) \in \text{int}(\mathcal{S}_3)$  and  $(x, z) \in \mathcal{C}_3$ , due to the result in Lemma 6.2. It thus remains to prove that  $\nabla_{x, z} \hat{v}$  is continuous across the boundary  $\partial\mathcal{C}_3$ . Let  $(x_0, z_0) \in \partial\mathcal{C}_3$  and take a sequence  $(x_n, z_n) \rightarrow (x_0, z_0)$  with  $\tau_n := \tau^*(x_n, z_n)$ . For a fixed  $t > 0$ , we notice  $(X_t, Z_t) \in \mathcal{C}_3$  on  $\{\tau_n > t\}$  and thus, upon using tower and Markov property, we obtain

$$\begin{aligned} \hat{v}_x(x_n, z_n) &= \mathbb{E}_{(x_n, z_n)}^{\mathbb{Q}} \left[ e^{-r\tau_n} \left( e^{X_{\tau_n}} (1 + e^{\frac{\gamma}{\sigma}(X_{\tau_n} + Z_{\tau_n})}) + \frac{\gamma}{\sigma} (e^{X_{\tau_n}} - \kappa) e^{\frac{\gamma}{\sigma}(X_{\tau_n} + Z_{\tau_n})} \right) \mathbb{1}_{\{\tau_n \leq t\}} \right] \\ &\quad + \mathbb{E}_{(x_n, z_n)}^{\mathbb{Q}} \left[ e^{-rt} \hat{v}_x(X_t, Z_t) \mathbb{1}_{\{\tau_n > t\}} \right]. \end{aligned}$$

Due to Assumption 4.1 we can invoke dominated convergence as well as Lemma A.6 to obtain

$$\lim_{n \rightarrow \infty} \hat{v}_x(x_n, z_n) = e^{x_0} (1 + e^{\frac{\gamma}{\sigma}(x_0 + z_0)}) + \frac{\gamma}{\sigma} (e^{x_0} - \kappa) e^{\frac{\gamma}{\sigma}(x_0 + z_0)} = \frac{\partial}{\partial x} \left( (e^x - \kappa) (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \right) \Big|_{(x_0, z_0)},$$

and hence, the continuity of  $\hat{v}_x$  across the optimal boundary. The continuity of  $\hat{v}_z$  across the free boundary follows similarly. For the last claim we observe that Lemma 6.2 implies

$$(A.11) \quad \frac{1}{2} \sigma^2 \hat{v}_{xx}(x, z) = r \hat{v}(x, z) - \mu_0 \hat{v}_x(x, z) + \frac{1}{2} (\mu_0 + \mu_1) \hat{v}_z(x, z),$$

for all  $(x, z) \in \mathcal{C}_3$ . But the right-hand side of (A.11) only involves functions which are continuous on  $\mathbb{R}^2$ , hence we deduce that  $\hat{v}_{xx}$  admits a continuous extension on  $\bar{\mathcal{C}}_3$  and is therefore bounded therein. It follows that  $\hat{v}_x(\cdot, z)$  is locally Lipschitz continuous on  $\bar{\mathcal{C}}_3$ , with a Lipschitz constant  $K(z)$  that is locally bounded on  $\mathbb{R}$ . Now, because  $\hat{v}_x(\cdot, z)$  is infinitely many times continuously differentiable in the stopping region  $\mathcal{S}_3$  (and hence locally bounded therein as well), we conclude that  $\hat{v}_{xx} \in L_{\text{loc}}^\infty(\mathbb{R}^2)$ .

## APPENDIX B. PROOF OF PROPOSITION 6.13

*Proof.* Let  $R > 0$  and define  $\tau_R := \inf\{t \geq 0 : |X_t| \geq R \text{ or } |Z_t| \geq R\}$ . Since  $\widehat{v} \in C^1(\mathbb{R}^2)$  and  $\widehat{v}_{xx} \in L_{\text{loc}}^\infty(\mathbb{R}^2)$ , we can apply a weak version of Ito's Lemma (see, e.g., Bensoussan and Lions [7], Lemma 8.1 and Th. 8.5, pp. 183-186) up to the stopping time  $\tau_R \wedge T$  for some  $T > 0$ , which results in

$$(B.1) \quad \widehat{v}(x, z) = \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-r(\tau_R \wedge T)} \widehat{v}(X_{\tau_R \wedge T}, Z_{\tau_R \wedge T}) - \int_0^{\tau_R \wedge T} e^{-rs} (\mathcal{L}_{X,Z} \widehat{v} - r) \widehat{v}(X_s, Z_s) ds \right].$$

The right-hand-side of (B.1) is well-defined, because  $Z$  is deterministic,  $X$  has an absolutely continuous transition density and  $\mathcal{L}_{X,Z} \widehat{v}$  is defined up to a set of zero Lebesgue measure. Since  $\widehat{v}$  solves the free-boundary problem (6.34), we have

$$(\mathcal{L}_{X,Z} \widehat{v} - r) \widehat{v}(x, z) = (\mathcal{L}_{X,Z} \widehat{v} - r) \widehat{v}(x, z) \mathbb{1}_{\{x < c(z)\}} + (\mathcal{L}_{X,Z} \widehat{v} - r) \widehat{v}(x, z) \mathbb{1}_{\{x \geq c(z)\}} = g(x, z) \mathbb{1}_{\{x \geq c(z)\}},$$

for almost all  $(x, z) \in \mathbb{R}^2$ . Using again that the transition density of  $X$  is absolutely continuous with respect to the Lebesgue measure, equation (B.1) becomes

$$\widehat{v}(x, z) = \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-r(\tau_R \wedge T)} \widehat{v}(X_{\tau_R \wedge T}, Z_{\tau_R \wedge T}) - \int_0^{\tau_R \wedge T} e^{-rs} g(X_s, Z_s) \mathbb{1}_{\{x \geq c(z)\}} ds \right].$$

Now, upon employing a change of measure as in Section 5, we obtain

$$(B.2) \quad \begin{aligned} \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-r(\tau_R \wedge T)} |\widehat{v}(X_{\tau_R \wedge T}, Z_{\tau_R \wedge T})| \right] &= \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-r(\tau_R \wedge T)} |\bar{v}(X_{\tau_R \wedge T}, e^{\frac{\gamma}{\sigma}(X_{\tau_R \wedge T} + Z_{\tau_R \wedge T})})| \right] \\ &\leq K_1 \mathbb{E}_{(x, \exp(\frac{\gamma}{\sigma}(x+z)))}^{\mathbb{Q}} \left[ e^{-r(\tau_R \wedge T)} e^{X_{\tau_R \wedge T}} (1 + \Phi_{\tau_R \wedge T}) \right] \\ &= K_1 (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \mathbb{E}_{(x,\pi)} \left[ e^{-r(\tau_R \wedge T)} e^{X_{\tau_R \wedge T}} \right], \end{aligned}$$

where  $\pi = e^{\frac{\gamma}{\sigma}(x+z)} / (1 + e^{\frac{\gamma}{\sigma}(x+z)})$ . Due to Assumption 4.1, it is easy to verify that taking limits in (B.2) yields

$$(B.3) \quad \lim_{T \uparrow \infty} \lim_{R \uparrow \infty} \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ e^{-r(\tau_R \wedge T)} \widehat{v}(X_{\tau_R \wedge T}, Z_{\tau_R \wedge T}) \right] = 0.$$

Furthermore,

$$(B.4) \quad \begin{aligned} \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ \int_0^{\tau_R \wedge T} e^{-rs} g(X_s, Z_s) \mathbb{1}_{\{x \geq c(z)\}} ds \right] &\leq \mathbb{E}_{(x,z)}^{\mathbb{Q}} \left[ \int_0^{\infty} e^{-rs} |g(X_s, Z_s)| ds \right] \\ &\leq \mathbb{E}_{(x, \exp(\frac{\gamma}{\sigma}(x+z)))}^{\mathbb{Q}} \left[ \int_0^{\infty} e^{-rs} \left( e^{X_s} \left( r - \frac{1}{2} \sigma^2 - \mu_0 \right) + rk + \Phi_s \left( e^{X_s} \left( r - \frac{1}{2} \sigma^2 - \mu_1 \right) + rk \right) \right) ds \right] \\ &\leq \mathbb{E}_{(x, \exp(\frac{\gamma}{\sigma}(x+z)))}^{\mathbb{Q}} \left[ \int_0^{\infty} e^{-rs} \left( e^{X_s} \left( r - \frac{1}{2} \sigma^2 - \mu_0 \right) + rk \right) ds \right] \\ &\quad + (1 + e^{\frac{\gamma}{\sigma}(x+z)}) \mathbb{E}_{(x,\pi)} \left[ \int_0^{\infty} e^{-rs} \left( e^{X_s} \left( r - \frac{1}{2} \sigma^2 - \mu_1 \right) + rk \right) ds \right] < \infty, \end{aligned}$$

where  $\pi = e^{\frac{\gamma}{\sigma}(x+z)} / (1 + e^{\frac{\gamma}{\sigma}(x+z)})$  and the last inequality follows again from Assumption 4.1. Hence, given the finiteness of the expectation in (B.4), we can apply dominated convergence theorem in order to interchange expectation and limits as  $R \uparrow \infty$  and  $T \uparrow \infty$ . Combining this result with (B.3) gives (6.35), which completes our proof.  $\square$

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F. DAMMANN: CENTER FOR MATHEMATICAL ECONOMICS (IMW), BIELEFELD UNIVERSITY, UNIVERSITÄTSSTRASSE 25, 33615, BIELEFELD, GERMANY  
 Email address: [dammann@uni-bielefeld.de](mailto:dammann@uni-bielefeld.de)

G. FERRARI: CENTER FOR MATHEMATICAL ECONOMICS (IMW), BIELEFELD UNIVERSITY, UNIVERSITÄTSSTRASSE 25, 33615, BIELEFELD, GERMANY  
 Email address: [giorgio.ferrari@uni-bielefeld.de](mailto:giorgio.ferrari@uni-bielefeld.de)