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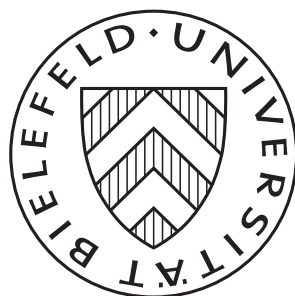


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Convex Monotone Semigroups and their Generators with Respect to Γ -Convergence

Jonas Blessing, Robert Denk, Michael Kupper and Max Nendel



CONVEX MONOTONE SEMIGROUPS AND THEIR GENERATORS WITH RESPECT TO Γ -CONVERGENCE

JONAS BLESSING^{*,1}, ROBERT DENK^{*,2}, MICHAEL KUPPER^{*,3}, AND MAX NENDEL^{**,4}

ABSTRACT. We study semigroups of convex monotone operators on spaces of continuous functions and their behaviour with respect to Γ -convergence. In contrast to the linear theory, the domain of the generator is, in general, not invariant under the semigroup. To overcome this issue, we consider different versions of invariant Lipschitz sets which turn out to be suitable domains for weaker notions of the generator. The so-called Γ -generator is defined as the time derivative with respect to Γ -convergence in the space of upper semicontinuous functions. Under suitable assumptions, we show that the Γ -generator uniquely characterizes the semigroup and is determined by its evaluation at smooth functions. Furthermore, we provide Chernoff approximation results for convex monotone semigroups and show that approximation schemes based on the same infinitesimal behaviour lead to the same semigroup. Our results are applied to semigroups related to stochastic optimal control problems in finite and infinite-dimensional settings as well as Wasserstein perturbations of transition semigroups.

Key words: Convex monotone semigroup, Γ -convergence, Lipschitz set, comparison principle, Chernoff approximation, optimal control, Wasserstein perturbation.

MSC 2020: Primary 47H20; 47J25; Secondary 35K55; 35B20; 49L20.

1. INTRODUCTION

The link between operator semigroups and abstract Cauchy problems through the infinitesimal behaviour of the semigroup is a classic question in the theory of partial differential equations. A fundamental result is that strongly continuous semigroups of bounded linear operators on Banach spaces are uniquely characterized by their infinitesimal generator. In a nonlinear setting, Alvarez et al. [1] provide an axiomatic foundation for viscosity solutions to fully nonlinear second-order partial differential equations based on monotone semigroups which are defined on the space of bounded uniformly continuous functions and satisfy suitable regularity and locality assumptions. This approach was picked up later by Biton [6] for semigroups on more general spaces of continuous functions with a certain behaviour at infinity. While these works mainly focus on the existence and axiomatization of second-order differential operators through semigroups, the uniqueness of the associated semigroups in terms of their generator is not yet fully clarified, cf. the discussion in [6, Section 5]. The key ideas of viscosity solutions are local comparisons with smooth functions and regularizations by introducing additional

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viscosity terms. We refer to Fleming and Soner [22, Chapter II.3] for an intuitive discussion on viscosity solutions in an operator-theoretic setting based on abstract dynamic programming principles. In the present paper we follow a different approach which is closer to the theory of linear semigroups and based on invariant sets which are suitable domains for a weaker definition of the generator. In order to provide uniqueness results for semigroups based on their infinitesimal behaviour, it is crucial that the domain of the generator is invariant under the semigroup. If we drop the linearity, the invariance may fail, see [17, Example 5.2]. Additionally, the domain might be even empty, see Crandall and Liggett [14, Section 4]. For convex semigroups, the invariance can, in general, only be guaranteed for spaces with order continuous norm, see [16]. In contrast to L^p -spaces and Orlicz hearts, which might be too large to handle the nonlinearity, spaces of continuous functions lack this property. Hence, we are looking for an invariant set on which we can define the generator in a weaker form such that the semigroup still represents the unique solution to the associated abstract Cauchy problem. Similar to Rademacher's theorem, it turns out that Lipschitz continuous paths of the semigroup are differentiable in a weaker sense, i.e., the generator can be defined with respect to Γ -convergence.

Let $S := (S(t))_{t \geq 0}$ be a strongly continuous semigroup of convex monotone operators on a suitable space of continuous functions. On the so-called upper Lipschitz, which is invariant under the semigroup, we can define the upper Γ -generator

$$A_{\Gamma}^{\dagger} f := \Gamma\text{-}\limsup_{h \downarrow 0} \frac{S(h)f - f}{h}.$$

Likewise we define the Γ -generator $A_{\Gamma} f$ if the limes superior in the previous equation can be replaced by a Γ -limit. Since the function $A_{\Gamma}^{\dagger} f$ is merely upper semicontinuous, an extension of S to the set of all upper semicontinuous functions is necessary in order to define the term $S(t)A_{\Gamma}^{\dagger} f$. For that purpose, we follow the ideas of Beer [4]. Under the assumption that S is continuous from above, there exists a unique extension which is upper semicontinuous with respect to Γ -convergence. The latter property is crucial for our work. Moreover, on the set of upper semicontinuous functions, the concept of Γ -convergence satisfies desirable stability properties, see, e.g., Dal Maso [15] and Rockafellar and Wets [38].

Our first main result is a comparison principle which implies, in particular, that convex monotone semigroups are uniquely determined by their upper Γ -generator on their upper Lipschitz set, see Theorem 2.10 and Corollary 2.11. Second, under additional assumptions, we show that $A_{\Gamma} f = \Gamma\text{-}\lim_{n \rightarrow \infty} A_{\Gamma} f_n$ if $(f_n)_{n \in \mathbb{N}}$ is a suitable approximating sequence for f , see Theorem 3.4. Typically, the approximating sequence can be constructed via convolutions with mollifiers or sup-inf-convolutions. In particular, we obtain an explicit description of $A_{\Gamma} f$ if, for smooth functions, the Γ -generator is given as a convex functional of certain partial derivatives. It was shown in Alvarez et al. [1] and Biton [6] that this is the case for typical fully nonlinear PDEs. Third, we study approximation schemes of the form

$$S(t)f = \lim_{l \rightarrow \infty} (I(2^{-n_l}t))^{2^{n_l}t} f$$

which are known as Chernoff approximations [11, 12] or Trotter formulae [42, 43]. In this case, key properties of S can be obtained from the corresponding properties of I , see Theorem 4.3 and Theorem 4.6.

The semigroups we consider are upper semicontinuous with respect to Γ -convergence and sequentially continuous in buc ¹, see Lemma 2.6 and Lemma 2.7. The idea of weakening topological properties of the semigroup is already present in the literature. Goldys and Kocan [24], van Casteren [44], Kunze [30] and Kraaij [28] study linear semigroups in strict topologies, see also Kraaij [26] and Yosida [46] for semigroups in locally convex spaces. Furthermore, equi-continuity in the strict topology is suitable for stability results. Kraaij [29] provides convergence results for nonlinear semigroups based on the connection between viscosity solutions to Hamilton–Jacobi equations and pseudo-resolvents and, in [27], Γ -convergence of functionals on path-spaces is established. A classical approach to nonlinear semigroups concentrates on the study of maximal monotone or m -accretive operators, see, e.g., Barbu [2], B enilan and Crandall [5], Br ezis [9], Kato [25] and the references therein. However, there exist simple examples of operators which are accretive but not m -accretive, see, e.g., [17, Example 5.2]. This obstacle was one of the motivations for the study of viscosity solutions to fully nonlinear equations, cf. Lions [32], Crandall et.al. [13] and Evans [19, Section 4].

In Section 5, the abstract results are applied to several classes of examples. In Subsection 5.1, we show that dynamic stochastic optimal control problems for drift and volatility controlled diffusions can be approximated by iterating a corresponding static control problem, where we only take simple deterministic controls. Moreover, the study of the so-called symmetric Lipschitz set yields a regularity result for the corresponding PDE even in the degenerate case. In Subsection 5.2, we show that, in the sublinear case, the previous approximation result can be lifted to an infinite-dimensional setting. In Subsection 5.3, we show that non-parametric Wasserstein perturbations of transition semigroups asymptotically coincide with perturbations which have a finite-dimensional parameter space. As a byproduct, we recover the Talagrand T_2 inequality for the normal distribution.

2. COMPARISON OF CONVEX MONOTONE SEMIGROUPS ON LIPSCHITZ SETS

2.1. Setup and Γ -convergence. Let (X, d) be a complete separable metric space and denote by $B(x, r) := \{y \in X : d(x, y) \leq r\}$ the closed ball with radius $r \geq 0$ around $x \in X$. We endow the set of all functions with the pointwise order, i.e., $f \leq g$ if and only if $f(x) \leq g(x)$ for all $f, g : X \rightarrow [-\infty, \infty)$ and $x \in X$. All order-related notions (sup, inf, max, min, lim sup, etc.) for such functions are understood with respect to this order. We define $f \vee g := \max\{f, g\}$, $f \wedge g := \min\{f, g\}$, $f^+ := f \vee 0$ and $f^- := -(f \wedge 0)$ for all $f, g : X \rightarrow [-\infty, \infty)$, where $f^-(x) := \infty$ if $f(x) = -\infty$.

We slightly relax the supremum norm in order to include unbounded functions with controlled growth behaviour at infinity. For this, we fix a bounded continuous function $\kappa : X \rightarrow (0, \infty)$ and consider the κ -weighted supremum norm

$$\|f\|_\kappa := \sup_{x \in X} |f(x)|\kappa(x) \in [0, \infty] \quad \text{for all } f : X \rightarrow [-\infty, \infty).$$

Let C_κ be the space of all continuous functions $f : X \rightarrow \mathbb{R}$ with $\|f\|_\kappa < \infty$ and U_κ be the set of all upper semicontinuous functions $f : X \rightarrow [-\infty, \infty)$ with $\|f^+\|_\kappa < \infty$. If $\kappa \equiv 1$, then $\|\cdot\|_\kappa$ is the usual supremum norm $\|\cdot\|_\infty$, C_κ coincides with the space C_b of all bounded continuous functions and U_κ is the set U_b of all upper semicontinuous functions which are bounded above by a real constant. Since the mapping

$$C_\kappa \rightarrow C_b, \quad f \mapsto f\kappa$$

¹A sequence converges buc if and only if it is bounded and converges uniformly on compacts. On Polish spaces, a sequence converges buc if and only if it converges in the strict topology, see [39].

is an order-preserving linear isometric isomorphism, the space C_κ is a Banach lattice. Note that

$$U_\kappa = (C_\kappa)_\delta := \left\{ \inf_{n \in \mathbb{N}} f_n : (f_n)_{n \in \mathbb{N}} \subset C_\kappa \right\}^2.$$

Let $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ be a sequence and $f \in U_\kappa$. We write $f_n \downarrow f$ if $(f_n)_{n \in \mathbb{N}}$ decreases pointwise to f . If f_n and f are real-valued, we say that $f_n \rightarrow f$ uniformly on compacts if $\sup_{x \in K} |f(x) - f_n(x)| \rightarrow 0$ for every compact set $K \subset X$. Furthermore, a set $F \subset U_\kappa$ is called bounded if $\sup_{f \in F} \|f\|_\kappa < \infty$ and bounded above if $\sup_{f \in F} \|f^+\|_\kappa < \infty$.

Definition 2.1. For every sequence $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$, which is bounded above, we define

$$\left(\Gamma\text{-lim sup}_{n \rightarrow \infty} f_n \right)(x) := \sup \left\{ \limsup_{n \rightarrow \infty} f_n(x_n) : (x_n)_{n \in \mathbb{N}} \subset X \text{ with } x_n \rightarrow x \right\} \in [-\infty, \infty)$$

for all $x \in X$. Moreover, we say that $f = \Gamma\text{-lim}_{n \rightarrow \infty} f_n$ with $f \in U_\kappa$ if, for every $x \in X$,

- $f(x) \geq \limsup_{n \rightarrow \infty} f_n(x_n)$ for every sequence $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \rightarrow x$,
- $f(x) = \lim_{n \rightarrow \infty} f_n(x_n)$ for some sequence $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \rightarrow x$.

For every $t \geq 0$ and $(f_s)_{s \geq 0} \subset U_\kappa$, which bounded above, we define

$$\Gamma\text{-lim sup}_{s \rightarrow t} f_s := \sup \left\{ \Gamma\text{-lim sup}_{n \rightarrow \infty} f_{s_n} : s_n \rightarrow t \right\} \in U_\kappa.$$

Furthermore, we say that $f = \Gamma\text{-lim}_{s \rightarrow t} f_s$ with $f \in U_\kappa$ if $f = \Gamma\text{-lim}_{n \rightarrow \infty} f_{s_n}$ for all sequences $(s_n)_{n \in \mathbb{N}} \subset [0, \infty)$ with $s_n \rightarrow t$.

For further details and a summary of basic results on Γ -convergence, we refer to Appendix A. The following geometric characterization of the Γ -lim sup is based on the work of Beer [4]. We will use it throughout this work to link Γ -convergence with monotone convergence and Γ -upper semicontinuity with continuity from above.

Remark 2.2. For every $f \in U_\kappa$, there exists a family $(\bar{f}^\varepsilon)_{\varepsilon > 0} \subset U_\kappa$ with

$$\bar{f}^\varepsilon \downarrow f \quad \text{as } \varepsilon \downarrow 0 \quad \text{and} \quad -\frac{1}{\varepsilon} \leq \bar{f}^\varepsilon \leq \|f^+\|_\kappa + \varepsilon \quad \text{for all } \varepsilon > 0.$$

Furthermore, let $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ be bounded above. Then, it holds $\Gamma\text{-lim sup}_{n \rightarrow \infty} f_n \leq f$ if and only if, for every $\varepsilon > 0$ and $K \subset X$ compact, there exists $n_0 \in \mathbb{N}$ such that

$$f_n(x) \leq \bar{f}^\varepsilon(x) \quad \text{for all } x \in K \text{ and } n \geq n_0.$$

The family $(\bar{f}^\varepsilon)_{\varepsilon > 0}$ is explicitly constructed in Appendix A.

2.2. Convex monotone semigroups on Lipschitz sets. Let $S := (S(t))_{t \geq 0}$ be a family of operators $S(t): C_\kappa \rightarrow U_\kappa$. The *norm generator* is defined by

$$A: D(A) \rightarrow C_\kappa, \quad f \mapsto \lim_{h \downarrow 0} \frac{S(h)f - f}{h},$$

where the domain $D(A)$ consists of all $f \in C_\kappa$ such that $S(t)f \in C_\kappa$ for all $t \geq 0$, the previous limit exists with respect to the norm $\|\cdot\|_\kappa$ and $S(s+t)f = S(s)S(t)f$ for all $s, t \geq 0$. As previously discussed, the assumption $S(t): D(A) \rightarrow D(A)$ for all $t \geq 0$ is, in general, too restrictive. We point out that this problem can not be avoided by restricting the semigroup and the generator to a subspace of C_κ . Hence, instead of considering the largest set on which the right derivative of the trajectories $t \mapsto S(t)f$ exists with respect to the norm, we consider the largest set on which the trajectories are merely (upper) Lipschitz continuous. For these trajectories the right derivative can still be defined as a limes superior. This brings us to the following crucial definition.

²It holds $f(x) = \inf_{n \in \mathbb{N}} \sup_{y \in X} \frac{1}{\kappa(x)} (\max\{(f\kappa)(y), -n\} - n^2 d(x, y))$ for all $f \in U_\kappa$ and $x \in X$.

Definition 2.3. The *upper Lipschitz set* \mathcal{L}_+^S consists of all $f \in C_\kappa$ such that

- (i) $S(t)f \in C_\kappa$ for all $t \geq 0$,
- (ii) $S(s+t)f = S(s)S(t)f$ for all $s, t \geq 0$,
- (iii) there exist $h_0 > 0$ and $c \geq 0$ with $(S(h)f - f)_\kappa \leq ch$ or all $h \in [0, h_0]$.

Furthermore, we define *the upper Γ -generator* by

$$A_\Gamma^+ f := \Gamma\text{-lim sup}_{h \downarrow 0} \frac{S(h)f - f}{h} \in U_\kappa \quad \text{for all } f \in \mathcal{L}_+^S.$$

By definition, it holds $D(A) \subset \mathcal{L}_+^S$ and $Af = A_\Gamma^+ f$ for all $f \in D(A)$. Furthermore, we will see in Section 3 that, under additional assumptions, the limes superior in the previous definition can be replaced by a limit. The invariance of the upper Lipschitz set, i.e., $S(t): \mathcal{L}_+^S \rightarrow \mathcal{L}_+^S$ for all $t \geq 0$, does not require further assumptions and is shown in Remark 2.9 below. While $Af \in C_\kappa$ holds by definition, $A_\Gamma^+ f$ is not necessarily continuous and may take the value $-\infty$. Hence, in order to give the term $S(t)A_\Gamma^+ f$ a meaning, an extension of S from C_κ to U_κ is necessary. Moreover, the argumentation in this article relies heavily on the fact that the extension is Γ -upper semicontinuous in time and continuous from above. This can be achieved under the following assumption which holds throughout the rest of this section. Let $B_{C_\kappa}(0, r) := \{f \in C_\kappa: \|f\|_\kappa \leq r\}$ and $B_{U_\kappa}(0, r) := \{f \in U_\kappa: \|f\|_\kappa \leq r\}$ be the closed balls with radius $r \geq 0$ around zero in C_κ and U_κ , respectively.

Assumption 2.4. The family $S := (S(t))_{t \geq 0}$ of operators $S(t): C_\kappa \rightarrow U_\kappa$ satisfies the following conditions:

- (S1) The operator $S(t)$ is convex and monotone,³ and $S(t)0 = 0$ for all $t \geq 0$.
- (S2) For every $t \geq 0$, the operator $S(t)$ is continuous from above, i.e., $S(t)f_n \downarrow S(t)f$ for all sequences $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ and $f \in C_\kappa$ with $f_n \downarrow f$.
- (S3) $\Gamma\text{-lim sup}_{s \rightarrow t} S(s)f \leq S(t)f$ and $S(0)f = f$ for all $t \geq 0$ and $f \in C_\kappa$.
- (S4) $\sup_{s \in [0, t]} \sup_{f \in B_{C_\kappa}(0, r)} \|S(s)f\|_\kappa < \infty$ for all $r, t \geq 0$.

Since C_κ is a Banach lattice and $S(t)$ is convex, it is sufficient to require condition (S2) for some $f \in C_\kappa$, e.g., $f = 0$. Although, in many examples, S is a priori only defined on a closed subspace $\mathcal{C} \subset C_\kappa$, we assume that S is defined on the whole space C_κ . The reason behind this is the fact that the property $U_\kappa = (C_\kappa)_\delta$ together with condition (S2) ensures the existence of a pointwise extension to U_κ which is again continuous from above and satisfies the conditions (S1)-(S4) for all $f \in U_\kappa$. If $U_\kappa = \mathcal{C}_\delta$, it is sufficient to state Assumption 2.4 with \mathcal{C} instead of C_κ . We conclude this subsection with the crucial observation that S is simultaneously upper semicontinuous in the variables t and f , see Lemma 2.6 below. Moreover, we state some further comments about Definition 2.3 and Assumption 2.4. Let B_κ be the space of all Borel measurable functions $f: X \rightarrow [-\infty, \infty)$ with $\|f^+\|_\kappa < \infty$.

Remark 2.5. We discuss the extension of S from C_κ to U_κ . For a proof, we refer to Corollary C.3.

- (i) Fix $t \geq 0$. Since $S(t): C_\kappa \rightarrow U_\kappa$ is continuous from above and $U_\kappa = (C_\kappa)_\delta$, there exists a unique extension $S(t): U_\kappa \rightarrow U_\kappa$ which is continuous from above. The family of extended operators satisfies the conditions (S1)-(S4) with U_κ instead of C_κ . In addition, for every $x \in X$, the functional

$$U_\kappa \rightarrow [-\infty, \infty), f \mapsto (S(t)f)(x)$$

³The mapping $C_\kappa \rightarrow [-\infty, \infty)$, $f \mapsto (S(t)f)(x)$ is convex and monotone for all $x \in X$.

can even be extended to B_κ in such a way that, for every $\varepsilon > 0$ and $c \geq 0$, there exists a compact set $K \subset X$ with $(S(t)(\frac{c}{\kappa}\mathbb{1}_{K^c}))(x) \leq \varepsilon$. Here, $K^c := X \setminus K$.

(ii) Fix $t \geq 0$ and $K \subset X$ compact. Let $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ be a sequence with $f_n \downarrow 0$. By part (i), the mapping

$$[0, t] \times K \rightarrow \mathbb{R}, (s, x) \mapsto (S(s)f_n)(x)$$

is upper semicontinuous for all $n \in \mathbb{N}$ and decreases pointwise to zero as $n \rightarrow \infty$. It thus follows from Dini's theorem that

$$\sup_{(s,x) \in [0,t] \times K} (S(s)f_n)(x) \downarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover, for every $\varepsilon > 0$ and $c \geq 0$, there exists a compact set $K_1 \subset X$ with

$$\sup_{(s,x) \in [0,t] \times K} (S(s)(\frac{c}{\kappa}\mathbb{1}_{K_1^c}))(x) \leq \varepsilon.$$

Lemma 2.6. *Let $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ be bounded above and $(t_n)_{n \in \mathbb{N}} \subset [0, \infty)$ a convergent sequence. Define $f := \Gamma\text{-lim sup}_{n \rightarrow \infty} f_n$ and $t := \lim_{n \rightarrow \infty} t_n$. Then,*

$$\Gamma\text{-lim sup}_{n \rightarrow \infty} S(t_n)f_n \leq S(t)f.$$

Proof. Let $\varepsilon \in (0, 1]$ and $K \subset X$ compact. By Remark 2.2, there exists $n_0 \in \mathbb{N}$ with $f_n(x) \leq \bar{f}^\varepsilon(x)$ for all $x \in K$ and $n \geq n_0$. We use the fact that $\bar{f}^\varepsilon \geq -1/\varepsilon\kappa$ together with the monotonicity of $S(t_n)$ to come up with

$$S(t_n)f_n \leq S(t_n)(\bar{f}^\varepsilon + \frac{c_\varepsilon}{\kappa}\mathbb{1}_{K^c}) \quad \text{for all } n \geq n_0,$$

where $c_\varepsilon := \sup_{n \in \mathbb{N}} \|f_n^+\|_\kappa + 1/\varepsilon < \infty$. For all $\lambda \in (0, 1)$, the convexity of $S(t_n)$ implies

$$S(t_n)(\bar{f}^\varepsilon + \frac{c_\varepsilon}{\kappa}\mathbb{1}_{K^c}) \leq \lambda S(t_n)(\frac{1}{\lambda}\bar{f}^\varepsilon) + (1 - \lambda)S(t_n)(\frac{c_\varepsilon}{\kappa(1-\lambda)}\mathbb{1}_{K^c}).$$

Hence, it follows from Lemma A.1(iv) and Remark 2.5(i) that

$$\Gamma\text{-lim sup}_{n \rightarrow \infty} S(t_n)f_n \leq \lambda S(t)(\frac{1}{\lambda}\bar{f}^\varepsilon) + (1 - \lambda) \sup_{n \in \mathbb{N}} S(t_n)(\frac{c_\varepsilon}{\kappa(1-\lambda)}\mathbb{1}_{K^c}).$$

Since $K \subset X$ is arbitrary, we can use Remark 2.5(ii) to conclude that

$$\Gamma\text{-lim sup}_{n \rightarrow \infty} S(t_n)f_n \leq \lambda S(t)(\frac{1}{\lambda}\bar{f}^\varepsilon).$$

Furthermore, $\bar{f}^\varepsilon \kappa \leq \|f^+\|_\kappa + \varepsilon \leq c_\varepsilon$ and the monotonicity of $S(t)$ yield

$$\Gamma\text{-lim sup}_{n \rightarrow \infty} S(t_n)f_n \leq \lambda S(t)(\frac{1}{\lambda}\bar{f}^\varepsilon) \leq \lambda S(t)(\bar{f}^\varepsilon + (\frac{1}{\lambda} - 1)\frac{c_\varepsilon}{\kappa}).$$

Since $S(t)$ is continuous from above, the right-hand side converges to $S(t)\bar{f}^\varepsilon$ as $\lambda \uparrow 1$. Thus, it follows from $\bar{f}^\varepsilon \downarrow f$ that

$$\Gamma\text{-lim sup}_{n \rightarrow \infty} S(t_n)f_n \leq S(t)\bar{f}^\varepsilon \downarrow S(t)f \quad \text{as } \varepsilon \downarrow 0. \quad \square$$

Lemma 2.7. *Let $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ be a bounded sequence and $f \in C_\kappa$ with $f_n \rightarrow f$ uniformly on compacts. Then, $S(t)f_n \rightarrow S(t)f$ uniformly on compacts for all $t \geq 0$.*

Proof. Let $K \subset X$ be compact and $\varepsilon > 0$. Choose $\delta, \lambda \in (0, 1)$ with

$$(1 - \lambda)\|S(t)f\|_\kappa < \frac{\varepsilon}{3} \quad \text{and} \quad c\|(1 - \lambda)f + \delta\|_\kappa < \frac{\varepsilon}{3}, \quad (2.1)$$

where the constant $c \geq 0$ will be fixed later. By Remark 2.5(ii), there exists a compact set $K_1 \subset X$ with $K \subset K_1$ such that

$$\sup_{x \in K} \left((1 - \lambda) S(t) \left(\frac{c}{(1 - \lambda)\kappa} \mathbf{1}_{K_1^c} \right) \right) (x) < \frac{\varepsilon}{3}.$$

Since $f_n \rightarrow f$ uniformly on compacts, there exists $n_0 \in \mathbb{N}$ with $f_n \leq f + \delta + \frac{c_1}{\kappa} \mathbf{1}_{K_1^c}$ for all $n \geq n_0$, where $c_1 := \sup_{n \in \mathbb{N}} 2\|f_n\|_\kappa$. Monotonicity and convexity of $S(t)$ imply

$$S(t)f_n \leq \lambda S(t) \left(\frac{f + \delta}{\lambda} \right) + (1 - \lambda) S(t) \left(\frac{c_1}{(1 - \lambda)\kappa} \mathbf{1}_{K_1^c} \right) \quad \text{for all } n \geq n_0.$$

By Lemma B.2(ii), there exists a constant $c_2 \geq 0$ such that

$$\left\| \lambda S(t) \left(\frac{f + \delta}{\lambda} \right) - S(t)f \right\|_\kappa \leq c_2 \|(1 - \lambda)f + \delta\|_\kappa + (1 - \lambda)\|S(t)f\|_\kappa.$$

Now, choose $\delta, \lambda \in (0, 1)$ such that condition (2.1) is satisfied with $c := c_1 \vee c_2$. Then,

$$(S(t)f_n)(x) \leq (S(t)f)(x) + \varepsilon \quad \text{for all } n \geq n_0 \text{ and } x \in K.$$

Since $(f_n)_{n \in \mathbb{N}}$ is bounded, we can change the role of f and f_n in the previous considerations in order to obtain the reverse estimate. \square

The following lemma shows that the upper Lipschitz set is invariant.

Lemma 2.8. *It holds $S(t): \mathcal{L}_+^S \rightarrow \mathcal{L}_+^S$ for all $t \geq 0$.*

Proof. Let $f \in \mathcal{L}_+^S$. Choose $h_0 \in (0, 1]$ and $c \geq 0$ such that $(S(h)f - f)\kappa \leq ch$ for all $h \in [0, h_0]$. We use $S(h)S(t)f = S(h + t) = S(t + h) = S(t)S(h)f$, Lemma B.1, the monotonicity of $S(t)$ and Lemma B.2 to estimate

$$\begin{aligned} \left(\frac{S(h)S(t)f - S(t)f}{h} \right) \kappa &\leq \left(S(t) \left(f + \frac{(S(h)f - f)^+}{h} \right) - S(t)f \right) \kappa \\ &\leq c' \left\| \frac{(S(h)f - f)^+}{h} \right\|_\kappa \leq c'ch \quad \text{for all } h \in (0, h_0], \end{aligned}$$

where $c' \geq 0$ is a constant independent of $h \in (0, h_0]$. \square

In the next remark, we give an outlook on stronger versions of the Lipschitz set. Furthermore, we discuss some results about the Lipschitz set from the theory of linear semigroups. Most of these results rely on the reflexivity of the underlying Banach space, a property which C_κ does not have.

Remark 2.9.

- (i) Similar to \mathcal{L}_+^S , one can define the *Lipschitz set* \mathcal{L}^S , consisting of all $f \in C_\kappa$ with
- $S(t)f \in C_\kappa$ for all $t \geq 0$,
 - $S(s + t)f = S(s)S(t)f$ for all $s, t \geq 0$,
 - there exists $h_0 > 0$ and $c \geq 0$ with $\|S(h)f - f\|_\kappa \leq ch$ for all $h \in [0, h_0]$.

Furthermore, we define the *symmetric Lipschitz set* $\mathcal{L}_{\text{sym}}^S := \{f \in \mathcal{L}^S : -f \in \mathcal{L}^S\}$. While $S(t): \mathcal{L}^S \rightarrow \mathcal{L}^S$ for all $t \geq 0$ follows by a similar argumentation as in the proof of Lemma 2.8, the invariance of $\mathcal{L}_{\text{sym}}^S$ can, in general, not be guaranteed. However, in several examples, the symmetric Lipschitz set is invariant and can, in contrast to \mathcal{L}_+^S and \mathcal{L}^S , be determined explicitly. This leads to regularity results for the associated semigroup, see [7, 8] and Subsection 5.1.

- (ii) Let $T := (T(t))_{t \geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space \mathcal{X} . For simplicity, we assume that the growth bound of T is negative, i.e., there exist $c \geq 0$ and $\omega < 0$ with $\|T(t)x\| \leq ce^{\omega t}\|x\|$ for all $x \in \mathcal{X}$. Then, the set

$$F_1 := \left\{ x \in \mathcal{X} : \sup_{h>0} \left\| \frac{T(h)x - x}{h} \right\| < \infty \right\}$$

is called the *Favard space* or the *saturation class* of T , a notion coming from approximation theory, see, e.g., [10, Section 2.1] and [18, Section II.5.b]. Denoting by A the norm generator of T , it is known that $F_1 = D(A)$ holds if \mathcal{X} is reflexive, see [10, Theorem 2.1.2]. If T is even holomorphic, then $F_1 = (\mathcal{X}, D(A))_{1,\infty}$, see [33, Proposition 2.2.2]. Here, $(\cdot, \cdot)_{1,\infty}$ stands for the real interpolation functor. However, for non-reflexive \mathcal{X} , an explicit description of F_1 seems to be unknown in many cases.

2.3. Comparison and uniqueness. Based on the preliminary work of the previous two subsections, we can now prove the following comparison principle which is the main result of Section 2. Basically, a semigroup satisfying Assumption 2.4 can be seen as the minimal Γ -supersolution of the associated abstract Cauchy problem. Furthermore, the semigroup is uniquely determined by its upper Γ -generator. In the next section, we will prove approximation results which show that, in many examples, the evaluation of the generator at smooth functions is sufficient to determine the Γ -generator and thus the semigroup.

Theorem 2.10. *Fix $T \geq 0$ and $f \in \mathcal{L}_+^S$. Let $u : [0, T] \rightarrow \mathcal{L}_+^S$ be a bounded function with $u(0) = f$ and Γ - $\limsup_{s \rightarrow t} u(s) \leq u(t)$ for all $t \in [0, T]$. Assume that, for all $t \in [0, T]$,*

$$\limsup_{h \downarrow 0} \left\| \left(\frac{u(t+h) - u(t)}{h} \right)^- \right\|_\kappa < \infty, \quad (2.2)$$

$$\Gamma\text{-}\limsup_{h \downarrow 0} \left(A_\Gamma^+ u(t) - \frac{u(t+h) - u(t)}{h} \right) \leq 0. \quad (2.3)$$

Then, it holds $S(t)f \leq u(t)$ for all $t \in [0, T]$.

Proof. Fix $t \in [0, T]$. We prove that the function $v : [0, t] \rightarrow \mathcal{L}_+^S$, $s \mapsto S(t-s)u(s)$ satisfies $v(0) \leq v(s)$ for all $s \in [0, t]$. First, we show that

$$\liminf_{h \downarrow 0} \frac{v(s+h) - v(s)}{h} \geq 0 \quad \text{for all } s \in [0, t]. \quad (2.4)$$

Let $s \in [0, t]$. Because of $u(s) \in \mathcal{L}_+^S$ and condition (2.2), there exists $h_0 > 0$ with

$$c := \sup_{h \in (0, h_0]} \max \left\{ \left\| \left(\frac{S(h)u(s) - u(s)}{h} \right)^+ \right\|_\kappa, \left\| \left(\frac{u(s+h) - u(s)}{h} \right)^- \right\|_\kappa \right\} < \infty. \quad (2.5)$$

For every $h \in (0, h_0]$, we define

$$f_h := \max \left\{ \frac{S(h)u(s) - u(s)}{h}, -\frac{c}{\kappa} \right\} \quad \text{and} \quad g_h := \max \left\{ -\frac{u(s+h) - u(s)}{h}, -\frac{c}{\kappa} \right\}.$$

It follows from Lemma A.1(vi) that

$$f := \Gamma\text{-}\limsup_{h \downarrow 0} f_h = \max \left\{ A_\Gamma^+ u(s), -\frac{c}{\kappa} \right\}.$$

Furthermore, we can use inequality (2.3), equation (2.5) and Lemma A.1(vi) to estimate

$$\Gamma\text{-}\limsup_{h\downarrow 0} (f + g_h) \leq 0. \quad (2.6)$$

Let $\varepsilon > 0$. By the boundedness of u and condition (S4), there exists $\lambda \in (0, 1]$ such that

$$\sup_{a,b \in [0,t]} \lambda \|S(a)u(b)\|_\kappa < \varepsilon. \quad (2.7)$$

Lemma B.1 and inequality (2.7) imply

$$\begin{aligned} & -\liminf_{h\downarrow 0} \frac{v(s+h) - v(s)}{h} = \limsup_{h\downarrow 0} \frac{v(s) - v(s+h)}{h} \\ & = \limsup_{h\downarrow 0} \frac{S(t-s-h)S(h)u(s) - S(t-s-h)u(s+h)}{h} \\ & \leq \limsup_{h\downarrow 0} \lambda \left(S(t-s-h) \left(\frac{S(h)u(s) - u(s+h)}{\lambda h} + u(s+h) \right) - S(t-s-h)u(s+h) \right) \\ & \leq \limsup_{h\downarrow 0} \lambda S(t-s-h) \left(\frac{f_h + g_h}{\lambda} + u(s+h) \right) + \frac{\varepsilon}{\kappa}. \end{aligned}$$

Furthermore, the boundedness of $(g_h)_{h \in (0, h_0]}$, $(f_h)_{h \in (0, h_0]}$, and u together with condition (S1), condition (S4) and Remark 2.5(ii) ensure that we can apply Lemma C.4 to estimate

$$\limsup_{h\downarrow 0} \lambda S(t-s-h) \left(\frac{f_h + g_h}{\lambda} + u(s+h) \right) \leq \limsup_{h\downarrow 0} \lambda S(t-s-h) \left(\frac{f + g_h}{\lambda} + u(s+h) \right).$$

Lemma 2.6, inequality (2.6), Lemma A.1(iv), $\Gamma\text{-}\limsup_{h\downarrow 0} u(s+h) \leq u(s)$ and inequality (2.7) yield

$$\limsup_{h\downarrow 0} \lambda S(t-s-h) \left(\frac{f + g_h}{\lambda} + u(s+h) \right) \leq \lambda S(t-s)u(s) \leq \frac{\varepsilon}{\kappa}.$$

We combine the previous estimates and let $\varepsilon \downarrow 0$ to obtain the inequality in (2.4).

Second, we adapt the proof of [36, Lemma 1.1 in Chapter 2] to show $v(0) \leq v(s)$ for all $s \in [0, t]$. Let $x \in X$ and $\varepsilon > 0$. Define $v_\varepsilon(s, x) := (v_\varepsilon(s))(x)$ and

$$v_\varepsilon(\cdot, x): [0, t] \rightarrow \mathbb{R}, \quad s \mapsto v(s, x) + \varepsilon s.$$

Moreover, let $s_0 := \sup\{s \in [0, t]: v_\varepsilon(0, x) \leq v_\varepsilon(s, x)\}$. From $\Gamma\text{-}\limsup_{r \rightarrow s} u(r) \leq u(s)$ and Lemma 2.6, we obtain that $v_\varepsilon(\cdot, x)$ is upper semicontinuous. In particular, it holds $v_\varepsilon(0, x) \leq v_\varepsilon(s_0, x)$. By contradiction, we assume that $s_0 < t$. Let $(s_n)_{n \in \mathbb{N}} \subset (s_0, t]$ be a sequence with $s_n \downarrow s_0$. It follows from $v_\varepsilon(s_n, x) < v_\varepsilon(0, x) \leq v_\varepsilon(s_0, x)$ and inequality (2.4) that

$$0 \geq \limsup_{n \rightarrow \infty} \frac{v_\varepsilon(s_n, x) - v_\varepsilon(s_0, x)}{s_n - s_0} = \limsup_{n \rightarrow \infty} \frac{v(s_n, x) - v(s_0, x)}{s_n - s_0} + \varepsilon \geq \varepsilon > 0.$$

This implies $v_\varepsilon(0, x) \leq v_\varepsilon(t, x)$ and therefore $v(0, x) \leq v(t, x)$ as $\varepsilon \downarrow 0$. In particular, we obtain $S(t)f = v(0) \leq v(t) = u(t)$. \square

Having a close look at the proof of the previous theorem, it seems natural to replace the conditions (2.2) and (2.3) by the assumption that

$$\limsup_{h\downarrow 0} \left\| \left(\frac{S(h)u(t) - u(t+h)}{h} \right)^+ \right\|_\kappa < \infty \quad \text{and} \quad \Gamma\text{-}\limsup_{h\downarrow 0} \frac{S(h)u(t) - u(t+h)}{h} \leq 0.$$

Indeed, the previous theorem remains valid and the proof simplifies. In particular, we do not need the technical Lemma C.4. However, in examples, this assumption is not verifiable. Lemma A.1(iv) implies that condition (2.3) is satisfied if

$$A_{\Gamma}^{+}u(t) \leq \Gamma\text{-}\liminf_{h \downarrow 0} \frac{u(t+h) - u(t)}{h} := - \left(\Gamma\text{-}\limsup_{h \downarrow 0} - \frac{u(t+h) - u(t)}{h} \right).$$

Furthermore, it follows from Remark 2.2 that condition (2.3) is satisfied if

$$\lim_{h \downarrow 0} \left(A_{\Gamma}^{+}u(t) - \frac{u(t+h) - u(t)}{h} \right)^{+} = 0$$

uniformly on compacts. We conclude this subsection with a comparison result for convex monotone semigroups and a remark on the concept of a Γ -supersolution.

Corollary 2.11. *Let $(T(t))_{t \geq 0}$ be another family of operators $T(t): C_{\kappa} \rightarrow U_{\kappa}$ with Lipschitz set \mathcal{L}^T and upper Γ -generator B_{Γ}^{+} . Suppose that T satisfies Assumption 2.4. Furthermore, let $\mathcal{D} \subset \mathcal{L}^T \cap \mathcal{L}_{+}^S$ satisfy $T(t): \mathcal{D} \rightarrow \mathcal{D}$ for all $t \geq 0$ and*

$$A_{\Gamma}^{+}f \leq B_{\Gamma}^{+}f \quad \text{for all } f \in \mathcal{D}. \quad (2.8)$$

Then, it holds $S(t)f \leq T(t)f$ for all $t \geq 0$ and $f \in \mathcal{D}$ such that the limit

$$B_{\Gamma}^{+}f = \lim_{h \downarrow 0} \frac{T(h)f - f}{h} \in C_{\kappa}$$

exists uniformly on compacts.

Proof. Define $u(t) := T(t)f$ for all $t \geq 0$ and $f \in \mathcal{D}$. Assumption 2.4 and the invariance of $\mathcal{D} \subset \mathcal{L}_{+}^S$ imply that $u: [0, \infty) \rightarrow \mathcal{L}_{+}^S$ is a well-defined mapping with $u(0) = 0$ which is bounded on compact intervals and satisfies $\Gamma\text{-}\limsup_{s \rightarrow t} u(s) \leq u(t)$ for all $t \geq 0$. Condition (2.2) also holds due to the invariance of $\mathcal{D} \subset \mathcal{L}^T$.

It remains to verify condition (2.3). For every $t \geq 0$ and $h > 0$, we use $u(t) \in \mathcal{D}$ and inequality (2.8) to estimate

$$A_{\Gamma}^{+}u(t) - \frac{u(t+h) - u(t)}{h} \leq B_{\Gamma}^{+}u(t) - \frac{u(t+h) - u(t)}{h}.$$

Let $(h_n)_{n \in \mathbb{N}} \subset (0, \infty)$ be a sequence with $h_n \downarrow 0$. For every $n \in \mathbb{N}$, we have

$$B_{\Gamma}^{+}u(t) - \frac{u(t+h_n) - u(t)}{h_n} = B_{\Gamma}^{+}u(t) - g_n + g_n - \frac{u(t+h_n) - u(t)}{h_n}, \quad (2.9)$$

where $g_n := \frac{1}{h_n}(T(t)(f + h_n B_{\Gamma}^{+}f) - T(t)f)$. It follows from Lemma B.1 that

$$\begin{aligned} & T(t)(f + h_n B_{\Gamma}^{+}f) - T(t) \left(\frac{T(h_n)f - f}{h_n} - B_{\Gamma}^{+}f + f + h_n B_{\Gamma}^{+}f \right) \\ & \leq g_n - \frac{u(t+h_n) - u(t)}{h_n} = \frac{T(t)(f + h_n B_{\Gamma}^{+}f) - T(t)T(h_n)f}{h_n} \\ & \leq T(t) \left(- \left(\frac{T(h_n)f - f}{h_n} - B_{\Gamma}^{+}f \right) + T(h_n)f \right) - T(t)T(h_n)f. \end{aligned}$$

Combining the previous estimate with Lemma 2.7 yields

$$g_n - \frac{u(t+h_n) - u(t)}{h_n} \rightarrow 0$$

uniformly on compacts. Furthermore, since $T(t)$ is convex, the sequence $(g_n)_{n \in \mathbb{N}}$ is non-increasing. Hence, there exists a function $g \in U_{\kappa}$ with $g_n \downarrow g$. We use Lemma A.1(iii)

and (v) to conclude $g = B_\Gamma^+ u(t)$. It follows from inequality (2.9), $B_\Gamma^+ u(t) - g_n \leq 0$ and Lemma A.1(iv) that condition (2.3) is satisfied. Theorem 2.10 yields $S(t)f \leq T(t)f$. \square

Remark 2.12. Let $T \geq 0$ and $f \in \mathcal{L}_+^S$. A function $u: [0, T] \rightarrow \mathcal{L}_+^S$ can be seen as a Γ -supersolution of the equation

$$\partial_t u(t) = A_\Gamma^+ u(t) \quad \text{for all } t \in [0, T], \quad u(0) = f, \quad (2.10)$$

if u satisfies the conditions from Theorem 2.10. Let $f \in \mathcal{L}^S$ such that the limit

$$A_\Gamma^+ f = \lim_{h \downarrow 0} \frac{S(h)f - f}{h} \in C_\kappa$$

exists uniformly on compacts. Define $u_0(t) := S(t)f$ for all $t \in [0, T]$. It follows from the proof of Corollary 2.11 with $T := S$ and Theorem 2.10 that u_0 is the smallest Γ -supersolution of equation (2.10). In this way, for a large class of fully nonlinear equations, we obtain a solution concept which is directly related to the idea of semigroups and their generators.

3. APPROXIMATION OF THE Γ -GENERATOR

Throughout this section, let $S := (S(t))_{t \geq 0}$ be a family of operators $S(t): C_\kappa \rightarrow U_\kappa$ satisfying Assumption 2.4. A priori it is not clear how the upper Γ -generator can be computed and whether the limes superior in the definition of A_Γ^+ can be replaced by a limit. In many examples, the norm generator can be computed explicitly for smooth functions as a differential operator. Furthermore, we want to recall that linear differential operators are typically closed and uniquely determined by the evaluation at smooth functions. Here, we do not claim that A_Γ is closed, i.e., that the graph of A_Γ is a closed subset of $C_\kappa \times U_\kappa$. However, under additional assumptions, we obtain approximation results for the Γ -generator.

Definition 3.1. The Γ -generator is defined by

$$A_\Gamma: D(A_\Gamma) \rightarrow U_\kappa, \quad f \mapsto \Gamma\text{-}\lim_{h \downarrow 0} \frac{S(h)f - f}{h},$$

where the domain $D(A_\Gamma)$ consists of all $f \in \mathcal{L}_+^S$ such that the previous limit exists.

In the sequel, we denote by S the family of extended operators $S(t): U_\kappa \rightarrow U_\kappa$ satisfying condition (S1)-(S4) with U_κ instead of C_κ , see Remark 2.5. For every $t \geq 0$, $f \in U_\kappa$ and $x \in X$, we define the pointwise integral

$$\left(\int_0^t S(s)f \, ds \right) (x) := \int_0^t (S(s)f)(x) \, ds.$$

3.1. General approximation results. The goal of this subsection is to prove an approximation result of the following form: for every $f \in \mathcal{L}_+^S$ and $(f_n)_{n \in \mathbb{N}} \subset D(A_\Gamma)$, which is bounded above such that $(A_\Gamma f_n)_{n \in \mathbb{N}}$ is also bounded above and $f_n \rightarrow f$ uniformly on compacts, it holds $A_\Gamma^+ f = \Gamma\text{-}\limsup_{n \rightarrow \infty} A_\Gamma f_n$. While the inequality $A_\Gamma^+ f \leq \Gamma\text{-}\limsup_{n \rightarrow \infty} A_\Gamma f_n$ is always satisfied, the reverse inequality only holds under an additional assumption on S and for special choices of the sequence $(f_n)_{n \in \mathbb{N}}$. This is, for example, the case if S is translation invariant and $(f_n)_{n \in \mathbb{N}}$ is generated by convolution, see Subsection 3.2. We need the following auxiliary estimate.

Lemma 3.2. *For every $r, t_0 \geq 0$ and $\varepsilon > 0$, there exists $\lambda_0 \in (0, 1]$ such that*

$$S(t)f - f \leq \lambda \int_0^t S(s) \left(\frac{1}{\lambda} A_\Gamma^+ f + f \right) ds + \frac{\varepsilon t}{\kappa}$$

for all $t \in [0, t_0]$, $f \in B_{C_\kappa}(0, r) \cap \mathcal{L}_+^S$ and $\lambda \in (0, \lambda_0]$.

Proof. Fix $r, t_0 \geq 0$ and $\varepsilon > 0$. By condition (S4) we can choose $\lambda_0 \in (0, 1]$ such that

$$\sup_{s \in [0, t_0]} \sup_{f \in B_{C_\kappa}(0, r)} \lambda_0 \|S(s)f\|_\kappa \leq \varepsilon \quad \text{and} \quad \lambda_0 t_0 \leq 1. \quad (3.1)$$

In the sequel, we fix $t \in [0, t_0]$, $f \in B_{C_\kappa}(0, r) \cap \mathcal{L}_+^S$ and $\lambda \in (0, \lambda_0]$. Define $h_n := 2^{-n}t$ and $t_n^k := k2^{-n}t$ for all $k, n \in \mathbb{N}_0$. For every $n \in \mathbb{N}$, it follows from the semigroup property of S on \mathcal{L}_+^S , inequality (3.1) and Lemma B.1 that

$$\begin{aligned} S(t)f - f &= \sum_{k=1}^{2^n} (S(t_n^k)f - S(t_n^{k-1})f) = \sum_{k=1}^{2^n} (S(t_n^{k-1})S(h_n)f - S(t_n^{k-1})f) \\ &\leq \lambda h_n \sum_{k=1}^{2^n} \left(S(t_n^{k-1}) \left(\frac{S(h_n)f - f}{\lambda h_n} + f \right) - S(t_n^{k-1})f \right) \\ &\leq \lambda h_n \sum_{k=1}^{2^n} S(t_n^{k-1}) \left(\frac{S(h_n)f - f}{\lambda h_n} + f \right) + \frac{\varepsilon t}{\kappa} \\ &= \lambda \int_0^t \sum_{k=1}^{2^n} S(t_n^{k-1}) \left(\frac{S(h_n)f - f}{\lambda h_n} + f \right) \mathbf{1}_{[t_n^{k-1}, t_n^k)}(s) ds + \frac{\varepsilon t}{\kappa}. \end{aligned}$$

We use Fatou's lemma and Lemma 2.6 to conclude that

$$\begin{aligned} S(t)f - f &\leq \lambda \int_0^t \limsup_{n \rightarrow \infty} \sum_{k=1}^n S(t_n^{k-1}) \left(\frac{S(h_n)f - f}{\lambda h_n} + f \right) \mathbf{1}_{[t_n^{k-1}, t_n^k)}(s) ds + \frac{\varepsilon t}{\kappa} \\ &\leq \lambda \int_0^t S(s) \left(\frac{1}{\lambda} A_\Gamma^+ f + f \right) ds + \frac{\varepsilon t}{\kappa}. \end{aligned}$$

Note that the sequence inside the integral, to which we apply Fatou's lemma, is bounded from above, because of $f \in \mathcal{L}_+^S$ and condition (S4). \square

Theorem 3.3. *Let $(f_n)_{n \in \mathbb{N}} \subset \mathcal{L}_+^S$ be a bounded sequence and $f \in C_\kappa$ with $f_n \rightarrow f$ uniformly on compacts and $(A_\Gamma^+ f_n)_{n \in \mathbb{N}}$ bounded above. Then,*

$$f \in \mathcal{L}_+^S \quad \text{and} \quad A_\Gamma^+ f \leq \Gamma\text{-}\limsup_{n \rightarrow \infty} A_\Gamma^+ f_n.$$

Proof. Let $\varepsilon > 0$. Choose $\lambda_0 \in (0, 1]$ such that Lemma 3.2 holds with $r := \sup_{n \in \mathbb{N}} \|f_n\|_\kappa$ and $t_0 := 1$. For every $h \in (0, 1]$ and $\lambda \in (0, \lambda_0]$, we use Lemma 2.7, Fatou's lemma,

Lemma 2.6 and Lemma A.1(iv) to conclude that

$$\begin{aligned} \frac{S(h)f - f}{h} &= \lim_{n \rightarrow \infty} \frac{S(h)f_n - f_n}{h} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\lambda}{h} \int_0^h S(s) \left(\frac{1}{\lambda} A_\Gamma^+ f_n + f_n \right) ds + \frac{\varepsilon}{\kappa} \\ &\leq \frac{\lambda}{h} \int_0^h \limsup_{n \rightarrow \infty} S(s) \left(\frac{1}{\lambda} A_\Gamma^+ f_n + f_n \right) ds + \frac{\varepsilon}{\kappa} \\ &\leq \frac{\lambda}{h} \int_0^h S(s) \left(\frac{1}{\lambda} g + f \right) ds + \frac{\varepsilon}{\kappa}, \end{aligned}$$

where $g := \Gamma\text{-lim sup}_{n \rightarrow \infty} A_\Gamma^+ f_n$. In particular, condition (S4) implies $f \in \mathcal{L}_+^S$. Moreover, it follows from Lemma A.1(iv), Lemma 2.6 and $S(0) = \text{id}_{U_\kappa}$ that

$$A_\Gamma^+ f \leq \Gamma\text{-lim sup}_{h \downarrow 0} \frac{\lambda}{h} \int_0^h S(s) \left(\frac{1}{\lambda} g + f \right) ds + \frac{\varepsilon}{\kappa} \leq g + \lambda f + \frac{\varepsilon}{\kappa}.$$

Letting $\varepsilon \downarrow 0$ and $\lambda \downarrow 0$, we obtain $A_\Gamma^+ f \leq \Gamma\text{-lim sup}_{n \rightarrow \infty} A_\Gamma^+ f_n$. \square

Under additional assumptions on the approximating sequence $(f_n)_{n \in \mathbb{N}}$, we also obtain the reverse estimate for the upper Γ -generator and that $f \in D(A_\Gamma)$.

Theorem 3.4. *Let $(f_n)_{n \in \mathbb{N}} \subset D(A_\Gamma)$ be a bounded sequence and $f \in C_\kappa$ with $f_n \rightarrow f$ uniformly on compacts and $(A_\Gamma f_n)_{n \in \mathbb{N}}$ is bounded above. Assume that, for every $\varepsilon > 0$, there exists a sequence $(r_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $r_n \rightarrow 0$ such that*

$$\left(\frac{S(h)f_n - f_n}{h} \right) (x) \leq \sup_{y \in B(x, r_n)} \left(\frac{S(h)f - f}{h} \right) (y) + \frac{\varepsilon}{\kappa(x)} \quad (3.2)$$

for all $h \in (0, 1]$, $n \in \mathbb{N}$ and $x \in X$. Then, $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-lim}_{n \rightarrow \infty} A_\Gamma f_n$.

Proof. First, let $(h_m)_{m \in \mathbb{N}} \subset (0, 1]$ be a sequence with $h_m \rightarrow 0$. By Theorem 3.3 and Lemma A.1(i), there exists a subsequence, which is still denoted by $(h_m)_{m \in \mathbb{N}}$, such that

$$g_1 := \Gamma\text{-lim}_{m \rightarrow \infty} \frac{S(h_m)f - f}{h_m} \in U_\kappa \quad \text{exists.} \quad (3.3)$$

Moreover, since $(A_\Gamma f_n)_{n \in \mathbb{N}}$ is bounded above and due to Lemma A.1(i), every subsequence $(f_{n_k})_{k \in \mathbb{N}}$ has a further subsequence $(f_{n_{k_l}})_{l \in \mathbb{N}}$ such that

$$g_2 := \Gamma\text{-lim}_{l \rightarrow \infty} A_\Gamma f_{n_{k_l}} \in U_\kappa \quad \text{exists.} \quad (3.4)$$

To simplify the notation, we subsequently write $f_l := f_{n_{k_l}}$. Theorem 3.3 implies

$$g_1 \leq A_\Gamma^+ f \leq \Gamma\text{-lim}_{l \rightarrow \infty} A_\Gamma f_l = g_2.$$

Second, we show that $g_1 \geq g_2$. To do so, let $x \in X$ and $\varepsilon > 0$. By definition of the Γ -limit, we can choose a sequence $(x_l)_{l \in \mathbb{N}} \subset X$ with $x_l \rightarrow x$ such that

$$\left(\Gamma\text{-lim}_{l \rightarrow \infty} A_\Gamma f_l \right) (x) = \lim_{l \rightarrow \infty} A_\Gamma f_l(x_l).$$

In addition, there exist sequences $(m_l)_{l \in \mathbb{N}}$ and $(y_l)_{l \in \mathbb{N}}$ with $d(x_l, y_l) \rightarrow 0$ such that

$$A_\Gamma f_l(x_l) \leq \left(\frac{S(h_{m_l})f_l - f_l}{h_{m_l}} \right) (y_l) + \varepsilon \quad \text{for all } l \in \mathbb{N}.$$

Let $(r_l)_{l \in \mathbb{N}} \subset (0, \infty)$ be a sequence satisfying condition (3.2). Then, for every $l \in \mathbb{N}$,

$$\begin{aligned} A_\Gamma f_l(x_l) &\leq \sup_{z \in B(y_l, r_l)} \left(\frac{S(h_{m_l})f - f}{h_{m_l}} \right) (z) + \frac{\varepsilon}{\kappa(y_l)} + \varepsilon \\ &\leq \sup_{z \in B(x, \delta_l)} \left(\frac{S(h_{m_l})f - f}{h_{m_l}} \right) (z) + \sup_{l \in \mathbb{N}} \frac{\varepsilon}{\kappa(y_l)} + \varepsilon, \end{aligned}$$

where $\delta_l := d(x, y_l) + r_l \rightarrow 0$. Since $\kappa > 0$ is continuous, we have $\inf_{l \in \mathbb{N}} \kappa(y_l) > 0$. It follows from Lemma A.1(vii) that

$$g_2(x) = \left(\Gamma\text{-}\lim_{l \rightarrow \infty} A_\Gamma f_l \right) (x) = \lim_{l \rightarrow \infty} A_\Gamma f_l(x_l) \leq \left(\Gamma\text{-}\lim_{l \rightarrow \infty} \frac{S(h_{m_l})f - f}{h_{m_l}} \right) (x) = g_1(x).$$

Third, we show that $f \in D(A_\Gamma)$ with $A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A_\Gamma f_n$. From the first part, we know that every sequence $(h_m)_{m \in \mathbb{N}} \subset (0, \infty)$ with $h_m \rightarrow 0$ has a subsequence which satisfies equation (3.3). A priori the choice of the subsequence and the limit g_1 depend on the choice of the sequence $(h_m)_{m \in \mathbb{N}}$. However, we have $g_1 = g_2$ and the function g_2 is independent of $(h_m)_{m \in \mathbb{N}}$. Hence, Lemma A.1(ii) implies

$$g_1 = \Gamma\text{-}\lim_{h \downarrow 0} \frac{S(h)f - f}{h},$$

i.e., $f \in D(A_\Gamma)$ with $A_\Gamma f = g_1$. Since the limit in equation (3.4) is also independent of the choice of of subsequence, we obtain $A_\Gamma f = \lim_{n \rightarrow \infty} A_\Gamma f_n$. \square

3.2. Convolution. In this subsection, we study two particular convolution schemes to generate approximating sequences $(f_n)_{n \in \mathbb{N}}$ which satisfy condition (3.2). The first one, convolution with probability measures, works particularly well if $X = \mathbb{R}^d$ and the measure has a smooth density with respect to the Lebesgue measure. In this case, $(f_n)_{n \in \mathbb{N}}$ is a sequence of smooth functions and the generator can be a differential operator of arbitrary order. However, the condition $f_n \in D(A_\Gamma)$ is no longer verifiable if X is infinite-dimensional. The second one, sup-inf-convolution, is restricted to first-order equations but can be applied in separable Hilbert spaces.

3.2.1. Convolution with mollifiers. Let X be a separable Banach space. Suppose that there exists $\delta_0 > 0$ with

$$c := \sup_{x \in X} \sup_{y \in B(x, \delta_0)} \frac{\kappa(x)}{\kappa(y)} < \infty. \quad (3.5)$$

To simplify the notation, we assume, w.l.o.g., that $\delta_0 := 1$. In the sequel, we fix a sequence $(\mu_n)_{n \in \mathbb{N}}$ of probability measures on the Borel σ -algebra $\mathcal{B}(X)$ of X which concentrate in the sense that

$$\mu_n(B(0, 1/n)^c) = 0 \quad \text{for all } n \in \mathbb{N}. \quad (3.6)$$

For every $n \in \mathbb{N}$, $f \in U_\kappa$ and $x \in X$, we define the convolution by

$$(f * \mu_n)(x) := \int_X f(x - y) \mu_n(dy) \in [-\infty, \infty).$$

Condition (3.5) ensures that the previous integral is well-defined.

Lemma 3.5. *For every $n \in \mathbb{N}$, the mapping $U_\kappa \rightarrow U_\kappa$, $f \mapsto f * \mu_n$ is well defined and upper semicontinuous, i.e., for every sequence $(f_m)_{m \in \mathbb{N}} \subset U_\kappa$, which is bounded above,*

$$\Gamma\text{-}\limsup_{m \rightarrow \infty} (f_m * \mu_n) \leq \left(\Gamma\text{-}\limsup_{m \rightarrow \infty} f_m \right) * \mu_n.$$

*Furthermore, it holds $\Gamma\text{-}\limsup_{n \rightarrow \infty} (f * \mu_n) \leq f$ for all $f \in U_\kappa$.*

Proof. First, we show that $U_\kappa \rightarrow U_\kappa$, $f \mapsto f * \mu_n$ is well defined. Let $n \in \mathbb{N}$, $f \in U_\kappa$ and $x \in X$. It follows from condition (3.5) and condition (3.6) that

$$(f * \mu_n)(x)\kappa(x) = \int_{B(0,1)} f(x-y)\kappa(x-y) \frac{\kappa(x)}{\kappa(x-y)} \mu_n(dy) \leq c\|f^+\|_\kappa.$$

Moreover, let $(x_m)_{m \in \mathbb{N}} \subset X$ be a sequence with $x_m \rightarrow x$. Fatou's lemma implies

$$\limsup_{m \rightarrow \infty} (f * \mu_n)(x_m) \leq \int_X \limsup_{m \rightarrow \infty} f(x_m - y) \mu_n(dy) \leq \int_X f(x - y) \mu_n(dy)$$

Second, we show that, for fixed $n \in \mathbb{N}$, the convolution is upper semicontinuous. Let $(f_m)_{m \in \mathbb{N}} \subset U_\kappa$ be bounded above, $x \in X$ and $(x_m)_{m \in \mathbb{N}} \subset X$ with $x_m \rightarrow x$. Since $\sup_{m \in \mathbb{N}} \|f_m^+\|_\kappa < \infty$ and $\nu(A) := \int_A \frac{1}{\kappa} d\mu_n$ defines a finite Borel measure, we can apply Fatou's lemma to conclude that

$$\begin{aligned} \limsup_{m \rightarrow \infty} (f * \mu_n)(x_m) &= \limsup_{m \rightarrow \infty} \int_X f_m(x_m - y) \mu_n(dy) \\ &\leq \int_X \limsup_{m \rightarrow \infty} f_m(x_m - y) \mu_n(dy) \leq \int_X \left(\Gamma\text{-}\limsup_{m \rightarrow \infty} f_m \right)(x - y) \mu_n(dy). \end{aligned}$$

Third, we show that $\Gamma\text{-}\limsup_{n \rightarrow \infty} (f * \mu_n) \leq f$ for all $f \in U_\kappa$. Let $x \in X$ and $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \rightarrow x$. Since f is upper semicontinuous, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $f(x_n - y) \leq f(x) + \varepsilon$ for all $n \geq n_0$ and $y \in B(0, 1/n)$. Hence,

$$(f * \mu_n)(x_n) = \int_{B(0, 1/n)} f(x_n - y) \mu_n(dy) \leq f(x) + \varepsilon \quad \text{for all } n \geq n_0.$$

Letting $\varepsilon \downarrow 0$ yields $\limsup_{n \rightarrow \infty} (f * \mu_n)(x_n) \leq f(x)$. \square

For every $x \in X$, we define the shift operator $\tau_x : U_\kappa \rightarrow U_\kappa$ by

$$(\tau_x f)(y) := f(x + y) \quad \text{for all } y \in X.$$

Condition (3.7) in the following lemma is clearly satisfied if S is translation invariant, i.e., $\tau_x S(t)f = S(t)(\tau_x f)$. Moreover, in many examples, the condition holds at least for Lipschitz continuous functions.

Lemma 3.6. *Let $f \in \mathcal{L}_+^S$ such that $f_n := f * \mu_n \in D(A_\Gamma)$ for all $n \in \mathbb{N}$. Assume that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|\tau_x S(t)f - S(t)(\tau_x f)\|_\kappa \leq \varepsilon t \quad \text{for all } t \in [0, 1] \text{ and } x \in B(0, \delta). \quad (3.7)$$

Then, it holds $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A_\Gamma f_n$.

Proof. We verify the assumptions of Theorem 3.4. First, we show that the sequence $(f_n)_{n \in \mathbb{N}}$ is bounded and converges to f uniformly on compacts. It follows from condition (3.5) that $\|f_n^+\|_\kappa \leq c\|f^+\|_\kappa$ for all $n \in \mathbb{N}$. Moreover, for every compact set $K \subset X$, continuity of f implies

$$\sup_{x \in K} |f_n(x) - f(x)| \leq \sup_{x \in K} \int_{B(0, 1/n)} |f(x - y) - f(x)| \mu_n(dy) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Second, we verify condition (3.2). To do so, let $\varepsilon > 0$. By condition (3.7), there exists $n_0 \in \mathbb{N}$ such that

$$S(t)(\tau_{-y} f) \leq \tau_{-y} S(t)f + \frac{\varepsilon t}{\kappa} \quad \text{for all } t \in [0, 1] \text{ and } y \in B(0, \frac{1}{n_0}).$$

For every $h \in [0, 1]$, $n \geq n_0$ and $x \in X$, we use Jensen's inequality and the monotonicity of $S(h)$ to estimate

$$\begin{aligned} (S(h)f_n)(x) &= \left(S(h) \left(\int_{B(0,1/n)} (\tau_{-y}f)(\cdot) \mu_n(dy) \right) \right) (x) \\ &\leq \int_{B(0,1/n)} (S(h)(\tau_{-y}f))(x) \mu_n(dy) \leq \int_{B(0,1/n)} \left((\tau_{-y}S(h)f)(x) + \frac{\varepsilon h}{\kappa(x)} \right) \mu_n(dy) \\ &= \left(S(h)f_n + \frac{\varepsilon h}{\kappa} \right) (x). \end{aligned}$$

It follows from the linearity of the convolution and condition (3.6) that

$$\frac{S(h)f_n - f_n}{h} \leq \left(\frac{S(h)f - f}{h} \right) * \mu_n + \frac{\varepsilon}{\kappa} \leq \sup_{y \in B(\cdot, 1/n)} \left(\frac{S(h)f - f}{h} \right) (y) + \frac{\varepsilon}{\kappa}. \quad (3.8)$$

Third, we show that the sequence $(A_\Gamma f_n)_{n \in \mathbb{N}}$ is bounded above. By inequality (3.8) with $\varepsilon := 1$, Jensen's inequality and condition (3.6), there exists $n_0 \in \mathbb{N}$ with

$$\|(S(h)f_n - f_n)^+\|_\kappa \leq \|(S(h)f - f)^+ * \mu_n\|_\kappa + h \leq c\|(S(h)f - f)^+\|_\kappa + h.$$

for all $h \in [0, 1]$ and $n \geq n_0$. Since $f \in \mathcal{L}_+^S$, we can choose $h_0 > 0$ and $c' \geq 0$ such that

$$\sup_{n \in \mathbb{N}} \|(S(h)f_n - f_n)^+\|_\kappa \leq c\|(S(h)f - f)^+\|_\kappa + h \leq (cc' + 1)h \quad \text{for all } h \in (0, h_0].$$

This shows that $(A_\Gamma^+ f_n)_{n \in \mathbb{N}}$ is bounded above. Now, Theorem 3.4 yields the claim. \square

To discuss the assumption $f * \mu_n \in D(A_\Gamma)$, we consider only the finite-dimensional case and convolution with mollifiers. Denote by Lip_b the space of all bounded Lipschitz continuous functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ and by C_b^∞ the space of all bounded infinitely differentiable functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ such that all derivatives are bounded. Moreover, let C_c^∞ be the set of all infinitely differentiable functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ with compact support.

Remark 3.7. Let $X := \mathbb{R}^d$. Moreover, let $\eta \in C_c^\infty$ with $\eta \geq 0$, $\text{supp}(\eta) \subset B(0, 1)$ and $\int_{\mathbb{R}^d} \eta(x) dx = 1$. Define $\eta_n(x) := n^d \eta(nx)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. For the measure $\mu_n(A) := \int_A \eta_n(y) dy$, the convolution $f * \mu_n$ is given by

$$(f * \eta_n)(x) = \int_{\mathbb{R}^d} f(x - y) \eta_n(y) dy.$$

We have $f * \eta_n \in C_b^\infty$ for all $f \in \text{Lip}_b$ and $n \in \mathbb{N}$. Assume that $S(t): \text{Lip}_b \rightarrow \text{Lip}_b$ for all $t \geq 0$, $C_b^\infty \subset D(A_\Gamma)$ and that condition (3.7) is satisfied for all $f \in \text{Lip}_b$. Lemma 3.6 yields $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A_\Gamma(f * \eta_n)$ for all $f \in \mathcal{L}_+^S \cap \text{Lip}_b$. In particular, the Γ -generator is uniquely determined by the evaluation at smooth functions. Hence, the uniqueness result, Corollary 2.11, can be improved accordingly, i.e., equality of the generator on C_b^∞ ensures equality of the semigroups on Lip_b . Furthermore, Lemma 2.7 implies equality on C_κ .

3.2.2. Regularization with sup-inf-convolution. Let X be a separable Hilbert space with norm $|\cdot|$. We fix $\kappa \equiv 1$ and denote by BUC the space of all bounded uniformly continuous functions $f: X \rightarrow \mathbb{R}$. For every $n \in \mathbb{N}$ and $f \in \text{BUC}$, we define the sup-inf-convolution

$$(\theta_n f)(x) := \sup_{y \in X} \inf_{z \in X} \left(f(z) + \frac{n}{2} |y - z|^2 - n |y - x|^2 \right). \quad (3.9)$$

It is shown in [31] that $\theta_n f \in \text{Lip}_b^1$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} \|\theta_n f - f\|_\infty = 0$ for all $f \in \text{BUC}$. Here, Lip_b^1 denotes the space of all differentiable functions $f \in \text{Lip}_b$ such that the first derivative is again bounded and Lipschitz continuous.

Lemma 3.8. *For every $r > 0$, $f, g \in B_{\text{BUC}}(0, r)$, $n \in \mathbb{N}$ and $x \in X$,*

$$(\theta_n f - \theta_n g)(x) \leq \sup_{y \in B(x, r_n)} (f - g)(y), \quad \text{where } r_n := (\sqrt{2} + 2) \sqrt{\frac{r}{n}}.$$

Proof. In equation (3.9), the supremum can be taken over the ball $B(x, \sqrt{2r/n})$ and the infimum over the ball $B(y, \sqrt{4r/n})$. Hence, we can estimate

$$\begin{aligned} (\theta_n f - \theta_n g)(x) &\leq \sup_{y \in B(x, \sqrt{2r/n})} \sup_{z \in B(y, \sqrt{4r/n})} (F(y, z) - G(y, z)) \\ &\leq \sup_{y \in B(x, \sqrt{2r/n})} \sup_{z \in B(y, \sqrt{4r/n})} (f(z) - g(z)) = \sup_{z \in B(x, r_n)} (f(z) - g(z)), \end{aligned}$$

where $F(y, z) := f(z) + \frac{n}{2}|y-z|^2 - n|y-x|^2$ and $G(y, z) := g(z) + \frac{n}{2}|y-z|^2 - n|y-x|^2$. \square

As an application of Theorem 3.4, we obtain the following result. Condition (i) is typically satisfied for first order equations, where we have $\text{Lip}_b^1 \subset D(A_\Gamma)$.

Lemma 3.9. *Assume that $S(t): \text{BUC} \rightarrow \text{BUC}$ for all $t \geq 0$. Let $f \in \mathcal{L}_+^S \cap \text{BUC}$ with*

- (i) $\theta_n f \in D(A_\Gamma)$ for all $n \in \mathbb{N}$,
- (ii) $S(t)(\theta_n f) \leq \theta_n S(t)f$ for all $n \in \mathbb{N}$ and $t \geq 0$.

Then, it holds $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A_\Gamma(\theta_n f)$.

Proof. We verify the assumptions of Theorem 3.4. Define $f_n := \theta_n f$ for all $n \in \mathbb{N}$. By definition and [31], it holds $\sup_{n \in \mathbb{N}} \|f_n\|_\infty \leq \|f\|_\infty$ and $\|\theta_n f - f\|_\infty \rightarrow 0$. Moreover, we use condition (ii), condition (S4) and Lemma 3.8 to choose a sequence $(r_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $r_n \rightarrow 0$ such that

$$S(h)f_n - f_n \leq \theta_n S(h)f - f_n \leq \sup_{y \in B(\cdot, r_n)} (S(h)f - f)(y) \quad \text{for all } h \in [0, 1].$$

Hence, condition (3.2) is satisfied. Furthermore, since $f \in \mathcal{L}_+^S$, there exist $h_0 \in (0, 1]$ and $c \geq 0$ with

$$\frac{S(h)f_n - f_n}{h} \leq \sup_{y \in B(\cdot, r_n)} \frac{S(h)f - f}{h} \leq ch \quad \text{for all } h \in (0, h_0].$$

This shows that $(A_\Gamma^+ f_n)_{n \in \mathbb{N}}$ is bounded above. Theorem 3.4 yields the claim. \square

3.3. Connection to distributional derivative. Let $X := \mathbb{R}^d$. Moreover, let $\eta \in C_c^\infty$ with $\eta \geq 0$, $\text{supp}(\eta) \subset B(0, 1)$ and $\int_{\mathbb{R}^d} \eta(x) dx = 1$. Denote by $f * \eta_n$ the convolution, where $\eta_n(x) := n^d \eta(nx)$. Let $I \subset \mathbb{N}_0^d$ be an index set and $H: \mathbb{R}^I \rightarrow \mathbb{R}$ be a convex function. Our goal is to identify $A_\Gamma f$ with $g := H((D^\alpha f)_{\alpha \in I})$ if the partial derivatives $D^\alpha f$ exist as regular distributions for all $\alpha \in I$ and g is locally integrable. Since $A_\Gamma f$ is, in contrast to g , upper semicontinuous, we want to replace g by its upper semicontinuous hull \bar{g} . To do so, we have to choose a suitable representative of g .

Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally integrable function. We define X_f as the set of all $x \in \mathbb{R}^d$ such that the limit

$$\tilde{f}(x) := \lim_{r \downarrow 0} \int_{B(x, r)} f(y) dy$$

exists.⁴ We remark that the Lebesgue set of a function f , consisting of all $x \in \mathbb{R}^d$ with $\tilde{f}(x) = f(x)$, depends on the choice of the representative, while the set X_f does not.

⁴Denoting by λ the Lebesgue measure, the normalized integral is given by

$$\int_{B(x, r)} f(y) dy := \frac{1}{\lambda(B(x, r))} \int_{B(x, r)} f(y) dy.$$

Furthermore, by the Lebesgue differentiation theorem, the complement of the Lebesgue set has measure zero, see [40, Corollary 3.1.6]. This implies $\lambda(X_f^c) = 0$ and therefore X_f is dense in \mathbb{R}^d .

Theorem 3.10. *Let $I \subset \mathbb{N}_0^d$ be an index set and $H: \mathbb{R}^I \rightarrow \mathbb{R}$ be a convex function. Let $f \in \mathcal{L}_+^S$ satisfy condition (3.7). For every $n \in \mathbb{N}$, we define $f_n := f * \eta_n$ and assume that*

- (i) $D^\alpha f$ and $D^\alpha f_n$ exist as regular distributions for all $\alpha \in I$,
- (ii) $H((D^\alpha f)_{\alpha \in I})$ and $H((D^\alpha f_n)_{\alpha \in I})$ are locally integrable,
- (iii) $f_n \in D(A_\Gamma)$ and $A_\Gamma f_n = H((D^\alpha f_n)_{\alpha \in I})$.

Furthermore, we define the functions

$$\begin{aligned} g(x) &:= H((D^\alpha f(x))_{\alpha \in I}) \quad \text{for all } x \in \mathbb{R}^d, \\ \tilde{g}(x) &:= \lim_{r \downarrow 0} \int_{B(x,r)} g(y) \, dy \quad \text{for all } x \in X_g, \\ \bar{g}(x) &:= \limsup_{y \in X_g, y \rightarrow x} \tilde{g}(y) \quad \text{for all } x \in \mathbb{R}^d. \end{aligned}$$

Then, it holds $f \in D(A_\Gamma)$ and $(A_\Gamma f)(x) = \bar{g}(x)$ for all $x \in \mathbb{R}^d$. In particular, we have $\bar{g}(x) < \infty$ for all $x \in \mathbb{R}^d$.

Proof. It follows from Lemma 3.6 that $f \in D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A_\Gamma f_n$. First, we show $\bar{g} \leq A_\Gamma f$. Define $F := (D^\alpha f)_{\alpha \in I}$ and $F_n := (D^\alpha f_n)_{\alpha \in I}$ for all $n \in \mathbb{N}$. It holds $F_n = F * \eta_n$ for all $n \in \mathbb{N}$, where the convolution of the vector valued-function F with η_n is understood componentwise. Hence, we can use condition (i) and [40, Theorem 3.2.1] to obtain $F_n \rightarrow F$ almost everywhere. Continuity of H and condition (iii) imply

$$g = H(F) = \lim_{n \rightarrow \infty} H(F_n) = \lim_{n \rightarrow \infty} A_\Gamma f_n \leq \Gamma\text{-}\limsup_{n \rightarrow \infty} A_\Gamma f_n = A_\Gamma f$$

almost everywhere. This yields the estimate

$$\int_{B(x,r)} g(y) \, dy \leq \int_{B(x,r)} (A_\Gamma f)(y) \, dy \quad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0.$$

For every $x \in X_g$, it follows from the upper semicontinuity of $A_\Gamma f$ that

$$\tilde{g}(x) = \lim_{r \downarrow 0} \int_{B(x,r)} g(y) \, dy \leq \limsup_{r \downarrow 0} \int_{B(x,r)} (A_\Gamma f)(y) \, dy \leq (A_\Gamma f)(x).$$

In particular, \tilde{g} is bounded above and therefore $\bar{g}(x) < \infty$ for all $x \in \mathbb{R}^d$. We use again that $A_\Gamma f$ is upper semicontinuous in order to conclude that

$$\bar{g}(x) = \limsup_{y \in X_g, y \rightarrow x} \tilde{g}(y) \leq \limsup_{y \in X_g, y \rightarrow x} (A_\Gamma f)(y) \leq (A_\Gamma f)(x) \quad \text{for all } x \in \mathbb{R}^d.$$

Second, we show $A_\Gamma f \leq \bar{g}$. Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Because of $A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A_\Gamma f_n$, there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ with $x_n \rightarrow x$ and $(A_\Gamma f_n)(x_n) \rightarrow (A_\Gamma f)(x)$. In addition, since \bar{g} is upper semicontinuous, there exists $\delta > 0$ such that $\bar{g}(y) < \bar{g}(x) + \varepsilon$ for all $y \in B(x, \delta)$. Choose $n_0 \in \mathbb{N}$ with $B(x_n, 1/n) \subset B(x, \delta)$ for all $n \geq n_0$. Since g is

locally integrable, it holds $H(F) = g = \tilde{g} \leq \bar{g}$ almost everywhere. It follows from the previous considerations and Jensen's inequality that

$$\begin{aligned} (A_\Gamma f_n)(x_n) &= H(F_n(x_n)) = H\left(\int_{B(0,1/n)} F(x_n - y)\eta_n(y) \, dy\right) \\ &\leq \int_{B(0,1/n)} H(F(x_n - y))\eta_n(y) \, dy \\ &\leq \int_{B(0,1/n)} \bar{g}(x_n - y)\eta_n(y) \, dy \leq \bar{g}(x) + \varepsilon. \end{aligned}$$

We obtain $(A_\Gamma f)(x) = \lim_{n \rightarrow \infty} A_\Gamma f_n(x_n) \leq \bar{g}(x)$ for all $x \in \mathbb{R}^d$. \square

In several examples, the set $\mathcal{L}_{\text{sym}}^S \cap \text{Lip}_b$ is invariant under S and has an explicit representation by means of Sobolev spaces. In addition, it holds $C_b^\infty \subset D(A_\Gamma)$ and $A_\Gamma f = H((D^\alpha f)_{\alpha \in I})$ for all $f \in C_b^\infty$. Hence, for every $f \in \mathcal{L}_{\text{sym}}^S \cap \text{Lip}_b$, the previous theorem yields that $u(t) := S(t)f$ solves the equation

$$\Gamma\text{-}\lim_{h \downarrow 0} \frac{u(t+h) - u(t)}{h} = H((D^\alpha u(t))_{\alpha \in I}) \quad \text{for all } t \geq 0.$$

These ideas can be carried out by using quite elementary calculations, see Subsection 5.1. Nonetheless, we want to mention a general result that locality of the generator implies the existence of a function H with $Af = H((D^\alpha f)_{\alpha \in I})$ for sufficiently smooth f .

Remark 3.11. Fix $\kappa := 1$ and assume that S restricted to BUC is a strongly continuous semigroup. Furthermore, let $H: \mathbb{R}^I \rightarrow \mathbb{R}$ be a convex function satisfying

$$Af = H((D^\alpha)_{\alpha \in I}) \quad \text{for all } f \in \text{BUC}^\infty, \quad (3.10)$$

where $I := \{\alpha \in \mathbb{N}_0^d: |\alpha| \leq 2\}$ and BUC^∞ denotes the space of all infinitely differentiable functions $f \in \text{BUC}$ such that all partial derivatives are again in BUC . Then, A is local in sense that, for fixed $x \in \mathbb{R}^d$, it holds $Af(x) = Ag(x)$ if $f, g \in \text{BUC}^\infty$ coincide on an open neighbourhood of x . On the other hand, it was shown in [6] that locality and some (technical) regularity of the semigroup already imply the existence of a convex function H satisfying equation (3.10).

To formulate the conditions from [6], we denote by C^∞ the space of all infinitely differentiable functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$. For every sequence $r := (r_n)_{n \in \mathbb{N}} \subset [0, \infty)$, let Q_r be the set of all $f \in C_c^\infty$ such that $\|D^\alpha f\|_\infty \leq r_n$ for all $n \in \mathbb{N}$ and $|\alpha| \leq n$. We call S *regular* if the following holds: Let $(r_n)_{n \in \mathbb{N}} \subset [0, \infty)$, $f \in \text{BUC} \cap C^\infty$ and $K \subset \mathbb{R}^d$ compact. Then, for every $T \geq 0$ and $\varepsilon > 0$, there exists $\delta_0 > 0$ such that, for all $t \in [0, T]$, $\delta \in (0, \delta_0]$, $g \in Q_r$ and $x \in K$

$$|(S(t)(f + \delta g))(x) - (S(t))(x) - \delta g(x)| \leq \varepsilon t.$$

Moreover, we call S *local* if the following holds: Let $x \in \mathbb{R}^d$ and $f, g \in \text{BUC} \cap C^\infty$ which coincide on an open neighbourhood of x . Then, for all $\varepsilon > 0$, there exists $h_0 > 0$ with

$$|S(h)f - S(h)g|(x) < \varepsilon h \quad \text{for all } h \in [0, h_0].$$

In particular, this condition implies that $Af(x) = Ag(x)$ if $f, g \in D(A)$.

In the sequel, let S be regular and local. Then, by [6, Theorem 3.1], there exists a continuous function $H: \mathbb{R}^d \times \mathbb{R}^I \rightarrow \mathbb{R}$ such that

$$Af(x) = H(x, (D^\alpha f(x))_{\alpha \in I}) \quad \text{for all } f \in \text{BUC}^\infty \text{ and } x \in \mathbb{R}^d.$$

It follows from the proof of [6, Theorem 3.1] that convexity of S implies convexity of the mapping $H(x, \cdot): \mathbb{R}^I \rightarrow \mathbb{R}$ for all $x \in \mathbb{R}^d$. Furthermore, if S is translation invariant, i.e., $S(t)(\tau_x f) = \tau_x S(t)f$, then H does not depend on $x \in \mathbb{R}^d$, see [6, Proposition 4.1].

4. GENERATING FAMILIES

In many examples, S can be approximated by iterating another family $I := (I(t))_{t \geq 0}$ of operators which do not form a semigroup but have (in contrast to S) an explicit representation. This leads to approximation schemes of the form

$$S(t)f = \lim_{l \rightarrow \infty} (I(2^{-nl}t))^{2^{nl}} f. \quad (4.1)$$

Furthermore, they can be used in order to construct nonlinear semigroups, see [8]. In this case, we call I a *generating family* and S an *associated semigroup*. In view of the previous results, we expect that two generating families with the same infinitesimal behaviour lead to the same associated semigroup.

Let I be a family of operators $I(t): C_\kappa \rightarrow C_\kappa$. For every $t \geq 0$ and $n \in \mathbb{N}$, we define the partition $\pi_n^t := \{k2^{-n} \wedge t: k \in \mathbb{N}_0\}$ and the iterated operator

$$I(\pi_n^t) := I(2^{-n})^k I(t - k2^{-n}), \quad \text{where } k := \max\{n \in \mathbb{N}_0: k2^{-n} \leq t\}.$$

The Lipschitz set \mathcal{L}^I consists of all $f \in C_\kappa$ for which there exist $h_0 > 0$ and $c \geq 0$ such that $\|I(h)f - f\|_\kappa \leq ch$ for all $h \in [0, h_0]$. Define $\mathbb{R}_+ := \{x \in \mathbb{R}: x \geq 0\}$ and denote by $\mathcal{T} := \{k2^{-n}: k, n \in \mathbb{N}_0\}$ the set of all positive dyadic numbers.

Assumption 4.1. Suppose that I satisfies the following conditions:

- (i) $I(0) = \text{id}_{C_\kappa}$.
- (ii) The operator $I(t)$ is convex and monotone with $I(t)0 = 0$ for all $t \geq 0$.
- (iii) There exists a function $\alpha: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is non-decreasing in the second argument, such that, for all $r, s, t \geq 0$,

$$I(t): B_{C_\kappa}(0, r) \rightarrow B_{C_\kappa}(\alpha(r, t)) \quad \text{and} \quad \alpha(\alpha(r, s), t) \leq \alpha(r, s + t).$$

- (iv) For every $r \geq 0$, there exists $\omega_r \geq 0$ such that

$$\|I(t)f - I(t)g\|_\kappa \leq e^{t\omega_r} \|f - g\|_\kappa \quad \text{for all } t \in [0, 1] \text{ and } f, g \in B_{C_\kappa}(0, r).$$

W.l.o.g., we assume that the mapping $r \mapsto \omega_r$ is non-decreasing.

- (v) There exists a countable set $\mathcal{D} \subset \mathcal{L}^I$ such that the sequence $(I(\pi_n^t)f)_{n \in \mathbb{N}}$ is uniformly equicontinuous for all $(f, t) \in \mathcal{D} \times \mathcal{T}$. Moreover, for every $f \in C_\kappa$, there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ with $\|f_n\|_\kappa \leq \|f\|_\kappa$ for all $n \in \mathbb{N}$ and $f_n \rightarrow f$ uniformly on compacts.
- (vi) For every $t \geq 0$, $x \in X$ and $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ with $f_n \downarrow 0$,

$$\sup_{s \in [0, t]} \sup_{k \in \mathbb{N}} (I(\pi_k^s)f_n)(x) \downarrow 0 \quad \text{as } n \rightarrow \infty.$$

For every $(f, t) \in \mathcal{D} \times \mathcal{T}$, by condition (iii) and (v) and Lemma D.1, the sequence $(I(\pi_n^t)f)_{n \in \mathbb{N}}$ has a subsequence which converges uniformly on compacts. Hence, up to a subsequence, we can define S by equation (4.1). If I is translation-invariant and $\kappa \equiv 1$, the verification of condition (v) is particularly simple for Lipschitz continuous functions. While many properties of S follow from the conditions (i)-(v) or even weaker conditions, the semigroup property is rather delicate and requires an additional assumption. However, condition (vi), which implies continuity from above, also guarantees the semigroup property. Another possibility to ensure the semigroup property is to assume norm convergence or monotone convergence, see [8, 23, 34].

Remark 4.2. We mention two sufficient conditions for Assumption 4.1(vi).

- (i) Let $\tilde{\kappa}: X \rightarrow (0, \infty)$ be another bounded continuous function such that, for every $\varepsilon > 0$, there exists a compact set $K \subset X$ with $\sup_{x \in K^c} \frac{\tilde{\kappa}(x)}{\kappa(x)} \leq \varepsilon$. Furthermore, we assume that there exists a function $\tilde{\alpha}: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is non-decreasing in the second argument, such that, for all $r, s, t \geq 0$ and $f \in C_\kappa$ with $\|f\|_{\tilde{\kappa}} \leq r$,

$$\|I(t)f\|_{\tilde{\kappa}} \leq \tilde{\alpha}(r, t) \quad \text{and} \quad \tilde{\alpha}(\tilde{\alpha}(r, s), t) \leq \tilde{\alpha}(r, s+t). \quad (4.2)$$

Let $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ be a sequence with $f_n \downarrow 0$. Dini's theorem implies uniform convergence on compact sets and, therefore, $\|f_n\|_{\tilde{\kappa}} \rightarrow 0$. It follows from Assumption 4.1(ii), equation (4.2), [8, Lemma 2.7] and Lemma B.2(ii) that

$$\sup_{s \in [0, t]} \sup_{k \in \mathbb{N}} \|I(\pi_k^s) f_n\|_{\tilde{\kappa}} \leq \tilde{\alpha}(3\|f_1\|_{\tilde{\kappa}}, t) \|f_n\|_{\tilde{\kappa}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

- (ii) Assume that there exists a family J of operators $J(t): C_\kappa \rightarrow C_\kappa$, which are continuous from above, such that, for all $s, t \geq 0$ and $f \in C_\kappa$,

$$I(t)f \leq J(t)f \quad \text{and} \quad J(s)J(t)f \leq J(s+t)f.$$

In addition, we suppose that the mapping $[0, \infty) \rightarrow \mathbb{R}$, $f \mapsto (J(t)f)(x)$ is upper semicontinuous for all $f \in C_\kappa$ and $x \in X$. The monotonicity of I implies $I(\pi_n^t)f \leq J(t)f$ for all $t \geq 0$, $n \in \mathbb{N}$ and $f \in C_\kappa$. Let $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ with $f_n \downarrow 0$. Then, for every $t \geq 0$ and $x \in X$, Dini's theorem implies

$$\sup_{s \in [0, t]} \sup_{k \in \mathbb{N}} (I(\pi_k^s) f_n)(x) \leq \sup_{s \in [0, t]} (J(t) f_n)(x) \downarrow 0 \quad \text{as } n \rightarrow \infty.$$

The next statement is a consequence of the results in [8] and Appendix C. For the reader's convenience, we provide a proof in Appendix D.

Theorem 4.3. *Suppose that I satisfies Assumption 4.1. Then, there exist a family S of operators $S(t): C_\kappa \rightarrow C_\kappa$ and a subsequence $(n_l)_{l \in \mathbb{N}} \subset \mathbb{N}$ such that, uniformly on compacts,*

$$S(t)f = \lim_{l \rightarrow \infty} I(\pi_{n_l}^t) f \quad \text{for all } (f, t) \in C_\kappa \times \mathcal{T}. \quad (4.3)$$

Furthermore, S satisfies Assumption 2.4 and is a strongly continuous convex monotone semigroup on \mathcal{L}^S . It holds $\mathcal{L}^I \subset \mathcal{L}^S$. In addition, for every $f, g \in C_\kappa$,

$$\lim_{h \downarrow 0} \left\| \frac{I(h)f - f}{h} - g \right\|_\kappa \quad \text{implies} \quad \lim_{h \downarrow 0} \left\| \frac{S(h)f - f}{h} - g \right\|_\kappa = 0.$$

If the sequence $(I(\pi_n^t)f)_{n \in \mathbb{N}}$ is non-decreasing, it holds $S(t)f = \sup_{n \in \mathbb{N}} I(\pi_n^t)f$. In particular, the limit in equation (4.3) exists without choosing a convergent subsequence. This is, for instance, the case for Nisio semigroups, see [34, 35]. Furthermore, under mild assumptions, one can show that $S(t)f = \sup_{\pi \in \mathcal{P}_t} I(\pi)f$, where \mathcal{P}_t consists of all finite partitions of the interval $[0, t]$. For details we refer to [8, Lemma 2.15].

In order to apply the results from Subsection 3.2.1, we subsequently only consider the case $X := \mathbb{R}^d$. Furthermore, we assume that there exists $\delta_0 > 0$ with

$$\sup_{x \in \mathbb{R}^d} \sup_{y \in B(x, \delta_0)} \frac{\kappa(x)}{\kappa(y)} \leq 1. \quad (4.4)$$

To simplify the notation, we assume, w.l.o.g., that $\delta_0 := 1$. For every $r \geq 0$, let $\text{Lip}_b(r)$ be the space of all r -Lipschitz functions $f: X \rightarrow \mathbb{R}$ with $\|f\|_\infty \leq r$.

Assumption 4.4. Suppose that I satisfies the following conditions:

(i) There exist $c \geq 0$ and $\delta \in (0, 1]$ such that

$$\|I(t)(\tau_x f) - \tau_x I(t)f\|_\kappa \leq crt|x|$$

for all $t \in [0, 1]$, $x \in B_{\mathbb{R}^d}(0, \delta)$, $r \geq 0$ and $f \in \text{Lip}_b(r)$.

(ii) For every $f \in C_b^\infty$, we can define the norm limit

$$I'(0)f := \lim_{h \downarrow 0} \frac{I(h)f - f}{h} \in C_\kappa.$$

(iii) There exists a function $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is non-decreasing in the second argument, such that, for all $r, s, t \geq 0$,

$$I(t): \text{Lip}_b(r) \rightarrow \text{Lip}_b(\beta(r, t)) \quad \text{and} \quad \beta(\beta(r, s), t) \leq \beta(r, s + t).$$

If I satisfies Assumption 4.1 and Assumption 4.4(ii), Theorem 4.3 yields $C_b^\infty \subset D(A)$ and $Af = I'(0)f$ for all $f \in C_b^\infty$, where A denotes the norm generator of S . In the sequel, we fix $\eta \in C_c^\infty$ with $\eta \geq 0$, $\text{supp}(\eta) \subset B_{\mathbb{R}^d}(0, 1)$ and $\int_{\mathbb{R}^d} \eta(x) dx = 1$. For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we define $\eta_n(x) := n^d \eta(nx)$ and denote by $f * \eta_n$ the convolution, see Remark 3.7.

Theorem 4.5. *Suppose that I satisfies Assumption 4.1 and Assumption 4.4. Then, it holds $\mathcal{L}_+^S \cap \text{Lip}_b \subset D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A(f * \eta_n)$ for all $f \in \mathcal{L}_+^S \cap \text{Lip}_b$. In addition, we have $S(t): \text{Lip}_b \rightarrow \text{Lip}_b$ for all $t \geq 0$.*

Proof. We verify condition (3.7) for all $f \in \text{Lip}_b$. Fix $n \in \mathbb{N}$ and $x \in B_{\mathbb{R}^d}(0, \delta)$. By induction, we show that, for all $k \in \mathbb{N}$, $r \geq 0$ and $f \in B_{C_\kappa}(0, r) \cap \text{Lip}_b(r)$,

$$\|I(2^{-n})^k(\tau_x f) - \tau_x I(2^{-n})^k f\|_\kappa \leq c\beta(r, k2^{-n})e^{k2^{-n}\omega_{\alpha(r, k2^{-n})}}k2^{-n}|x|. \quad (4.5)$$

For $k = 1$, the claim holds by Assumption 4.4(i). For the induction step, we assume that inequality (4.5) holds for some fixed $k \in \mathbb{N}$. Let $r \geq 0$ and $f \in B_{C_\kappa}(0, r) \cap \text{Lip}_b(r)$. Assumption 4.1(iii), Assumption 4.4(iii) and condition (4.4) imply

$$I(2^{-n})f, I(2^{-n})(\tau_x f), \tau_x I(2^{-n})f \in B_{C_\kappa}(0, \alpha(r, 2^{-n})) \cap \text{Lip}_b(\beta(r, 2^{-n})).$$

We use [8, Lemma 2.7] and inequality (4.5) to conclude

$$\begin{aligned} & \|I(2^{-n})^{k+1}(\tau_x f) - \tau_x I(2^{-n})^{k+1}f\|_\kappa \\ & \leq \|I(2^{-n})^k I(2^{-n})(\tau_x f) - I(2^{-n})^k(\tau_x I(2^{-n})f)\|_\kappa \\ & \quad + \|I(2^{-n})^k(\tau_x I(2^{-n})f) - \tau_x I(2^{-n})^k I(2^{-n})f\|_\kappa \\ & \leq e^{k2^{-n}\omega_{\alpha(r, 2^{-n}), k2^{-n}}} \|I(2^{-n})(\tau_x f) - \tau_x I(2^{-n})f\|_\kappa \\ & \quad + c\beta(\beta(r, 2^{-n}), k2^{-n})e^{k2^{-n}\omega_{\alpha(r, 2^{-n}), k2^{-n}}}k2^{-n}|x| \\ & \leq ce^{k2^{-n}\omega_{\alpha(r, (k+1)2^{-n})}}\beta(r, 2^{-n})2^{-n}|x| + c\beta(r, (k+1)2^{-n})e^{k2^{-n}\omega_{\alpha(r, (k+1)2^{-n})}}k2^{-n}|x| \\ & \leq c\beta(r, (k+1)2^{-n})e^{k2^{-n}\omega_{\alpha(r, (k+1)2^{-n})}}(k+1)2^{-n}|x|. \end{aligned}$$

Equation (4.3), inequality (4.5) and equation (D.6) imply

$$\|S(t)(\tau_x f) - \tau_x S(t)f\|_\kappa \leq c\beta(r, t)e^{t\omega_{\alpha(r, t)}}t|x|$$

for all $t \geq 0$, $x \in B_{\mathbb{R}^d}(0, \delta)$, $r \geq 0$ and $f \in B_{C_\kappa}(0, r) \cap \text{Lip}_b(r)$. In particular, we obtain that condition (3.7) is satisfied for all $f \in \text{Lip}_b$. Lemma 3.6 implies $\mathcal{L}_+^S \cap \text{Lip}_b \subset D(A_\Gamma)$ and $A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A(f * \eta_n)$ for all $f \in \mathcal{L}_+^S \cap \text{Lip}_b$. The invariance of Lip_b is a result of Assumption 4.4(iii), equation (4.3) and equation (D.6). \square

If we define \mathcal{L}_+^I and $\mathcal{L}_{\text{sym}}^I$ analogously to \mathcal{L}_+^S and $\mathcal{L}_{\text{sym}}^S$, then, similar to [8, Lemma 2.8 and Lemma 2.9], one can show that $\mathcal{L}_+^I \subset \mathcal{L}_+^S$. Moreover, in many examples, it holds $I(t)f \leq S(t)f$ for all $t \geq 0$ and $f \in C_\kappa$ and thus $\mathcal{L}_+^I = \mathcal{L}_+^S$. Unfortunately, an explicit characterization of \mathcal{L}_+^I is, in general, not possible. For that purpose, the symmetric Lipschitz set is more suitable, see Subsection 5.1. In [8, Section 5], the authors provide conditions which guarantee that $\mathcal{L}_{\text{sym}}^I = \mathcal{L}_{\text{sym}}^S$ and $S(t): \mathcal{L}_{\text{sym}}^S \rightarrow \mathcal{L}_{\text{sym}}^S$ for all $t \geq 0$.

Theorem 4.6. *Let I and J be two generating families of operators on C_κ which satisfy Assumption 4.1 and Assumption 4.4. Denote by S and T , respectively, the associated semigroups which exist by Theorem 4.3.*

(i) *Let $\mathcal{C} \subset \mathcal{L}^T \cap \mathcal{L}_+^S \cap \text{Lip}_b$ such that $T(t): \mathcal{C} \rightarrow \mathcal{C}$ for all $t \geq 0$. Assume that*

$$I'(0)f \leq J'(0)f \quad \text{for all } f \in C_b^\infty.$$

Then, it holds $S(t)f \leq T(t)f$ for all $t \geq 0$ and $f \in C_\kappa$.

(ii) *Assume that $\mathcal{L}_+^J \cap \text{Lip}_b \subset \mathcal{L}_+^I$, $I'(0)f \leq J'(0)f$ for all $f \in C_b^\infty$ and*

$$J(s+t)f \leq J(s)J(t)f \quad \text{for all } s, t \geq 0 \text{ and } f \in C_\kappa.$$

Moreover, let the mapping $[0, \infty) \rightarrow \mathbb{R}$, $t \mapsto (J(t)f)(x)$ be lower semicontinuous for all $f \in C_\kappa$ and $x \in X$. Then, it holds $S(t)f \leq T(t)f$ for all $t \geq 0$ and $f \in C_\kappa$.

Proof. First, it follows from Theorem 4.3 and Assumption 4.4(ii) that

$$Af = I'(0)f \leq J'(0)f = Bf \quad \text{for all } f \in C_b^\infty,$$

where A and B denote the norm generators of S and T , respectively. Hence, for every $f \in \mathcal{C}$, Theorem 4.5 and Lemma A.1(iv) imply

$$A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A(f * \eta_n) \leq \Gamma\text{-}\lim_{n \rightarrow \infty} B(f * \eta_n) = B_\Gamma f.$$

We use Corollary 2.11 to conclude $S(t)f \leq T(t)f$ for all $t \geq 0$ and $f \in C_b^\infty$. Let $f \in C_\kappa$ be arbitrary. Condition (4.4) ensures that there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset C_b^\infty$ with $\|f_n\|_\kappa \leq \|f\|_\kappa$ for all $n \in \mathbb{N}$ and $f_n \rightarrow f$ uniformly on compacts. Lemma 2.7 implies that $S(t)f = \lim_{n \rightarrow \infty} S(t)f_n \leq \lim_{n \rightarrow \infty} T(t)f_n = T(t)f$.

Second, it follows from equation (4.3) that $J(t)f \leq \lim_{n \rightarrow \infty} J(\pi_n^t)f = T(t)f$ for all $t \in \mathcal{T}$ and $f \in C_\kappa$. For arbitrary times $t \geq 0$, the inequality $J(t)f \leq T(t)f$ follows from the required lower semicontinuity of J and condition (S3). We obtain

$$\mathcal{L}^T \cap \text{Lip}_b \subset \mathcal{L}_+^T \cap \text{Lip}_b = \mathcal{L}_+^J \cap \text{Lip}_b \subset \mathcal{L}_+^I \subset \mathcal{L}_+^S.$$

By Remark 2.9, Theorem 4.3 and Theorem 4.5, it holds

$$T(t): \mathcal{L}^T \cap \text{Lip}_b \rightarrow \mathcal{L}^T \cap \text{Lip}_b \quad \text{for all } t \geq 0.$$

Hence, the claim follows from part (i). \square

5. EXAMPLES

5.1. Control problems and upper semigroup envelopes. In this subsection, we show that value functions of stochastic optimal control problems can be approximated by a sequence of static optimization problems over increasingly finer partitions. The latter corresponds to the construction of so-called upper semigroup envelopes, cf. Nisio [35] and Nendel and Röckner [34]. For a representative class of optimal control problems, we thus show that the value function coincides with the upper envelope of a suitable family of penalized linear semigroups.

Let $(W_t)_{t \geq 0}$ be a d -dimensional standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. Denote by \mathbb{S}_+^d the

set of all symmetric positive semidefinite $d \times d$ -matrices. Throughout this subsection, we fix a measurable function $L: \mathbb{S}_+^d \times \mathbb{R}^d \rightarrow [0, \infty]$ satisfying the following assumption.

Assumption 5.1. Suppose that L satisfies the following conditions:

- (i) There exists $(a^*, b^*) \in \mathbb{S}_+^d \times \mathbb{R}^d$ with $L(a^*, b^*) = 0$.
- (ii) L grows superlinearly, i.e.,

$$\lim_{|a|+|b| \rightarrow \infty} \frac{L(a, b)}{|a| + |b|} = \infty.$$

In order to apply Itô's isometry, we endow $\mathbb{R}^{d \times d}$ with the Frobenius norm

$$|a| := \sqrt{\sum_{i,j=1}^d |a_{ij}|^2} \quad \text{for all } a \in \mathbb{R}^{d \times d}.$$

Let \mathcal{A} be the set of all predictable processes $(a, b): \Omega \times \mathbb{R}_+ \rightarrow \mathbb{S}_+^d \times \mathbb{R}^d$ with

$$\mathbb{E} \left[\int_0^t |a_s| + |b_s| ds \right] < \infty \quad \text{for all } t \geq 0.$$

For every $t \geq 0$, $f \in C_b$ and $x \in \mathbb{R}^d$, we define

$$(S(t)f)(x) := \sup_{(a,b) \in \mathcal{A}} \left(\mathbb{E} \left[f \left(x + \int_0^t \sqrt{a_s} dW_s + \int_0^t b_s ds \right) \right] - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \right),$$

where $\mathbb{E}[X]$ denotes the expectation of a random variable $X: \Omega \rightarrow \mathbb{R}$. Note that $S(t)f$ is the value function of a dynamic optimal control problem with finite time horizon $t \geq 0$. In addition, for every $t \geq 0$, $f \in C_b$ and $x \in \mathbb{R}^d$, we define the value function of the related static control problem

$$(J(t)f)(x) := \sup_{(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \left(\mathbb{E}[f(x + \sqrt{a}W_t + bt)] - L(a, b)t \right).$$

Subsequently, we use the following notations:

$$\begin{aligned} X_t^{a,b} &:= \int_0^t \sqrt{a_s} dW_s + \int_0^t b_s ds \quad \text{for all } (a, b) \in \mathcal{A} \text{ and } t \geq 0, \\ c_L &:= \sup_{(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \frac{|a| + |b|}{1 + L(a, b)}, \\ L^*(c) &:= \sup_{(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \left(c(|a| + |b|) - L(a, b) \right) \quad \text{for all } c \geq 0, \\ B_{a,b}f &:= \frac{1}{2} \text{tr}(aD^2f) + \langle \nabla f, b \rangle \quad \text{for all } f \in \text{BUC}^2 \text{ and } (a, b) \in \mathbb{S}_+^d \times \mathbb{R}^d, \end{aligned}$$

where BUC^2 denotes the space of all twice differentiable $f \in \text{BUC}$ such that the first and second derivative are in BUC . Assumption 5.1(ii) implies $c_L < \infty$ and $L^*(c) \rightarrow 0$ as $c \downarrow 0$. The proof of the following auxiliary estimates consists of straightforward but lengthy calculations, see Appendix E.

Lemma 5.2.

- (i) For every $c, t \geq 0$, $r > 0$ and $(a, b) \in \mathcal{A}$,

$$c\mathbb{P}(|X_t^{a,b}| \geq r) - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \leq \frac{c}{r} + L^*\left(\frac{c}{r}\right)t.$$

In particular, S is continuous from above.

(ii) For every $c, t \geq 0$, $\delta \in (0, 1]$, $x \in \mathbb{R}^d$, $(a, b) \in \mathcal{A}$ and $f \in \text{Lip}_b(c)$,

$$\mathbb{E} \left[\int_0^t f(x + X_s^{a,b}) ds \right] \leq (f(x) + c\delta)t + \frac{cc_L}{\delta^2} \mathbb{E} \left[\int_0^t \int_0^s 1 + L(a_u, b_u) du ds \right].$$

(iii) Let $t \geq 0$, $x \in \mathbb{R}^d$, $f \in \text{BUC}^2$ and $(a, b) \in \mathcal{A}$ with

$$(S(t)f)(x) \leq t + \mathbb{E}[f(x + X_t^{a,b})] - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right].$$

Then, for $c_f := 2(1 + \|B_{a^*, b^*} f\|_\infty) + L^*(\|D^2 f\|_\infty \vee 2\|\nabla f\|_\infty)$, it holds

$$\mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \leq c_f t.$$

The next theorem is a consequence of the results in Section 4 and a generalization of the example presented in [8, Section 6.1]. For the reader's convenience, we provide a proof in Appendix E. Denote by L^∞ the set of all bounded measurable functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ and by $W^{1,\infty}$ the corresponding first order Sobolev space. For $f \in W^{1,\infty}$ and $a \in \mathbb{S}_+^d$, we say that $\Delta_a f$ exists in L^∞ if there exists a function $g \in L^\infty$ with

$$\int_{\mathbb{R}^d} g\phi dx = - \int_{\mathbb{R}^d} \langle \sqrt{a}\nabla f, \sqrt{a}\nabla\phi \rangle dx \quad \text{for all } \phi \in C_c^\infty$$

In this case, since g is unique almost everywhere, we define $\Delta_a f := g$. Clearly, it holds $\Delta_a f = \sum_{i,j=1}^n a_{ij} \partial_{ij} f$ for all $f \in \text{BUC}^2$. Moreover, let $\nabla_b f := \langle b, \nabla f \rangle$ for all $f \in W^{1,\infty}$ and $b \in \mathbb{R}^d$. Define

$$\mathbb{S}_L := \left\{ a \in \mathbb{S}_+^d : \inf_{b \in \mathbb{R}^d} L(a, b) < \infty \right\}.$$

Theorem 5.3. *There exists a family S of operators $S(t): C_b \rightarrow C_b$, which satisfy Assumption 2.4, such that, uniformly on compacts,*

$$T(t)f = \lim_{n \rightarrow \infty} J(\pi_n^t) f = \sup_{n \in \mathbb{N}} J(\pi_n^t) f \quad \text{for all } f \in C_b \text{ and } t \geq 0.$$

Furthermore, T is a strongly continuous convex monotone semigroup on BUC . Denoting by B the norm generator of T , it holds $\text{BUC}^2 \subset D(B)$ with

$$Bf = \sup_{(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \left(\frac{1}{2} \Delta_a f + \nabla_b f - L(a, b) \right) \quad \text{for all } f \in \text{BUC}^2.$$

It holds $T(t): \mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b \rightarrow \mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b$ for all $t \geq 0$ and $\mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b$ has the explicit characterization

$$\mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b = \left\{ f \in \bigcap_{a \in \mathbb{S}_L} D(\Delta_a) \cap W^{1,\infty} : \sup_{(a,b) \in \mathbb{S}_L \times \mathbb{R}^d} (\|B_{a,b} f\|_\infty - L(a, b)) < \infty \right\}$$

Furthermore, we have $T(t)f \leq S(t)f$ for all $t \geq 0$ and $f \in C_b$.

We do not assume that the matrices $a \in \mathbb{S}_L$ are positive definite which is a common assumption for parabolic PDEs. Moreover, in the completely degenerate case $\mathbb{S}_L = \{0\}$, we have $\mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b = \text{Lip}_b$ and T is a shift semigroup corresponding to a first order PDE. If there exists a positive definite matrix $a \in \mathbb{S}_L$, then [33, Theorem 3.1.7] implies

$$\mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b \subset D(\Delta) = \bigcap_{p \geq 1} \{f \in W_{\text{loc}}^{2,p} \cap \text{BUC} : \Delta f \in L^\infty\}.$$

In particular, if the function

$$H: \mathbb{R}^{d \times d} \times \mathbb{R}^d, (x, y) \mapsto \sup_{(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \left(\frac{1}{2} \sum_{i,j=1}^d a_{ij} x_{ij} + \sum_{i=1}^d b_i y_i \right)$$

has at most polynomial growth, we can apply Theorem 3.10. Note that the operator B can still be degenerate, since positive definiteness is not required for all $a \in \mathbb{S}_L$. We also remark that if there exist $a \in \mathbb{S}_+^d$ and $\varepsilon > 0$ with

$$\sup_{\{b \in \mathbb{R}^d: |b|=\varepsilon\}} L(a, b) < \infty,$$

then one can show $\mathcal{L}_{\text{sym}}^T \subset \text{Lip}_b$. For details we refer to the proof of [8, Theorem 6.3]. However, without this additional assumption on L , the intersection with Lip_b is necessary. To our knowledge, an explicit description of $\mathcal{L}_{\text{sym}}^T$ seems to be unknown even in the linear case $B = \Delta$, where $\mathcal{L}_{\text{sym}}^T$ coincides with the Favard space which has been discussed in Remark 2.9. Denote by Lip_b^2 the space of all twice differentiable functions $f \in \text{Lip}_b$ such that the first and second derivative are in Lip_b . Moreover, we denote by A the norm generator of S .

Theorem 5.4. *The family S satisfies Assumption 2.4. Furthermore, we have $\text{Lip}_b^2 \subset D(A)$, $Af = Bf$ for all $f \in \text{Lip}_b^2$ and $\mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b \subset \mathcal{L}_+^S$. Hence,*

$$S(t)f = T(t)f \quad \text{for all } (f, t) \in C_b \times \mathbb{R}_+.$$

Proof. First, we note that $S(t)$ is convex and monotone with $S(t)0 = 0$ for all $t \geq 0$. By Lemma 5.2(i), S is continuous from above. It holds $\|S(t)f - S(t)g\|_\infty \leq \|f - g\|_\infty$ for all $t \geq 0$ and $f, g \in C_b$. Thus, we obtain $S(t): \text{Lip}_b(r) \rightarrow \text{Lip}_b(r)$ for all $r, t \geq 0$, since $S(t)$ is translation invariant. It follows that $S(t): \text{BUC} \rightarrow \text{BUC}$ and $S(t): C_b \rightarrow U_b$, because $\text{Lip}_b \subset \text{BUC}$ is dense and $U_b = (\text{BUC})_\delta$. Moreover, the dynamic programming principle, see, e.g., Fabbri et al. [20, Theorem 2.24] or Pham [37, Theorem 3.3.1], yields that S is a semigroup on BUC . Hence, in order to verify condition (S3), it suffices to show that $\lim_{t \downarrow 0} \|S(t)f - f\|_\infty = 0$ for a dense subset of BUC .

Second, we show that $\text{Lip}_b^2 \subset D(A)$ and $Af = Bf$ for all $f \in \text{Lip}_b^2$. Let $f \in \text{Lip}_b^2$, $\varepsilon \in (0, 1]$, $t > 0$ and $x \in \mathbb{R}^d$. Choose $(a, b) \in \mathcal{A}$ with

$$(S(t)f)(x) \leq \frac{\varepsilon t}{2} + \mathbb{E}[f(x + X_t^{a,b})] - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right].$$

By Theorem 5.3 and Assumption 5.1(ii), there exists $c > 0$ with $Bf \in \text{Lip}_b(c)$. Define $\delta := \varepsilon/2c$. We use Itô's formula, Theorem 5.3 and Lemma 5.2(ii) and (iii) to obtain

$$\begin{aligned} \left(\frac{S(t)f - f}{t} \right) (x) &\leq \frac{\varepsilon}{2} + \mathbb{E} \left[\frac{1}{t} \int_0^t B_{a_s, b_s} f(x + X_s^{a,b}) - L(a_s, b_s) ds \right] \\ &\leq \frac{\varepsilon}{2} + \mathbb{E} \left[\frac{1}{t} \int_0^t Bf(x + X_s^{a,b}) ds \right] \\ &\leq Bf(x) + \varepsilon + \frac{cL}{\delta^2} \mathbb{E} \left[\frac{1}{t} \int_0^t \int_0^s 1 + L(a_u, b_u) du ds \right] \\ &\leq Bf(x) + \varepsilon + \frac{cL}{\delta^2} \frac{(1 + c_f)t}{2}. \end{aligned}$$

It follows from $T(t)f \leq S(t)f$ and Theorem 5.3 that

$$0 = \lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left(\frac{T(t)f - f}{t} - Bf \right) (x) \leq \lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left(\frac{S(t)f - f}{t} - Bf \right) (x) \leq \varepsilon.$$

Letting $\varepsilon \downarrow 0$ yields $f \in D(A)$ and $Af = Bf$.

Third, we show that $\mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b \subset \mathcal{L}_+^S$. Let $f \in \mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b$. Theorem 5.3 implies $f \in \bigcap_{a \in \mathbb{S}_L} D(\Delta_a) \cap W^{1,\infty}$ and

$$C := \sup_{(a,b) \in \mathbb{S}_L \times \mathbb{R}^d} (\|B_{a,b}f\|_\infty - L(a,b)) < \infty.$$

Let $\eta \in C_c^\infty$ with $\eta \geq 0$, $\text{supp}(\eta) \subset B(0,1)$, and $\int_{\mathbb{R}^d} \eta(x) dx = 1$. Let $\eta_n(x) := n^d \eta(nx)$ and $f_n := f * \eta_n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. For every $(a,b) \in \mathcal{A}$, $t \geq 0$ and $x \in \mathbb{R}^d$, we use Itô's formula and $\|B_{a_s,b_s}f_n\|_\infty \leq \|B_{a_s,b_s}f\|_\infty$ to estimate

$$\begin{aligned} & \mathbb{E}[f(x + X_t^{a,b})] - f(x) - \mathbb{E} \left[\int_0^t L(a_s, a_s) ds \right] \\ &= \lim_{n \rightarrow \infty} (\mathbb{E}[f_n(x + X_t^{a,b})] - f_n(x)) - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t B_{a_s,b_s} f_n(x + X_s^{a,b}) - L(a_s, b_s) ds \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^t (B_{a_s,b_s} f_n(x + X_s^{a,b}) - L(a_s, b_s)) \mathbf{1}_{\{a_s \in \mathbb{S}_L\}} ds \right] \leq Ct. \end{aligned}$$

We obtain $S(t)f - f \leq Ct$ for all $t \geq 0$. It follows from Theorem 5.3 and Corollary 2.11 that $S(t)f = T(t)f$ for all $t \geq 0$ and $f \in \text{Lip}_b^2$. Now, let $f \in C_b$ by arbitrary. Choose a bounded sequence $(f_n)_{n \in \mathbb{N}} \subset \text{Lip}_b^2$ with $f_n \rightarrow f$ uniformly on compacts. Lemma 2.7 implies $S(t)f = \lim_{n \rightarrow \infty} S(t)f_n = \lim_{n \rightarrow \infty} T(t)f_n = T(t)f$. \square

5.2. Trace class Wiener processes with drift. Let X be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Denote by $\mathcal{S}_1(X)$ the space of all trace class operators endowed with the trace class norm $\|\cdot\|_{\mathcal{S}_1(X)}$.

Assumption 5.5. Let $B \times \mathcal{Q} \subset X \times \mathcal{S}_1(X)$ satisfy the following conditions:

- (i) Q is selfadjoint and positive semidefinite for all $Q \in \mathcal{Q}$.
- (ii) $Q_1 Q_2 = Q_2 Q_1$ for all $Q_1, Q_2 \in \mathcal{Q}$.
- (iii) $B \times \mathcal{Q}$ is bounded.

By the previous assumption and [47, Corollary 3.2.5], there exists a joint orthonormal basis of eigenvectors $(e_k)_{k \in \mathbb{N}} \subset X$, i.e., for every $Q \in \mathcal{Q}$ and $k \in \mathbb{N}$, there exists $\mu_{Q,k} \geq 0$ with $Qe_k = \mu_{Q,k}e_k$. For every $Q \in \mathcal{Q}$ and $x \in X$,

$$Qx = \sum_{k \in \mathbb{N}} \mu_{Q,k} \langle x, e_k \rangle e_k.$$

Hence, every $Q \in \mathcal{Q}$ can be identified with the sequence $\mu_Q = (\mu_{Q,k})_{k \in \mathbb{N}} \in \ell^1$ satisfying $\text{tr}(Q) = \|Q\|_{\mathcal{S}_1(X)} = \|\mu_Q\|_{\ell^1}$. Assumption 5.5(iii) implies that $\Lambda := B \times \mu(\mathcal{Q}) \subset X \times \ell^1$ is bounded, where $\mu(\mathcal{Q}) := \{\mu_Q : Q \in \mathcal{Q}\}$.

Let $(\xi^k)_{k \in \mathbb{N}}$ be a sequence of independent one-dimensional standard Brownian motions on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. For every $\lambda := (b, \mu) \in \Lambda$, the semigroup $(T_\lambda(t))_{t \geq 0}$ associated to the Q -Wiener

process with $\mu = \mu_Q$ and drift term b is given by

$$(T_\lambda(t)f)(x) := \mathbb{E} \left[f \left(x + tb + \sum_{k \in \mathbb{N}} \sqrt{\mu_k} \xi_t^k e_k \right) \right]$$

for all $t \geq 0$, $f \in \text{BUC}$ and $x \in X$. Furthermore, we define

$$J(t)f := \sup_{\lambda \in \Lambda} T_\lambda(t)f \quad \text{for all } t \geq 0 \text{ and } f \in \text{BUC}.$$

Let \mathcal{A} be the set of all predictable processes

$$(b, \mu): \Omega \times [0, \infty) \rightarrow \Lambda, \quad (\omega, t) \mapsto (b_t(\omega), (\mu_t^k(\omega))_{k \in \mathbb{N}}).$$

For every $t \geq 0$, $f \in \text{BUC}$ and $x \in X$, we define

$$(S(t)f)(x) := \sup_{(b, \mu) \in \mathcal{A}} \mathbb{E} \left[f \left(x + \int_0^t b_s \, ds + \sum_{k \in \mathbb{N}} \left(\int_0^t \sqrt{\mu_s^k} \, d\xi_s^k \right) e_k \right) \right].$$

Theorem 5.6. *There exists a strongly continuous convex monotone semigroup T on BUC such that, uniformly on compacts,*

$$T(t)f = \lim_{n \rightarrow \infty} J(\pi_n t) = \sup_{n \in \mathbb{N}} J(\pi_n^t) f \quad \text{for all } f \in \text{BUC} \text{ and } t \geq 0.$$

Furthermore, it holds $S(t)f = T(t)f$ for all $t \geq 0$ and $f \in \text{BUC}$ such that there exist a compact linear operator $K: X \rightarrow X$ and $g \in \text{BUC}$ with $f(x) := g(Kx)$ for all $x \in X$.

Proof. First, we show the existence of the associated semigroup T . Since

$$\sup_{(b, \mu) \in \Lambda} (|b| + \|\mu\|_{\ell^1}) < \infty,$$

we can apply the results from [34, Section 2 and Example 7.2]. Hence, there exists a strongly continuous convex monotone semigroup T on BUC such that $J(\pi_n^t)f \uparrow T(t)f$ for all $t \geq 0$ and $f \in \text{BUC}$. Dini's theorem implies uniform convergence on compacts. For every $t \geq 0$ and $f \in \text{BUC}$, it holds $J(t)f \leq S(t)f$ and thus $T(t)f \leq S(t)f$ by construction.

Second, we show that $S(t)f = T(t)f$ for all $t \geq 0$ and $f \in \text{BUC}$ depending only on finitely many coordinates, i.e., there exists $n \in \mathbb{N}$ such that

$$f(x) = f \left(\sum_{k=1}^n \langle x, e_k \rangle e_k \right) \quad \text{for all } x \in X.$$

Fix $n \in \mathbb{N}$ and define $X_n := \text{span}\{e_1, \dots, e_n\} \subset X$. For every $t \geq 0$, $f_n \in \text{C}_b(X_n)$ and $x \in X_n$, we define

$$(S_n(t)f_n)(x) := \sup_{(b, \mu) \in \mathcal{A}_n} \mathbb{E} \left[f_n \left(x + \int_0^t b_s \, ds + \sum_{k=1}^n \left(\int_0^t \sqrt{\mu_s^k} \, d\xi_s^k \right) e_k \right) \right],$$

where \mathcal{A}_n denotes the set of all predictable processes $(b, \mu): \Omega \times [0, \infty) \rightarrow \Lambda_n$ with

$$\Lambda_n := \left\{ ((b, e_k))_{k=1, \dots, n}, (\mu_k)_{k=1, \dots, n} : (b, \mu) \in \Lambda \right\} \subset \mathbb{R}^n \times \mathbb{R}_+^n.$$

In addition, for every $t \geq 0$, $f_n \in \text{C}_b(X_n)$ and $x \in X_n$, we define

$$(T_{\lambda, n}(t)f_n)(x) := \mathbb{E} \left[f_n \left(x + tb + \sum_{k=1}^n \sqrt{\mu_k} \xi_t^k e_k \right) \right] \quad \text{for all } \lambda := (b, \mu) \in \Lambda_n,$$

$$(I_n(t)f_n)(x) := \sup_{\lambda \in \Lambda_n} (T_{\lambda, n}(t)f_n)(x).$$

Note that Assumption 5.1 is satisfied with $L := \infty \cdot \mathbf{1}_{\Lambda_n^c}$. Hence, by Theorem 5.3, there exists a semigroup T_n on $C_b(X_n)$ associated to I_n . Furthermore, Theorem 5.4 implies

$$S_n(t)f_n = T_n(t)f_n \quad \text{for all } t \geq 0 \text{ and } f_n \in C_b(X_n).$$

Let $f \in \text{BUC}$ such that there exists $f_n \in \text{BUC}(X_n)$ with

$$f(x) = f_n \left(\sum_{k=1}^n \langle x, e_k \rangle e_k \right) \quad \text{for all } x \in X.$$

It follows from the definition of S_n and the construction of T_n that

$$S(t)f = S_n(t)f_n = T_n(t)f_n = T(t)f \quad \text{for all } t \geq 0.$$

Third, we show that $S(t)f = T(t)f$ for all $t \geq 0$ and all $f \in \text{BUC}$ such that there exist a compact linear operator $K: X \rightarrow X$ and $g \in \text{BUC}$ with $f(x) := g(Kx)$ for all $x \in X$. Since the finite rank operators are dense in the space of all compact linear operators w.r.t. the operator norm $\|\cdot\|_{L(X)}$, there exists a sequence $(K_n)_{n \in \mathbb{N}}$ of finite rank operators with $\|K - K_n\|_{L(X)} \rightarrow 0$. Let $f_n(x) := g(K_n x)$ for all $x \in X$ and $n \in \mathbb{N}$. It holds $\|f_n\|_\infty \leq \|g\|_\infty$ for all $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \sup_{x \in B(0,r)} |f(x) - f_n(x)| = 0 \quad \text{for all } r \geq 0.$$

Moreover, the boundedness of Λ yields

$$c := \sup_{(b,\mu) \in \mathcal{A}} \mathbb{E} \left[\left| \int_0^t b_s ds + \sum_{k=1}^{\infty} \left(\int_0^t \sqrt{\mu_s^k} d\xi_s^k \right) e_k \right| \right] < \infty.$$

For every $r > 0$ and $x \in X$, we use Chebyshev's inequality to estimate

$$\begin{aligned} & (S(t)|f - f_n|)(x) \\ & \leq \sup_{y \in B(0,r)} |f(y) - f_n(y)| \\ & \quad + 2\|g\|_\infty \sup_{(b,\mu) \in \mathcal{A}} \mathbb{P} \left(\left| x + \int_0^t b_s ds + \sum_{k=1}^{\infty} \left(\int_0^t \sqrt{\mu_s^k} d\xi_s^k \right) e_k \right| > r \right) \\ & \leq \sup_{y \in B(0,r)} |f(y) - f_n(y)| + \frac{2\|g\|_\infty}{r} \sup_{(b,\mu) \in \mathcal{A}} \mathbb{E} \left[\left| x + \int_0^t b_s ds + \sum_{k=1}^{\infty} \left(\int_0^t \sqrt{\mu_s^k} d\xi_s^k \right) e_k \right| \right] \\ & \leq \sup_{y \in B(0,r)} |f(y) - f_n(y)| + \frac{2\|g\|_\infty(|x| + c)}{r}. \end{aligned}$$

By choosing first $r > 0$ and then $n \in \mathbb{N}$ sufficiently large, we obtain

$$\begin{aligned} & \max\{|S(t)f - S(t)f_n|(x), |T(t)f - T(t)f_n|(x)\} \\ & \leq \max\{(S(t)|f - f_n|)(x), (T(t)|f - f_n|)(x)\} = (S(t)|f - f_n|)(x) \rightarrow 0. \end{aligned}$$

Hence, the second part implies $S(t)f = T(t)f$. \square

5.3. Wasserstein perturbation of linear transition semigroups. Let $X := \mathbb{R}^d$ and $\kappa \equiv 1$. Moreover, for a fixed $p \in (1, \infty)$, we denote by \mathcal{P}_p the set of all probability measures on the Borel- σ -algebra $\mathcal{B}(\mathbb{R}^d)$ with finite p -th moment. Let $(\mu_t)_{t \geq 0} \subset \mathcal{P}_p$ and $(\psi_t)_{t \geq 0}$ be a family of functions $\psi_t: \mathbb{R}^d \rightarrow \mathbb{R}^d$. Following the setting in [23], for every $t \geq 0$, $f \in C_b$ and $x \in \mathbb{R}^d$, we define

$$(R(t)f)(x) := \int_{\mathbb{R}^d} f(\psi_t(x) + y) d\mu_t(y).$$

Typical examples include Koopman semigroups and transition semigroups of Lévy and Ornstein–Uhlenbeck processes, see [23]. Denote by \mathcal{L}^R the Lipschitz set of R . Let C_0^∞ be the space of all infinitely differentiable functions $f \in C_0$ such that all derivatives are in C_0 , where C_0 consists of all continuous functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ with $\lim_{|x| \rightarrow \infty} f(x) = 0$.

Assumption 5.7. Suppose that R forms a semigroup. Furthermore, let $(\mu_t)_{t \geq 0}$ and $(\psi_t)_{t \geq 0}$ satisfy the following conditions:

- (i) $\lim_{t \downarrow 0} \int_{\mathbb{R}^d} |y|^p d\mu_t(y) = 0$.
- (ii) There exist $r > 0$ and $c \geq 0$ such that $\mu_t(B(0, r)^c) \leq ct$ for all $t \in [0, 1]$.
- (iii) $\psi_t(0) = 0$ for all $t \geq 0$.
- (iv) There exists $c \geq 0$ such that, for all $x, y \in \mathbb{R}^d$ and $t \in [0, 1]$,

$$|\psi_t(x) - \psi_t(y) - (x - y)| \leq ct|x - y|.$$

- (v) For every $f \in C_0^\infty \cap \mathcal{L}^R$, the limit

$$R'(0)f := \lim_{h \downarrow 0} \frac{R(h)f - f}{h} \in C_b$$

exists uniformly on compacts.

For every $x, y \in \mathbb{R}^d$ and $t \in [0, 1]$, the conditions (iii) and (iv) imply

$$|\psi_t(x) - x| \leq ct|x| \quad \text{and} \quad |\psi_t(x) - \psi_t(y)| \leq e^{ct}|x - y|. \quad (5.1)$$

Lemma 5.8. *It holds $C_c^\infty \subset \mathcal{L}^R$.*

Proof. Let $f \in C_c^\infty$. Choose $c, r \geq 0$ such that Assumption 5.7(ii) and (iv) are satisfied. In addition, let $\text{supp}(f) \subset B(0, r)$. By Assumption 5.7(v), there exist $t_0 \in (0, 1/2c]$ and $c' \geq 0$ with

$$\sup_{x \in B(0, 4r)} |(R(t)f - f)(x)| \leq c't \quad \text{for all } t \in [0, t_0].$$

Inequality (5.1) implies $|\psi_t(x) + y| \geq r$ for all $x \in B(0, 4r)^c$, $y \in B(0, r)$ and $t \in [0, t_0]$. Hence, for every $x \in B(x, 4r)^c$ and $t \in [0, t_0]$, it follows from Assumption 5.7(ii) that

$$|(R(t)f)(x) - f(x)| \leq \|f\|_\infty \mu_t(B(0, r)^c) \leq c\|f\|_\infty t.$$

We obtain $\|R(t)f - f\|_\infty \leq \max\{c', c\|f\|_\infty\}t$ for all $t \in [0, t_0]$. \square

In the sequel, we consider a perturbation of the linear transition semigroup R , where we take the supremum over all probabilities which are sufficiently close to the reference measure μ_t . Recall that, in Subsection 5.1, we considered a Brownian motion with uncertain drift and volatility, where the uncertainty was parametrized by a finite-dimensional parameter space. Here, the non-parametric uncertainty is instead given by an infinite-dimensional ball of probability measures. We define the p -Wasserstein distance by

$$W_p(\mu, \nu) := \left(\inf_{\pi \in \text{Cpl}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - z|^p d\pi(y, z) \right)^{\frac{1}{p}} \quad \text{for all } \mu, \nu \in \mathcal{P}_p,$$

where $\text{Cpl}(\mu, \nu)$ consists of all probability measures on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ with first marginal μ and second marginal ν . Let $\varphi: [0, \infty) \rightarrow [0, \infty]$ be a convex lower semicontinuous function with $\varphi(0) = 0$ and $\varphi(v) > 0$ for some $v > 0$. Furthermore, we assume that that the mapping $[0, \infty) \rightarrow [0, \infty]$, $v \mapsto \varphi(v^{1/p})$ is convex. The previous assumptions ensure that

$$\varphi^*(w) := \sup_{v \geq 0} (vw - \varphi(v)) < \infty \quad \text{for all } w \geq 0.$$

For every $t \geq 0$, $f \in C_b$ and $x \in \mathbb{R}^d$, we define

$$(I(t)f)(x) := \sup_{\nu \in \mathcal{P}_p} \left(\int_{\mathbb{R}^d} f(\psi_t(x) + z) d\nu(z) - \varphi_t(\mathcal{W}_p(\mu_t, \nu)) \right),$$

where $\varphi_t: [0, \infty) \rightarrow [0, \infty]$ denotes the rescaled function

$$\varphi_t(v) := \begin{cases} t\varphi\left(\frac{v}{t}\right), & t > 0, v \geq 0, \\ 0, & t = v = 0, \\ +\infty, & t = 0, v \neq 0. \end{cases}$$

The following result is a consequence of [23, Theorem 3.13] and Lemma 3.6.

Theorem 5.9. *There exists a semigroup S on C_b , which satisfies Assumption 2.4, such that, uniformly on compacts,*

$$S(t)f = \lim_{n \rightarrow \infty} I(\pi_n)f = \inf_{n \in \mathbb{N}} I(\pi_n)f \quad \text{for all } f \in C_b \text{ and } t \geq 0. \quad (5.2)$$

Denoting by A_Γ the Γ -generator of S , it holds $C_0^\infty \cap \mathcal{L}^R \subset D(A_\Gamma)$ and

$$A_\Gamma f = R'(0)f + \varphi^*(|\nabla f|) = \lim_{h \downarrow 0} \frac{S(h)f - f}{h}$$

uniformly on compacts for all $f \in C_0^\infty \cap \mathcal{L}^R$. Condition (3.7) is satisfied for all $f \in \text{Lip}_b$. Furthermore, it holds $S(t): \text{Lip}_b \rightarrow \text{Lip}_b$ and $S(t): C_0 \rightarrow C_0$ for all $t \geq 0$.

Proof. By [23, Theorem 3.13 and Remark 3.14], there exists a semigroup S on C_b which satisfies equation (5.2), Assumption 2.4 and the statement about the Γ -generator. Choose $c \geq 0$ such that Assumption 5.7(iv) is satisfied. It follows from the proof of Theorem 4.5 that condition (3.7) holds for all $f \in \text{Lip}_b$ if we verify the following conditions:

- $I(t): B_{C_b}(0, r) \rightarrow B_{C_b}(0, r)$ for all $r, t \geq 0$,
- $\|I(t)f - I(t)g\|_\infty \leq \|f - g\|_\infty$ for all $t \geq 0$ and $f, g \in C_b$,
- $I(t): \text{Lip}_b(r) \rightarrow \text{Lip}_b(e^{ct}r)$ for all $r, t \geq 0$,
- $\|\tau_x(I(t)f) - I(t)(\tau_x f)\|_\infty \leq crt|x|$ for all $r, t \geq 0$, $f \in \text{Lip}_b(r)$ and $x \in \mathbb{R}^d$.

The first two statements follow immediately from the definition of I . For every $r, t \geq 0$, $f \in \text{Lip}_b(r)$ and $x, y \in \mathbb{R}^d$, inequality (5.1) implies

$$\begin{aligned} |(I(t)f)(x) - (I(t)f)(y)| &\leq \sup_{\nu \in \mathcal{P}_p} \int_{\mathbb{R}^d} |f(\psi_t(x) + z) - f(\psi_t(y) + z)| d\nu(z) \\ &\leq e^{ct}r|x - y|. \end{aligned} \quad (5.3)$$

Furthermore, we use Assumption 5.7(iv) to estimate

$$\begin{aligned} |(\tau_x(I(t)f) - I(t)(\tau_x f))(y)| &\leq \sup_{\nu \in \mathcal{P}_p} \int_{\mathbb{R}^d} |f(\psi_t(x + y) + z) - f(\psi_t(y) + x + z)| d\nu(z) \\ &\leq r|\psi_t(x + y) - \psi_t(y) - x| \leq crt|x|. \end{aligned}$$

It remains to show $S(t): C_0 \rightarrow C_0$ for all $t \geq 0$. Since S is semigroup, it suffices to show $S(t)f \in C_0$ for all $t \in [0, 1/2c]$ and $f \in C_0$. Since $S(t)$ is monotone and convex with $S(t)0 = 0$, it holds $|S(t)f| \leq S(t)|f| \leq I(t)|f|$. By [23, Lemma 3.4], there exists a constant $c' \geq 0$ with

$$(I(t)|f|)(x) \leq \sup_{\{v \in \mathcal{P}_p: \mathcal{W}_p(\mu_t, \nu) \leq c'\}} \int_{\mathbb{R}^d} |f(\psi_t(x) + y)| d\nu(y).$$

Let $\varepsilon > 0$. Since $f \in C_0$ and the Wasserstein ball $\{v \in \mathcal{P}_p: \mathcal{W}_p(\mu_t, \nu) \leq c'\}$ is tight, we can choose $r \geq 0$ with

$$\sup_{x \in B(0, r)^c} |f(x)| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{\mathcal{W}_p(\mu_t, \nu) \leq c'} \nu(B(0, r)^c) \|f\|_\infty \leq \frac{\varepsilon}{2}.$$

For every $x \in B(0, 4r)^c$ and $y \in B(0, r)$, inequality (5.1) implies $|\psi_t(x) + y| \geq r$. Thus,

$$\begin{aligned} (I(t)|f|)(x) &\leq \sup_{\mathcal{W}_p(\mu_t, \nu) \leq c'} \left(\int_{B(0, r)} |f(\psi_t(x) + y)| d\nu(y) + \int_{B(0, r)^c} |f(\psi_t(x) + y)| d\nu(y) \right) \\ &\leq \sup_{\mathcal{W}_p(\mu_t, \nu) \leq c'} \left(\frac{\varepsilon}{2} + \nu(B(0, r)^c) \|f\|_\infty \right) \leq \varepsilon. \end{aligned}$$

We obtain $S(t): C_0 \rightarrow C_0$ for all $t \geq 0$. \square

In addition to the Wasserstein perturbation, we consider a perturbation which is parametrized only by drifts $b \in \mathbb{R}^d$. For every $t \geq 0$, $f \in C_b$ and $x \in \mathbb{R}^d$, we define

$$(J(t)f)(x) := \sup_{b \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(\psi_t(x) + y + b) d\mu_t(y) - \varphi_t(|b|t) \right).$$

We remark that $\mathcal{W}_p(\mu_t, \nu_b) = |b|t$ for all $b \in \mathbb{R}^d$ and $t \geq 0$, where

$$\nu_b(A) := \int_{\mathbb{R}^d} \mathbf{1}_A(b + y) d\mu_t(y) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d).$$

Furthermore, the previous definition of J is consistent with the one in Subsection 5.1 if we fix the volatility matrix a and choose $\mu_t := \mathbb{P} \circ (\sqrt{a}W_t)^{-1}$ and $L(b) := \varphi(|b|)$. Note that uncertainty in the volatility is not included in the setting of this subsection. Define $\text{Lip}_0 := \text{Lip}_b \cap C_0$.

Theorem 5.10. *There exists a family T of operators $T(t): C_b \rightarrow C_b$, which satisfy Assumption 2.4, such that, uniformly on compacts,*

$$T(t)f = \lim_{n \rightarrow \infty} J(\pi_n^t)f \quad \text{for all } (f, t) \in C_b \times \mathcal{T}. \quad (5.4)$$

Furthermore, T is a strongly continuous convex monotone semigroup on C_0 . Denoting by B_Γ the Γ -generator of T , it holds $C_0^\infty \cap \mathcal{L}^R \subset D(B_\Gamma)$ and

$$B_\Gamma f = R'(0)f + \varphi^*(|\nabla f|) = \lim_{h \downarrow 0} \frac{T(h)f - f}{h}$$

uniformly on compacts for all $f \in C_0^\infty \cap \mathcal{L}^R$. Condition (3.7) is satisfied for all $f \in \text{Lip}_0$. Furthermore, we have $T(t): \text{Lip}_0 \rightarrow \text{Lip}_0$ for all $t \geq 0$.

Proof. First, we verify Assumption 4.1. Clearly, the conditions (i)-(iv) are satisfied. Choose $c \geq 0$ such that Assumption 5.7(iv) is satisfied. Similar to inequality (5.3), one can show that $J(t): \text{Lip}_b(r) \rightarrow \text{Lip}_b(e^{ct}r)$ for all $r, t \geq 0$. In particular, the sequence $(J(\pi_n^t)f)_{n \in \mathbb{N}}$ is uniformly Lipschitz continuous for all $t \geq 0$ and $f \in \text{Lip}_b$. Assumption 4.1(vi) follows from Remark 4.2(ii), since $J(t)f \leq S(t)f$ for all $t \geq 0$ and $f \in C_b$. Next, we show that $J(t): C_b \rightarrow C_b$ for all $t \geq 0$. Let $t \geq 0$ and $f \in C_b$. Choose a bounded sequence $(f_n)_{n \in \mathbb{N}} \subset \text{Lip}_b$ with $f_n \rightarrow f$ uniformly on compacts. Lemma 2.7 yields $J(t)f_n \rightarrow J(t)f$ uniformly on compacts. Indeed, the corresponding proof only relies on the fact that $J(t)$ is convex, monotone and continuous from above. Since $J(t)f_n \in \text{Lip}_b \subset C_b$ for all $n \in \mathbb{N}$, we obtain $J(t)f \in C_b$. Moreover, it holds $|J(t)f| \leq J(t)|f| \leq S(t)|f|$ and thus $|J(\pi_n^t)f| \leq S(t)|f| \in C_0$ for all $t \geq 0$, $n \in \mathbb{N}$ and

$f \in C_0$. It remains to show that $\mathcal{L}^J \subset C_0$ is dense. For every $c, t \geq 0$ and $f \in \text{Lip}_b(c)$, it follows from [23, Equation (3.7)] that

$$0 \leq J(t)f - R(t)f \leq I(t)f - R(t)f \leq t\varphi^*(c). \quad (5.5)$$

Hence, Lemma 5.8 implies $C_c^\infty \subset \mathcal{L}^R \subset \mathcal{L}^J$. By Theorem 4.3, there exists a family T of operators $T(t): C_b \rightarrow C_b$ and a subsequence $(n_l)_{l \in \mathbb{N}}$, which satisfy Assumption 2.4, such that, uniformly on compacts,

$$T(t)f = \lim_{l \rightarrow \infty} J(\pi_{n_l}^t)f \quad \text{for all } (f, t) \in C_b \times \mathcal{T}.$$

Since $(J(\pi_n^t)f)_{n \in \mathbb{N}}$ is non-decreasing, the convergence holds without choosing a subsequence. In addition, T is a strongly continuous convex monotone semigroup on C_0 and it holds $T(t): \text{Lip}_0 \rightarrow \text{Lip}_0$ for all $t \geq 0$.

Second, we show that $C_0^\infty \cap \mathcal{L}^R \subset D(B_\Gamma)$ and

$$B_\Gamma f = R'(0)f + \varphi^*(|Df|) = \lim_{h \downarrow 0} \frac{T(h)f - f}{h}$$

uniformly on compacts for all $f \in C_0^\infty \cap \mathcal{L}^R$. For every $h > 0$ and $b, x, y \in \mathbb{R}^d$, we use Taylor's formula to obtain

$$|f(\psi_h(x) + y + bh) - f(\psi_h(x) + y) - \langle \nabla f(\psi_h(x) + y), bh \rangle| \leq \|D^2 f\|_\infty |b|^2 h^2.$$

As seen in the proof of [23, Lemma 3.6], it holds $\lim_{h \downarrow 0} |R(h)g - g|(x) = 0$ for all $g \in \text{Lip}_b$. Hence, we can estimate

$$\begin{aligned} \liminf_{h \downarrow 0} \left(\frac{J(h)f - R(h)f}{h} \right) (x) &\geq \liminf_{h \downarrow 0} \int_{\mathbb{R}^d} \langle \nabla f(\psi_h(x) + y, b) d\mu_h(y) - \varphi(|b|) \\ &= \liminf_{h \downarrow 0} R(h)(\langle \nabla f, b \rangle)(x) - \varphi(|b|) \\ &= \langle \nabla f(x), b \rangle - \varphi(|b|). \end{aligned}$$

Taking the supremum over all $b \in \mathbb{R}^d$ yields

$$\liminf_{h \downarrow 0} \left(\frac{J(h)f - R(h)f}{h} \right) (x) \geq \varphi^*(|\nabla f|).$$

We use the previous inequality, $J(h)f \leq T(h)f \leq S(h)f$, Assumption 5.7(v) and Theorem 5.9 to conclude

$$\lim_{h \downarrow 0} \frac{T(h)f - f}{h} = R'(0)f + \varphi^*(|\nabla f|)$$

uniformly on compacts.

Third, we remark that the verification of condition (3.7) for all $f \in \text{Lip}_0$ is similar to the proof of Theorem 5.9. Moreover, it follows from $J(t): \text{Lip}_0(r) \rightarrow \text{Lip}_0(e^{ct}r)$ and the construction of T that $T(t): \text{Lip}_0(r) \rightarrow \text{Lip}_0(e^{ct}r)$ for all $r, t \geq 0$. \square

Theorem 5.11. *It holds $S(t)f = T(t)f$ for all $t \geq 0$ and $f \in C_b$.*

Proof. By construction, it holds $0 \leq T(t)f - R(t)f \leq S(t)f - R(t)f \leq I(t)f - R(t)f$ for all $t \geq 0$ and $f \in \text{Lip}_0$. Hence, inequality (5.5) implies

$$\mathcal{L}^R \cap \text{Lip}_0 = \mathcal{L}^S \cap \text{Lip}_0 = \mathcal{L}^T \cap \text{Lip}_0.$$

Let $\mathcal{D} := \mathcal{L}^R \cap \text{Lip}_0$. It follows from Remark 2.9(i) and Theorem 5.10 that $T(t): \mathcal{D} \rightarrow \mathcal{D}$ for all $t \geq 0$. Next, we show that $A_\Gamma f = B_\Gamma f$ for all $f \in \mathcal{D}$. Let $\eta \in C_c^\infty$ with $\eta \geq 0$, $\text{supp}(\eta) \subset B(0, 1)$, and $\int_{\mathbb{R}^d} \eta(x) dx = 1$. Define $\eta_n(x) := n^d \eta(nx)$ and $f_n := f * \eta_n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Then, it holds $f_n \in C_0^\infty$ for all $n \in \mathbb{N}$. Choose $c, r \geq 0$ such that

Assumption 5.7(iv) is satisfied and $f \in \text{Lip}_0(r)$. For every $t \geq 0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we use Fubini's theorem to estimate

$$\begin{aligned} & |R(t)f_n - (R(t)f) * \eta_n|(x) \\ & \leq \int_{B(0,1)} \left(\int_{\mathbb{R}^d} |f(\psi_t(x) + y - z) - f(\psi_t(x - z) + y)| d\mu_t(y) \right) \eta_n(z) dz \leq crt. \end{aligned}$$

We obtain $\|R(t)f_n - f_n\|_\infty \leq \|R(t)f - f\|_\infty + crt$ and, therefore, $f_n \in \mathcal{L}^R$. It follows from Lemma 3.6, Theorem 5.9 and Theorem 5.10 that

$$A_\Gamma f = \Gamma\text{-}\lim_{n \rightarrow \infty} A_\Gamma f_n = \Gamma\text{-}\lim_{n \rightarrow \infty} B_\Gamma f_n = B_\Gamma f.$$

Hence, Corollary 2.11 implies $S(t)f \leq T(t)f$ for all $t \geq 0$ and $f \in C_0^\infty \cap \mathcal{L}^R$ while the inequality $S(t)f \geq T(t)f$ holds by construction. Now, let $f \in C_b$ be arbitrary and choose a bounded sequence $(f_n)_{n \in \mathbb{N}} \subset C_c^\infty$ with $f_n \rightarrow f$ uniformly on compacts. We use Lemma 2.7 and Lemma 5.8 to conclude

$$S(t)f = \lim_{n \rightarrow \infty} S(t)f_n = \lim_{n \rightarrow \infty} T(t)f_n = T(t)f. \quad \square$$

In the particular case that S coincides with the entropic semigroup, as a byproduct of Theorem 5.11, we recover that μ_t satisfies the Talagrand T_2 inequality, see [45, Chapter 22] and [41]. Denote by $\mathcal{N}(0, t\mathbb{1})$ the d -dimensional normal distribution with mean zero and covariance matrix $t\mathbb{1}$, where $\mathbb{1} \in \mathbb{R}^{d \times d}$ is the identity matrix.

Corollary 5.12. *It holds $\mathcal{W}_2(\nu, \mu_t) \leq \sqrt{2tH(\nu|\mu_t)}$ for all $t \geq 0$ and $\nu \in \mathbb{P}_2$, where $H(\nu|\mu_t)$ denotes the relative entropy of ν w.r.t. $\mu_t := \mathcal{N}(0, t)$.*

Proof. Choose $\psi_t := \text{id}_{\mathbb{R}^d}$, $\mu_t := \mathcal{N}(0, t\mathbb{1})$ and $\varphi(v) := v^2/2$ for all $t, v \geq 0$. Moreover, let $(W_t)_{t \geq 0}$ be a d -dimensional standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. We show that

$$(T(t)f)(x) = (\tilde{S}(t)f)(x) := \frac{1}{2} \log (\mathbb{E}[\exp(2f(x + W_t))])$$

for all $t \geq 0$, $f \in C_b$ and $x \in \mathbb{R}^d$. By elementary calculation, see [7, Lemma 4.4] and the proof of [7, Theorem 4.5], one can show that $J(t)f \leq \tilde{S}(t)f$ and thus $T(t)f \leq \tilde{S}(t)f$ for all $t \geq 0$ and $f \in C_b$. To show the reverse inequality, we want to apply Corollary 2.11. By straightforward computations, one can show that \tilde{S} satisfies Assumption 2.4 and $\text{BUC}^2 \subset D(\tilde{A})$ with

$$\tilde{A}f = \frac{1}{2}(\Delta f + |\nabla f|^2) \quad \text{for all } f \in \text{BUC}^2.$$

Moreover, it follows from Itô's formula and Theorem 5.3 that $\mathcal{D} := \mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b \subset \mathcal{L}_+^{\tilde{S}}$. Since \tilde{S} and T are translation invariant, Lemma 3.6 yields $\tilde{A}_\Gamma f = B_\Gamma f$ for all $f \in \mathcal{D}$. Hence, Corollary 2.11 implies $\tilde{S}(t)f = T(t)f$ for all $t \geq 0$ and $f \in \text{BUC}^2$. For arbitrary $f \in C_b$, the equality follows by approximation. In addition, it follows from Theorem 5.11 that $\tilde{S}(t)f = S(t)f \leq I(t)f$ for all $t \geq 0$ and $f \in C_b$. Fenchel–Moreaus's theorem yields

$$\begin{aligned} \frac{\mathcal{W}_2(\nu, \mu_t)^2}{2t} &= \sup_{f \in C_b} \left(\int_{\mathbb{R}^d} f d\nu - (I(t)f)(0) \right) \\ &\leq \sup_{f \in C_b} \left(\int_{\mathbb{R}^d} f d\nu - (\tilde{S}(t)f)(0) \right) = H(\nu|\mu_t). \quad \square \end{aligned}$$

APPENDIX A. Γ -CONVERGENCE

Following the works of Beer [4], Dal Maso [15] and Rockafellar and Wets [38], we gather some basics about Γ -convergence. We remark that, in [15] and [38], all results are formulated for extended real-valued lower semicontinuous functions. However, a function $f: X \rightarrow [-\infty, \infty)$ is upper semicontinuous if and only if $-f: X \rightarrow (-\infty, \infty]$ is lower semicontinuous, and all results immediately transfer to our setting.

Lemma A.1. *Let $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ and $(g_n)_{n \in \mathbb{N}} \subset U_\kappa$ be bounded above, and $f, g \in U_\kappa$.*

- (i) *It holds $\Gamma\text{-}\limsup_{n \rightarrow \infty} f_n \in U_\kappa$. Furthermore, $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ has a Γ -convergent subsequence, i.e., $\Gamma\text{-}\lim_{k \rightarrow \infty} f_{n_k} \in U_\kappa$ exists for a subsequence $(n_k)_{k \in \mathbb{N}}$.*
- (ii) *We have $f = \Gamma\text{-}\lim_{n \rightarrow \infty} f_n$ if and only if every subsequence $(n_k)_{k \in \mathbb{N}}$ has another subsequence $(n_{k_l})_{l \in \mathbb{N}}$ with $f = \Gamma\text{-}\lim_{l \rightarrow \infty} f_{n_{k_l}}$.*
- (iii) *If $f_n \downarrow f$, then $f = \Gamma\text{-}\lim_{n \rightarrow \infty} f_n$.*
- (iv) *It holds $\Gamma\text{-}\limsup_{n \rightarrow \infty} (f_n + g_n) \leq \Gamma\text{-}\limsup_{n \rightarrow \infty} f_n + \Gamma\text{-}\limsup_{n \rightarrow \infty} g_n$. Moreover, we have $\Gamma\text{-}\limsup_{n \rightarrow \infty} f_n \leq \Gamma\text{-}\limsup_{n \rightarrow \infty} g_n$ if $f_n \leq g_n$ for all $n \in \mathbb{N}$.*
- (v) *Assume that $f \in C_\kappa$, $f_n \rightarrow f$ uniformly on compacts and $g = \Gamma\text{-}\lim_{n \rightarrow \infty} g_n$. Then, it holds $f + g = \Gamma\text{-}\lim_{n \rightarrow \infty} (f_n + g_n)$.*
- (vi) *If $f = \Gamma\text{-}\limsup_{n \rightarrow \infty} f_n$ and $g \in C_\kappa$, then $f \vee g = \Gamma\text{-}\limsup_{n \rightarrow \infty} (f_n \vee g)$.*
- (vii) *It holds $(\Gamma\text{-}\limsup_{n \rightarrow \infty} f_n)(x) \geq \limsup_{n \rightarrow \infty} \sup_{y \in B(x, \delta_n)} f_n(y)$ for all $x \in X$ and $(\delta_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $\delta_n \rightarrow 0$.*

Proof. Part (i) follows from [15, Remark 4.11], [15, Theorem 4.16], and [15, Theorem 8.4]. For part (ii) and (iii), we refer to [15, Proposition 8.3] and [15, Proposition 5.4], respectively. Part (iv) and (vii) are direct consequences of the definition of the Γ -lim sup.

In order to show part (v), let $x \in X$ and $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \rightarrow x$ and $g(x) = \lim_{n \rightarrow \infty} g_n(x_n)$. For every $n \in \mathbb{N}$,

$$|(f + g)(x) - (f_n + g_n)(x_n)| \leq |f(x) - f(x_n)| + \sup_{y \in K} |f(y) - f_n(y)| + |g(x) - g_n(x_n)|,$$

where $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact. Since $f_n \rightarrow f$ uniformly on compact and f is continuous, the right-hand side converges to zero. We obtain $f + g \leq \Gamma\text{-}\lim_{n \rightarrow \infty} (f_n + g_n)$ and the reverse inequality follows from part (iv).

It remains to show part (vi). The inequality $f \vee g \leq \Gamma\text{-}\limsup_{n \rightarrow \infty} (f_n \vee g)$ follows from part (iv). Let $(x_n)_{n \in \mathbb{N}} \subset X$ and $x \in X$ with $x_n \rightarrow x$. Continuity of g implies

$$\limsup_{n \rightarrow \infty} (f_n \vee g)(x) = \lim_{k \rightarrow \infty} (f_{n_k} \vee g)(x_k) = \left(\limsup_{k \rightarrow \infty} f_{n_k}(x_{n_k}) \right) \vee g(x) \leq (f \vee g)(x),$$

where $(x_{n_k})_{k \in \mathbb{N}}$ is a suitable subsequence approximating the limes superior. \square

Let $f \in U_\kappa$ and $\varepsilon > 0$. Following [4], the *upper ε -parallel function* to f is defined by

$$f^\varepsilon: X \rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{\kappa(x)} \left(\sup_{y \in B(x, \varepsilon)} \max \{ f(y) \kappa(y), -\frac{1}{\varepsilon} \} + \varepsilon \right).$$

If closed bounded sets in X are compact (e.g., if $X = \mathbb{R}^d$ or X is compact), one can show that f^ε is upper semicontinuous, cf. the proof of [4, Lemma 1.3]. For a general metric space (X, d) , however, the upper semicontinuity of f^ε cannot be guaranteed. For this reason, we consider the upper semicontinuous envelope of f^ε , which is defined by

$$\overline{f}^\varepsilon: X \rightarrow \mathbb{R}, \quad x \mapsto \limsup_{y \rightarrow x} f^\varepsilon(y) = \inf_{\delta > 0} \sup_{y \in B(x, \delta)} f^\varepsilon(y).$$

Note that, for all $x \in X$,

$$\bar{f}^\varepsilon(x) = \sup \left\{ \limsup_{n \rightarrow \infty} f^\varepsilon(x_n) : (x_n)_{n \in \mathbb{N}} \subset X \text{ with } x_n \rightarrow x \right\} = \Gamma\text{-}\limsup_{n \rightarrow \infty} f^\varepsilon,$$

where in the last expression f^ε stands for the constant sequence $(f^\varepsilon)_{n \in \mathbb{N}}$. We have the following geometric description of the Γ -lim sup.

Lemma A.2.

(i) For every $f \in U_\kappa$ and $\varepsilon > 0$,

$$f^\varepsilon \leq \bar{f}^\varepsilon \leq \inf_{\varepsilon' > \varepsilon} f^{\varepsilon'} \quad \text{and} \quad \|(\bar{f}^\varepsilon)^+\|_\kappa \leq \|f^+\|_\kappa + \varepsilon. \quad (\text{A.1})$$

Furthermore, it holds $\bar{f}^\varepsilon \in U_\kappa$ for all $\varepsilon > 0$ and $\bar{f}^\varepsilon \downarrow f$ as $\varepsilon \downarrow 0$.

(ii) Let $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ be bounded above and $f \in U_\kappa$. Then, $\Gamma\text{-}\limsup_{n \rightarrow \infty} f_n \leq f$ if and only if, for every $\varepsilon > 0$ and compact set $K \subset X$, there exists $n_0 \in \mathbb{N}$ with

$$f_n(x) \leq \bar{f}^\varepsilon(x) \quad \text{for all } x \in K \text{ and } n \geq n_0. \quad (\text{A.2})$$

Proof. First, we show inequality A.1. Fix $f \in U_\kappa$, $\varepsilon > 0$ and $x \in X$. For every $\varepsilon' > \varepsilon$,

$$\begin{aligned} f^\varepsilon(x) &\leq \bar{f}^\varepsilon(x) \leq \sup_{y \in B(x, \varepsilon' - \varepsilon)} f^\varepsilon(y) \\ &= \sup_{y \in B(x, \varepsilon' - \varepsilon)} \frac{1}{\kappa(y)} \left(\sup_{z \in B(y, \varepsilon)} \max \left\{ f(z)\kappa(z), -\frac{1}{\varepsilon} \right\} + \varepsilon \right) \\ &\leq \frac{c_{\varepsilon'}(x)}{\kappa(x)} \left(\sup_{y \in B(x, \varepsilon' - \varepsilon)} \sup_{z \in B(y, \varepsilon)} \max \left\{ f(z)\kappa(z), -\frac{1}{\varepsilon} \right\} + \varepsilon \right) \\ &\leq \frac{c_{\varepsilon'}(x)}{\kappa(x)} \left(\sup_{z \in B(x, \varepsilon')} \max \left\{ f(z)\kappa(z), -\frac{1}{\varepsilon'} \right\} + \varepsilon' \right) = c_{\varepsilon'}(x) f^{\varepsilon'}(x), \end{aligned}$$

where $c_{\varepsilon'}(x) := \sup_{y \in B(x, \varepsilon' - \varepsilon)} \frac{\kappa(x)}{\kappa(y)}$. Continuity of κ implies $c_{\varepsilon'}(x) \downarrow 1$ as $\varepsilon' \downarrow \varepsilon$. Hence, taking the infimum over $\varepsilon' > \varepsilon$ in the previous estimate yields the first part of inequality (A.1). Furthermore, we can estimate

$$(\bar{f}^\varepsilon \kappa)^+(x) \leq \inf_{\varepsilon' > \varepsilon} (f^{\varepsilon'} \kappa)^+(x) \leq \inf_{\varepsilon' > \varepsilon} \left(\sup_{y \in X} (f\kappa)^+(y) + \varepsilon' \right) = \|f^+\|_\kappa + \varepsilon.$$

In particular, we obtain $\bar{f}^\varepsilon \in U_\kappa$, because \bar{f}^ε is upper semicontinuous by definition.

Second, we show that $\bar{f}^\delta \downarrow f$ as $\delta \downarrow 0$. Due to inequality (A.1) it is sufficient to prove $f^\delta \downarrow f$ as $\delta \downarrow 0$. Since $f\kappa$ is upper semicontinuous, for every $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $(f\kappa)(y) \leq (f\kappa)(x) + \varepsilon$ for all $y \in B(x, \delta)$. We obtain

$$f(x) \leq f^\delta(x) = \frac{1}{\kappa(x)} \sup_{y \in B(x, \delta)} \max \left\{ (f\kappa)(y), -\frac{1}{\delta} \right\} + \delta \leq \max \left\{ f(x), -\frac{1}{\delta} \right\} + \delta + \varepsilon.$$

This implies $f(x) \leq \inf_{\delta > 0} f^\delta(x) \leq f(x) + \varepsilon \downarrow f(x)$ as $\varepsilon \downarrow 0$.

Third, let $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ be bounded above and $f \in U_\kappa$ with $\Gamma\text{-}\limsup_{n \rightarrow \infty} f_n \leq f$. We follow the proof of [4, Lemma 1.5] to verify inequality (A.2). Let $K \subset X$ be compact and $\varepsilon > 0$. Since $\Gamma\text{-}\limsup_{n \rightarrow \infty} f_n \leq f$ and $\kappa > 0$ is continuous, we obtain

$$\limsup_{n \rightarrow \infty} (f_n \kappa)(x_n) \leq (f\kappa)(x) \quad \text{for all } x \in K \text{ and } (x_n)_{n \in \mathbb{N}} \subset X \text{ with } x_n \rightarrow x.$$

Hence, for every $x \in K$, there exist $n_x \in \mathbb{N}$ and $r_x \in (0, \varepsilon)$ such that

$$(f_n \kappa)(y) \leq \max \left\{ (f\kappa)(x), -\frac{1}{\varepsilon} \right\} + \varepsilon \quad \text{for all } n \geq n_x \text{ and } y \in B(x, r_x).$$

By compactness of K , we can choose $x_1, \dots, x_k \in K$ with $K \subset \bigcup_{i=1}^k B(x_i, r_{x_i})$. Define $n_0 := n_{x_1} \vee \dots \vee n_{x_k}$. Let $x \in K$ and $i \in \{1, \dots, k\}$ with $d(x, x_i) < r_{x_i} < \varepsilon$. We obtain

$$f_n(x) \leq \frac{1}{\kappa(x)} \left(\max \{ (f\kappa)(x_i), -\frac{1}{\varepsilon} \} + \varepsilon \right) \leq \bar{f}^\varepsilon(x) \quad \text{for all } n \geq n_0.$$

Fourth, let $(f_n)_{n \in \mathbb{N}} \subset \mathcal{U}_\kappa$ be bounded above and $f \in \mathcal{U}_\kappa$ such that inequality (A.2) holds. Let $x \in X$ and $(x_n)_{n \in \mathbb{N}} \subset X$ with $x_n \rightarrow x$. Since $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $f_n(x_n) \leq \bar{f}^\varepsilon(x_n)$ for all $n \geq n_0$. We obtain $\limsup_{n \rightarrow \infty} f_n(x_n) \leq \bar{f}^\varepsilon(x)$ for all $\varepsilon > 0$, because \bar{f}^ε is upper semicontinuous. Hence, part (i) implies $\Gamma\text{-lim sup}_{n \rightarrow \infty} f_n \leq \inf_{\varepsilon > 0} \bar{f}^\varepsilon = f$. \square

We conclude this section by noting how the definition of \bar{f}^ε simplifies if (X, d) satisfies an additional geometric property.

Lemma A.3. *Assume that (X, d) has midpoints, i.e., for every $x, z \in X$ and $\lambda \in [0, 1]$, there exists $y_\lambda \in X$ with*

$$d(x, y_\lambda) = \lambda d(x, z) \quad \text{and} \quad d(y_\lambda, z) = (1 - \lambda)d(x, z).$$

Then, it holds $\bar{f}^\varepsilon = \inf_{\varepsilon' > \varepsilon} f^{\varepsilon'}$ for all $f \in \mathcal{U}_\kappa$ and $\varepsilon > 0$.

Proof. Let $x \in X$ and $\varepsilon' > \varepsilon$. Since (X, d) has midpoints, for every $z \in B(x, \varepsilon')$ there exists $y \in X$ with $d(x, y) \leq \varepsilon' - \varepsilon$ and $d(y, z) \leq \varepsilon$. Hence, we can estimate

$$\begin{aligned} f^{\varepsilon'}(x) &= \frac{1}{\kappa(x)} \sup_{z \in B(x, \varepsilon')} \left(\max \{ f(z)\kappa(z), -\frac{1}{\varepsilon'} \} + \varepsilon' \right) \\ &= \frac{1}{\kappa(x)} \sup_{y \in B(x, \varepsilon' - \varepsilon)} \sup_{z \in B(y, \varepsilon)} \left(\max \{ f(z)\kappa(z), -\frac{1}{\varepsilon'} \} + \varepsilon' \right) \\ &\leq c_{\varepsilon'}(x) \sup_{y \in B(x, \varepsilon' - \varepsilon)} \frac{1}{\kappa(y)} \sup_{z \in B(y, \varepsilon)} \left(\max \{ f(z)\kappa(z), -\frac{1}{\varepsilon'} \} + \varepsilon' \right), \end{aligned}$$

where $c_{\varepsilon'}(x) := \sup_{y \in B(x, \varepsilon' - \varepsilon)} \frac{\kappa(y)}{\kappa(x)}$. Continuity of κ implies $c_{\varepsilon'} \downarrow 1$ as $\varepsilon' \downarrow \varepsilon$. We obtain

$$\inf_{\varepsilon' > \varepsilon} f^{\varepsilon'}(x) \leq \inf_{\varepsilon' > \varepsilon} \sup_{y \in B(x, \varepsilon' - \varepsilon)} f^\varepsilon(y) = \bar{f}^\varepsilon(x).$$

The reverse estimate follows from inequality (A.1). \square

APPENDIX B. BASIC CONVEXITY ESTIMATES

Lemma B.1. *Let $\phi : C_\kappa \rightarrow \mathbb{R}$ be a convex functional. Then,*

$$\phi(f) - \phi(g) \leq \lambda \left(\phi \left(\frac{f - g}{\lambda} + g \right) - \phi(g) \right) \quad \text{for all } f, g \in C_\kappa \text{ and } \lambda \in (0, 1].$$

The previous statement remains valid if we replace C_κ by B_κ .

Proof. We use the convexity to estimate

$$\begin{aligned} \phi(f) - \phi(g) &= \phi \left(\lambda \left(\frac{f - g}{\lambda} + g \right) + (1 - \lambda)g \right) - \phi(g) \\ &\leq \lambda \phi \left(\frac{f - g}{\lambda} + g \right) + (1 - \lambda)\phi(g) - \phi(g) \\ &= \lambda \left(\phi \left(\frac{f - g}{\lambda} + g \right) - \phi(g) \right). \end{aligned} \quad \square$$

Let F_κ be the space of all functions $f: X \rightarrow [-\infty, \infty)$ with $\|f^+\|_\kappa < \infty$.

Lemma B.2. *Let $\Phi: C_\kappa \rightarrow F_\kappa$ be a convex monotone operator with $\Phi(0) = 0$.*

(i) *For every $r \geq 0$, there exists $c \geq 0$ such that*

$$\|\Phi(f)\|_\kappa \leq c\|f\|_\kappa \quad \text{for all } f \in B_{C_\kappa}(0, r).$$

One can choose $c := \frac{1}{r} \|(\Phi(\frac{r}{\kappa}))^+\|_\kappa$. In particular, the function $\Phi(f)$ is real-valued, i.e., $\Phi(f): X \rightarrow \mathbb{R}$ for all $f \in C_\kappa$.

(ii) *For every $r \geq 0$, there exists $c \geq 0$ such that*

$$\|\Phi(f) - \Phi(g)\|_\kappa \leq c\|f - g\|_\kappa \quad \text{for all } f, g \in B_{C_\kappa}(0, r).$$

One can choose $c := \frac{1}{r} \sup_{f' \in B_{C_\kappa}(0, 3r)} \|\Phi(f')\|_\kappa < \infty$.

The previous statements remain valid if we replace C_κ by B_κ .

Proof. First, let $r > 0$, $f \in B_{C_\kappa}(0, r)$ and $\lambda := \|f\|_\kappa/r$. We use the fact that Φ is convex and monotone with $\Phi(0) = 0$ to estimate

$$\Phi(f) = \Phi(\lambda \frac{1}{\lambda} f + (1 - \lambda)0) \leq \lambda \Phi(\frac{1}{\lambda} f) \leq \lambda \Phi(\frac{r}{\kappa}) = \frac{\|f\|_\kappa}{r} \Phi(\frac{r}{\kappa}).$$

Moreover, it follows from the convexity of Φ and $\Phi(0) = 0$ that

$$0 = \Phi(0) = \Phi(\frac{1}{2}f + \frac{1}{2}(-f)) \leq \frac{1}{2}\Phi(f) + \frac{1}{2}\Phi(-f).$$

We conclude $(\Phi(\pm f))(x) > -\infty$ for all $x \in X$ and $-\Phi(-f) \leq \Phi(f)$. Combining the previous estimates yields

$$-\frac{\|f\|_\kappa}{r} \Phi(\frac{r}{\kappa}) \leq -\Phi(-f) \leq \Phi(f) \leq \frac{\|f\|_\kappa}{r} \Phi(\frac{r}{\kappa}).$$

Hence, it holds $\|\Phi(f)\|_\kappa \leq c\|f\|_\kappa$ with $c := \frac{1}{r} \|(\Phi(\frac{r}{\kappa}))^+\|_\kappa < \infty$. For $r = 0$, the claim follows from $\Phi(0) = 0$.

Second, let $r \geq 0$ and $f, g \in B_{C_\kappa}(0, r)$. We define

$$\Phi_f: C_\kappa \rightarrow F_\kappa, \quad f' \mapsto \Phi(f + f') - \Phi(f) \quad \text{for all } f' \in C_\kappa.$$

Note, that $\|\Phi(f)\|_\kappa < \infty$ by the first part and therefore $\Phi(f') \in F_\kappa$ for all $f' \in C_\kappa$. Furthermore, it follows from the first part that

$$\|\Phi(f) - \Phi(g)\|_\kappa = \|\Phi_f(f - g)\|_\kappa \leq \frac{1}{2r} \|(\Phi_f(\frac{2r}{\kappa}))^+\|_\kappa \|f - g\|_\kappa \leq c\|f - g\|_\kappa,$$

where $c := \frac{1}{r} \sup_{f' \in B_{C_\kappa}(0, 3r)} \|\Phi(f')\|_\kappa < \infty$. □

APPENDIX C. EXTENSION OF CONVEX MONOTONE FUNCTIONS

Denote by ca_κ^+ the set of all Borel measures $\mu: \mathcal{B}(X) \rightarrow [0, \infty]$ with $\int_X \frac{1}{\kappa} d\mu < \infty$. Let $\phi: C_\kappa \rightarrow \mathbb{R}$ be a convex monotone functional with $\phi(0) = 0$. We define the convex conjugate of ϕ by

$$\phi^*: \text{ca}_\kappa^+ \rightarrow [0, \infty], \quad \mu \mapsto \sup_{f \in C_\kappa} (\mu f - \phi(f)), \quad \text{where } \mu f := \int_X f d\mu.$$

Let B_κ be the space of all Borel measurable functions $f: X \rightarrow [-\infty, \infty)$ such that $\|f^+\|_\kappa < \infty$. Denote by $B_{B_\kappa}(0, r) := \{f \in B_\kappa: \|f\|_\kappa \leq r\}$ the closed ball with radius $r \geq 0$ around zero. Using the ideas from [3], we obtain the following extension and dual representation result.

Theorem C.1. *Let $\phi: C_\kappa \rightarrow \mathbb{R}$ be a convex monotone functional with $\phi(0) = 0$, which is continuous from above. Then, the following statements hold:*

(i) For every $r \geq 0$, there exists a $\sigma(\text{ca}_\kappa^+, \text{C}_\kappa)$ -compact convex set $M_r \subset \text{ca}_\kappa^+$ with

$$\phi(f) = \max_{\mu \in M_r} (\mu f - \phi^*(\mu)) \quad \text{for all } f \in B_{\text{C}_\kappa}(0, r).$$

One can choose $M_r := \{\mu \in \text{ca}_\kappa^+ : \phi^*(\mu) \leq \phi(2r/\kappa) - 2\phi(-r/\kappa)\}$.

(ii) Define $\phi_1: \text{U}_\kappa \rightarrow [-\infty, \infty)$, $f \mapsto \inf\{\phi(g) : g \in \text{C}_\kappa, g \geq f\}$. The functional ϕ_1 is convex, monotone and the unique extension of ϕ , which is continuous from above. In addition, ϕ_1 admits the dual representation

$$\phi_1(f) = \max_{\mu \in M_r} (\mu f - \phi^*(\mu)) \quad \text{for all } r \geq 0 \text{ and } f \in B_{\text{U}_\kappa}(0, r).$$

(iii) Define $\phi_2: \text{B}_\kappa \rightarrow [-\infty, \infty)$, $f \mapsto \lim_{c \rightarrow \infty} \sup_{\mu \in \text{ca}_\kappa^+} (\mu(\max\{f, -\frac{c}{\kappa}\}) - \phi^*(\mu))$. The functional ϕ_1 is convex, monotone and an extension of ϕ . In addition, ϕ_1 admits the dual representation

$$\phi_2(f) = \sup_{\mu \in M'_r} (\mu f - \phi^*(\mu)) \quad \text{for all } r \geq 0 \text{ and } f \in B_{\text{B}_\kappa}(0, r),$$

where $M'_r := \bigcap_{\varepsilon > 0} \{\mu \in \text{ca}_\kappa^+ : \phi^*(\mu) \leq \phi(2r/\kappa) - 2\phi(-r/\kappa) + \varepsilon\}$ is $\sigma(\text{ca}_\kappa^+, \text{C}_\kappa)$ -compact and convex. In particular, for all $\varepsilon > 0$ and $r \geq 0$, there exists a compact set $K \subset X$ with $\phi_2(\frac{r}{\kappa} \mathbf{1}_{K^c}) < \varepsilon$.

Proof. First, we apply [3, Theorem 2.2] to obtain

$$\phi(f) = \max_{\mu \in \text{ca}_\kappa^+} (\mu f - \phi^*(\mu)) \quad \text{for all } f \in \text{C}_\kappa.$$

Let $r \geq 0$ and $f \in B_{\text{C}_\kappa}(0, r)$. Choose $\mu \in \text{ca}_\kappa^+$ with $\phi(f) = \mu f - \phi^*(\mu)$. It follows from the definition of ϕ^* and the monotonicity of ϕ that

$$\mu \frac{2r}{\kappa} - \phi\left(\frac{2r}{\kappa}\right) \leq \phi^*(\mu) = \mu f - \phi(f) \leq \mu \frac{r}{\kappa} - \phi\left(-\frac{r}{\kappa}\right).$$

We obtain $\mu \frac{r}{\kappa} \leq \phi\left(\frac{2r}{\kappa}\right) - \phi\left(-\frac{r}{\kappa}\right)$ and therefore $\phi^*(\mu) \leq \phi\left(\frac{2r}{\kappa}\right) - 2\phi\left(-\frac{r}{\kappa}\right)$. Hence,

$$\phi(f) = \max_{\mu \in M_r} (\mu f - \phi^*(\mu)) \quad \text{for all } f \in B_{\text{C}_\kappa}(0, r),$$

where $M_r := \{\mu \in \text{ca}_\kappa^+ : \phi^*(\mu) \leq \phi(2r/\kappa) - 2\phi(-r/\kappa)\}$. Moreover, the set M_r is convex and $\sigma(\text{ca}_\kappa^+, \text{C}_\kappa)$ -compact, see [3, Theorem 2.2].

Second, by monotonicity of ϕ , the functional ϕ_1 is monotone and an extension of ϕ . We show that $\phi_1(f) = \lim_{n \rightarrow \infty} \phi(f_n)$ for all $(f_n)_{n \in \mathbb{N}} \subset \text{C}_\kappa$ and $f \in \text{U}_\kappa$ with $f_n \downarrow f$. By definition of the infimum, there exists a sequence $(g_k)_{k \in \mathbb{N}} \subset \text{C}_\kappa$ such that $\phi(g_k) \rightarrow \phi_1(f)$ as $k \rightarrow \infty$. Let $g_n^k := f_n \vee g_k$ for all $k, n \in \mathbb{N}$. Since $g_n^k \downarrow g_k$ as $n \rightarrow \infty$, and ϕ is monotone and continuous from above, we obtain

$$\lim_{n \rightarrow \infty} \phi(f_n) \leq \lim_{n \rightarrow \infty} \phi(g_n^k) = \phi(g_k) \quad \text{for all } k \in \mathbb{N}.$$

Monotonicity of ϕ_1 implies $\phi_1(f) \leq \lim_{n \rightarrow \infty} \phi(f_n) \leq \lim_{k \rightarrow \infty} \phi(g_k) = \phi_1(f)$. In particular, it follows that ϕ_1 is convex. Indeed, let $f, g \in \text{U}_\kappa$ and $\lambda \in [0, 1]$. Since $\text{U}_\kappa = (\text{C}_\kappa)_\delta$ and C_κ is directed downwards, there exist sequences $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ in C_κ with $f_n \downarrow f$ and $g_n \downarrow g$. We obtain

$$\begin{aligned} \phi_1(\lambda f + (1 - \lambda)g) &= \lim_{n \rightarrow \infty} \phi(\lambda f_n + (1 - \lambda)g_n) \leq \lim_{n \rightarrow \infty} (\lambda \phi(f_n) + (1 - \lambda)\phi(g_n)) \\ &= \lambda \phi_1(f) + (1 - \lambda)\phi_1(g). \end{aligned}$$

Third, we show that ϕ_1 is continuous from above. Let $(f_n)_{n \in \mathbb{N}} \subset U_\kappa$ and $f \in U_\kappa$ with $f_n \downarrow f$. Since $U_\kappa = (C_\kappa)_\delta$, for every $k \in \mathbb{N}$, there exists a sequence $(f_k^n)_{n \in \mathbb{N}} \subset C_\kappa$ with $f_k^n \downarrow f_k$ as $n \rightarrow \infty$. Define $\tilde{f}_n := \min\{f_1^n, \dots, f_n^n\} \in C_\kappa$ for all $n \in \mathbb{N}$. It holds

$$\begin{aligned}\tilde{f}_{n+1} &= \min\{f_1^{n+1}, \dots, f_n^{n+1}, f_{n+1}^{n+1}\} \leq \min\{f_1^n, \dots, f_n^n\} = \tilde{f}_n \quad \text{for all } n \in \mathbb{N}, \\ f_n &= \min\{f_1, \dots, f_n\} \leq \min\{f_1^n, \dots, f_n^n\} = \tilde{f}_n \quad \text{for all } n \in \mathbb{N}, \\ \tilde{f}_n &= \min\{f_1^n, \dots, f_k^n, \dots, f_n^n\} \leq f_k^n \quad \text{for all } k, n \in \mathbb{N} \text{ with } k \leq n.\end{aligned}$$

We conclude that $f = \lim_{n \rightarrow \infty} f_n \leq \lim_{n \rightarrow \infty} \tilde{f}_n \leq \lim_{n \rightarrow \infty} f_k^n = f_k$ for all $k \in \mathbb{N}$. Thus, it follows from the monotonicity of ϕ_1 and the second part of this proof that $\phi_1(f) \leq \lim_{n \rightarrow \infty} \phi_1(f_n) \leq \lim_{n \rightarrow \infty} \phi(\tilde{f}_n) = \phi_1(f)$. We have shown that ϕ_1 is continuous from above. Hence, [3, Theorem 2.2] implies

$$\phi_1(f) = \max_{\mu \in \text{ca}_\kappa^+} (\mu f - \phi^*(\mu)) \quad \text{for all } f \in U_\kappa.$$

By the same arguments as in the first step, the maximum in the previous equation can be taken over the set M_r for all $r \geq 0$ and $f \in B_{U_\kappa}(0, r)$. The uniqueness of ϕ_1 as extension, which is continuous from above, follows from $U_\kappa = (C_\kappa)_\delta$.

Fourth, by definition, the functional ϕ_2 is convex and monotone. Since ϕ_1 is continuous from above, we obtain from the dual representation of ϕ_1 that

$$\phi_1(f) = \lim_{c \rightarrow \infty} \phi_1(\max\{f, -\frac{c}{\kappa}\}) = \phi_2(f) \quad \text{for all } f \in U_\kappa.$$

Similar to the first part of this proof, it follows that the supremum in the definition of ϕ_2 can be taken over M'_r for all $r \geq 0$ and $f \in B_{B_\kappa}(0, r)$. By [3, Theorem 2.2], the set M'_r is $\sigma(\text{ca}_\kappa^+, C_\kappa)$ -compact and convex. The last statement follows from $\phi^* \geq 0$ and the fact that, by Prokhorov's theorem, the set $\{\mu_\kappa : \mu \in M'_r\}$ is tight for all $r \geq 0$, where $\mu_\kappa(A) := \int_A \frac{1}{\kappa} d\mu$ for all $A \in \mathcal{B}(X)$. \square

Remark C.2. Let $\phi: C_\kappa \rightarrow \mathbb{R}$ be a convex monotone functional with $\phi(0) = 0$, which is continuous from above at zero, i.e., $\phi(f_n) \downarrow 0$ for all $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ with $f_n \downarrow 0$. Then, it follows from the proof of [3, Theorem 2.2] that ϕ is continuous from above, i.e., $\phi(f_n) \downarrow \phi(f)$ for all $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ and $f \in C_\kappa$ with $f_n \downarrow f$. However, if we replace C_κ by U_κ , this statement does not remain valid, because U_κ is not a vector space.

With the help of Theorem C.1, we can show the assertions from Remark 2.5(i) and (ii).

Corollary C.3. *Let $(S(t))_{t \geq 0}$ be a family of operators $S(t): C_\kappa \rightarrow U_\kappa$ which satisfy the conditions (S1)-(S4). Then, the following statements hold:*

- (i) *For every $t \geq 0$, there exists a unique extension $S(t): U_\kappa \rightarrow U_\kappa$, which is continuous from above. The family of extended operators satisfies the conditions (S1)-(S4) with U_κ instead of C_κ . In addition, for every $x \in X$, the functional*

$$U_\kappa \rightarrow [-\infty, \infty), f \mapsto (S(t)f)(x)$$

can be even extended to the space B_κ such that, for every $\varepsilon > 0$ and $c \geq 0$, there exists a compact set $K \subset X$ with $(S(t)(\frac{c}{\kappa} \mathbb{1}_{K^c}))(x) < \varepsilon$.

- (ii) *Let $t \geq 0$ and $K \subset X$ compact. Then, for every $\varepsilon > 0$ and $c \geq 0$, there exists a compact set $K_1 \subset X$ such that*

$$\sup_{(s,x) \in [0,t] \times K} (S(t)(\frac{c}{\kappa} \mathbb{1}_{K_1^c}))(x) \leq \varepsilon.$$

Proof. First, we extend $S(t)$ from C_κ to U_κ . Let $t \geq 0$ and $x \in X$. By Lemma B.2, it holds $(S(t)f)(x) \in \mathbb{R}$ for all $f \in C_\kappa$. Thus, the conditions (S1) and (S2) yield

$$\phi^{t,x}: C_\kappa \rightarrow \mathbb{R}, f \mapsto (S(t)f)(x)$$

satisfies the assumptions of Theorem C.1. Define $(S(t)f)(x) := \phi_1^{t,x}(f)$ for all $f \in U_\kappa$, where $\phi_1^{t,x}$ denotes the extension of $\phi^{t,x}$ from Theorem C.1(ii). The family of extended operators satisfies the conditions (S1), (S2) and (S4) with U_κ instead of C_κ and is the unique extension which is continuous from above. Next, we verify condition (S3), i.e.,

$$\Gamma\text{-}\limsup_{s \rightarrow t} S(s)f \leq S(t)f \quad \text{for all } t \geq 0 \text{ and } f \in U_\kappa.$$

Fix $t \geq 0$, $f \in U_\kappa$ and $x \in X$. Let $(t_n)_{n \in \mathbb{N}} \subset [0, \infty)$ and $(x_n)_{n \in \mathbb{N}} \subset X$ with $t_n \rightarrow t$ and $x_n \rightarrow x$. We use Theorem C.1(ii) and condition (S3) to estimate

$$\begin{aligned} \limsup_{n \rightarrow \infty} (S(t_n)f)(x_n) &= \inf_{n \in \mathbb{N}} \sup_{k \geq n} \inf \{ (S(t_k)g)(x_k) : g \in C_\kappa, g \geq f \} \\ &\leq \inf \left\{ \limsup_{n \rightarrow \infty} (S(t_n)g)(x_n) : g \in C_\kappa, g \geq f \right\} \\ &\leq \inf \{ (S(t)g)(x) : g \in C_\kappa, g \geq f \} = (S(t)f)(x). \end{aligned}$$

The statement about the extension to B_κ holds because of Theorem C.1(iii).

Second, let $c, t \geq 0$, $K \subset X$ compact and $\varepsilon > 0$. By part (i), we can define

$$\phi_{s,x}: B_\kappa \rightarrow [-\infty, \infty), f \mapsto (S(s)f)(x) \quad \text{for all } (s, x) \in [0, t] \times K.$$

For every $(s, x) \in [0, t] \times K$ and $K_1 \subset X$ compact, Theorem C.1(iii) implies

$$0 \leq \phi_{s,x} \left(\frac{c}{\kappa} \mathbf{1}_{K_1^c} \right) \leq \sup_{\mu \in M} \mu \left(\frac{c}{\kappa} \mathbf{1}_{K_1^c} \right),$$

where $M := \bigcup_{(s,x) \in [0,t] \times K} \{ \mu \in \text{ca}_\kappa^+ : \phi_{s,x}^*(\mu) \leq \phi_{s,x}(2c/\kappa) - 2\phi_{s,x}(-c/\kappa) \}$. Furthermore, by Dini's theorem, the functional

$$\phi: C_\kappa \rightarrow \mathbb{R}, f \mapsto \sup_{(s,x) \in [0,t] \times K} \phi_{s,x}(f)$$

is well-defined, convex, monotone and continuous from above. Hence, [3, Theorem 2.2] and condition (S4) imply that the set

$$M \subset \left\{ \mu \in \text{ca}_\kappa^+ : \phi^*(\mu) \leq \sup_{(s,x) \in [0,t] \times K} \left| \phi_{s,x} \left(\frac{2c}{\kappa} \right) - 2\phi_{s,x} \left(-\frac{c}{\kappa} \right) \right| \right\}$$

is $\sigma(\text{ca}_\kappa^+, C_\kappa)$ -relatively compact. The claim follows from Prokhorov's theorem which ensures that $\{ \mu_\kappa : \mu \in M \}$ is tight, where $\mu_\kappa(A) := \int_A \frac{1}{\kappa} d\mu$ for all $A \in \mathcal{B}(X)$. \square

Lemma C.4. *Let $(\phi_n)_{n \in \mathbb{N}}$ be a sequence of functionals $\phi_n: C_\kappa \rightarrow \mathbb{R}$ which satisfy the following conditions:*

- ϕ_n is convex, monotone and $\phi_n(0) = 0$ for all $n \in \mathbb{N}$,
- $\sup_{n \in \mathbb{N}} \sup_{f \in B_{C_\kappa}(0,r)} |\phi_n(f)| < \infty$ for all $r \geq 0$,
- $\sup_{n \in \mathbb{N}} \phi_n(f_k) \downarrow 0$ as $k \rightarrow \infty$ for all $(f_k)_{k \in \mathbb{N}} \subset C_\kappa$ with $f_k \downarrow 0$.

For every $n \in \mathbb{N}$, by Theorem C.1, there exists a unique extension of $\phi_n: C_\kappa \rightarrow \mathbb{R}$ to a functional $\phi_n: U_\kappa \rightarrow [-\infty, \infty)$ which is continuous from above. Let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be bounded sequences in U_κ . Then,

$$\limsup_{n \rightarrow \infty} \phi_n(f_n + g_n) \leq \limsup_{n \rightarrow \infty} \phi_n(f + g_n), \quad \text{where } f := \Gamma\text{-}\limsup_{n \rightarrow \infty} f_n.$$

Proof. First, we show that $\limsup_{n \rightarrow \infty} \phi_n(f_n + g_n) \leq \limsup_{n \rightarrow \infty} \phi_n(\bar{f}^\varepsilon + g_n)$ for all $\varepsilon \in (0, 1]$. Since the functionals $(\phi_n)_{n \in \mathbb{N}}$ are uniformly bounded, the functional

$$\phi: C_\kappa \rightarrow \mathbb{R}, f \mapsto \sup_{n \in \mathbb{N}} \phi_n(f)$$

is well-defined, convex, monotone and satisfies $\phi(0) = 0$. Furthermore, it follows from Theorem C.1 and $\phi^* \leq \phi_n^*$ that

$$\phi_n(f) = \max_{\mu \in M_r} (\mu f - \phi_n^*(\mu)) \quad \text{for all } n \in \mathbb{N}, r \geq 0 \text{ and } f \in B_{U_\kappa}(0, r),$$

where $M_r := \{\mu \in \text{ca}_\kappa^+ : \phi^*(\mu) \leq \sup_{n \in \mathbb{N}} |\phi_n(2r/\kappa) - 2\phi_n(-r/\kappa)|\}$. In the sequel, we fix $r := \sup_{n \in \mathbb{N}} (\|f_n\|_\kappa + \|g_n\|_\kappa + 1)$ and $M := M_r$. Let $\varepsilon > 0$. Since ϕ is continuous from above, by [3, Theorem 2.2] and Prokhorov's theorem, the set $\{\mu_\kappa : \mu \in M\}$ is tight, where $\mu_\kappa(A) := \int_A \frac{1}{\kappa} d\mu$ for all $A \in \mathcal{B}(X)$. Hence, there exists $K \subset X$ compact with $\sup_{\mu \in M} \mu_\kappa(K^c)(r + 1/\varepsilon) < \varepsilon$. Moreover, by Lemma A.2, we can choose $n_0 \in \mathbb{N}$ such that

$$f_n(x) \leq \bar{f}^\varepsilon(x) \quad \text{for all } x \in K \text{ and } n \geq n_0.$$

For every $n \geq n_0$, it follows from $\bar{f}^\varepsilon \geq -\varepsilon/\kappa$ that

$$\begin{aligned} \phi_n(f_n + g_n) &= \max_{\mu \in M} (\mu(f_n + g_n) - \phi_n^*(\mu)) \\ &\leq \max_{\mu \in M} \left(\mu \left(\bar{f}^\varepsilon + g_n + \left(\frac{r+1/\varepsilon}{\kappa} \right) \mathbb{1}_{K^c} \right) - \phi_n^*(\mu) \right) \\ &\leq \max_{\mu \in M} (\mu(\bar{f}^\varepsilon + g_n) - \phi_n^*(\mu)) + \varepsilon = \phi_n(\bar{f}^\varepsilon + g_n) + \varepsilon. \end{aligned}$$

We obtain $\limsup_{n \rightarrow \infty} \phi_n(f_n + g_n) \leq \limsup_{n \rightarrow \infty} \phi_n(\bar{f}^\varepsilon + g_n)$ for all $\varepsilon > 0$.

Second, we show $\inf_{\varepsilon \in (0, 1]} \limsup_{n \rightarrow \infty} \phi_n(\bar{f}^\varepsilon + g_n) \leq \limsup_{n \rightarrow \infty} \phi_n(f + g_n)$. Using the dual representation from the first part, we obtain

$$\begin{aligned} \inf_{\varepsilon \in (0, 1]} \limsup_{n \rightarrow \infty} \phi_n(\bar{f}^\varepsilon + g_n) &= \inf_{n \in \mathbb{N}} \inf_{\varepsilon \in (0, 1]} \max_{\mu \in M} \sup_{k \geq n} (\mu(\bar{f}^\varepsilon + g_k) - \phi_k^*(\mu)) \\ &= \inf_{n \in \mathbb{N}} \inf_{\varepsilon \in (0, 1]} \max_{\mu \in M} (\mu \bar{f}^\varepsilon - \alpha_n(\mu)), \end{aligned}$$

where $\alpha_n(\mu) := \inf_{k \geq n} (\phi_k^*(\mu) - \mu g_k)$. For fixed $n \in \mathbb{N}$, we want to interchange the maximum over $\mu \in M$ with the infimum over $\varepsilon \in (0, 1]$ by using [21, Theorem 2]. To do so, we have to replace α_n by a function which is convex and lower semicontinuous. It holds $\inf_{\mu \in M} \alpha_n(\mu) > -\infty$, because $\phi_k^* \geq 0$ and $(g_k)_{k \in \mathbb{N}}$ is bounded. Hence, we can define $\bar{\alpha}_n: M \rightarrow \mathbb{R}$ as the lower semicontinuous convex hull of α_n , i.e., the supremum over all lower semicontinuous convex functions which are dominated by α_n . Fenchel–Moreau's theorem, [21, Theorem 2] and Lemma A.2 imply

$$\begin{aligned} \inf_{n \in \mathbb{N}} \inf_{\varepsilon \in (0, 1]} \max_{\mu \in M} (\mu \bar{f}^\varepsilon - \alpha_n(\mu)) &= \inf_{n \in \mathbb{N}} \inf_{\varepsilon \in (0, 1]} \max_{\mu \in M} (\mu \bar{f}^\varepsilon - \bar{\alpha}_n(\mu)) \\ &= \inf_{n \in \mathbb{N}} \max_{\mu \in M} (\mu f - \bar{\alpha}_n(\mu)) = \inf_{n \in \mathbb{N}} \max_{\mu \in M} (\mu f - \alpha_n(\mu)) = \limsup_{n \rightarrow \infty} \phi_n(f + g_n). \quad \square \end{aligned}$$

APPENDIX D. PROOFS OF SECTION 4

We need the following version of Arzéla–Ascoli's theorem. A sequence $(f_n)_{n \in \mathbb{N}}$ of functions $f_n: X \rightarrow \mathbb{R}$ is called uniformly equicontinuous if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $d(x, y) < \delta$.

Lemma D.1. *Let $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ be bounded and uniformly equicontinuous. Then, there exist a function $f \in C_\kappa$ and a subsequence $(n_l)_{l \in \mathbb{N}}$ with $f_{n_l} \rightarrow f$ uniformly on compacts.*

Proof. Let $D \subset X$ be countable and dense. By the Bolzano–Weierstrass theorem and a diagonalization argument, there exists a subsequence $(n_l)_{l \in \mathbb{N}}$ such that the limit

$$f(x) := \lim_{l \rightarrow \infty} f_{n_l}(x) \in \mathbb{R}$$

exists for all $x \in D$. Moreover, the mapping $D \rightarrow \mathbb{R}$, $x \mapsto f(x)$ is uniformly continuous and satisfies $\sup_{x \in D} |f(x)|\kappa(x) < \infty$. Hence, there exists a unique extension $f \in C_\kappa$. Since $(f_n)_{n \in \mathbb{N}}$ is uniformly equicontinuous, it holds $f_{n_l}(x) \rightarrow f(x)$ for all $x \in X$. In addition, we have $f_{n_l} \rightarrow f$ uniformly on compacts. Indeed, assume by contradiction that there exist $\varepsilon > 0$, $K \subset X$ compact and a subsequence $(k_l)_{l \in \mathbb{N}}$ of $(n_l)_{l \in \mathbb{N}}$ with

$$\sup_{x \in K} |f_{k_l}(x) - f(x)| \geq \varepsilon \quad \text{for all } l \in \mathbb{N}. \quad (\text{D.1})$$

By Arzela–Ascoli’s theorem, the sequence $(f_{k_l})_{l \in \mathbb{N}}$ has a subsequence which converges uniformly on K to a continuous function $g: K \rightarrow \mathbb{R}$. Due to the pointwise convergence $f_{k_l}(x) \rightarrow f(x)$ for all $x \in K$, we obtain $f = g$. This shows that

$$\lim_{l \rightarrow \infty} \sup_{x \in K} |f(x) - f_{k_l}(x)| = 0,$$

which contradicts inequality (D.1). \square

Proof of Theorem 4.3. First, as a consequence of Lemma D.1 and the results in [8, Section 2.1], there exist a family $(S(t))_{t \geq 0}$ of operators $S(t): \overline{\mathcal{D}} \rightarrow C_\kappa$ and a subsequence $(n_l)_{l \in \mathbb{N}} \subset \mathbb{N}$ which satisfy the following conditions:

- $S(t)f = \lim_{l \rightarrow \infty} I(\pi_{n_l}^t)f$ uniformly on compacts for all $(f, t) \in \overline{\mathcal{D}} \times \mathcal{T}$.
- $S(0) = \text{id}_{\overline{\mathcal{D}}}$.
- $S(t)$ is convex and monotone with $S(t)0 = 0$ for all $t \geq 0$.
- The mapping $[0, \infty) \rightarrow C_\kappa$, $t \mapsto S(t)f$ is continuous for all $f \in \overline{\mathcal{D}}$.
- $S(t): B_{\overline{\mathcal{D}}}(0, r) \rightarrow B_{C_\kappa}(0, \alpha(r, t))$ for all $r, t \geq 0$.
- $\|S(t)f - S(t)g\|_\kappa \leq e^{t\omega_\alpha(r, t)}\|f - g\|_\kappa$ for all $t \geq 0$ and $f, g \in B_{\overline{\mathcal{D}}}(0, r)$.

We remark that the results [8] are formulated with the assumption that $(I(\pi_n^t)f)_{n \in \mathbb{N}}$ is relatively compact w.r.t. the norm for all $(f, t) \in \mathcal{D} \times \mathcal{T}$. However, it follows immediately from the corresponding proofs that the existence of a subsequence $(I(\pi_{n_l}^t)f)_{l \in \mathbb{N}}$ which converges uniformly on compacts is sufficient for the previous statements. The latter is ensured by Assumption 4.1(iii) and (v) and Lemma D.1. Define $\|f\|_{\infty, Y} := \|f\mathbf{1}_Y\|_\infty$ for all $f: X \rightarrow \mathbb{R}$ and $Y \subset X$.

Second, we show that, for every $\varepsilon > 0$, $r, t \geq 0$ and compact set $K \subset X$, there exist another compact set $K_1 \subset X$ and a constant $c \geq 0$ with

$$\|I(\pi_n^t)f - I(\pi_n^t)g\|_{\infty, K} \leq c\|f - g\|_{\infty, K_1} + \varepsilon \quad (\text{D.2})$$

for all $n \in \mathbb{N}$ and $f, g \in B_{C_\kappa}(0, r)$. Assumption 4.1(iii) and [8, Lemma 2.7] imply

$$I(\pi_n^t): B_{C_\kappa}(0, r) \rightarrow B_{C_\kappa}(0, \alpha(r, t)) \quad \text{for all } n \in \mathbb{N}.$$

Moreover, the operators $I(\pi_n^t)$ are convex and monotone with $I(\pi_n^t)0 = 0$. Hence, by Assumption 4.1(vi) and Theorem C.1, they can be extended to B_κ in such a way that

$$I(\pi_n^t): B_{B_\kappa}(0, r) \rightarrow B_{B_\kappa}(0, \alpha(r, t)) \quad \text{for all } n \in \mathbb{N}. \quad (\text{D.3})$$

Indeed, the previous equation follows from

$$-\alpha(r, t) \leq I(\pi_n^t)\left(-\frac{r}{\kappa}\right) \leq I(\pi_n^t)f \leq I(\pi_n^t)\left(\frac{r}{\kappa}\right) \leq \alpha(r, t) \quad \text{for all } f \in B_{B_\kappa}(0, r).$$

Hence, for every $n \in \mathbb{N}$ and $f, g \in B_{B_\kappa}(0, r)$, Lemma B.2(ii) yields

$$\|I(\pi_n^t)f - I(\pi_n^t)g\|_\kappa \leq \alpha(3r, t)\|f - g\|_\kappa. \quad (\text{D.4})$$

Due to inequality (D.3) and $\inf_{x \in K} \kappa(x) > 0$, we can choose $\lambda \in (0, 1]$ with

$$\sup_{n \in \mathbb{N}} \sup_{f' \in B_{B_\kappa}(0, 2r)} \lambda \|I(\pi_n^t) f'\|_{\infty, K} \leq \frac{\varepsilon}{6}.$$

Furthermore, by Assumption 4.1(vi), inequality (D.3), and Theorem C.1, there exists a compact set $K_1 \subset X$ with

$$\sup_{n \in \mathbb{N}} \lambda \|I(\pi_n^t) \left(\frac{2r}{\lambda \kappa} \mathbf{1}_{K_1^c} \right)\|_{\infty, K} \leq \frac{\varepsilon}{3}.$$

For details, we refer to the proof of Corollary C.3. We use Lemma B.1, the previous to inequalities and the fact that $I(\pi_n^t)$ is convex and monotone to conclude

$$\begin{aligned} I(\pi_n^t) f - I(\pi_n^t)(f \mathbf{1}_{K_1}) &\leq \lambda I(\pi_n^t) \left(\frac{f}{\lambda} \mathbf{1}_{K_1^c} + f \mathbf{1}_{K_1} \right) - \lambda I(\pi_n^t)(f \mathbf{1}_{K_1}) \\ &\leq \frac{\lambda}{2} I(\pi_n^t) \left(\frac{2f}{\lambda} \mathbf{1}_{K_1^c} \right) + \frac{\lambda}{2} I(\pi_n^t)(2f \mathbf{1}_{K_1}) + \frac{\varepsilon}{6} \\ &\leq \frac{\lambda}{2} I(\pi_n^t) \left(\frac{2r}{\lambda \kappa} \mathbf{1}_{K_1^c} \right) + \frac{\varepsilon}{3} \leq \frac{\varepsilon}{2}, \end{aligned}$$

where all functions are evaluated at a fixed point $x \in K$. Interchanging the roles of f and $f \mathbf{1}_{K_1}$ in the previous estimate yields

$$\|I(\pi_n^t) f - I(\pi_n^t)(f \mathbf{1}_{K_1})\|_{\infty, K} \leq \frac{\varepsilon}{2}.$$

Moreover, the same inequality holds with g instead of f . Hence, inequality (D.4) implies

$$\begin{aligned} \|I(\pi_n^t) f - I(\pi_n^t) g\|_{\infty, K} &\leq \|I(\pi_n^t)(f \mathbf{1}_{K_1}) - I(\pi_n^t)(g \mathbf{1}_{K_1})\|_{\infty, K} + \varepsilon \\ &\leq \sup_{x \in K} \frac{1}{\kappa(x)} \|I(\pi_n^t)(f \mathbf{1}_{K_1}) - I(\pi_n^t)(g \mathbf{1}_{K_1})\|_{\kappa} + \varepsilon \\ &\leq \sup_{x \in K} \frac{\alpha(3r, t)}{\kappa(x)} \|f \mathbf{1}_{K_1} - g \mathbf{1}_{K_1}\|_{\kappa} + \varepsilon \leq c \|f - g\|_{\infty, K_1} + \varepsilon, \end{aligned}$$

where $c := \alpha(3r, t) \|\kappa\|_{\infty} \sup_{x \in K} 1/\kappa(x)$.

Third, we extend S to C_κ . It follows from inequality (D.2) and the first part that, for every $\varepsilon > 0$, $r, t \geq 0$ and compact set $K \subset X$, there exist another compact set $K_1 \subset X$ and a constant $c \geq 0$ such that, for all $f, g \in B_{C_\kappa}(0, r) \cap \overline{\mathcal{D}}$,

$$\|S(t)f - S(t)g\|_{\infty, K} \leq c \|f - g\|_{\infty, K_1} + \varepsilon. \quad (\text{D.5})$$

Let $t \geq 0$ and $f \in C_\kappa$. By Assumption 4.1(v), we can choose a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{D}$ with $\|f_n\|_{\kappa} \leq \|f\|_{\kappa}$ for all $n \in \mathbb{N}$ and $f_n \rightarrow f$ uniformly on compacts. Inequality (D.5) ensures that the limit

$$S(t)f := \lim_{n \rightarrow \infty} S(t)f_n \in C_\kappa$$

exists, uniformly on compacts, and is independent of the choice of the approximating sequence $(f_n)_{n \in \mathbb{N}}$. Moreover, we remark that inequality (D.5) is preserved in the limit, i.e., we can replace $B_{C_\kappa}(0, r) \cap \overline{\mathcal{D}}$ by $B_{C_\kappa}(0, r)$. In particular, for every bounded sequence $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ and $f \in C_\kappa$ with $f_n \rightarrow f$ uniformly on compacts, it holds $S(t)f_n \rightarrow S(t)f$ uniformly on compacts.

Fourth, we verify equation (4.3) and the conditions (S1)-(S4). Let $t \in \mathcal{T}$, $f \in C_\kappa$, $\varepsilon > 0$ and $K \subset X$ compact. Choose a further compact set $K_1 \subset X$ and $c \geq 0$ such

that inequality (D.2) and inequality (D.5) are satisfied with $r := \|f\|_\kappa$. Moreover, by Assumption 4.1(v), there exists $g \in B_{C_\kappa}(0, r) \cap \mathcal{D}$ with $\|f - g\|_{\infty, K_1} < \varepsilon$. We obtain

$$\begin{aligned} \|S(t)f - I(\pi_{n_l}^t)f\|_{\infty, K} &\leq \|S(t)f - S(t)g\|_{\infty, K} + \|S(t)g - I(\pi_{n_l}^t)g\|_{\infty, K} \\ &\quad + \|I(\pi_{n_l}^t)g - I(\pi_{n_l}^t)f\|_{\infty, K} \\ &\leq 2c\|f - g\|_{\infty, K_1} + 2\varepsilon + \|S(t)g - I(\pi_{n_l}^t)g\|_{\infty, K} \\ &\leq 2(c+1)\varepsilon + \|S(t)g - I(\pi_{n_l}^t)g\|_{\infty, K}. \end{aligned}$$

It follows from $g \in \mathcal{D}$ and the first part that $\lim_{l \rightarrow \infty} \|S(t)f - I(\pi_{n_l}^t)f\|_{\infty, K} = 0$. Condition (S1) follow from Assumption 4.1(ii) and the construction of S . Let $t \geq 0$, $f \in C_\kappa$, $K \subset X$ compact and $\varepsilon > 0$. Since the constant c in inequality (D.5) can be chosen uniformly for compact time intervals, there exists $g \in \mathcal{D}$ with

$$\|S(s)f - S(t)f\|_{\infty, K} \leq \|S(s)g - S(t)g\|_{\infty, K} + \varepsilon \quad \text{for all } s \in [0, t+1].$$

It follows the continuity of the mapping $[0, \infty) \rightarrow C_\kappa$, $g \mapsto S(t)g$ that

$$\lim_{s \rightarrow t} \|S(s)f - S(t)f\|_{\infty, K} = 0. \quad (\text{D.6})$$

Now, the conditions (S2)-(S4) follows immediately from what we have shown so far. Indeed, for every $r, t \geq 0$ and $f \in B_{C_\kappa}(0, r)$, we use inequality (D.3) to estimate

$$\|S(t)f\|_\kappa \leq \sup_{s \in [0, t] \cap \mathcal{T}} \|S(s)f\|_\kappa \leq \sup_{s \in [0, t] \cap \mathcal{T}} \sup_{n \in \mathbb{N}} \|I(\pi_n^s)f\|_\kappa \leq \alpha(r, t).$$

Moreover, Lemma A.1(v) implies $\Gamma\text{-}\lim_{s \rightarrow t} S(s)f = S(t)f$. Let $t \geq 0$ and $(f_n)_{n \in \mathbb{N}} \subset C_\kappa$ with $f_n \downarrow 0$. It follows from Assumption 4.1(vi) that

$$0 \leq S(t)f_n \leq \sup_{s \in [0, t] \cap \mathcal{T}} S(s)f_n \leq \sup_{s \in [0, t] \cap \mathcal{T}} \sup_{k \in \mathbb{N}} I(\pi_k^s)f_n \downarrow 0 \quad \text{as } n \rightarrow \infty.$$

Fifth, we show that S forms a semigroup. Let $s, t \in \mathcal{T}$, $n \in \mathbb{N}$ with $2^n s, 2^n t \in \mathbb{N}_0$ and $f \in B_{C_\kappa}(0, r)$ for some $r \geq 0$. It holds

$$\begin{aligned} S(s+t)f - S(s)S(t)f &= (S(s+t)f - I(\pi_n^{s+t})f) + (I(\pi_n^s)I(\pi_n^t)f - I(\pi_n^s)S(t)f) \\ &\quad + (I(\pi_n^s)S(t)f - S(s)S(t)f). \end{aligned}$$

By equation (4.3), the first and third term on the right-hand side convergence to zero uniformly on compacts along the subsequence $(n_l)_{l \in \mathbb{N}}$. It remains to show

$$I(\pi_{n_l}^t)I(\pi_{n_l}^s)f - I(\pi_{n_l}^s)S(t)f \rightarrow 0$$

pointwise as $l \rightarrow \infty$. In the sequel, we suppress the subsequence in the notation and evaluate all functions at a fixed point $x \in X$. Let $\varepsilon > 0$. Choose $\lambda \in (0, 1)$ with

$$\lambda\alpha(r, s+t) < \varepsilon \quad \text{and} \quad \lambda\alpha(4\alpha(r, t), s) < \varepsilon.$$

By Assumption 4.1(vi), inequality (D.3) and Theorem C.1, there exists a compact set $K \subset X$ with

$$\sup_{n \in \mathbb{N}} I(\pi_n^s)(\frac{4c}{\kappa}\mathbb{1}_{K^c}) < \frac{\varepsilon}{\kappa}, \quad \text{where} \quad c := \frac{\alpha(r, t)}{\lambda}. \quad (\text{D.7})$$

In addition, for $\delta := \lambda r$, we can choose $n_0 \in \mathbb{N}$ with

$$\sup_{x \in K} |I(\pi_n^t)f - S(t)f|(x) < \frac{\delta}{\kappa} \quad \text{for all } n \geq n_0. \quad (\text{D.8})$$

For every $n \geq n_0$, we use Lemma B.1, inequality (D.3), inequality (D.8), the monotonicity of $I(\pi_n^s)$ and $r \leq \alpha(r, t)$ to estimate

$$\begin{aligned} I(\pi_n^s)I(\pi_n^t)f - I(\pi_n^s)S(t)f &\leq \lambda I(\pi_n^s) \left(\frac{I(\pi_n^t)f - S(t)f}{\lambda} + S(t)f \right) - \lambda I(\pi_n^s)S(t)f \\ &\leq \lambda I(\pi_n^s) \left(\frac{\delta}{\lambda\kappa} + \frac{2\alpha(r, t)}{\lambda\kappa} \mathbb{1}_{K^c} + \frac{\alpha(r, t)}{\kappa} \right) + \frac{\lambda\alpha(r, s+t)}{\kappa} \\ &\leq \lambda I(\pi_n^s) \left(\frac{2\alpha(r, t)}{\kappa} + \frac{2c}{\kappa} \mathbb{1}_{K^c} \right) + \frac{\varepsilon}{\kappa}. \end{aligned}$$

Furthermore, convexity of $I(\pi_n^s)$, inequality (D.3) and inequality (D.7) imply

$$\begin{aligned} \lambda I(\pi_n^s) \left(\frac{2\alpha(r, t)}{\kappa} + \frac{2c}{\kappa} \mathbb{1}_{K^c} \right) + \frac{\varepsilon}{\kappa} &\leq \frac{\lambda}{2} I(\pi_n^s) \left(\frac{4\alpha(r, t)}{\kappa} \right) + \frac{\lambda}{2} I(\pi_n^s) \left(\frac{4c}{\kappa} \mathbb{1}_{K^c} \right) + \frac{\varepsilon}{\kappa} \\ &\leq \frac{\lambda\alpha(4\alpha(r, t), s)}{2\kappa} + \frac{\varepsilon}{2\kappa} + \frac{\varepsilon}{\kappa} \leq \frac{2\varepsilon}{\kappa}. \end{aligned}$$

Interchanging the roles of f and g in the previous estimate yields

$$|I(\pi_n^s)I(\pi_n^t)f - I(\pi_n^s)S(t)f| < \frac{2\varepsilon}{\kappa(x)} \quad \text{for all } n \geq n_0.$$

We conclude $S(s+t)f = S(s)S(t)f$ for all $s, t \in \mathcal{T}$ and $f \in C_\kappa$. Now, let $s, t \geq 0$ be arbitrary. Choose $(s_n)_{n \in \mathbb{N}} \subset [0, s] \cap \mathcal{T}$ and $(t_n)_{n \in \mathbb{N}} \subset [0, t] \cap \mathcal{T}$ with $s_n \rightarrow s$ and $t_n \rightarrow t$. It follows from condition (S4), inequality (D.5) and equation (D.6) that

$$\begin{aligned} S(s+t)f - S(s)S(t)f &= (S(s+t)f - S(s_n+t_n)f) + (S(s_n)S(t_n)f - S(s_n)S(t)f) \\ &\quad + (S(s_n)S(t)f - S(s)S(t)f) \rightarrow 0 \end{aligned}$$

uniformly on compacts.

Sixth, for every $f, g \in C_\kappa$, it follows from [8, Theorem 4.3 and Lemma 4.4] that

$$\lim_{h \downarrow 0} \left\| \frac{I(h)f - f}{h} - g \right\|_\kappa \quad \text{implies} \quad \lim_{h \downarrow 0} \left\| \frac{S(h)f - f}{h} - g \right\|_\kappa = 0.$$

In [8] it is assumed that the semigroup S is strongly continuous. However, we obtain immediately from the proof of [8, Theorem 4.3] that the inequality

$$\left\| \frac{S(t)f - f}{t} - g \right\|_\kappa \leq \limsup_{s \rightarrow t} \left\| \frac{S(s)f - f}{s} - g \right\|_\kappa \quad \text{for all } t > 0$$

is sufficient. The latter is ensured by equation (D.6). Furthermore, by Remark 2.9(i), it holds $S(t): \mathcal{L}^S \rightarrow \mathcal{L}^S$ for all $t \geq 0$. Hence, the previous results yield that S is a strongly continuous convex monotone semigroup on \mathcal{L}^S . The inclusion $\mathcal{L}^I \subset \mathcal{L}^S$ follows from [8, Lemma 2.13]. \square

APPENDIX E. PROOFS OF SECTION 5.1

Proof of Lemma 5.2. First, for every $c, t \geq 0$, $r > 0$ and $(a, b) \in \mathcal{A}$, we show that

$$c\mathbb{P}(|X_t^{a,b}| \geq r) - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \leq \frac{c}{r} + L^* \left(\frac{c}{r} \right) t.$$

Chebyshev's inequality, Jensen's inequality and Itô's isometry imply

$$\begin{aligned} \mathbb{P}(|X_t^{a,b}| \geq r) &\leq \frac{\mathbb{E}[|X_t^{a,b}|]}{r} \leq \frac{1}{r} \left(\mathbb{E} \left[\left| \int_0^t \sqrt{a_s} dW_s \right|^2 \right]^{\frac{1}{2}} + \mathbb{E} \left[\int_0^t |b_s| ds \right] \right) \\ &= \frac{1}{r} \left(\mathbb{E} \left[\int_0^t |a_s| ds \right]^{\frac{1}{2}} + \mathbb{E} \left[\int_0^t |b_s| ds \right] \right) \leq \frac{1}{r} \left(1 + \mathbb{E} \left[\int_0^t |a_s| + |b_s| ds \right] \right). \end{aligned}$$

Using the definition of L^* , we conclude

$$c\mathbb{P}(|X_t^{a,b}| \geq r) - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \leq \frac{c}{r} + L^* \left(\frac{c}{r} \right) t.$$

Second, we show that S is continuous from above. Let $t \geq 0$, $x \in \mathbb{R}^d$ and $(f_n)_{n \in \mathbb{N}} \subset \text{BUC}$ be a sequence with $f_n \downarrow 0$. For every $r > 0$ and $(a, b) \in \mathcal{A}$, we use the first part to estimate

$$\begin{aligned} &\mathbb{E}[f_n(x + X_t^{a,b})] - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \\ &= \mathbb{E} \left[f_n(x + X_t^{a,b}) \mathbf{1}_{\{|X_t^{a,b}| < r\}} \right] + \mathbb{E} \left[f_n(x + X_t^{a,b}) \mathbf{1}_{\{|X_t^{a,b}| \geq r\}} \right] - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \\ &\leq \sup_{y \in B(x,r)} f_n(y) + \|f_1\|_\infty \cdot \mathbb{P}(|X_t^{a,b}| \geq r) - \mathbb{E} \left[\int_0^t L(a_s, b_s) ds \right] \\ &\leq \sup_{y \in B(x,r)} f_n(y) + \frac{\|f_1\|_\infty}{r} + L^* \left(\frac{\|f_1\|_\infty}{r} \right) t. \end{aligned}$$

It follows from $L^*(\varepsilon) \rightarrow 0$ as $\varepsilon \downarrow 0$ and Dini's theorem that $(S(t)f_n)(x) \downarrow 0$. Since $C_b \subset U_b = (\text{BUC})_\delta$, the previous statement remains valid if we replace BUC by C_b .

Third, for every $c, t \geq 0$, $\delta \in (0, 1]$, $x \in \mathbb{R}^d$, $(a, b) \in \mathcal{A}$ and $f \in \text{Lip}_b(c)$, we show that

$$\mathbb{E} \left[\int_0^t f(x + X_s^{a,b}) ds \right] \leq (f(x) + c\delta)t + \frac{cL}{\delta^2} \mathbb{E} \left[\int_0^t \int_0^s 1 + L(a_u, b_u) du ds \right].$$

We use Chebyshev's inequality, Itô's isometry and the definition of c_L to estimate

$$\begin{aligned} &\mathbb{E} \left[\int_0^t f(x + X_s^{a,b}) ds \right] \\ &\leq \mathbb{E} \left[\int_0^t \left(f(x + X_s^{a,0}) + c \int_0^s |b_u| du \right) ds \right] \\ &= \mathbb{E} \left[\int_0^t \left(f(x + X_s^{a,0}) \mathbf{1}_{\{|X_s^{a,0}| < \delta\}} + f(x + X_s^{a,0}) \mathbf{1}_{\{|X_s^{a,0}| \geq \delta\}} + c \int_0^s |b_u| du \right) ds \right] \\ &\leq (f(x) + c\delta)t + c\mathbb{E} \left[\int_0^t \left(\mathbb{P}(|X_s^{a,0}| \geq \delta) + \int_0^t |b_u| du \right) ds \right] \\ &\leq (f(x) + c\delta)t + \frac{c}{\delta^2} \mathbb{E} \left[\int_0^t \int_0^s |a_u| + |b_u| du ds \right] \\ &\leq (f(x) + c\delta)t + \frac{cL}{\delta^2} \mathbb{E} \left[\int_0^t \int_0^s 1 + L(a_u, b_u) du ds \right]. \end{aligned}$$

Fourth, let $t \geq 0$, $x \in \mathbb{R}^d$, $f \in \text{BUC}^2$ and $(a, b) \in \mathcal{A}$ with

$$(S(t)f)(x) \leq t + \mathbb{E}[f(x + X_t^{a,b})] - \mathbb{E} \left[\int_0^t L(a_s, b_s) \, ds \right].$$

Assumption 5.1(i) implies $\mathbb{E}[f(x + X_t^{a^*, b^*})] \leq (S(t)f)(x)$. Define

$$c_f := 2(1 + \|B_{a^*, b^*} f\|_\infty) + L^*(\|D^2 f\|_\infty \vee 2\|\nabla f\|_\infty).$$

We use Itô's formula and the definition of L^* to estimate

$$\begin{aligned} 2\mathbb{E} \left[\int_0^t L(a_s, b_s) \, ds \right] &\leq 2t + 2\mathbb{E}[f(x + X_t^{a,b})] - 2\mathbb{E}[f(x + X_t^{a^*, b^*})] \\ &\leq 2t + 2\mathbb{E} \left[\int_0^t B_{a_s, b_s} f(x + X_s^{a,b}) \, ds \right] - 2\mathbb{E} \left[\int_0^t B_{a_s^*, b_s^*} f(x + X_s^{a^*, b^*}) \, ds \right] \\ &\leq 2(1 + \|B_{a^*, b^*} f\|_\infty)t + \mathbb{E} \left[\int_0^t |a_s| \cdot \|D^2 f\|_\infty + 2|b_s| \cdot \|\nabla f\|_\infty \, ds \right] \\ &\leq \left(2(1 + \|B_{a^*, b^*} f\|_\infty) + L^*(\|D^2 f\|_\infty \vee 2\|\nabla f\|_\infty) \right) t + \mathbb{E} \left[\int_0^t L(a_s, b_s) \, ds \right] \\ &= c_f t + \mathbb{E} \left[\int_0^t L(a_s, b_s) \, ds \right]. \end{aligned}$$

Rearranging the previous inequality yields

$$\mathbb{E} \left[\int_0^t L(a_s, b_s) \, ds \right] \leq c_f t. \quad \square$$

Proof of Theorem 5.3. First, we show that J satisfies Assumption 4.1 and Assumption 4.4. Clearly, $J(t)$ is convex and monotone for all $t \geq 0$. Moreover, Assumption 5.1(i) implies $J(0) = \text{id}_{C_b}$ and $J(t)0 = 0$ for all $t \geq 0$. For every $t \geq 0$ and $f, g \in C_b$,

$$\|J(t)f - J(t)g\|_\infty \leq \sup_{(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \|T_{a,b}(t)f - T_{a,b}(t)g\|_\infty \leq \|f - g\|_\infty,$$

where $(T_{a,b}(t))_{t \geq 0}$ denotes the linear semigroup given by

$$(T_{a,b}(t)f)(x) := \mathbb{E}[f(x + \sqrt{a}W_t + bt)] \quad \text{for all } t \geq 0, f \in C_b \text{ and } x \in \mathbb{R}^d.$$

Since $J(t)$ is translation invariant, we obtain $J(t): \text{Lip}_b(r) \rightarrow \text{Lip}_b(r)$ for all $r, t \geq 0$. In particular, it follows by induction that the sequence $(I(\pi_n^t)f)_{n \in \mathbb{N}}$ is uniformly Lipschitz continuous for all $t \geq 0$ and $f \in \text{Lip}_b$. Let $t \geq 0$, $x \in \mathbb{R}^d$ and $(f_n)_{n \in \mathbb{N}} \subset \text{BUC}$ be a sequence with $f_n \downarrow 0$. Lemma 5.2(i) and Dini's theorem imply

$$0 \leq \sup_{s \in [0, t] \cap \mathcal{T}} \sup_{k \in \mathbb{N}} (J(\pi_k^t)f_n)(x) \leq \sup_{s \in [0, t]} (S(s)f_n)(x) \downarrow 0 \quad \text{as } n \rightarrow \infty.$$

Indeed, by Theorem 5.4, the mapping $t \mapsto (S(t)f_n)(x)$ is continuous. Furthermore, it holds $J(t)f \leq S(t)f$ for all $t \geq 0$ and $f \in \text{BUC}$ and S is a semigroup on BUC . Since $C_b \subset U_b = (\text{BUC})_\delta$, the previous statement remains valid if we replace BUC by C_b . Next, for every $f \in \text{BUC}^2$, we show that

$$\lim_{h \downarrow 0} \left\| \frac{J(h)f - f}{h} - \sup_{(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \left(\frac{1}{2} \Delta_a f + \nabla_b f - L(a, b) \right) \right\|_\infty = 0.$$

By Assumption 5.1(ii) and Itô's formula, there exists $r \geq 0$ such that

$$\begin{aligned} & \left\| \frac{J(h)f - f}{h} - \sup_{(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d} \left(\frac{1}{2} \Delta_a f + \nabla_b f - L(a,b) \right) \right\|_\infty \\ & \leq \sup_{\{|a|, |b| \leq r\}} \left\| \frac{T_{a,b}(h)f - f}{h} - \frac{1}{2} \Delta_a f - \nabla_b f \right\|_\infty \\ & \leq \sup_{\{|a|, |b| \leq r\}} \int_0^h \left(\|\nabla_b f(\cdot + X_s^{a,b}) - \nabla_b f\|_\infty + \frac{1}{2} \|\Delta_a f(\cdot + X_s^{a,b}) - \Delta_a f\|_\infty \right) ds. \end{aligned}$$

For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d$ with $|a|, |b| \leq r$ and $s \geq 0$,

$$\left(\|\nabla_b f(\cdot + X_s^{a,b}) - \nabla_b f\|_\infty + \frac{1}{2} \|\Delta_a f(\cdot + X_s^{a,b}) - \Delta_a f\|_\infty \right) \mathbf{1}_{\{|X_s^{a,b}| < \delta\}} < \varepsilon.$$

Furthermore, Chebyshev's inequality implies

$$\sup_{\{|a|, |b| \leq r\}} \mathbb{P}(|X_s^{a,b}| \geq \delta) \leq \sup_{\{|a|, |b| \leq r\}} \frac{2}{\delta^2} (\mathbb{E}[|\sqrt{a}W_s|^2] + |b|^2 s^2) \rightarrow 0 \quad \text{as } s \downarrow 0.$$

It remains to show that $J(t): C_b \rightarrow C_b$ for all $t \geq 0$. Let $t \geq 0$ and $f \in C_b$. Choose a bounded sequence $(f_n)_{n \in \mathbb{N}} \subset \text{Lip}_b$ with $f_n \rightarrow f$ uniformly on compacts. Lemma 2.7 implies $J(t)f_n \rightarrow J(t)f$ uniformly on compacts. Indeed, the corresponding proof only relies on the fact that $J(t)$ is convex, monotone and continuous from above. Since $J(t)f_n \in \text{Lip}_b \subset C_b$ for all $n \in \mathbb{N}$, we obtain $J(t)f \in C_b$. By Theorem 4.3, there exists a family T of operators $T(t): C_b \rightarrow C_b$ which satisfy the conditions from Theorem 5.3 except for the statement about the symmetric Lipschitz set.

Second, we show that $\mathcal{L}_{\text{sym}}^T \cap \text{BUC} = \mathcal{L}_{\text{sym}}^J \cap \text{BUC}$ and

$$T(t): \mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b \rightarrow \mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b \quad \text{for all } t \geq 0.$$

By straightforward computations, one can verify the conditions (i)-(iv) from [8, Assumption 5.2] with $J^+(t)f := J(t)f$ and $J^-(t)f := -J(t)(-f)$ for all $t \geq 0$ and $f \in \text{BUC}$. Moreover, it holds $T(t)f = \sup_{n \in \mathbb{N}} J(\pi_n^t)f$ for all $(f, t) \in \text{BUC} \times \mathcal{T}$ and $J(t)f \leq T(t)f$ for all $(f, t) \in \text{BUC} \times \mathbb{R}_+$. We remark that in [8] a slightly stronger assumption has been made, but it follows immediately from the corresponding proof that [8, Theorem 5.3] is still applicable. We obtain $\mathcal{L}_{\text{sym}}^T \cap \text{BUC} = \mathcal{L}_{\text{sym}}^J \cap \text{BUC}$ and

$$T(t): \mathcal{L}_{\text{sym}}^T \cap \text{BUC} \rightarrow \mathcal{L}_{\text{sym}}^T \cap \text{BUC} \quad \text{for all } t \geq 0.$$

In addition, the invariance of Lip_b holds by Theorem 4.5. It remains to show the explicit representation of $\mathcal{L}_{\text{sym}}^T \cap \text{Lip}_b = \mathcal{L}_{\text{sym}}^J \cap \text{Lip}_b$.

Third, let $f \in \mathcal{L}_{\text{sym}}^J \cap \text{Lip}_b$. Choose $c \geq 0$ and $t_0 > 0$ with

$$\|J(t)f - f\|_\infty \leq ct \quad \text{and} \quad \|J(t)(-f) + f\|_\infty \leq ct \quad \text{for all } t \in [0, t_0].$$

Fix $(a,b) \in \mathbb{S}_+^d \times \mathbb{R}^d$ with $L(a,b) < \infty$ and $t \in [0, t_0]$. It holds

$$\begin{aligned} -(c + L(a,b))t & \leq -(J(t)(-f) + f + L(a,b)t) \leq -(T_{a,b}(t)(-f) + f) \\ & = T_{a,b}(t)f - f \leq J(t)f - f + L(a,b)t \leq (c + L(a,b))t \end{aligned}$$

and thus $\|T_{a,b}(t)f - f\|_\infty \leq (c + L(a,b))t$. Let $\eta \in C_c^\infty$ with $\eta \geq 0$, $\text{supp}(\eta) \subset B(0, 1)$ and $\int_{\mathbb{R}^d} \eta(x) dx = 1$. Define $\eta_n(x) := n^d \eta(nx)$ and $f_n := f * \eta_n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$.

We use Fubini's theorem to estimate

$$\begin{aligned} |T_{a,b}(t)f_n - f_n|(x) &= \left| \mathbb{E} \left[\int_{\mathbb{R}^d} f(x + \sqrt{a}W_t + bt - y) \eta_n(y) dy \right] - f_n(x) \right| \\ &= \left| \int_{\mathbb{R}^d} \mathbb{E}[f(x + \sqrt{a}W_t + bt - y)] \eta_n(y) dy - f_n(x) \right| \\ &= |(T_{a,b}(t)f - f) * \eta_n|(x) \leq \|T_{a,b}(t)f - f\|_\infty \leq (c + L(a, b))t. \end{aligned}$$

It follows from $f_n \in \text{BUC}^2 \subset D(B_{a,b})$ that

$$\|B_{a,b}f_n\|_\infty \leq c + L(a, b) \quad \text{for all } n \in \mathbb{N}. \quad (\text{E.1})$$

In addition, for every $n \in \mathbb{N}$,

$$\Delta_a f_n = \frac{1}{2} \Delta_a f_n + \nabla_b f_n + \frac{1}{2} \Delta_a f_n + \nabla_{-b} f_n = B_{a,b}f_n + B_{a,-b}f_n. \quad (\text{E.2})$$

Inequality (E.1) and equation (E.2) imply

$$\sup_{n \in \mathbb{N}} \|\Delta_a f_n\|_\infty \leq \|B_{a,b}f_n\|_\infty + \|B_{a,-b}f_n\|_\infty \leq 2(c + L(a, b)).$$

By Banach-Alaoglu's theorem, there exists $g \in L^\infty$ such that $\Delta_a f_{n_k} \rightarrow g$ in the weak*-topology for a suitable subsequence. Moreover, it follows from $f \in \text{Lip}_b = W^{1,\infty}$ that $\sqrt{a}^T \nabla f_n = (\sqrt{a}^T \nabla f) * \eta_n \rightarrow \sqrt{a}^T \nabla f$ and thus $f \in D(\Delta_a)$ with $\Delta_a f = g$. Since the supremum norm is lower semicontinuous w.r.t. weak*-topology, inequality (E.1) yields $\|B_{a,b}f\|_\infty \leq c + L(a, b)$. We obtain

$$f \in \bigcap_{a \in \mathbb{S}_L} D(\Delta_a) \cap W^{1,\infty} \quad \text{and} \quad \sup_{(a,b) \in \mathbb{S}_L \times \mathbb{R}^d} (\|B_{a,b}f\|_\infty - L(a, b)) < \infty.$$

Fourth, let $f \in \bigcap_{a \in \mathbb{S}_L} D(\Delta_a) \cap W^{1,\infty}$ and assume that

$$c := \sup_{(a,b) \in \mathbb{S}_L \times \mathbb{R}^d} (\|B_{a,b}f\|_\infty - L(a, b)) < \infty.$$

Let $t \geq 0$, $(a, b) \in \mathbb{S}_L \times \mathbb{R}^d$ and define $f_n := f * \eta_n$ for all $n \in \mathbb{N}$. It follows from Itô's formula and $B_{a,b}f_n = (B_{a,b}f) * \eta_n$ that

$$\begin{aligned} T_{a,b}(t)f - f - L(a, b)t &= \lim_{n \rightarrow \infty} (T_{a,b}(t)f_n - f_n - L(a, b)t) \\ &\leq \sup_{n \in \mathbb{N}} (\|B_{a,b}f_n\|_\infty - L(a, b))t \\ &\leq (\|B_{a,b}f\|_\infty - L(a, b))t \leq ct. \end{aligned}$$

This implies $J(t)f - f \leq ct$. For the lower bound, we use $L(a^*, b^*) = 0$ to estimate

$$J(t)f - f \geq T_{a^*, b^*}(t)f - f \geq -\|B_{a^*, b^*}f\|_\infty t.$$

We obtain $f \in \mathcal{L}^J$. Applying the previous estimate on $-f$ yields $f \in \mathcal{L}_{\text{sym}}^J$. \square

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