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Evolutionary multi-objective optimisation for large-scale portfolio selection with both random and uncertain returns

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ABSTRACT

With the advent of Big Data, managing large-scale portfolios of thousands of securities is one of the most challenging tasks in the asset management industry. This study uses an evolutionary multi-objective technique to solve large-scale portfolio optimisation problems with both long-term listed and newly listed securities. The future returns of long-term listed securities are defined as random variables whose probability distributions are estimated based on sufficient historical data, while the returns of newly listed securities are defined as uncertain variables whose uncertainty distributions are estimated based on experts' knowledge. Our approach defines security returns as theoretically uncertain random variables and proposes a three-moment optimisation model with practical trading constraints. In this study, a framework for applying arbitrary multi-objective evolutionary algorithms to portfolio optimisation is established, and a novel evolutionary algorithm based on large-scale optimisation techniques is developed to solve the proposed model. The experimental results show that the proposed algorithm outperforms state-of-the-art evolutionary algorithms in large-scale portfolio optimisation.

1. Introduction

In 2022, the global asset management industry hit a new high of 126 trillion of assets under management(AUM). This figure represents 28 percent of global financial assets, up from 23 percent a decade ago (McKinsey, 2022). The computational complexity of optimal portfolio construction, which simultaneously balances risk minimisation with return maximisation, is perhaps the most intrinsic and recurrent financial problem in the asset management industry. Modern portfolio theory (MPT), first introduced by Markowitz (1952), extols the virtues of the first two moments of the Gaussian distribution (the so-called mean-variance model) as sufficient to solve the problem of optimal portfolio allocation based on practitioners' views on risk and return. In academia, the MPT continues to be challenged from various perspectives. Some scholars have chosen to extend the mean-variance approach in non-trivial directions (Lassance, 2022; Zhen & Chen, 2022; Li et al., 2022b), while others have attacked the statistical validity of the mean-variance model and proposed practical extension to accommodate fat-tailedness of risk factors (Samuelson, 1975; Theodossiou & Savva, 2015; Sahamkhadam et al., 2022), and or improvements to the distribution of speculative assets (Mandelbrot, 1963; Jensen, 1968; Hsieh, 1991; Carr & Wu, 2003; Borovicka et al., 2016). In this paper, we consider the practical challenge of large-scale portfolio optimisation where historical data for some eligible securities is limited.

Most existing portfolio optimisation models view financial asset returns as random variables whose distributional characteristics can be extracted from historical data. However, when there is a lack of sufficient historical data for newly listed securities, scholars have proposed the use of fuzzy variables estimated by experts' judgement, leading to the development of fuzzy portfolio optimisation theory (Tanaka & Guo, 1999; Gupta et al., 2008; Zhang et al., 2007; Mehlawat et al., 2020; Zhang et al., 2022). Also, Liu (2007) proposes the uncertainty theory as another alternative tool for modelling indeterministic quantities that are subject to experts' estimates. Based on the uncertainty theory, portfolio optimisation problems are able to be solved in uncertain environments (Huang, 2012; Huang & Di, 2016; Zhai & Bai, 2018; Dai & Qin, 2021).

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This study addresses the real-world challenge of large-scale portfolio optimisation problems when the eligible basket of securities includes both established (long-term listed) and newly listed entities. In a situation where a large number of securities are of interest, portfolio selection is essentially considered as a large-scale optimisation task. In such a scenario, both long-term listed securities and newly listed securities are usually included in the portfolio. Since sufficient historical samples are available for the long-term listed securities, it is usually assumed that the security returns are random variables whose distributions are statistically traceable. In contrast, since sufficient historical data is not available for newly listed securities, security returns are usually defined as uncertain variables whose distributions are estimated based on experts' estimates. In this hybrid environment, uncertain random variables are introduced to deal with the complex system with randomness and uncertainty (Liu, 2013a,b). Qin (2015) first proposes a mean-variance portfolio optimisation model to address this problem. Since then, a large body of work has been put into solving portfolio optimisation problems under uncertain random environments (Qin et al., 2017; Ahmadzade et al., 2020; Li & Teo, 2021; Mehlatat et al., 2021).

In practise, trading restrictions add fractions to real-world portfolio optimisation and inhibit the use of classical MPT approaches. Investment restrictions such as cardinality and minimum transaction lot are important factors that should be taken into account (Branke et al., 2009; Liu & Zhang, 2015; Dai & Qin, 2021; Kobayashi et al., 2023). As is often the case in practise, investors prefer a portfolio with a reasonable number of securities, especially when the investment capital is large. The cardinality constraint helps investors construct a portfolio with an appropriate size. Most existing models have been studied under the assumption of perfect fractionability of investment, which is difficult to implement in the real world. In the financial market, securities often have to be traded in integer multiples of a round lot, where a lot is a fixed quantity of units in a transaction. Therefore, it is necessary to include the constraint of a minimum transaction lot in the portfolio optimisation problem.

The introduction of complex constraints makes solving portfolio optimisation models challenging, especially for large-scale investments. Multi-Objective Evolutionary Algorithms (MOEA) have been shown to cope with constrained portfolio optimisation models. This has spurred research on alternative algorithms, allowing MOEA to shine in the field of portfolio optimisation (Woodside-Oriakhi et al., 2011; Chen et al., 2018; Wang et al., 2018; Chen et al., 2019; Petchrompo et al., 2022). Most existing MOEAs have shown promising performance in solving complex optimisation problems, but their performance may deteriorate when they process a large number of decision variables (Tian et al., 2021). In recent years, three types of Large-Scale Multi-objective Evolutionary Algorithms (LSMOEA) have been developed based on decision variable grouping, decision space reduction and novel search strategies to address the challenges of large-scale problems (Tian et al., 2021). Most LSMOEAs assume that the problem contains no or few simple constraints, so they are not directly applicable to large-scale portfolio optimisation problems with realistic constraints. Although evolutionary algorithms have achieved remarkable success in optimising portfolios with constraints, research on the application of MOEAs to large-scale portfolio optimisation tasks is still limited.

To illustrate the above view, the performance of representative MOEAs on the mean-variance model is discussed. The mean-variance model is a quadratic programming problem whose true efficient front can be easily obtained (e.g., with the function 'quadprog' in MATLAB). Two portfolio problems with 30 securities (small-scale case) and 1000 securities (large-scale case) are considered. Four representative MOEAs, NSGA-II (Deb et al., 2002), WOF (Zille et al., 2018), LSMOF (He et al., 2019) and LMOEADS (Qin et al., 2021), are used to find 100 efficient solutions to the mean-variance models, the first one being the best known multi-objective algorithm and the next three being representatives of the outstanding LSMOEAs. The parameters associated with the algorithms are set according to the original literature, and the termination condition is set at 30000 function evaluations. The efficient fronts determined by the algorithms are shown in Figure 1. As can be seen in Figure 1(a), all MOEAs converge to the true effective frontier in the small-scale case, with the three LSMOEAs achieving better diversity than NSGA-II. However, in the large-scale case, Figure 1(b) shows that their performance deteriorates dramatically, with NSGA-II and WOF showing poor convergence and diversification, and LSMOF and LMOEADS performing much worse than in the small-scale case. This observation suggests that the application of MOEAs in large-scale portfolio optimisation still holds considerable challenges and potential for improvement.

This study addresses a large-scale portfolio optimisation problem involving long-term and newly listed securities in an uncertain random environment. A multi-objective portfolio optimisation model with realistic constraints is proposed. In this model, the mean, variance and skewness of the portfolio return serve as decision criteria and are complemented by constraints such as cardinality, bounding, minimum transaction lot and no short selling to make the model more comprehensive and applicable to real investment scenarios. To solve the proposed model, a novel

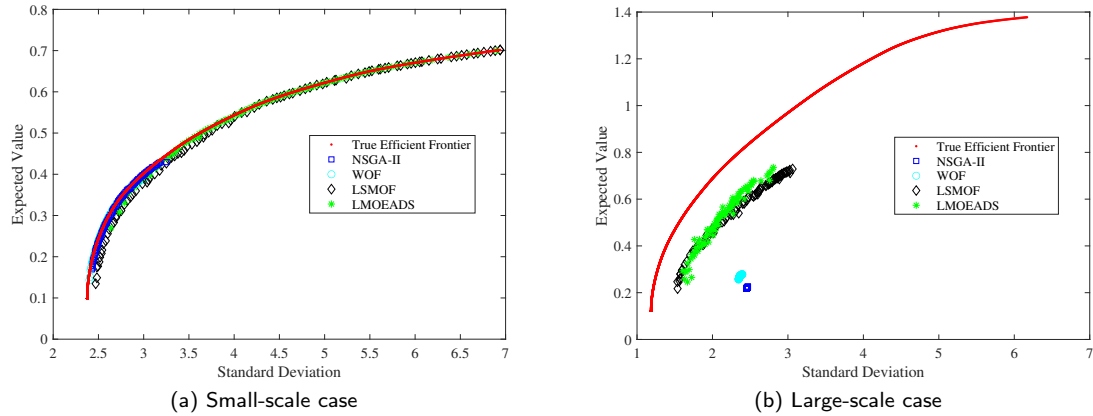


Figure 1: Efficient fronts of the mean-variance models obtained by the NSGA-II and LMOEADS.

Table 1
Feature comparison with existing approaches.

Feature	Qin (2015)	Chen et al. (2019)	Chang et al. (2020)	Dai & Qin (2021)	Huang & Jiang (2021)	Mehlawat et al. (2021)	Li & Teo (2021)	Our
Skewness	×	✓	×	×	×	✓	✓	✓
Environment	Uncertain Random	Uncertain	Uncertain	Uncertain	Uncertain	Uncertain Random	Uncertain Random	Uncertain Random
Cardinality	×	✓	✓	×	×	✓	✓	✓
Transaction Lot	×	×	×	✓	×	×	×	✓
Bounding	×	✓	×	✓	×	✓	×	✓
No Short-selling	×	✓	✓	✓	✓	✓	✓	✓
Risk-free Asset	×	×	×	×	×	×	×	✓
Large-scale	×	×	×	×	×	×	×	✓
Solution	CQP	ICA-FA	LP	GA	LP	GA	ASM	LSWEOA

CQP: Convex Quadratic Programming; ICA-FA: Imperialist Competitive Algorithm and Firefly Algorithm; LP: Linear Programming; GA: Genetic Algorithm; IFMOICA: Improved Multi-objective Imperialist Competitive; ASM: Active-Set Method; LSWEOA: Lager-scale Weighting optimisation Evolutionary Algorithm.

evolutionary algorithm based on large-scale multi-objective optimisation techniques is proposed. First, an encoder-decoder method is developed to deal with the constraints and convert the proposed model into a model without constraints. Then, an optimisation framework based on variable space reduction is designed to solve the converted model. Finally, a novel search strategy is developed to improve the operational efficiency of the algorithm.

Based on the above discussion, a comparison of the features with some important related works is given in Table 1. The main highlights and innovations of this study are summarised below:

1. A mean-variance-skewness model for large-scale portfolio optimisation problems with long-term listed and newly listed securities have been formulated in an uncertain random environment.
2. In order to improve the practicality of the proposed model, realistic constraints have been included. In particular, the constraints of cardinality, minimum transaction lot, bounding and prohibition of short selling are considered and explained in detail.
3. An encoder-decoder method is presented to convert the proposed model into an unconstrained multi-objective model. The developed constraint-handling method enables the application of arbitrary MOEAs to the proposed model, which greatly extends the capabilities of existing evolutionary algorithms in portfolio optimisation.
4. To solve the proposed model, a novel evolutionary algorithm based on large-scale multi-objective optimisation techniques is developed. The proposed algorithm is extended with a decision space reduction method and a novel search strategy to solve large-scale portfolio optimisation problems efficiently.
5. The proposed model and algorithm are implemented on two datasets of the Chinese stock market. Numerical analysis shows that the proposed algorithm outperforms the existing MOEAs on large-scale portfolio optimisation problems.

The rest of this paper is structured as follows: Section 2 reviews the relevant literature. Section 3 briefly introduces the properties of the theory of uncertainty and uncertain random variables. Section 4 describes the proposed model. Section 5 explains the solution algorithm. Section 6 conducts two case studies. Finally, Section 7 provides a summary and conclusion.

2. Literature Review

Markowitz (1952) proposes the mean-variance portfolio optimisation model, which formulates the problem in a mathematical framework for the first time. The model uses mean and variance to characterise return and risk, respectively. However, variance treats investment returns above and below the mean as equivalent to increased risk, which is unrealistic when security returns are asymmetrically distributed. To address this challenge, some scholars have replaced variance with a measure of downside risk (Markowitz, 1959; Ling et al., 2020; Rigamonti & Lucivjanska, 2022); others have used skewness, the third central moment, to measure the degree of asymmetry in the distribution of returns (Samuelson, 1975; Li et al., 2010; Theodossiou & Savva, 2015; Zhou & Palomar, 2021). In these approaches, only the randomness of financial markets is taken into account, and security returns are treated as random variables whose distributions can be derived from historical data. However, in some emerging markets, there may be a lack of sufficient trading data. In this case, some researchers consider security returns as fuzzy variables whose distributions are estimated by experts' knowledge. Zadeh (1965) introduced the concept of fuzzy sets, which later became the basis of fuzzy techniques. With the development of fuzzy techniques, researchers began to use fuzzy numbers to formulate payoff distributions and to study fuzzy portfolio optimisation problems based on three different approaches: fuzzy set theory (Gupta et al., 2008), possibility theory (Tanaka & Guo, 1999; Zhang et al., 2007; Tsaur, 2013; Liu et al., 2020) and credibility theory (Huang, 2008; Mehlawat & Gupta, 2014; Guo et al., 2016; Mehlawat et al., 2020).

However, fuzzy theory has been criticised for the paradoxes associated with describing security returns using fuzzy numbers (Huang & Ying, 2013). To better describe subjectively imprecise quantities, Liu (2007) proposes an uncertainty theory that deals with uncertain quantities estimated by experts. Based on uncertainty theory, many works have been done to solve related portfolio optimisation problems. Huang (2011) develops a mean-risk model for uncertain portfolio optimisation. Huang (2012) proposes a mean-variance method for portfolio optimisation with returns based on experts' estimates. Yao (2014) introduces sinusoidal entropy for uncertain sets and explores its applications in portfolio optimisation. Some scholars discuss uncertain portfolio optimisation problems considering background risk from different perspectives (Huang & Di, 2016; Huang & Jiang, 2021; Huang et al., 2021). Zhai & Bai (2018) propose a mean-risk model for uncertain portfolio optimisation with background risk. Zhai et al. (2018) design an uncertain mean-risk skewness model for portfolio optimisations. Chang et al. (2020) deal with uncertain portfolio optimisation problems with mental accounts and realist constraints. Dai & Qin (2021) proposes an uncertain portfolio optimisation model with minimal transaction lots and dynamic risk preference. Li et al. (2022a) provide an analytical solution to a multi-period uncertain portfolio optimisation problem.

In most real-world situations, it is doubtful whether there is sufficient historical data for all securities or for any securities. Consequently, portfolio optimisation problems in the real world are usually simultaneously associated with random and uncertain returns. Liu (2013a,b) proposes to use uncertain random variables to model systems with randomness and uncertainty. Qin (2015) is the first to study portfolio optimisation problems in hybrid environments using uncertain random variables. Qin et al. (2017) develop a VaR model with uncertain random mean for portfolio optimisation problems. Ahmadzade et al. (2020) introduce the partial divergence measure of uncertain random variables and apply it to portfolio optimisation problems. Mehralizade et al. (2020) design a new risk criterion for the uncertain random portfolio optimisation problem. Li & Teo (2021) present a mean-variance-skewness model for the uncertain random portfolio optimisation problem. Mehlawat et al. (2021) propose a portfolio optimisation model using higher moments in uncertain random environments.

In addition to uncertainty and randomness, real-world constraints are also important factors in portfolio optimisations. The studies conducted in this area are very active, have various constraints such as cardinality and transaction lots, and are integrated into the existing portfolio optimisation models (Lin & Liu, 2008; Li & Xu, 2013; Liu & Zhang, 2015; Mehlawat et al., 2021; Li & Teo, 2021). However, the introduction of realistic constraints has turned the models into NP-hard problems that can be computationally very challenging. MOEA is a good candidate to solve the models. Numerous works have been carried out with the aim of using MOEAs to solve multi-objective portfolio optimisation models with realistic constraints. Chen et al. (2018) design a novel hybrid MOEA to solve the multi-period mean-variance-skewness model. Wang et al. (2018) present a fuzzy simulation-based particle swarm optimisation algorithm

for the bi-objective portfolio optimisation model. Chen et al. (2019) present a novel hybrid ICA-FA algorithm for solving multi-period uncertain portfolio optimisation. Li et al. (2021) develop an improved imperialistic competition algorithm for solving the fuzzy random portfolio optimisation model.

Although MOEAs are widely used in portfolio optimisation, it is still a challenge to use them to solve large-scale portfolio optimisation models with thousands of securities. MOEAs have shown promise in solving various optimisation problems, but their performance may deteriorate when tackling problems with a large number of decision variables (Tian et al., 2021). In recent years, significant efforts have been made to solve large multi-objective optimisation problems, with performance improved mainly by three techniques: grouping of decision variables (Ma et al., 2016; Cao et al., 2017; Du et al., 2019), decision space reduction (Zille et al., 2018; He et al., 2019; Chang et al., 2020; Qin et al., 2021; Tian et al., 2022) and novel search strategies (Hong et al., 2019; Tian et al., 2020; He et al., 2022). Some MOEAs have been proposed to solve constrained multi-objective optimisation models with several decision variables based on co-evolutionary or multi-stage frameworks (Fan et al., 2019; Li et al., 2019). However, most existing methods for dealing with constraints need to test the feasibility of potential solutions, which makes the algorithms inefficient for large-scale optimisation models.

In summary, portfolio optimisation as a fundamental financial problem has been the focus of interest of scholars in various fields. Based on the analysis of the existing literature, this study addresses the following four main challenges:

1. How to account for uncertainty and randomness in optimising large portfolios of long-term listed and newly listed securities?
2. How to develop investment strategies for such large-scale investments that are closer to investors' decision criteria and real investment scenarios?
3. How can existing advanced evolutionary algorithms be used to solve portfolio optimisation problems with realistic constraints?
4. How to improve the performance of evolutionary algorithms in solving large-scale portfolio optimisation problems?

To overcome these challenges, a large-scale uncertain random mean-variance-skewness portfolio optimisation model is proposed. In this model, the future returns of long-term listed and newly listed securities are treated as random and uncertain variables, respectively. Additionally, the constraints of cardinality, minimum transaction lot, bounding and prohibition on short selling are considered to fit the real world of investment. Then, an encoder-decoder method is developed to convert the proposed model into a constraint-free model so that any MOEA can be applied to solve the proposed model. Besides, a novel evolutionary algorithm based on large-scale multi-objective optimisation techniques is proposed to solve the transformed model.

3. Preliminaries

This section introduces uncertainty theory and uncertain random variables as a precursor to this study.

3.1. Uncertainty Theory

Liu (2007) proposes uncertainty theory as a branch of axiomatic mathematics to study uncertainty in relation to the degree of human belief.

Definition 1. (Liu, 2007). Let Γ be a non-empty set and ℓ be a σ -algebra over Γ . An element $\Lambda \in \ell$ is called an event. A function $\mathcal{M} : \ell \rightarrow [0, 1]$ is called an uncertain measure if it satisfies: (1) $\mathcal{M}(\Gamma) = 1$; (2) $\mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1$ for any event $\Lambda \in \ell$; (3) For every countable sequence of events $\Lambda_1, \Lambda_2, \dots$, we have $\mathcal{M}(\bigcup_i \Lambda_i) \leq \sum_i \mathcal{M}(\Lambda_i)$. The triple $(\Gamma, \ell, \mathcal{M})$ is called an uncertain space.

Definition 2. (Liu, 2007). Let $(\Gamma_k, \ell_k, \mathcal{M}_k)$ be uncertain space for $k = 1, 2, \dots$. The product uncertain measure \mathcal{M} is an uncertain measure satisfying $\mathcal{M}\{\prod_k \mathcal{M}_k\{\Lambda_k\}\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$, where Λ_k are arbitrarily chosen events from ℓ_k .

Definition 3. (Liu, 2007). An uncertain variable is a function ξ from an uncertain space $(\Gamma_k, \ell_k, \mathcal{M}_k)$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B .

Definition 4. (Liu, 2007). The uncertain distribution Φ of an uncertain variable ξ is defined by $\Phi_\xi(x) = \mathcal{M}\{\xi \leq x\}$ for any real number x .

Definition 5. (Liu, 2007). The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\} - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\},$$

provided that at least one of the two integrals exists.

Definition 6. (Liu, 2010). The uncertain variables ξ_1, \dots, ξ_n are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^n \{\xi_i \in B_i\}\right\} = \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\},$$

for any Borel sets B_1, \dots, B_n of real numbers.

3.2. Uncertain Random Variable

The uncertain variable is developed by Liu (2013a) for modelling complex systems with uncertainty and randomness. Let $(\Gamma, \ell, \mathcal{M})$ and (Ω, \mathcal{A}, P) be uncertain space and probability space, respectively. The product $(\Gamma, \ell, \mathcal{M}) \times (\Omega, \mathcal{A}, P)$ is called a chance space.

Definition 7. (Liu, 2013a). Let $(\Gamma, \ell, \mathcal{M}) \times (\Omega, \mathcal{A}, P)$ be a chance space and $\Theta \in \ell \times \mathcal{A}$ be an uncertain random event. The chance measure of Θ is defined by

$$\text{Ch}\{\Theta\} = \int_0^1 P\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \geq x\} \quad (1)$$

Definition 8. (Liu, 2013a). An uncertain random variable is a function ξ from a chance space $(\Gamma, \ell, \mathcal{M}) \times (\Omega, \mathcal{A}, P)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\ell \times \mathcal{A}$ for any Borel set B . Its chance distribution is defined by $\Psi(x) = \text{Ch}\{\xi \leq x\}$.

Theorem 1. (Liu, 2013a). Let ξ_1 be a random variable and ξ_2 be an uncertain variable. ξ_1 or ξ_2 can be regarded as a special uncertain random variable, and $\xi_1 + \xi_2$ is also an uncertain random variable.

Definition 9. (Liu, 2013b). Let ξ be an uncertain random variable. Its expected value is defined by

$$E[\xi] = \int_0^{+\infty} \text{Ch}\{\xi \geq x\} dx - \int_{-\infty}^0 \text{Ch}\{\xi \leq x\} dx,$$

provided that at least one of the two integrals exists.

Theorem 2. (Liu, 2013a). Let η be a random variable and ξ be an uncertain variable. The uncertain random variable $\eta + \xi$ has an expected value

$$E[\eta + \xi] = E[\eta] + E[\xi].$$

Definition 10. (Liu, 2013a). Let ξ be an uncertain random variable with a finite mean value e . Its variance is defined by

$$V[\xi] = E[(\xi - e)^2].$$

Theorem 3. (Guo & Wang, 2014). Let ξ be an uncertain random variable with chance distribution $\Psi(x)$. If ξ has a finite mean value e , then

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 d\Psi(x).$$

Table 2

Key notations used in the model derivation.

Category	Symbolization	Annotation
Indices	n	Number of long-term listed securities
	m	Number of newly listed securities
	i	Index for long-term listed securities, $i = 1, \dots, n$
	j	Index for newly listed securities, $j = 1, \dots, m$
Decision variables	x_0	Investment proportion of the risk-free asset
	x_i	Investment proportion of long-term listed securities
	y_j	Investment proportion of newly listed securities
Modeling variables	K_{\min}	Minimum number of the securities held
	K_{\max}	Maximum number of the securities held
	l_i^1	Lower bound of the investment proportion in i -th long-term listed security if it is held
	u_i^1	Upper bound of the investment proportion in i -th long-term listed security if it is held
	l_j^2	Lower bound of the investment proportion in j -th newly listed security if it is held
	u_j^2	Upper bound of the investment proportion in j -th newly listed security if it is held
	W_0	Total capital at the beginning of investment
	P_i^1	Price of a round lot of i -th long-term listed security
	P_j^2	Price of a round lot of j -th newly listed security
	r_0	Risk-free return
	η_i	Random return of i -th long-term listed security
	ξ_j	Uncertain return of j -th newly listed security
Υ_i	Probability distribution of random return ξ_i	
Φ_j	Uncertain distribution of uncertain return η_j	

Definition 11. (Li & Shu, 2022; Mehlawat et al., 2021; Zhai et al., 2022). Let ξ be an uncertain random variable with a finite mean value e . Its skewness is defined by

$$S[\xi] = E[(\xi - e)^3].$$

Theorem 4. (Li & Shu, 2022; Mehlawat et al., 2021; Zhai et al., 2022). Let ξ be an uncertain random variable with chance distribution $\Psi(x)$. If ξ has a finite mean value e , then

$$S[\xi] = \int_{-\infty}^{+\infty} (x - e)^3 d\Psi(x).$$

4. Constrained multi-objective portfolio optimisation model

We consider a multi-objective portfolio optimisation problem with large-scale securities and a risk-free asset, where long-term listed and newly listed securities are represented simultaneously. For simplicity, the notations used are listed in Table 2.

Let $\mathbf{x} = (x_1, \dots, x_n)'$ and $\mathbf{y} = (y_1, \dots, y_m)'$ be the portfolio vectors of long-term listed securities and newly listed securities, respectively, where $'$ is the transpose operator. The portfolio vector for a risk-free asset and all securities can be represented by $X = (x_0, \mathbf{x}', \mathbf{y}')$. Then, the portfolio return is given by

$$R(X) = x_0 r_0 + \mathbf{x}' \boldsymbol{\eta} + \mathbf{y}' \boldsymbol{\xi} = x_0 r_0 + x_1 \eta_1 + \dots + x_n \eta_n + y_1 \xi_1 + \dots + y_m \xi_m.$$

where $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)'$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)'$ are the return vector of long-term listed and newly listed securities, respectively. It is worth noting that $\mathbf{x}' \boldsymbol{\eta}$ is a random variable and $\mathbf{y}' \boldsymbol{\xi}$ is an uncertain variable, so $R(X)$ can be considered an uncertain random variable according to Theorem 1.

4.1. Objectives

In a risk-return framework, the returns and risks of investments are naturally measured by the expected value and variance of $R(X)$. Since skewness is an important statistical indicator of asymmetric characteristics, it would be a

valuable extension to add maximising skewness as an additional objective. To simplify the presentation, the following two assumptions from Qin (2015) are used.

Assumption 1. The random vector $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)'$ has a multivariate normal distribution with the following probability density function

$$\Upsilon(\mathbf{z}) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp\left(-\frac{1}{2}(\mathbf{z} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{z} - \boldsymbol{\mu})\right), \forall \mathbf{z} \in \mathbb{R}^n,$$

where $\boldsymbol{\mu}$ and Σ are the expected vector and covariance matrix of $\boldsymbol{\eta}$, respectively.

Assumption 2. The uncertain returns of the newly listed securities are independent in the sense of the uncertain measure, i.e., ξ_1, \dots, ξ_m are independent uncertain variables that follow Definition 6.

By Assumption 1, the random variable $\mathbf{x}'\boldsymbol{\eta}$ is normally distributed with mean $\mathbf{x}'\boldsymbol{\mu}$ and variance $\mathbf{x}'\Sigma\mathbf{x}$. The probability distribution of $\mathbf{x}'\boldsymbol{\eta}$ is thus given by

$$\Upsilon(z) = \frac{1}{\sqrt{2\pi\mathbf{x}'\Sigma\mathbf{x}}} \exp\left(-\frac{(z - \mathbf{x}'\boldsymbol{\mu})^2}{2\mathbf{x}'\Sigma\mathbf{x}}\right), \forall z \in \mathbb{R},$$

According to Theorem 2, the expected value of portfolio return $R(X)$ is given by

$$E[R(X)] = x_0 r_0 + E[\mathbf{x}'\boldsymbol{\eta}] + E[\mathbf{y}'\boldsymbol{\xi}] = x_0 r_0 + \mathbf{x}'\boldsymbol{\mu} + \mathbf{y}'\boldsymbol{v},$$

where $\boldsymbol{v} = (E[\xi_1], \dots, E[\xi_m])'$ is the expected vector of $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)'$.

It follows from Theorems 3 and 4 that the variance and skewness of uncertain random variable $R(X)$ are

$$V[R(X)] = V[R(X) - x_0 r_0] = \int_{-\infty}^{+\infty} (z - (\mathbf{x}'\boldsymbol{\mu} + \mathbf{y}'\boldsymbol{v}))^2 d\Psi(z)$$

and

$$S[R(X)] = S[R(X) - x_0 r_0] = \int_{-\infty}^{+\infty} (z - (\mathbf{x}'\boldsymbol{\mu} + \mathbf{y}'\boldsymbol{v}))^3 d\Psi(z),$$

respectively, where $\Psi(z)$ is the chance distribution of $R(X)$ determined by

$$\Psi(z) = \int_{-\infty}^{+\infty} \Phi(z - u) d\Upsilon(u)$$

and $\Phi(\cdot)$ is the uncertain distribution of $\mathbf{y}'\boldsymbol{\xi}$.

Similar to treating random returns as normally distributed, uncertain returns ξ_1, \dots, ξ_m are usually assumed to follow the same type of uncertain distribution. Here, zigzag uncertain variables are used to deal with the uncertain returns of newly listed securities.

Assumption 3. The uncertain return ξ_j ($j = 1, \dots, m$) is assumed to be a zigzag uncertain variable $\mathcal{Z}(a_j, b_j, c_j)$ with uncertain distribution

$$\Phi_j(z) = \begin{cases} 0, & \text{if } z \leq a_j \\ \frac{z - a_j}{2(b_j - a_j)}, & \text{if } a_j < z \leq b_j \\ \frac{z + c_j - 2b_j}{2(c_j - b_j)}, & \text{if } b_j < z \leq c_j \\ 1, & \text{otherwise} \end{cases}.$$

In this case, the expected value of the portfolio return can be determined as follows

$$E[R(X)] = x_0 r_0 + \mathbf{x}' \boldsymbol{\mu} + \mathbf{y}' \cdot \frac{\mathbf{a} + 2\mathbf{b} + \mathbf{c}}{4} \quad (2)$$

where $\mathbf{a} = (a_1, \dots, a_m)'$, $\mathbf{b} = (b_1, \dots, b_m)'$ and $\mathbf{c} = (c_1, \dots, c_m)'$.

As can be seen from the discussion in Qin (2015), the variance of the portfolio return $R(X)$ has an analytic expression

$$V[R(X)] = \mathbf{x}' \Sigma \mathbf{x} + \frac{5(\mathbf{y}' \mathbf{b} - \mathbf{y}' \mathbf{a})^2 + 5(\mathbf{y}' \mathbf{c} - \mathbf{y}' \mathbf{b})^2 + 6(\mathbf{y}' \mathbf{b} - \mathbf{y}' \mathbf{a})(\mathbf{y}' \mathbf{c} - \mathbf{y}' \mathbf{b})}{48} \quad (3)$$

From the discussions in (Li & Teo, 2021; Mehlawat et al., 2021; Zhai et al., 2022) it follows that the skewness of the portfolio return $R(X)$ has an analytical expression

$$S[R(X)] = \frac{(\mathbf{y}' \mathbf{a} - 2\mathbf{y}' \mathbf{b} + \mathbf{y}' \mathbf{c})(\mathbf{y}' \mathbf{c} - \mathbf{y}' \mathbf{a})^2}{32} \quad (4)$$

So far, the three objective functions of the mean-variance-skewness portfolio optimisation model have been constructed in Equations (2)-(4). Next, the constraints of the model are described in detail.

4.2. Constraints

(1) Cardinality Constraint:

As is often the case in practice, investors prefer to manage a portfolio of a reasonable size, especially when faced with large-scale investments. On the one hand, it is impractical to invest in thousands of securities simultaneously, which can significantly increase subsequent management costs. On the other hand, investing in too few securities can lead to a concentration of capital, which does not achieve the purpose of risk diversification. Therefore, the cardinality constraint is presented to limit the number of securities held in the portfolio to a certain interval $[K_{\min}, K_{\max}]$, i.e.,

$$K_{\min} \leq \sum_{i=1}^n \text{sgn}(x_i) + \sum_{j=1}^m \text{sgn}(y_j) \leq K_{\max} \quad (5)$$

where $\text{sgn}(\cdot)$ is the signum function.

(2) Minimum Transaction Lot Constraint:

Most existing portfolio optimisation models assume that investments are perfectly fractionable so that the investment portion of each security can be represented by any real number on $[0, 1]$. However, the transaction is usually associated with minimum transaction lots or rounds. A lot in securities represents the number of units of a financial instrument traded on an exchange. To incorporate minimum transaction lot constraint, the number of transaction lots invested in each security must be an integer. For example, a round lot in the China Stock Exchange is 100 shares. Suppose the market price of a security is a . In this case, the price of a round lot of this security is equal to $P = 100a$, and the capital invested in this security must be an integral multiple of P . Based on the above discussion, the constraint of minimum transaction lots is formulated as

$$\frac{W_0 x_i}{P_i^1} \in \mathbb{N}, \quad i = 1, \dots, n \quad (6)$$

$$\frac{W_0 y_j}{P_j^2} \in \mathbb{N}, \quad j = 1, \dots, m \quad (7)$$

where W_0 is the total capital, and P_i^1 and P_j^2 are the price of a round lot of i -th long-term listed security and j -th newly listed security, respectively. It should be noted that the risk-free asset is assumed to be perfectly fractionable.

(3) Bounding Constraint:

To obtain a diversified portfolio, it is assumed that the investment share of each held security must be limited in an interval. The bounding constraints ensure that neither a large nor a very small portion of the capital is invested

in a single security. If an asset is held in the portfolio, its investment share is limited to the interval. Otherwise, the investment share is set to zero. Therefore, the bounding constraints are formulated as follows

$$\text{sgn}(x_i)l_i^1 \leq \text{sgn}(x_i)x_i \leq \text{sgn}(x_i)u_i^1, \quad i = 1, \dots, n \quad (8)$$

$$\text{sgn}(y_j)l_j^2 \leq \text{sgn}(y_j)y_j \leq \text{sgn}(y_j)u_j^2, \quad j = 1, \dots, m \quad (9)$$

It is worth pointing out that the condition $\sum_{k=1}^{K_{\max}} l_k < 1$ holds to ensure the validity of the cardinality constraint¹, where l_1, \dots, l_{n+m} are the descending order of the sequence $l_1^1, \dots, l_n^1, l_1^2, \dots, l_m^2$.

(4) Budget Constraint:

In general, an investor's available capital is finite. Let $\mathbf{1}_n$ and $\mathbf{1}_m$ be $n \times 1$ and $m \times 1$ matrices with each entry 1, respectively. Suppose that all the available capital must be invested. The investment threshold is thus formulated as

$$x_0 + \mathbf{x}'\mathbf{1}_n + \mathbf{y}'\mathbf{1}_m = 1 \quad (10)$$

(5) No Short-selling Constraint:

It is assumed that short selling is not permitted, and therefore we have

$$x_0 \geq 0, \quad x_i \geq 0, \quad y_i \geq 0, \quad i = 1, \dots, n, \quad j = 1, \dots, m \quad (11)$$

4.3. Model Formulation

Based on the above discussion, the three-moment portfolio selection model can be constructed by optimising the objectives in Equations (2)-(4) subject to the constraints in Equations (5)-(11), i.e.,

$$P_1 \left\{ \begin{array}{l} \max_X E[R(X)] = x_0 r_0 + \mathbf{x}'\boldsymbol{\mu} + \mathbf{y}' \cdot \frac{\mathbf{a} + 2\mathbf{b} + \mathbf{c}}{4} \\ \min_X V[R(X)] = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} + \frac{6(\mathbf{y}'\mathbf{b} - \mathbf{y}'\mathbf{a})(\mathbf{y}'\mathbf{c} - \mathbf{y}'\mathbf{b}) + 5(\mathbf{y}'\mathbf{b} - \mathbf{y}'\mathbf{a})^2 + 5(\mathbf{y}'\mathbf{c} - \mathbf{y}'\mathbf{b})^2}{48} \\ \max_X S[R(X)] = \frac{(\mathbf{y}'\mathbf{a} - 2\mathbf{y}'\mathbf{b} + \mathbf{y}'\mathbf{c})(\mathbf{y}'\mathbf{c} - \mathbf{y}'\mathbf{a})^2}{32} \\ \text{s.t.} \quad K_{\min} \leq \sum_{i=1}^n \text{sgn}(x_i) + \sum_{j=1}^m \text{sign}(y_j) \leq K_{\max} \\ \frac{W_0 x_i}{P_i^1} \in \mathbb{N} \\ \frac{W_0 y_j}{P_j^2} \in \mathbb{N} \\ \text{sgn}(x_i)l_i^1 \leq \text{sgn}(x_i)x_i \leq \text{sgn}(x_i)u_i^1 \\ \text{sgn}(y_j)l_j^2 \leq \text{sgn}(y_j)y_j \leq \text{sgn}(y_j)u_j^2 \\ x_0 + \mathbf{x}'\mathbf{1}_n + \mathbf{y}'\mathbf{1}_m = 1 \\ x_0 \geq 0, \quad x_i \geq 0, \quad y_i \geq 0 \\ i = 1, \dots, n, \quad j = 1, \dots, m \end{array} \right.$$

5. Solution Algorithm

Considering that the proposed model P_1 is a multi-objective programming model with complex constraints, it would be tedious to solve it using conventional optimisation approaches. MOEA is a suitable alternative solution tool. Despite the complex technical details, most of the MOEAs share a general framework as presented in Figure 2 (Eiben & Smith, 2015). In this study, an encoder-decoder method is first developed to convert the model P_1 into an unconstrained model. Then, a novel MOEA method based on large-scale optimisation techniques is proposed to solve the converted model. The optimisation framework of the proposed solution algorithm is presented in Figure 3. Next, we introduce the essential components of the solution algorithm.

¹This condition guarantees that at least one feasible portfolio exists when any $K \in [K_{\min}, K_{\max}]$ securities are selected to construct the portfolio.

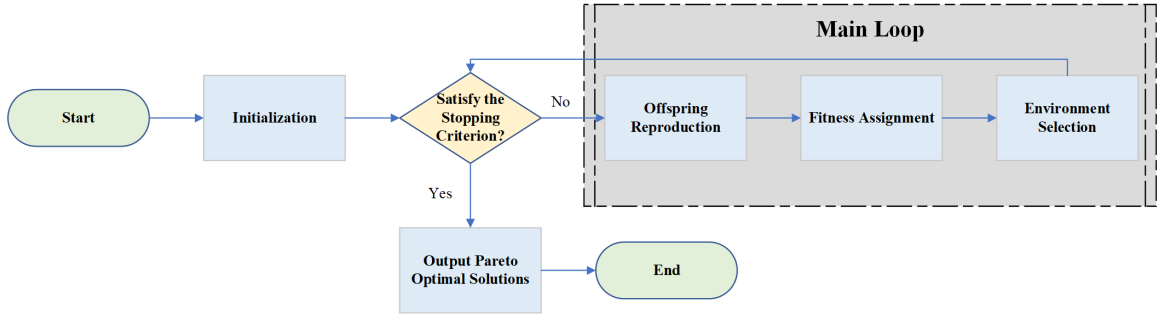


Figure 2: General framework of MOEAs.

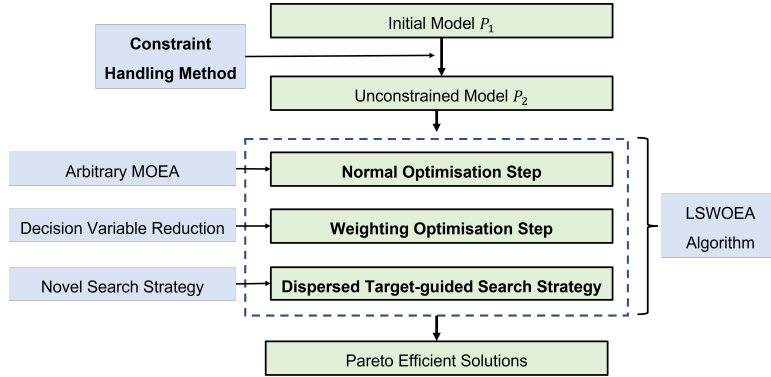


Figure 3: Optimisation framework of the proposed solution algorithm.

5.1. Constraint Handling Method

A portfolio vector $X = (x_0, \mathbf{x}', \mathbf{y}')'$ is encoded by a real value vector \mathbf{p} in the following search space

$$\Delta = \{ \mathbf{p} = (p_0, p_1, \dots, p_{n+m}, p_{n+m+1})' : 0 \leq p_k \leq 1, k = 0, \dots, n+m+1 \}.$$

Given a representative vector $\mathbf{p} \in \Delta$, a unique solution X of the model P_1 satisfying all constraints is obtained by a decoding method. Next, we introduce the decoding method in detail.

First, the element p_{n+m+1} is used to indicate the number of the held securities in the portfolio. Specifically, the cardinality of the portfolio X is formulated as follows

$$K = \text{Round} (K_{\min} + p_{n+m+1} (K_{\max} - K_{\min})) \quad (12)$$

where $\text{Round}(\cdot)$ is the round function.

Second, the elements p_1, \dots, p_{n+m} are sorted in descending order to $\tilde{p}_1, \dots, \tilde{p}_{n+m}$. Denote the serial number of the element p_k in the new order sequence by $q_k, k = 1, \dots, n+m$. The first K elements in the new order sequence, i.e. $\tilde{p}_1, \dots, \tilde{p}_K$, are selected to form a potential portfolio. The index sets of the selected long-term listed and newly listed securities are thus respectively represented by

$$S_1 = \{ k : q_k \leq K, k = 1, \dots, n \} \quad (13)$$

$$S_2 = \{ k - n : q_k \leq K, k = n+1, \dots, n+m \} \quad (14)$$

Then, a potential portfolio $\tilde{X} = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n, \tilde{y}_1, \dots, \tilde{y}_m)'$ is formulated by the normalisation operation as follows

$$\tilde{x}_0 = \frac{p_0}{p_0 + \sum_{k=1}^K \tilde{p}_k} \quad (15)$$

$$\tilde{x}_i = \begin{cases} \frac{\tilde{p}_{q_i}}{p_0 + \sum_{k=1}^K \tilde{p}_k}, & \text{if } i \in S_1 \\ 0, & \text{if } i \notin S_1 \end{cases} \quad (16)$$

$$\tilde{y}_j = \begin{cases} \frac{\tilde{p}_{q_{n+j}}}{p_0 + \sum_{k=1}^K \tilde{p}_k}, & \text{if } j \in S_2 \\ 0, & \text{if } j \notin S_2 \end{cases} \quad (17)$$

Note that the potential solution meets the constraints of cardinality, budget and no short sales.

The next step is to check whether the potential solution satisfies the bounding constraints. If it does not, it will be moved into the feasible space. Let $\tilde{l}_i^1 = l_i^1 / (\sum_{i \in S_1} l_i^1 + \sum_{j \in S_2} l_j^2)$ and $\tilde{l}_j^2 = l_j^2 / (\sum_{i \in S_1} l_i^1 + \sum_{j \in S_2} l_j^2)$ for $i \in S_1$ and $j \in S_2$, respectively. The feasibility of potential solution \tilde{X} is tested under the bounding constraints by the following Equation (18)

$$\begin{cases} \theta_1 = \max_{i \in S_1} \left\{ \frac{\max\{l_i^1 - \tilde{x}_i, 0\} + \min\{u_i^1 - \tilde{x}_i, 0\}}{\tilde{l}_i^1 - \tilde{x}_i} \right\} \\ \theta_2 = \max_{j \in S_2} \left\{ \frac{\max\{l_j^2 - \tilde{y}_j, 0\} + \min\{u_j^2 - \tilde{y}_j, 0\}}{\tilde{l}_j^2 - \tilde{y}_j} \right\} \\ \theta = \max\{\theta_1, \theta_2\} \end{cases} \quad (18)$$

If $\theta = 0$, the potential solution satisfies all the bounding constraints; otherwise, $\theta > 0$, and the potential solution is revised to

$$\begin{cases} \tilde{x}_i \leftarrow \tilde{x}_i + \theta \cdot (\tilde{l}_i^1 - \tilde{x}_i), & \text{if } i \in S_1 \\ \tilde{y}_j \leftarrow \tilde{y}_j + \theta \cdot (\tilde{l}_j^2 - \tilde{y}_j), & \text{if } j \in S_2 \end{cases} \quad (19)$$

Finally, to satisfy the minimum transaction lot constraint, the potential solution \tilde{X} is transformed into a feasible solution of Model P_1 as follows:

$$\begin{cases} x_i = \frac{P_i^1}{W_0} \cdot \left\lfloor \frac{W_0 \tilde{x}_i}{P_i^1} \right\rfloor, & i = 1, \dots, n \\ y_j = \frac{P_j^2}{W_0} \cdot \left\lfloor \frac{W_0 \tilde{y}_j}{P_j^2} \right\rfloor, & j = 1, \dots, m \\ x_0 = \tilde{x}_0 + \sum_{i=1}^n (\tilde{x}_i - x_i) + \sum_{j=1}^m (\tilde{y}_j - y_j) \end{cases} \quad (20)$$

where $\lfloor \cdot \rfloor$ is the floor function. Hereby, we assume $W_0 l_i^1$ and $W_0 l_i^2$ are an integer multiple of P_i^1 and P_i^2 , respectively, to ensure the cardinality constraint always holds².

In summary, the pseudocode of the decoding method is shown in Algorithm 1. Then the model P_1 can be transformed into the following model:

$$P_2 \begin{cases} \min_p \{-E[R(X)], V[R(X)], -S[R(X)]\} \\ \text{s.t. } X = \text{Decoder}(p) \\ 0 \leq p_k \leq 1, k = 0, \dots, n + m + 1 \end{cases}$$

It is clear that the transformed model is an unconstrained optimisation model in the search space Δ . In the next subsection, a novel MOEA based on large-scale optimisation techniques called LMWOEA is proposed to solve it in the case of large-scale investment.

²This assumption ensures that when a security is held, its investment amount after adjusted by the floor function (see Equation (20)) is still greater than the given lower bound.

Algorithm 1 Decoder(p)

Require: Representative vector $p = (p_0, \dots, p_{n+m+1})'$.

Ensure: Portfolio vector $X = (x_0, \mathbf{x}', \mathbf{y}')'$.

- 1: $\tilde{p}_1, \dots, \tilde{p}_{n+m} \leftarrow$ Sort the elements p_1, \dots, p_{n+m} in descending order.
 - 2: $q_1, \dots, q_{n+m} \leftarrow$ Record the serial number of elements p_1, \dots, p_{n+m} in the new sequence $\tilde{p}_1, \dots, \tilde{p}_{n+m}$.
 - 3: $K \leftarrow$ Determine the cardinality of the portfolio. //See Equation (12).
 - 4: $S_1, S_2 \leftarrow$ Determine the index sets of the selected long-term listed and newly listed securities. //See Equations (13)-(14).
 - 5: $\tilde{X} = (\tilde{x}_0, \tilde{\mathbf{x}}', \tilde{\mathbf{y}}')' \leftarrow$ Construct a potential portfolio. //See Equations (15)-(17).
 - 6: $\theta \leftarrow$ Check the bounding constraints of the potential solution \tilde{X} . //See Equation (18).
 - 7: **if** $\theta > 0$ **then**
 - 8: $\tilde{X} \leftarrow$ Revise the potential solution based on Equation (19).
 - 9: **end if**
 - 10: $X \leftarrow$ Revise the potential solution based on Equation (20).
 - 11: **return** X
-

5.2. Large-scale Evolutionary Algorithm

Although the constraint handling method allows for the easy application of arbitrary MOEAs to the proposed models, there are still significant challenges in terms of computational accuracy and time, especially given the large-scale securities. To efficiently solve the model P_2 , a novel evolutionary algorithm named LSWOEA based on large-scale multi-objective optimisation techniques is designed. The algorithm first applies a weighting optimisation-based decision variable reduction technique and then develops a novel dispersed target-guided search strategy to improve the search performance. LSWOEA is briefly introduced in Algorithm 2.

Algorithm 2 LSWOEA(P_2, B, A_1)

Require: Problem P_2 , Multi-objective optimiser A , Single-objective optimiser B .

Ensure: Solution Population S .

- 1: $FE_{\max}, g_1, ps \leftarrow$ Parameter Setting. // FE_{\max} : Maximum number of function evaluations; g_1 : Number of maximum iterations in the normal optimisation step; ps : Population size.
 - 2: $S \leftarrow$ Randomly initialize population of size ps for Problem P_2 .
 - 3: **repeat**
 - 4: $S \leftarrow$ Weighting_Optimisation(S, P_2, B). //See Algorithm 3.
 - 5: $g \leftarrow 1$.
 - 6: **while** $g \leq g_1$ **do**
 - 7: $S_1 \leftarrow$ Generate a offspring population of S by the multi-objective optimiser A .
 - 8: $S \leftarrow$ Conduct the environment selection on $S \cup S_1$.
 - 9: $R \leftarrow$ Create a set of reference vectors. //See Subsection 5.2.2.
 - 10: $S \leftarrow$ Dispersed_Target_Guided_Strategy(S, P_2, R). //See Algorithm 4.
 - 11: $g \leftarrow g + 1$.
 - 12: **end while**
 - 13: **until** All function evaluations are used.
 - 14: **return** S
-

It should be noted that a population-based multi-objective optimiser and a single-objective optimiser must be included in the optimisation process. In this study, at each generation, the multi-objective optimiser A is randomly selected from the traditional MOEAs, including NSGA-II (Deb et al., 2002), MOEA-D (Zhang & Li, 2007), SMPSO (Nebro et al., 2009) and NSGA-III (Deb & Jain, 2014), and the single-objective optimiser B is implemented by the widely used DE (Zhang & Sanderson, 2009).

5.2.1. Decision Space Reduction Method

The decision space reduction-based technique (Zille et al., 2018; He et al., 2019; Qin et al., 2021) is a main category of existing MOEAs for large-scale optimisation, which transforms the original problem into a simpler one with a small number of decision variables. The main idea of problem transformation in this study is inspired by He et al. (2019).

To illustrate, consider a general multi-objective problem with M objectives and N decision variables

$$P \begin{cases} \min_{\mathbf{x}} \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_M(\mathbf{x}))' \\ \text{s.t. } \mathbf{x} = (x_1, \dots, x_N)' \in \mathbb{R}^N \\ 0 \leq x_j \leq 1, j = 1, \dots, N \end{cases}$$

Taking a fixed solution $\tilde{\mathbf{x}}$ as a reference point, two direction vectors are defined as follows

$$\begin{cases} \mathbf{v}_l = \tilde{\mathbf{x}} - \mathbf{0}_N \\ \mathbf{v}_u = \mathbf{1}_N - \tilde{\mathbf{x}} \end{cases} \quad (21)$$

where $\mathbf{0}_N$ and $\mathbf{1}_N$ are the lower and upper boundary points of the search space. Given two weight variables w_1 and w_2 between 0 and 0.5, two corresponding points in the search space are determined by

$$\begin{cases} \mathbf{x}_1^{new} = \mathbf{0}_N + w_1 \frac{\mathbf{v}_l}{\|\mathbf{v}_l\|} l_{\max} \\ \mathbf{x}_2^{new} = \mathbf{1}_N - w_2 \frac{\mathbf{v}_u}{\|\mathbf{v}_u\|} l_{\max} \end{cases} \quad (22)$$

where $l_{\max} = \|\mathbf{1}_N - \mathbf{0}_N\| = \sqrt{N}$ is the maximum diagram length in the search space. Then the objective values associated with the weight variables w_1 and w_2 can be calculated as follows

$$\begin{cases} \mathbf{g}_l(\tilde{\mathbf{x}}, w_1) = \mathbf{f}(\mathbf{x}_1^{new}) \\ \mathbf{g}_u(\tilde{\mathbf{x}}, w_2) = \mathbf{f}(\mathbf{x}_2^{new}) \end{cases} \quad (23)$$

Given a set of reference solutions of size h , once each of them is associated with two weighting variables, a total number of $2h$ new solutions can be constructed. Specifically, denote the set of reference solutions by $\tilde{X} = \{\tilde{\mathbf{x}}_1', \dots, \tilde{\mathbf{x}}_h'\}$ and the weight vector by $\mathbf{w} = (w_{11}, w_{12}, \dots, w_{h1}, w_{h2})'$. The corresponding $2h$ solutions, denoted by $\mathbf{x}_{11}^{new}, \mathbf{x}_{12}^{new}, \dots, \mathbf{x}_{h1}^{new}, \mathbf{x}_{h2}^{new}$, can be generated according to Equations (21) and (22) and their objective values can be calculated according to Equation (23). Assume that the set of reference points \tilde{X} is given. In this case, the optimisation of the decision vector \mathbf{x} in the original problem can be converted into an optimisation of the weight vector \mathbf{w} to find a set of superior solutions for the original problem. Here the metric of hypervolume (HV) (While et al., 2006) is used to evaluate the quality of a set of solutions. Denote the HV of $2h$ solutions associated with the weight vector \mathbf{w} and the reference point set \tilde{X} by $H(\mathbf{w}, \tilde{X})$. Then, for an arbitrary but fixed set of reference solutions \tilde{X} , the original problem P can be reconstructed as the following one-objective model

$$P_3(\tilde{X}) \begin{cases} \max_{\mathbf{w}} f(\mathbf{w}) = H(\mathbf{w}, \tilde{X}) \\ \text{s.t. } \mathbf{w} = (w_{11}, w_{12}, \dots, w_{h1}, w_{h2})' \in \mathbb{R}^{2h} \\ 0 \leq w_{j1}, w_{j2} \leq 0.5, j = 1, \dots, h \end{cases}$$

It can be seen that the weighting optimisation problem $P_3(\tilde{X})$ has only $2h < N$ decision variables, which serves the purpose of decision space reduction by bounding the search space. Model $P_3(\tilde{X})$ is optimised with a single objective optimiser with population size ps_w . In each iteration, up to $2h \cdot ps_w$ new solutions (de-duplicated) to the original problem P can be obtained, which are collected as candidate solutions.

Obviously, the original problem P and the weight optimisation problem $P_3(\tilde{X})$ are complementary. On the one hand, Model P can reach all possible solutions, but it can converge very slowly in a large-dimensional space. On the other hand, Model $P_3(\tilde{X})$ has the disadvantage of restricting the search space and the advantage of searching a smaller space more thoroughly. To exploit the synergy of these two formulations, two different optimisation phases are alternated: a normal optimisation step and a weight optimisation step. The original problem P is optimised in the normal optimisation step for fixed function evaluations of g_1 . Then, the weighting optimisation step is performed for h different reference solutions, as shown in Algorithm 3.

Algorithm 3 Weighting_Optimisation(S, P, B)

Require: Population S , Original Problem P , Single-objective optimiser B .

Ensure: New Population S .

- 1: $h, ps_w, g_2 \leftarrow$ Parameter setting. // h : Number of reference solutions; ps_w : Population size; g_2 : Maximum number of iterations.
 - 2: $\tilde{X} = \{\tilde{x}_1, \dots, \tilde{x}_h\} \leftarrow$ Conduct the environment selection on S to select h reference solutions.
 - 3: $P_3(\tilde{X}) \leftarrow$ Construct a weighting optimisation problem.
 - 4: $H \leftarrow$ Randomly initialise population of size ps_w for problem $P_3(\tilde{X})$.
 - 5: $S_1 \leftarrow$ Collect the generated candidate solutions based on the initial population H . // See Equations (21) and (22).
 - 6: **for all** $g = 1, \dots, g_2$ **do**
 - 7: $H \leftarrow$ Optimize Problem $P_3(\tilde{X})$ with optimiser B .
 - 8: $S_2 \leftarrow$ Collect the generated candidate solutions based on the current population H . // See Equations (21) and (22).
 - 9: $S_1 \leftarrow S_1 \cup S_2$.
 - 10: **end for**
 - 11: $S \leftarrow$ Conduct the environment selection on $S \cup S_1$ to select ps solutions.
 - 12: **return** S
-

5.2.2. A novel search strategy

The search space increases exponentially as the number of decision variables increases. This dramatically degrades the search performance and convergence ability of evolutionary algorithms. To solve this problem, a dispersed target-guided search strategy is proposed to identify promising search directions in the decision space.

First, a set of reference vectors $R = \{r_1, \dots, r_{N_f}\}$ on a normalised $(M - 1)$ -dimensional hyperplane is established for the M -objective problem. In this study, a method from Das & Dennis (1998) is used to generate the N_f reference vectors widely distributed on the entire normalised hyperplane.

Second, the objective vectors of the parent population are normalised. Suppose that N_f solutions are selected from the current population to form the parent population denoted by $S_p = \{x_1, \dots, x_{N_f}\}$. The ideal point and the anti-ideal point of the population are determined by the minimum and maximum values for each objective function, respectively, i.e., $z^{\min} = (z_1^{\min}, \dots, z_M^{\min})'$ and $z^{\max} = (z_1^{\max}, \dots, z_M^{\max})'$. Then the normalised objective functions for each solution are defined by

$$f_i^n(x|S_p) = \frac{f_i(x) - z_i^{\min}}{z_i^{\max} - z_i^{\min}}, \quad i = 1, \dots, M \quad (24)$$

Then each member of the parent population is associated with a unique reference vector. The matching problem is considered as a classical assignment problem where the objective is to minimise the total distance between individuals and relevant reference points. Denote the decision variable as $D = \{d_{ij}\} \in \mathbb{R}^{N_f \times N_f}$. Then the assignment model is formulated as follows

$$P_4 \left\{ \begin{array}{l} \min_D \sum_{i=1}^{N_f} \sum_{j=1}^{N_f} d_{ij} \cdot \|f^n(x_i|S_p) - r_j\| \\ \text{s.t.} \sum_{i=1}^{N_f} d_{ij} = 1, \quad j = 1, \dots, N_f \\ \sum_{j=1}^{N_f} d_{ij} = 1, \quad i = 1, \dots, N_f \\ d_{ij} \in \{0, 1\} \end{array} \right.$$

where $\|\cdot\|$ is the L_2 - norm function and $d_{ij} = 1$ indicates that the solution \mathbf{x}_i is ordered to the reference point \mathbf{r}_j . It can be seen that the model P_3 is an integer linear programming problem that can be solved with the function 'intlinprog' in MATLAB.

Next, the search direction for each parent solution is constructed. Define a target function for the parent solution \mathbf{x}_i

$$f_i^t(\mathbf{x}|D) = \sum_{j=1}^{N_f} d_{i,j} \mathbf{r}_j' f^n(\mathbf{x}|S_p), \quad i = 1, \dots, N_f \quad (25)$$

Then the search direction $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,N})$ of the parent solution \mathbf{x}_i is defined as

$$v_{ij} = f_i^t(\mathbf{x}_i + \delta \mathbf{e}_j | D) - f_i^t(\mathbf{x}_i - \delta \mathbf{e}_j | D), \quad j = 1, \dots, N \quad (26)$$

where δ is a fully small positive number and \mathbf{e}_j is a unit vector whose j th element is 1. $v_{i,j} > 0$ indicates that slightly enlarging the value of x_{ij} tends to increase the objective value $f_i^t(\mathbf{x}_i | D)$, and vice versa.

Finally, the offspring solution \mathbf{x}_i^{new} of the parent solution \mathbf{x}_i is generated along the search direction $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,N})'$:

$$\mathbf{x}_i^{new} = \mathbf{x}_i + rand \cdot (0.5 - 0.5 \text{sgn}(v_i) - \mathbf{x}_i), \quad i = 1, \dots, N_f \quad (27)$$

where $rand \in [0, 1]$ is a random coefficients. Equation 26 drives solution \mathbf{x}_i in the direction of decreasing the objective value and ensures that its elements remain between $[0, 1]$.

In summary, Algorithm 4 represents the pseudocode of the dispersed target-guided search strategy. To simply exhibit how the search directions are defined, an example with two objectives is given in Figure 4. As shown in Figure 4, a parent population with six solutions is selected, denoted by blue dots. Six reference points represented by red dots are uniformly generated and linked to a separate solution. Guided by the reference points, the ideal search directions for the parent solutions are shown as the green arrows in Figure 4. It can be seen that the proposed search strategy encourages both convergence and diversification.

Algorithm 4 Dispersed_Target_Guided_Strategy(S, P, R)

Require: Population S , Problem P , A Set of Reference Solutions $R = (\mathbf{r}'_1, \dots, \mathbf{r}'_{N_f})$.

Ensure: New Population S .

- 1: $S_p \leftarrow$ Randomly select N_f solutions from Population S as parents.
 - 2: $f_1^n(\mathbf{x}|S_p), \dots, f_M^n(\mathbf{x}|S_p) \leftarrow$ Define the normalize functions for the parent population. //See Equation (24).
 - 3: $D \leftarrow$ Assign each parent to a unique reference vector by solving Problem P_4 .
 - 4: **for all** $i = 1, \dots, N_f$ **do**
 - 5: $f_i^t(\mathbf{x}|D) \leftarrow$ Define the target objective function for the parent solution \mathbf{x}_i . //See Equation (25).
 - 6: $\mathbf{v}_i \leftarrow$ Build the search direction for the parent solution \mathbf{x}_i . //See Equation (26).
 - 7: $\mathbf{x}_i^{new} \leftarrow$ Generate a offspring population for the parent solution \mathbf{x}_i . //See Equation (27).
 - 8: **end for**
 - 9: $S \leftarrow$ Conduct the environment selection on $S \cup \{\mathbf{x}_i^{new}\}_{i=1}^{N_f}$.
 - 10: **return** S
-

6. Numerical experiments

In this section, two numerical experiments are presented to illustrate the applicability and effectiveness of the proposed model and algorithm. First, a small-scale case with only 30 securities is conducted to illustrate the proposed approach comprehensively. Then, a large-scale case with thousands of securities is discussed to demonstrate the superiority of the proposed LSWOEA in large-scale portfolio optimisation.

6.1. Small-scale Case

Now consider a portfolio optimisation problem with 20 long-term listed securities and 10 newly listed securities. The securities are randomly selected from the Shanghai Stock Exchange (SSE), and their codes and market prices are

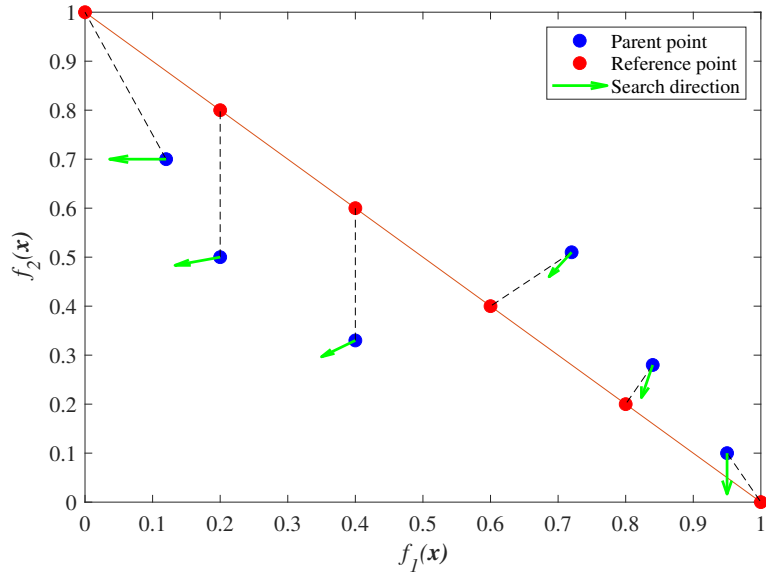


Figure 4: Diagram of the search direction.

Table 3

Codes and market prices of the selected securities.

Label	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15
Code	600000	600004	600006	600007	600008	600009	600010	600011	600012	600015	600016	600017	600018	600019	600020
Price	10.5371	17.3769	4.4009	16.0374	2.8041	76.4827	1.3095	5.197	4.9876	6.5335	5.3314	2.7994	5.1843	4.5603	4.067
Label	A16	A17	A18	A19	A20	A21	A22	A23	A24	A25	A26	A27	A28	A29	A30
Code	600021	600022	600026	600027	600028	600025	600901	600903	600929	600933	600938	601016	601028	601038	601068
Price	7.7203	1.3299	5.9047	3.0495	3.991	3.8486	5.3257	14.8497	6.2256	12.7783	4.4331	5.8268	27.8886	5.57	4.0658

Table 4

Sample means of the returns for the long-term listed securities.

Label	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12	A13	A14	A15
Mean	0.1773	0.1791	0.0755	0.1311	0.0502	0.2662	0.0644	0.0133	0.1279	0.1137	0.1382	0.0653	0.1436	-0.0019	0.0805
Label	A16	A17	A18	A19	A20										
Mean	0.1862	-0.1203	-0.0769	-0.0057	-0.0321										

listed in Table 3. For the long-term listed securities (A1-A20), historical data of 614 weekly returns from 1 January 2008 to 31 December 2019 are used to calculate the sample mean vector μ and the covariance matrix Σ , which are reported in Tables 4 and 5, respectively. For the newly listed securities (A21-A30), the historical data of 51 weekly returns from 1 January 2019 to 31 December 2019 is used as reference information for the experts' estimation. In this experiment, the uncertain return of a newly listed security is simply estimated as a zigzag uncertain variable $\mathcal{Z}(a, b, c)$, where a , b and c are the minimum, average and maximum of the historical returns, respectively. The uncertain parameters for the newly listed securities are listed in Table 6. It should be noted that in practice, these uncertain parameters should be estimated more accurately based on experts' knowledge and opinion.

The initial wealth W_0 is set at CNY 1 million. A round lot of the securities on the SSE is 100 shares. The minimum and maximum number of the held securities are set to $K_{\min} = 8$ and $K_{\max} = 16$, respectively. The minimum investment amount of securities is five lots, and the upper limits of investment proportions are set to 0.5 for all securities. Additionally, some general parameters for the proposed LSWOEA are given as follows: $EF_{\max} = 30000$, $ps = 100$, $g_1 = 30$, $h = 10$, $ps_w = 10$ and $g_2 = 30$. The LSWOEA is applied to solve the problem to obtain a set of diversified efficient solutions. For brevity, only 15 solutions with the largest crowding distances are listed in Table 7. Table 7

Table 5

Sample covariance matrix of the returns for the long-term listed securities.

24.4	8.6	8.3	10.2	11.2	11.3	11.1	9.8	9.2	19.3	16.5	9.7	13.0	13.3	8.7	8.9	11.3	17.3	9.8	11.7
8.6	20.1	11.2	12.5	14.1	14.2	12.6	7.4	12.6	8.4	7.4	14.5	11.7	10.8	11.6	10.9	11.9	14.1	8.7	7.8
8.3	11.2	33.9	12.6	18.6	11.7	14.4	8.4	14.2	9.5	7.9	16.6	13.1	10.4	14.7	16.8	14.1	16.5	12.7	8.5
10.2	12.5	12.6	29.1	18.0	12.1	12.0	7.8	13.4	10.5	8.6	14.5	13.4	11.5	13.3	13.7	13.5	15.9	9.4	8.4
11.2	14.1	18.6	18.0	43.8	15.3	18.4	12.3	13.4	11.9	11.5	20.9	15.0	14.1	17.9	18.0	17.4	20.3	16.2	11.3
11.3	14.2	11.7	12.1	15.3	23.5	12.0	9.3	10.3	10.9	9.3	14.3	15.8	11.4	11.6	12.1	11.1	15.9	10.7	9.3
11.1	12.6	14.4	12.0	18.4	12.0	38.6	12.6	12.1	11.8	10.7	15.3	17.1	17.8	13.9	16.6	18.1	17.1	13.3	10.3
9.8	7.4	8.4	7.8	12.3	9.3	12.6	23.4	5.9	9.9	9.6	10.0	10.9	11.2	8.2	12.2	11.0	11.4	18.1	10.2
9.2	12.6	14.2	13.4	13.4	10.3	12.1	5.9	28.2	9.0	7.3	15.0	11.4	8.8	15.0	12.5	12.4	13.9	9.3	7.2
19.3	8.4	9.5	10.5	11.9	10.9	11.8	9.9	9.0	22.9	17.3	10.0	13.2	12.7	9.7	8.8	9.9	16.3	10.0	12.0
16.5	7.4	7.9	8.6	11.5	9.3	10.7	9.6	7.3	17.3	19.4	8.0	11.6	12.3	7.7	7.4	8.3	14.2	9.6	11.3
9.7	14.5	16.6	14.5	20.9	14.3	15.3	10.0	15.0	10.0	8.0	30.6	17.7	12.0	16.2	15.7	17.3	19.0	13.4	9.1
13.0	11.7	13.1	13.4	15.0	15.8	17.1	10.9	11.4	13.2	11.6	17.7	37.2	14.1	15.0	15.4	11.9	19.1	11.4	11.1
13.3	10.8	10.4	11.5	14.1	11.4	17.8	11.2	8.8	12.7	12.3	12.0	14.1	23.9	10.3	11.6	16.3	17.0	11.2	11.4
8.7	11.6	14.7	13.3	17.9	11.6	13.9	8.2	15.0	9.7	7.7	16.2	15.0	10.3	23.2	15.6	14.0	15.5	11.5	8.2
8.9	10.9	16.8	13.7	18.0	12.1	16.6	12.2	12.5	8.8	7.4	15.7	15.4	11.6	15.6	32.8	13.8	14.6	14.7	8.7
11.3	11.9	14.1	13.5	17.4	11.1	18.1	11.0	12.4	9.9	8.3	17.3	11.9	16.3	14.0	13.8	36.6	17.7	13.6	9.4
17.3	14.1	16.5	15.9	20.3	15.9	17.1	11.4	13.9	16.3	14.2	19.0	19.1	17.0	15.5	14.6	17.7	39.6	15.2	13.0
9.8	8.7	12.7	9.4	16.2	10.7	13.3	18.1	9.3	10.0	9.6	13.4	11.4	11.2	11.5	14.7	13.6	15.2	23.7	9.5
11.7	7.8	8.5	8.4	11.3	9.3	10.3	10.2	7.2	12.0	11.3	9.1	11.1	11.4	8.2	8.7	9.4	13.0	9.5	19.7

Table 6

Uncertain returns for the newly listed securities.

Label	A21	A22	A23	A24	A25	A26	A27	A28	A29	A30
<i>a</i>	-8.4967	-8.5106	-8.2014	-11.2266	-8.8620	-10.3896	-6.5934	-15.8198	-13.2426	-8.8388
<i>b</i>	0.7201	0.2341	0.1650	-0.3225	1.2628	0.0261	-0.2097	2.6569	0.4138	0.0475
<i>c</i>	11.9512	14.1780	17.6014	14.4050	18.2809	9.5142	6.7416	34.5289	29.7723	6.6667

shows that the solutions satisfy all realistic constraints, indicating that the proposed approach is practical to solve the portfolio optimisation problem with realistic constraints.

To empirically investigate the performance of LSWOEA, six existing MOEAs are selected as the baselines for the experiments, namely, SMPSO (Nebro et al., 2009), NSGA-II (Deb et al., 2002), MOEA-D (Zhang & Li, 2007), WOF-NSGA-II (Zille et al., 2018), LSWOF-NSGA-II (He et al., 2019) and LMOEADS (Qin et al., 2021). The first three are widely known MOEAs using particle swarm optimisation, genetic algorithm and decomposition, respectively, and the last three are state-of-the-art LSMOEAs. To allow fair comparisons, the population size and the maximum number of objective evaluations for all algorithms are set to 100 and 30000, respectively. Other recommended parameter settings for the compared algorithms are taken from the original literature. All compared algorithms are reproduced based on the PlatEMO (Tian et al., 2017). Since the true Pareto front of the problem is unknown, the hypervolume (HV) indicator (While et al., 2006), which measures both convergence and diversity, is used to measure the performance of each solution set. For a more accurate calculation, the objective space is normalised by Equation (28)

$$f_i = \frac{f_i - f_i^{\min}}{f_i^{\max} - f_i^{\min}} \quad (28)$$

where f_i^{\max} and f_i^{\min} denote the maximum and minimum values of the i th objective for all solutions in the test. The reference point is set to the maximum values of the normalised objectives, i.e. $(1, 1, 1)'$.

In the comparisons, each algorithm is run 20 times on the proposed model to obtain the statistical results. Figure 5 shows the comparison results of the Pareto fronts obtained by the proposed LSWOEA and each compared algorithm in the last test. Table 8 lists the comparisons of the best, worst and mean HVs obtained by the seven algorithms in the 20

Table 7
Optimal asset allocations for the small-scale case.

Label	Investment Proportion (Number of Shares / 100)														
A0	0.0020	0.1679	0.0214	0.0057	0.0474	0.0304	0.0044	0.0392	0.0126	0.0099	0.0760	0.0027	0.0042	0.0068	0.0068
A1	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A2	0.0000(0)	0.1355(78)	0.0000(0)	0.0070(4)	0.0539(31)	0.0104(6)	0.0000(0)	0.0730(42)	0.2294(132)	0.0000(0)	0.1425(82)	0.0608(35)	0.0000(0)	0.0122(7)	0.0000(0)
A3	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0356(81)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0304(69)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0044(10)
A4	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0096(6)	0.0289(18)	0.0000(0)	0.0385(24)	0.0000(0)	0.0000(0)	0.0000(0)	0.0112(7)	0.0000(0)	0.0064(4)
A5	0.0000(0)	0.0000(0)	0.0269(96)	0.0000(0)	0.0000(0)	0.0059(21)	0.0533(190)	0.0000(0)	0.0000(0)	0.0286(102)	0.0000(0)	0.0000(0)	0.0000(0)	0.0017(6)	0.0000(0)
A6	0.0000(0)	0.0000(0)	0.0382(5)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0024(324)	0.0000(0)	0.0306(4)	0.0382(5)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A7	0.0013(10)	0.0000(0)	0.0000(0)	0.0042(32)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0016(12)	0.0000(0)	0.0000(0)
A8	0.0026(5)	0.1050(202)	0.0000(0)	0.0000(0)	0.0946(182)	0.0000(0)	0.0000(0)	0.1455(280)	0.0935(180)	0.0000(0)	0.0000(0)	0.0457(88)	0.0229(44)	0.0000(0)	0.0000(0)
A9	0.0000(0)	0.0000(0)	0.0264(53)	0.0075(15)	0.0000(0)	0.0000(0)	0.0080(16)	0.0464(93)	0.0928(186)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A10	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0444(68)	0.0046(7)	0.0105(16)	0.0000(0)	0.0000(0)	0.0477(73)	0.0000(0)	0.0235(36)	0.0000(0)	0.0000(0)	0.0000(0)
A11	0.0000(0)	0.1109(208)	0.0347(65)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0901(169)	0.0528(99)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0277(52)
A12	0.0000(0)	0.0476(170)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0031(11)
A13	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0073(14)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A14	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A15	0.0000(0)	0.0000(0)	0.0468(115)	0.0000(0)	0.0000(0)	0.0000(0)	0.0321(79)	0.0000(0)	0.0000(0)	0.0313(77)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A16	0.0000(0)	0.0579(75)	0.0000(0)	0.0039(5)	0.0000(0)	0.0000(0)	0.0000(0)	0.0479(62)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A17	0.0012(9)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A18	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0384(65)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A19	0.0024(8)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0604(198)	0.0000(0)	0.0000(0)	0.0015(5)	0.0000(0)
A20	0.0000(0)	0.0822(206)	0.0000(0)	0.0068(17)	0.0000(0)	0.0000(0)	0.0000(0)	0.0686(172)	0.0000(0)	0.0000(0)	0.1489(373)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A21	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0362(94)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0250(65)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A22	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0256(48)	0.0000(0)	0.0000(0)	0.0000(0)
A23	0.0000(0)	0.0000(0)	0.1203(81)	0.0074(5)	0.1604(108)	0.0089(6)	0.0000(0)	0.1812(122)	0.0431(29)	0.0000(0)	0.0817(55)	0.1515(102)	0.0089(6)	0.1960(132)	0.0074(5)
A24	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A25	0.0000(0)	0.0000(0)	0.0588(46)	0.0000(0)	0.1763(138)	0.0524(41)	0.0000(0)	0.0741(58)	0.0000(0)	0.0000(0)	0.0869(68)	0.1802(141)	0.0115(9)	0.0396(31)	0.0473(37)
A26	0.0022(5)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A27	0.0000(0)	0.1987(341)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0623(107)	0.1101(189)	0.0000(0)	0.1503(258)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)
A28	0.4964(178)	0.0000(0)	0.3124(112)	0.4797(172)	0.1952(70)	0.3960(142)	0.4267(153)	0.0446(16)	0.1785(64)	0.3988(143)	0.0558(20)	0.2315(83)	0.4602(165)	0.3821(137)	0.4518(162)
A29	0.4902(880)	0.0000(0)	0.3141(564)	0.4779(858)	0.1560(280)	0.4818(865)	0.4289(770)	0.0462(83)	0.1487(267)	0.3977(714)	0.1593(286)	0.2785(500)	0.4779(858)	0.3559(639)	0.4450(799)
A30	0.0016(4)	0.0943(232)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0000(0)	0.0016(4)	0.0000(0)	0.0000(0)
Cardinality	8.0000	8.0000	9.0000	8.0000	9.0000	8.0000	8.0000	13.0000	9.0000	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
Mean	5.1070	-0.0052	3.7313	4.9759	2.8302	4.6521	4.4361	1.1674	1.8815	4.1716	1.5071	3.5586	4.8946	4.4374	4.8092
Variance	181.7515	4.8239	98.7724	172.0954	59.1661	152.9261	136.5120	15.5258	30.7568	121.6968	24.0647	93.2201	167.4917	141.3012	160.5001
Skewness	954.3545	-0.0534	382.0130	881.8210	167.4952	744.9057	620.4033	15.1647	52.2265	516.8136	38.0050	348.1535	847.5590	659.7523	789.4525

Table 8
Best, worst and mean HVs of the algorithms on the small-scale case.

Algorithm	SMP SO	NSGA-II	MOEA-D	WOF-NSGA-II	LSMOF-NSGA-II	LMOEADS	LSWOEA
Best	0.3305	0.2297	0.3082	0.3711	0.4244	0.4212	0.4251
Worst	0.1172	0.1392	0.1608	0.1580	0.3588	0.3827	0.4180
Mean	0.1589	0.1700	0.2227	0.2471	0.4134	0.4088	0.4215

runs, and Figure 6 shows the boxplots of the statistical results. As can be seen from Figure 5, the LSWOEA achieves higher quality and more distributionally efficient solutions compared to the traditional MOEAs (SMP SO, NSGA-II and MOEA-D), while the LSWOEA performs comparably to the existing LSMOEA's (WOF-NSGA-II, LSMOEF-NSGA-II and LMOEADS). Furthermore, it can be seen from Table 8 that the best, worst and mean HVs of the LSWOEA are slightly higher than those of the other LSMOEA's, indicating that the LSWOEA is slightly superior to them. Additionally, Figure 6 suggests that the LSWOEA is more stable than the other algorithms. In summary, the LSMOEA's generally perform better than the traditional algorithms on small portfolio optimisation problems, with the LSWOEA proposed in this study performing slightly better than the other large algorithms.

6.2. Large-scale case

In this section, a case study of 1,063 securities is conducted to illustrate the effectiveness of the proposed methodology in optimising large portfolios. The securities used for the experiment are selected from the SSE's A-Shares, of which 792 securities listed prior to 2008 are considered long-term listed and 271 securities listed in 2018 are considered newly listed. Weekly 12-year data from 1 January 2008 to 31 December 2019 is used for long-term listed securities and weekly 1-year data from 1 January 2019 to 31 December 2019 is used for newly listed securities. The initial assets are set at CNY 100 million and the minimum and maximum number of the held securities are set at

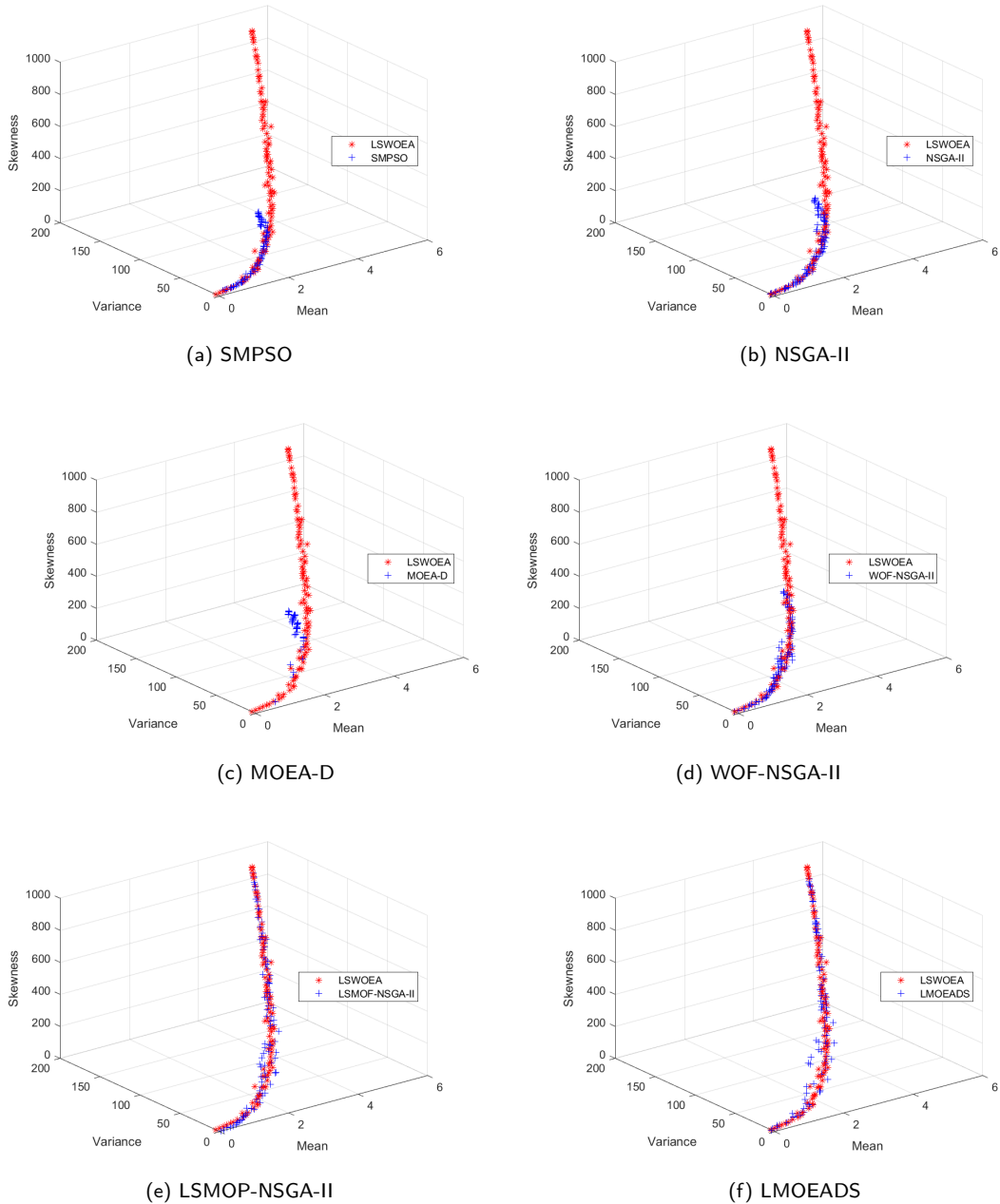


Figure 5: Pareto fronts obtained by the LSWOEA and each compared algorithm on the small-scale case.

80 and 600, respectively. The other parameters of the model and the algorithm are set according to the same rules as in the small scale.

Each algorithm is run 20 times on the proposed model for 30000 function evaluations. Additionally, the experimental results are recorded for 60000 function evaluations to test the convergence speed of the algorithms. Figure 5 shows the comparative Pareto frontier results obtained in the final test for the proposed LSWOEA and each comparison algorithm, where A-3 and A-6 denote Algorithm A for 30000 and 60000 function evaluations, respectively. Table 8 lists the comparison of the best, worst and average HV obtained by the seven algorithms over 20 runs, and Figure 6 shows the boxplots of the statistical results.

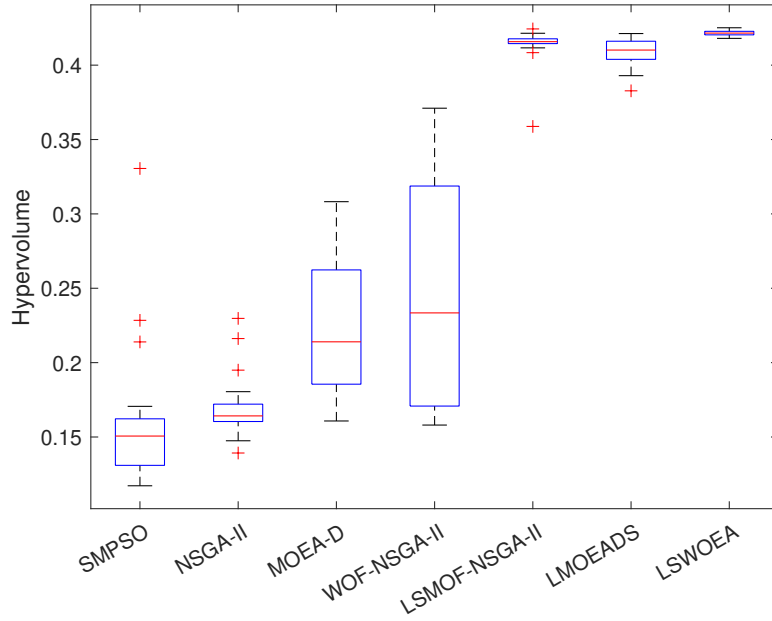


Figure 6: Boxplots of the HVs on the small-scale case.

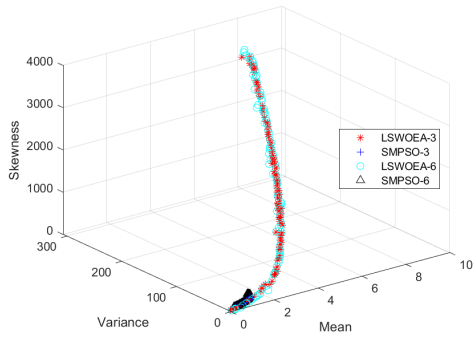
Table 9

Best, worst and mean HVs of the algorithms on the large-scale case.

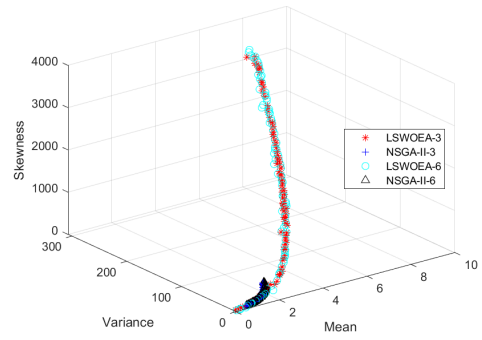
FE_{max}	Algorithm	SMPSO	NSGA-II	MOEA-D	WOF-NSGA-II	LSMOF-NSGA-II	LMOEADS	LSWOEA
30000	Best	0.0285	0.0474	0.0369	0.0470	0.4194	0.1123	0.4575
	Worst	0.0201	0.0321	0.0159	0.0313	0.0249	0.0261	0.4218
	Mean	0.0230	0.0418	0.0267	0.0407	0.2822	0.0411	0.4299
60000	Best	0.0300	0.0545	0.0427	0.0544	0.4258	0.3461	0.4602
	Worst	0.0219	0.0387	0.0234	0.0397	0.0548	0.0302	0.4229
	Mean	0.0257	0.0446	0.0333	0.0460	0.3281	0.0686	0.4313

As shown in Figure 7, LSWOEA shows excellent performance and obtains a set of high-quality diversified efficient solutions. Among the compared algorithms, the traditional algorithms (SMPSO, NSGA-II and MOEA-D) continue to perform poorly, while the existing LSMOEA (WOF-NSGA-II, LSMOF-NSGA-II and LMOEADS), which are comparable to LSWOEA in the small-scale case, perform intensely worse in this case. When the number of function evaluations is increased from 30000 to 60000, only LSMOF-NSGA-II achieves a significant improvement, but its performance is still worse than that of the proposed LSWOEA.

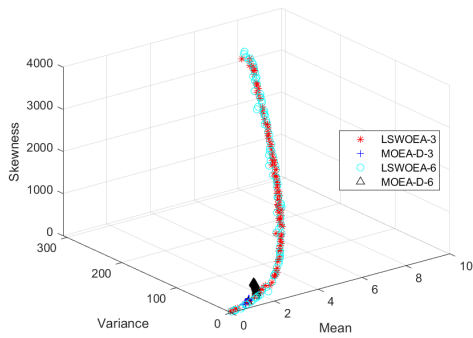
As seen in Table 7 and Figure 8, the best, worst and mean HVs obtained by LSWOEA are much better than those of the other algorithms, indicating that LSWOEA has an outstanding advantage in dealing with large-scale portfolio optimisation problems. Since the performance of LSMOEA (WOF-NSGA-II, LSMOF-NSGA-II and LMOEADS), which are comparable to LSWOEA in the small-scale case, perform intensely worse in this case, this section specifically compares their performance. With 30000 function evaluations, the best HV of LSMOF-NSGA-II is 0.4194, which is comparable to LSWOEA. However, the worst score of HV is only 0.0249, compared to the worst score of HV of 0.4218, indicating that LSMOEA has excellent robustness that guarantees superiority in the worst case. The performance of LSMOF-NSGA-II improves when the number of function evaluations is increased to 60000, and the average performance of HV increases by 0.0459. However, its worst-case value HV remains low, suggesting that it may converge prematurely to a poor solution set when solving large-scale cases. LSWOEA shows the slightest improvement of all the algorithms when the number of function evaluations increases. This indicates that LSWOEA has good convergence efficiency and approaches convergence at 30000 function evaluations.



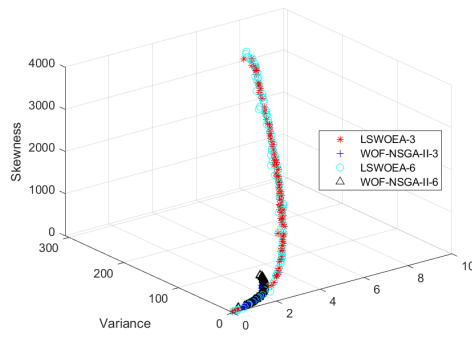
(a) SMPSO



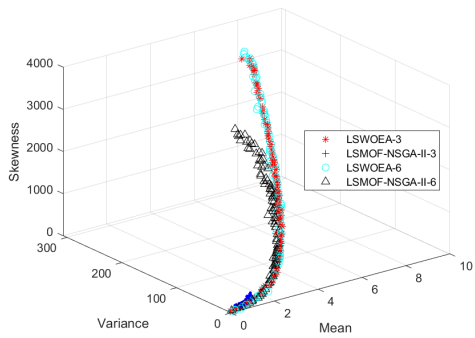
(b) NSGA-II



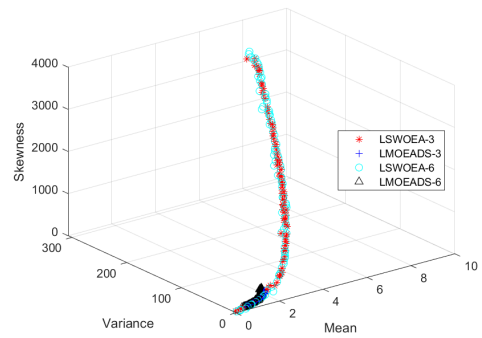
(c) MOEA-D



(d) WOF-NSGA-II



(e) LSMOF-NSGA-II



(f) LMOEADS

Figure 7: Pareto fronts obtained by the LSWOE and each compared algorithm on the large-scale case.

In summary, existing evolutionary algorithms effectively solve small-scale portfolio optimisation problems but face significant challenges when dealing with large-scale cases. Even the advanced LSMOEAs (e.g. WOF, LSMOF and LMOEADS) perform poorly when the number of securities increases to more than a thousand. The proposed LSMOEAs provides an effective solution tool for large-scale portfolio optimisation and enriches the practical relevance of evolutionary algorithms and portfolio optimisation in the context of Big Data.

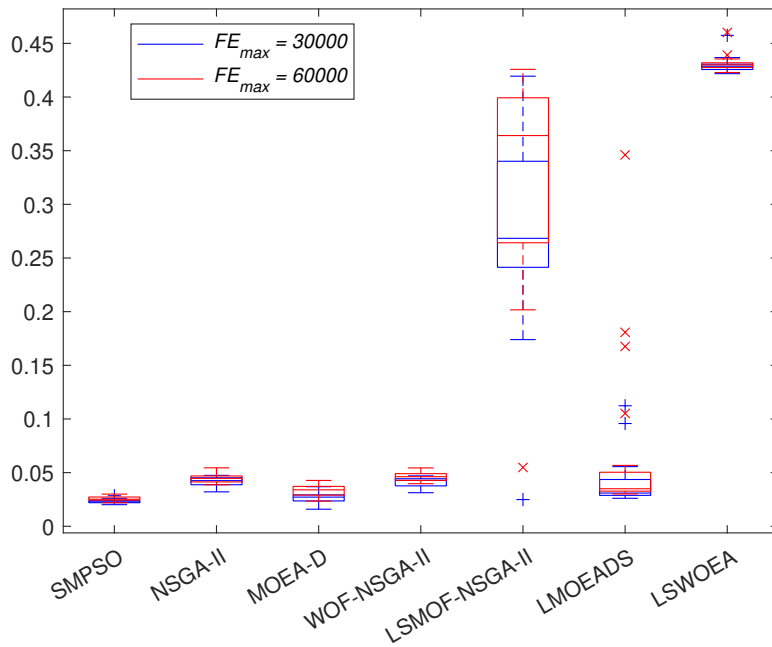


Figure 8: Boxplots of the HVs on the large-scale case.

7. Conclusion

This study discusses the application of evolutionary multi-objective optimisation to a large-scale portfolio selection problem with long-term listed and newly listed securities. A model of multi-objective portfolio optimisation with real constraints is proposed. In this model, the future returns of long-term listed securities are quantified by random variables, while newly listed securities are uncertain variables. The portfolio return is then considered as an uncertain random variable with mean, variance and skewness as triple criteria for decision-making. Additionally, realistic constraints such as cardinality, minimum transaction lot, bounds and no short selling are introduced to match real investment conditions. In terms of solution algorithms, this study addresses two dilemmas of MOEA in large-scale portfolio optimisation problems. On the one hand, an encoder-decoder method is developed to handle the complex constraints, providing a solution framework for applying arbitrary MOEAs to portfolio selection problems. On the other hand, a novel MOEA for large-scale portfolio optimisation is proposed, enriching the practice of evolutionary algorithms in the portfolio optimisation community.

To evaluate the effectiveness of the proposed model and algorithm, a numerical experiment analysis is performed for a small and a large-scale portfolio optimisation problem. In the small-scale case, the application of the proposed method is systematically presented to illustrate the practicality of the proposed approach. The comparison with some state-of-the-art algorithms shows that the proposed LSWOEA can significantly outperform the traditional MOEAs, but only slightly outperform the existing LSMOEAs. In the large-scale case, LSWOEA still maintains excellent performance while all the compared algorithms (including the existing LSMOEAs) drop sharply. The experimental results show that the proposed algorithm outperforms some benchmark MOEAs in large-scale portfolio optimisation and can effectively solve portfolio selection problems with thousands of securities.

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