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Public Good Provision with a Distributor^{*†}

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October 20, 2022

Abstract

We present a model of public good provision with a distributor. Our main result describes a symmetric mixed-strategy equilibrium, where all agents contribute to a common fund with probability p and the distributor provides either a particular amount of public goods or nothing. A corollary of this finding is the efficient public good provision equilibrium where all agents contribute to the common fund, all agents are expected to contribute, and the distributor spends the entire common fund for the public good provision.

Keywords : Public goods; Embezzlement; Distributor.

JEL Codes: D73, H40

***Conflict of Interest:** None

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1 INTRODUCTION

Free-riding is a chronic problem that is rampant in many economic situations where rational agents have to contribute to a common fund in order to obtain public goods. At the same time, resource enablers (distributors), such as government officers and/or authorities, responsible for redistribution of the collected common fund can act in their own interests, leading to a lower provision of the public goods. In these situations, the actions of the agents and the distributor are interlinked and influence each other. However, the public good literature often considers models only with the contributing agents, ignoring the distributor.¹

In this paper, we bring together the contributing agents and the distributor. First, rational agents have to decide whether or not to make a contribution to a common fund. Then, a random audit of a fraction of agents takes place and any non-contributor, if audited, has to contribute and also has to pay a penalty. The distributor gets the common fund. He spends a part of the common fund on the public good provision and embezzles the rest. Finally, after observing the public good provision, agents can express their discontent and punish the distributor.

Our model has several real-life applications: fare evasions in public transport, TV licenses, and so on. For example, fare evasion is a problem commonly faced by public transport companies, particularly those which have a ‘self-service proof of payment’ (SS-PoP) system in place.² In the case of the SS-PoP, each passenger must validate her ticket before using the public transport. Random spot-checks are held by inspectors to verify if the correct fares are paid by passengers. The fare collected goes towards providing better transport facilities, improving connectivity, etc., and in case of sub-standard facilities, delays, etc., passengers can complain. Another example is TV licenses in the UK, which are required by the law.³ The annual license fee is £159. Random audits are conducted in the UK, and if found guilty of license fee evasion, the evader has to pay the fee and an additional penalty of up to £1000. The income

¹The literature on public goods started from Samuelson (1954). J. Ledyard (1995) and Chaudhuri (2011) provide reviews of public goods literature.

²See Barabino et al. (2020) for a detailed survey on the topic.

³There are more than 20 countries where TV licenses are required to watch TV.

from the licenses goes towards funding BBC television and radio and more than 70% of BBC's income comes from these licensing fees. Consumers expect BBC to produce good content, and the quality of the content directly affects the viewership.

Our model captures situations, similar to the above examples, where the provision of public goods is affected by the actions of both the agents and the distributor.⁴ There are two types of punishments in our model. First, each agent can be punished (with a positive probability) for trying to evade the contribution.⁵ Since it is hard to monitor all the agents, only a fraction of agents is monitored. Second, each agent forms expectations about the public good provision, and if the actual provision is below these expectations, then the agent punishes the distributor.

Our main result is that for any rational agents' expectations, there always exists a symmetric (mixed-strategy) equilibrium, where the distributor uses the following cutoff strategy: he either exactly provides the expected level of public goods if he collects enough funds, or the distributor embezzles the entire common fund if he does not collect enough funds to match the expectations. A corollary of our main result is the efficient public good provision equilibrium. We show that if the punishment for embezzlement of common funds is high enough, there exists an equilibrium where all agents make contributions and expect the efficient public good provision from the distributor. If this level of public goods is not provided by the distributor, then all agents punish him. This leads to a situation where the distributor either spends the entire common fund on the public goods, or he embezzles the entire common fund if not enough funds are collected. In this situation, each agent is pivotal for public good provision, and the efficient level of public good provision can surprisingly be achieved without any penalties to agents for not contributing.⁶

Note that agents' expectations of the public good provision are a measure of ac-

⁴See Discussion section for another example.

⁵Some recent work is focused on improving the mechanisms of redistributing public funds and decreasing the free-rider problem. The most success was probably achieved with different punishment mechanisms. For example, Fehr and Gächter (2000) and Fehr and Gächter (2002) consider a peer punishment (i.e. decentralised or informal punishment). Andreoni and Bergstrom (1996), Baldassarri and Grossman (2012), and Markussen et al. (2014, 2016) analyze central sanction mechanisms to tackle the free-rider problem.

⁶See J. O. Ledyard (1984), Palfrey and Rosenthal (1983) for some relevant literature.

countability for the distributor. Agents could use a multitude of social accountability mechanisms to put pressure on the distributor. When agents perceive their rights to be violated and/or they receive inadequate public goods, they challenge the distributor. Some examples of social accountability measures include the monitoring of public sector performance, social media shaming, protests, complaints, and claim-making.⁷

The outline of the paper is as follows: Section 2 describes the model. The main results are discussed in Section 3. We conclude in Section 4. The proofs have been relegated to the Appendix A.

2 MODEL

We consider a four-stage sequential-move game with n agents and a distributor, G . First, each agent decides whether or not to make a contribution to a common fund. Then, Nature - an independent agent, which acts as a non-strategic player of the game - selects $k \leq n$ agents to audit at random. If the audited agent did not contribute, then this agent has to make her contribution and is also penalized. The total contributions (without penalties) go into the common fund. The distributor decides how to allocate this common fund: what goes to the public good provision and what to keep (embezzle) for himself. Finally, agents voice their opinion about the distributor by punishing him in the case of lower provision of public goods than what they expected. We will formally describe the game now.

In stage 1, each agent i simultaneously chooses an action t_i , where $t_i = 0$ ($= 1$) implies no contribution is made (contribution is made) by agent i . We assume that the contribution is 1 unit for each agent and the total amount collected goes towards the common fund.

In stage 2, Nature randomly selects k (out of n) agents to audit, and each agent has the same chance, $\frac{k}{n}$, to be audited. If a non-contributing agent is audited, she will need to contribute 1 unit and pay a penalty ($z \geq 0$) for her free-riding.⁸

⁷See Fox (2015) for a meta-analysis on social accountability.

⁸Depending on the context, penalties can be of different form (emotional cost, monetary cost, etc.).

In stage 3, the distributor observes the total common fund $X \in \{k, k + 1, \dots, n\}$. Since k agents are audited, the common fund has at least k units. The distributor decides about public good provision by choosing any amount, $g \leq X$, from the common fund. In this case, each agent receives ag and the distributor gets $ag + (X - g)$, where $0 < a < 1$ is the marginal per capita return from the public good, and $(X - g)$ is the amount which the distributor embezzles. Note that the distributor also benefits from the public good provision.

In stage 4, each agent can complain if the public good provision was below her expectations. Formally, each agent i forms expectations, $\tau_i \in \{k, k + 1, \dots, n\}$, of the total common fund X available to the distributor. Agents observe the level of public good, g , being provided by the distributor, and in the case of $g < \tau_i$, agent i punishes the distributor for not meeting her expectations.⁹ The punishment means that the distributor's payoff decreases by $b \geq 0$ from every agent complaint. We assume that agents care about getting their opinion across in the case when their expectations are not met, or agent i gets some disutility if $\tau_i > g$ and she does not complain. This assumption means that each agent has a dominant action at stage 4. Our results also hold if we assume that only a particular share of agents behaves that way. Figure 1 summarizes the four stages of the game.

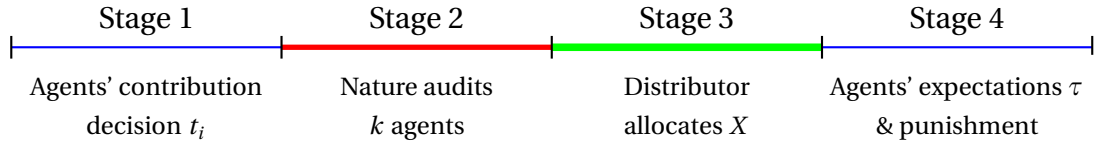


Figure 1: Timeline

We can describe payoffs of all players now. Agent i receives

$$u_i((1, \tau_i), \dots; g) = ag - 1,$$

We assume that penalties are not available to be used for the public good provision. In certain contexts, it can be assumed that penalties are used to fund the audit.

⁹For example, the distributor loses the confidence of agent i .

where agent i contributed in stage 1, the distributor provided g units of public goods in stage 3, and agent i expected τ_i units of public goods from the distributor. Analogously, agent i receives

$$u_i((0, \tau_i), \dots; g) = \begin{cases} ag, & \text{if } i \text{ was not audited,} \\ ag - 1 - z, & \text{if } i \text{ was audited,} \end{cases}$$

where agent i did not contribute in stage 1, the distributor provided g units of public goods in stage 3, and agent i expected τ_i units of public goods from the distributor.

The distributor gets

$$u_G((1, \tau_1), \dots, (0, \tau_n); g) = X - g + ag - [\text{\#complaints}]b,$$

where the total common fund is X , the distributor provided g units of public goods, agent i expected τ_i units of public goods from the distributor, and $[\text{\#complaints}]$ is the number of agents whose public good expectations are above g .

3 ANALYSIS

Agents and the distributor have many strategies in the model. However, it is enough to restrict our attention to symmetric (mixed and pure) strategies for agents and cut-off strategies for the distributor in order to obtain our main results.

We will consider symmetric mixed strategies $\sigma = (p, \tau)$ where all agents randomize over the first stage actions and contribute towards the common fund with the same probability $0 \leq p \leq 1$, and expect τ units of public goods. Symmetric pure strategies $(1, \tau)$ and $(0, \tau)$ specify whether agents contribute or not in stage 1, and τ describes their public good provision expectations. Since agents have the dominant action in stage 4, they complain if $\tau > g$ and they do not complain if $\tau \leq g$.

A cutoff strategy $\langle g \rangle$ means that the distributor provides exactly $g \geq 0$ units of public goods if the total common fund is at or above this cutoff level, or $g \leq X$. Otherwise, if $g > X$, no public good is produced. Note that the distributor, who uses the cut-

off strategy, embezzles funds more often than not: he plunders whatever is left in the common fund after the cutoff level is satisfied.

We will be looking for symmetric (pure and mixed) equilibria. Our results depend on the important element of the model – agents' expectations, τ . In the next lemma, we characterize these expectations, assuming that agents are rational.

LEMMA 1. *Suppose that there are n agents, and k of them are audited. If each agent contributes 1 unit with probability p at stage 1, then the expected number of units, τ , in the common fund is*

$$\tau = k + p(n - k). \quad (1)$$

This result is intuitive: rational agents expect k agents to be audited and contribute to the public fund for sure, and each of the $(n - k)$ non-audited agents contribute to the common fund with probability p . The formal proof of Lemma 1 is in the Appendix.

We consider a class of symmetric mixed-strategy equilibria $((p, \tau), \dots, (p, \tau); \langle \tau \rangle)$, where each agent contributes towards the common fund with probability p and expects, from Lemma 1, τ units of public goods, and the distributor uses the cutoff strategy, $\langle \tau \rangle$, for public good provision: he either provides τ units if the common fund $X \geq \tau$, or 0 units if the common fund $X < \tau$. Our results depend on the four parameters of the model: marginal per capita return a , punishment for embezzlement b , penalty for free-riding z , and agents' expectations about the common fund τ .

THEOREM 1 (Symmetric Mixed-Strategy Equilibrium). *Suppose that $0 < a < 1$, n , and k are given. Then, for any agents' expectations, $\tau \in \{k, k + 1, \dots, n\}$, there exists a symmetric mixed strategy equilibrium $((p, \tau), \dots, (p, \tau); \langle \tau \rangle)$, where*

- *all agents contribute with probability $p = \frac{\tau - k}{n - k}$ and expect τ units of public goods;*
- *the distributor uses the cutoff strategy, $\langle \tau \rangle$, for public good provision, where*

$$\langle \tau \rangle = \begin{cases} 0, & \text{if } X < \tau, \\ \tau, & \text{if } X \geq \tau. \end{cases}$$

In the equilibrium, the penalty for free-riding, $z^ = z(p, a)$, is uniquely determined, and the penalty for embezzlement, $b \geq b^*$, has to be above the threshold level, $b^* = b(\tau, a)$.*

Theorem 1 shows the importance of agents' expectations. Higher (lower) expectations lead to a higher (lower) level of contributions and higher (lower) public good provision by the distributor.

If agents have the lowest expectations about the public good provision, or $\tau = k$, then every agent free-rides with probability 1, which leads to the *free-riding equilibrium*, where each agent does not contribute, or $p = 0$, and the distributor provides exactly k units of public goods in the equilibrium. This is a typical outcome in the public good games.

COROLLARY 1 (Free-Riding Equilibrium). *If $\tau = k$, then $p = 0$. There exists the threshold level, $b^* = b(n, a)$, and the penalty for free-riding, $z^* = z(n, a)$, such that for any penalty for the embezzlement, $b \geq b^*$, $((0, k), \dots, (0, k); \langle k \rangle)$ is the free-riding equilibrium.*

Corollary 1 is intuitive: if the punishment for free-riding is relatively small, then all agents can free-ride and expect the minimal level of public good provision, k units, from the distributor, who in turn does not have any incentives to produce more than k units of public goods. These agents' expectations are self-enforced in the equilibrium. Moreover, punishment conditions for the free-riding equilibrium depend on the population size, n , and the number of audited agents, k . If the audit level, k , and the punishment for free-riding, z , are fixed, then increasing the population size, n , makes it easier to sustain the free-riding equilibrium.

If agents have the highest expectations about the public good provision, or $\tau = n$, then every agent contributes with probability 1, which leads to the *efficient public good provision equilibrium*, where the distributor provides exactly n units of public goods in the equilibrium. The *efficient public good provision equilibrium* is the surprising outcome in the public good games.

COROLLARY 2 (Efficient Public Good Provision Equilibrium). *If $\tau = n$, then $p = 1$. There exists the threshold level, $b^* = b(n, a)$, such that for any penalty for the embezzlement, $b \geq b^*$, $((1, n), \dots, (1, n); \langle n \rangle)$ is the efficient public good provision equilibrium.*

Corollary 2 demonstrates that the efficient public good provision can be achieved without punishment for free-riding if each agent is *pivotal* in the following sense: each agent contributes and expects provision of all n units from the distributor. If the distributor does not provide exactly n units, then each agent punishes him. If the number of agents, n , is large, then this punishment is severe, and the distributor prefers to avoid it. This means that the distributor will only consider two options: either provide all n units of public goods or embezzle the whole common fund. Therefore, each agent is pivotal for the efficient public good provision: she expects that her deviation (free-riding) leads to no public good provision (most likely, unless she is audited). The distributor executes the punishment and the reward here. Hence, surprisingly, there is no need to impose any punishment for the individual free-riding. It is interesting to emphasize that the rational distributor embezzles the common fund in almost all situations but the equilibrium.

4 DISCUSSION

We develop a model of public good provision with the distributor, where provision of public goods depends on the actions of both agents and the distributor. Another application of our model is tax evasion and the public good provision. Tax is a mandatory financial charge levied upon citizens to fund public expenditures including provision of public goods. In most societies, elected political leaders (governor, mayor, etc.) control the public funds and decide how to redistribute them. Empirical evidence shows that political corruption (such as embezzlement of public funds by governmental officials for private gain) exists both in developed and developing countries.¹⁰

Our model can connect tax evasion, political corruption, and public good provision by capturing the actions of all citizens and the governor. The main problem of the public good provision is free-riding. Each citizen decides whether or not to evade

¹⁰See, for example, Costas-Pérez et al. (2012), Ferraz and Finan (2008), Reinikka and Svensson (2004).

taxes and free-ride on the public good provision. The governor collects taxes in the common fund and decides how to allocate it. We show that the governor can provide the efficient public good provision *without punishments for tax evasion* because each citizen is pivotal in this equilibrium and does not evade taxes.

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APPENDIX A

Proof of Lemma 1. Let j denote the number of agents contributing. Let k_s denote the number of successful audits. Let K be a random variable whose outcome is k_s . Here K follows a hyper-geometric distribution whose probability mass function (p.m.f) is given by

$$Pr(K = k_s) = \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} \quad (2)$$

Let τ represent the expected number of units that the distributor has.

$$\begin{aligned} \tau &= \sum_{j=0}^n \sum_{k_s=0}^k (j + k_s) \binom{n}{j} p^j (1-p)^{n-j} \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} \\ &= \sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=0}^k (j + k_s) \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} \right) \\ &= \sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=0}^k j \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} + \sum_{k_s=0}^k k_s \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} \right) \end{aligned} \quad (3)$$

We apply *absorption identity* (Graham et al., 1994, p. 157) on the last term of (3) and we get the following:

$$\tau = \sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=0}^k j \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} + \sum_{k_s=1}^k (n-j) \frac{\binom{j}{k-k_s} \binom{n-j-1}{k_s-1}}{\binom{n}{k}} \right) \quad (4)$$

Then we use *Vandermonde's identity* on the last two terms in (4)

$$\begin{aligned} \tau &= \sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \left(j \frac{\binom{n}{k}}{\binom{n}{k}} + (n-j) \frac{\binom{n-1}{k-1}}{\binom{n}{k}} \right) \\ &= \sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \left(j + (n-j) \frac{k}{n} \right) \\ &= \sum_{j=0}^n (j) \binom{n}{j} p^j (1-p)^{n-j} + k \sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} - \frac{k}{n} \sum_{j=0}^n (j) \binom{n}{j} p^j (1-p)^{n-j} \\ &= \sum_{j=1}^n (n) \binom{n-1}{j-1} p^j (1-p)^{n-j} + k(1) - \frac{k}{n} \sum_{j=1}^n (n) \binom{n-1}{j-1} p^j (1-p)^{n-j} \\ &= \sum_{j=1}^n (np) \binom{n-1}{j-1} p^{j-1} (1-p)^{n-j} + k(1) - \frac{k}{n} \sum_{j=1}^n (np) \binom{n-1}{j-1} p^{j-1} (1-p)^{n-j} \\ &= np + k - kp \\ &= k + p(n-k) \end{aligned} \quad (5)$$

□

Proof of Theorem 1. Consider a mixed strategy profile $\sigma_{-i} = (p, \tau)$ where all agents but agent i contribute with probability p and expect τ units to be provided by the distributor. Let us assume that the distributor plays a cutoff strategy, $\langle \tau \rangle$, for public good provision, where

$$\langle \tau \rangle = \begin{cases} 0, & \text{if } X < \tau, \\ \tau, & \text{if } X \geq \tau. \end{cases}$$

The expected utility of agent i when contributing is given by

$$\begin{aligned} E u_i((1, \tau), \sigma_{-i}; \langle \tau \rangle) &= \frac{k}{n} \left[\sum_{j=\tau-1}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (-1 + \tau a) \right. \\ &\quad + \sum_{j=\tau-k}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-1-j}^{k-1} \frac{\binom{j}{k-1-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k-1}} \right) (-1 + \tau a) \\ &\quad + \sum_{j=\tau-k}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=0}^{\tau-2-j} \frac{\binom{j}{k-1-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k-1}} \right) (-1) \\ &\quad + \sum_{j=0}^{\tau-k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (-1) \Big] \\ &\quad + \left(1 - \frac{k}{n}\right) \left[\sum_{j=\tau-1}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (-1 + \tau a) \right. \\ &\quad + \sum_{j=\tau-k-1}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-1-j}^k \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right) (-1 + \tau a) \\ &\quad + \sum_{j=\tau-k-1}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=0}^{\tau-2-j} \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right) (-1) \\ &\quad + \sum_{j=0}^{\tau-k-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} (-1) \Big]. \end{aligned} \tag{6}$$

Similarly, the expected utility of agent i when not contributing is

$$\begin{aligned}
Eu_i((0, \tau), \sigma_{-i}; \langle \tau \rangle) = & \frac{k}{n} \left[\sum_{j=\tau-1}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (-1 + \tau a - z) \right. \\
& + \sum_{j=\tau-k}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-1-j}^{k-1} \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k-1}} \right) (-1 + \tau a - z) \\
& + \sum_{j=\tau-k}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=0}^{\tau-2-j} \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k-1}} \right) (-1 - z) \\
& + \sum_{j=0}^{\tau-k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (-1 - z) \Big] \\
& + (1 - \frac{k}{n}) \left[\sum_{j=\tau}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (\tau a) \right. \\
& + \sum_{j=\tau-k}^{\tau-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-j}^k \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right) (\tau a) \\
& + \sum_{j=\tau-k}^{\tau-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=0}^{\tau-1-j} \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right) (0) \\
& + \sum_{j=0}^{\tau-k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (0) \Big]. \tag{7}
\end{aligned}$$

In the mixed-strategy equilibrium, agent i has to be indifferent between her two actions, or $Eu_i((1, \tau), \sigma_{-i}, \langle \tau \rangle) = Eu_i(0, \tau); \sigma_{-i}; \langle \tau \rangle)$. Therefore,

$$\begin{aligned}
\frac{k}{n}(-z) = & (1 - \frac{k}{n}) \left[(-1) + \sum_{j=\tau-1}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (\tau a) \right. \\
& + \sum_{j=\tau-k-1}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-1-j}^k \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right) (\tau a) \\
& - \sum_{j=\tau}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} (\tau a) \\
& - \sum_{j=\tau-k}^{\tau-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-j}^k \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right) (\tau a) \Big] \\
= & (1 - \frac{k}{n}) \left[(-1) + \binom{n-1}{\tau-1} p^{\tau-1} (1-p)^{n-\tau} (\tau a) \right. \\
& + \sum_{j=\tau-k-1}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-1-j}^k \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right) (\tau a) \\
& - \sum_{j=\tau-k}^{\tau-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-j}^k \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right) (\tau a) \Big].
\end{aligned}$$

Solving for z , we get

$$z = \left(\frac{n-k}{k} \right) \left[1 - A\tau a - B\tau a + C\tau a \right], \tag{8}$$

where

$$\begin{aligned}
A &= \binom{n-1}{\tau-1} p^{\tau-1} (1-p)^{n-\tau}, \\
B &= \sum_{j=\tau-k-1}^{\tau-2} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-1-j}^k \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right), \\
C &= \sum_{j=\tau-k}^{\tau-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \left(\sum_{k_s=\tau-j}^k \frac{\binom{j}{k-k_s} \binom{n-1-j}{k_s}}{\binom{n-1}{k}} \right).
\end{aligned}$$

For z to be non-negative, it has to be

$$a \leq \frac{1}{\tau(A+B-C)}. \quad (9)$$

From Lemma 1 we have

$$p = \frac{\tau-k}{n-k}. \quad (10)$$

Therefore, for any fix τ, k , and n , the probability p is uniquely determined. From (8), a unique value of z can be found.

The only thing we have to check now is that the distributor does not have a profitable deviation, or

$$Eu_G(\sigma_i, \sigma_{-i}; \langle \tau \rangle) \geq Eu_G(\sigma_i, \sigma_{-i}; \langle 0 \rangle), \quad (11)$$

where

$$\begin{aligned}
Eu_G(\sigma_i, \sigma_{-i}; \langle \tau \rangle) &= \sum_{j=\tau}^n \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=0}^k \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} (\tau a + j + k_s - \tau) \right) \\
&+ \sum_{j=0}^{\tau-1} \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=\tau-j}^k \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} (\tau a + j + k_s - \tau) \right) \\
&+ \sum_{j=0}^{\tau-1} \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=0}^{\tau-j-1} \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} (j + k_s - nb) \right).
\end{aligned} \quad (12)$$

We rewrite the last expression in the following way:

$$\begin{aligned}
Eu_G(\sigma_i, \sigma_{-i}; \langle \tau \rangle) &= k + (n-k)p + \sum_{j=\tau}^n \binom{n}{j} p^j (1-p)^{n-j} (\tau a - \tau) \\
&+ \sum_{j=0}^{\tau-1} \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=\tau-j}^k \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} (\tau a - \tau) + \sum_{k_s=0}^{\tau-j-1} \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} (-nb) \right).
\end{aligned} \quad (13)$$

The right-hand side of the inequality (11) is

$$Eu_G(\sigma_i, \sigma_{-i}; 0) = k + p(n-k) - nb. \quad (14)$$

From expressions (13), (14), and (11), we get

$$b \geq \frac{(\tau - \tau a)(D + E)}{n(1 - F)},$$

where

$$\begin{aligned} D &= \sum_{j=\tau}^n \binom{n}{j} p^j (1-p)^{n-j}, \\ E &= \sum_{j=0}^{\tau-1} \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=\tau-j}^k \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} \right), \\ F &= \sum_{j=0}^{\tau-1} \binom{n}{j} p^j (1-p)^{n-j} \left(\sum_{k_s=0}^{\tau-j-1} \frac{\binom{j}{k-k_s} \binom{n-j}{k_s}}{\binom{n}{k}} \right). \end{aligned}$$

□