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Generalised Gately Values of Cooperative Games*

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September 2022

Abstract

We investigate Gately’s solution concept for cooperative games with transferable utilities. Gately’s conception is a bargaining solution and minimises the maximal quantified “propensity to disrupt” the negotiation of the players over the allocation of the generated collective payoffs. Gately’s solution concept is well-defined for a broad class of games. We consider a generalisation based on a parameter-based quantification of the propensity to disrupt. Furthermore, we investigate the relationship of these generalised Gately values with the Core. Gately’s solution is in the Core for all regular 3-player games. We identify precise conditions under which generalised Gately values are Core imputations for arbitrary regular cooperative games.

We devise an axiomatisation of the Gately value for the class of regular cooperative games. We conclude the paper with an application of the Gately value to the measurement of power in hierarchical social networks.

Keywords: Cooperative TU-game; sharing values; Gately point; Core.

JEL classification: C71

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1 Introduction: Gately's solution method

Gately (1974) seminally considered an allocation method founded on individual players' opportunities to disrupt the negotiations regarding the allocation of the generated collective payoffs. He introduced the notion of an individual player's *propensity to disrupt*, expressing the relative disruption an individual player causes when leaving the negotiations. In fact, Gately formulated this "propensity to disrupt" as the ratio of the other players' collective loss and the individual player's loss due to disruption of the negotiations. The prevailing solution method aims to minimise the maximal propensity to disrupt over all imputations and players in the game. Staudacher and Anwander (2019) show that for most cooperative games this solution method results in a unique imputation, which we can denote as the *Gately value* of the game under consideration.

Clearly, Gately's solution concept falls within the category of a *bargaining-based* solution concepts, including the bargaining set (Aumann and Maschler, 1964), the Kernel (Davis and Maschler, 1965) and the nucleolus (Schmeidler, 1969).¹ Contrary to many of these bargaining-based solution concepts, Gately's conception results in an easily to compute allocation rule that can also be categorised as a *compromise value* such as the CIS-value (Driessen and Funaki, 1991) and the τ -value (Tijs, 1981). These solution concepts have a fundamentally different axiomatic foundation than the *fairness-based* allocation rules such as the egalitarian solution, the Shapley value (Shapley, 1953), the Banzhaf value (Banzhaf, 1965; Lehrer, 1988), and related notions.

Gately (1974) investigated his conception in the setting of one particular 3-player cost game only. Gately's notion was extended to arbitrary n -player cooperative games by Littlechild and Vaidya (1976).² Charnes et al. (1978) introduced various concepts that are closely related to an extended notion of the propensity to disrupt. They introduced *mollifiers* and *homomollifiers*, measuring the disparities emerging from abandonment of negotiations as differences rather than ratios. These formulations result in associated games with a given cooperative game. Charnes et al. (1978) primarily investigated the properties of these associated games.

Gately points: Existence, uniqueness and relationship with the Core Staudacher and Anwander (2019) point out that the original research questions as posed by Gately (1974) were never properly investigated and answered in the literature. In particular, Staudacher and Anwander focussed on one particular application within the broad range of possibilities in Gately's approach, namely the so-called *Gately point*—defined as an imputation in which all propensities to disrupt are balanced and minimal. The Gately point is a solution to a minimax problem and Staudacher and Anwander show that every standard cooperative game has a unique Gately point. This settles indeed the most basic question concerning Gately's original conception.

There are significant advantages of Gately's conception over related bargaining solutions such as the Bargaining Set, the Kernel and the Nucleolus (Maschler, 1992). Indeed, the computing of these solutions is rather difficult, while Gately points can be derived easily through a well-structured

¹For an overview of these solution concepts we also refer to textbooks such as Moulin (1986, 2004), Maschler et al. (2013), Owen (2013) and Gilles (2010).

²Littlechild and Vaidya used their extended notion of propensity to disrupt to define the *disruption nucleolus* in which these propensities are lexicographically minimised.

formula based on only a few coalitional worths in the game under consideration. This reduces the computational complexity significantly.

Gately (1974) states clearly that he views the conception of the Gately point and related concepts based on his notion of “propensity to disrupt” as Core selectors. In particular, this extends to games with large Cores, such as the cost game considered by Gately. Indeed, there is an interesting and yet unexplored relationship between Gately points and the Core. In this paper we investigate this relationship further. In particular, we show that the unique Gately point is a Core selector for every regular 3-player cooperative game.

Littlechild and Vaidya (1976) already showed that this cannot be extended to n -player games by constructing a 4-player game in which the Gately point is not in the Core. Nevertheless we are able to establish the exact conditions for which the unique Gately point is in the Core of a regular n -player game. We refer to this condition as *top dominance* in which the grand coalition generates the largest net benefit in relation to the marginal player contributions. This condition reduces to *top convexity* for zero-normalised games (Shubik, 1982; Jackson and van den Nouweland, 2005).

We also provide an axiomatisation of the Gately value: The Gately value is the unique allocation rule that satisfies three properties, namely efficiency, a reduced form of additivity known as the compromise property, and the property that the assigned allocation in a zero-normalised game is a multiple of the vector of marginal contributions or “utopia values”. This axiomatisation shows the functioning of the Gately approach to value allocation from a different perspective, in particular through the decomposition of games in the individual worth vector and a zero-normalised game.

Finally, we introduce the *dual Gately value* as the Gately point of the dual of a given cooperative game. We show that the dual Gately value is identical to the Gately value for the broad class of regular games.

Generalised Gately points Gately’s definition of his propensity to disrupt puts equal weight on assessing the loss or gain of the other players versus the loss or gain of the player under consideration. We consider a parametric formulation in which a weight is attached to the relative importance of the gain or loss of the individual player in comparison with the weight attached to the gain or loss of all other players in the game. The higher the assigned weight, the more an individual’s loss or gain due to disruption is taken into account.

The imputations that balance these weighted propensities to disrupt are now referred to as generalised α -Gately points, where $\alpha > 0$ is the weight put on an individual’s loss or gain due to disruption. It is clear that $\alpha = 1$ refers to the original Gately point. We show that for all $\alpha > 0$, all regular cooperative games admit a unique α -Gately point, generalising the insight of Staudacher and Anwander (2019).

For any $\alpha > 0$, the unique α -Gately point is in the Core of the game if and only if the game satisfies α -Top Dominance, a parametric variant of the top convexity condition. In particular, the α -Top Dominance condition implies that the game is regular as well as partitionally superadditive. However, counterexamples show that there exist superadditive games with non-empty Cores that do not contain any α -Gately point.

We conclude this paper with an application of the Gately value to the measurement of centrality

in directed social networks. In particular, a (directed) network can be converted into a cooperative game to which a well-chosen game-theoretic solution concept can then be applied. In most cases this refers to the application of the Shapley value.³

Interpreting a directed network as a map that describes social superior-subordinate relationships, we convert a directed network into a so-called *score game*, which assigns to every coalition of players the number of players that it subordinates. There are different ways to count these subordinates, leading to different score games. The standard method is to apply the Shapley value to these score games, leading a family of *β -centrality measures* (Gilles, 2010, Chapter 5).

Here we apply the Gately value to these score games. This alternative approach emphasises the propensity by a player to disrupt the structure of a network. The resulting *Gately centrality measure* assigns a numerical value to the power of an individual player to disrupt the functioning of the network. We give an explicit mathematical form for the Gately centrality measure and, through some examples, we investigate the ranking of players based on this Gately measure in comparison to the ranking based on the β -measure. We believe that this method provides a valid alternative to the Shapley-value based approaches that are common in this field of social network analysis.

Structure of the paper We introduce and illustrate Gately’s approach through a simple application to a trade problem in Section 2. Section 3 develops the formal treatment of Gately’s approach, defines generalised Gately points and values, develops an axiomatisation of the Gately value, and discusses the dual of the Gately value. Section 4 is devoted to the investigation of the relationship of Gately points and the Core. We conclude the paper in Section 5 with an application of the Gately value to the measurement of centrality of nodes in a directed network.

2 An illustrative example: Bargaining of a sale

To illustrate the ideas behind Gately (1974)’s conception, consider a situation with one seller—denoted as player S—and two buyers—denoted, respectively, as players B1 and B2. The seller owns an indivisible object that has value to all three players. For the seller S, the object has an intrinsic value of 3, introducing this as a reservation value in any bargained trade between the seller S and any buyer. Buyer B1 assigns a value of 9 to the object, while buyer B2 attributes a value of 6 to the object. Hence, the buyers have a ceiling on their bids of 9 and 6, respectively.

Clearly, if there is no trade and seller S retains the ownership rights, he retains the intrinsic value assigned to the object. Hence, seller S by herself generates a personal value of $v(S) = 3$.

If seller S and buyer B1 negotiate a settlement of $P \geq 0$, the result is a gain from trade of $P - 3$ for seller S and a gain from trade of $9 - P$ for buyer B1. (Note that buyer B2 would have no gains from this trade.) Clearly, the only viable trades are made at prices that would result in non-negative gains from trade for both seller S and buyer B1. Therefore, the viable range of trades would be $3 \leq P \leq 9$. The total value generated from a trade between seller S and buyer B1 at a viable price P is now the sum of the intrinsic value of the object plus the total gains from trade. Hence, the generated value is

³For a recent overview of these game-theoretic approaches we also refer to Gómez et al. (2003), Pozo et al. (2011) and Tarkowski et al. (2018).

given by $v(S, B1) = 3 + (P - 3) + (9 - P) = 9$ irrespective of the agreed terms of trade in the viable range.

Similarly, a trade between seller S and buyer B2 results in gains from trade of $P - 3$ and $6 - P$, respectively. Note that the viable range for such a trade is given as $3 \leq P \leq 6$ and the total generated value from a trade between seller S and buyer B2 at a viable price P is computed as $v(S, B2) = 3 + (P - 3) + (6 - P) = 6$.

If seller S negotiates trade with both buyers B1 and B2 simultaneously, there would be no further gains than the gains from trade generated from a trade between seller S and buyer B1. Hence, the total value from a trade resulting from the grand negotiation would be $v(S, B1, B2) = v(S, B1) = 9$.

Representation as a cooperative game This trade situation can be represented as a cooperative game that is fully described by the values generated by the various potential trades between trading partners. In particular, we determine this cooperative game is fully described by $v(S) = 3$, $v(B1) = v(B2) = 0$, $v(S, B1) = 9$, $v(S, B2) = 6$, $v(B1, B2) = 0$ and $v(S, B1, B2) = 9$.

An imputation in this game theoretic representation is a triple $(x_s, x_1, x_2) \in \mathbb{R}_+^3$ such that $x_s + x_1 + x_2 = v(S, B1, B2) = 9$, where $x_s \geq 3$ represents the payout assigned to seller S, $x_1 \geq 0$ the payout made to buyer B1, and $x_2 \geq 0$ the payout made to buyer B2, respectively. These payouts are assumed to be negotiated between the three trading parties and, therefore, add up to the total value generated from their interaction.

The Core of this simple trade situation consists of all potential divisions of the total generated value between seller S and the prime buyer B1. Here the value generated between from the trade between seller S and buyer B2 is used as an outside option for seller S, providing a lower bound of $v(S, B2) = 6$ on the compensation that has to be afforded to seller S. The Core can therefore be described by the set of Core allocations $C(v) = \{(t, 9 - t, 0) \mid 6 \leq t \leq 9\}$.

We compute that the Shapley value of this simple trade situation is given by $\varphi(v) = (6\frac{1}{2}, 2, \frac{1}{2})$. We remark that in the Shapley value the trade between seller S and buyer B2 is used as an outside option that also leads to added value, of which a certain portion is attributed to buyer B2. We remark that the Shapley value is *not* a Core allocation, i.e., $\varphi(v) \notin C(v)$.

Gately's solution concept Gately (1974) introduced the idea that during a negotiation between the seller and the two buyers, each of these three parties can disrupt the proceedings by departing the negotiations. This innovative analysis was introduced to delineate and focus on a particular Core selector in the application explored in Gately (1974). This can be measured through a formalisation of each player's "propensity to disrupt". Gately (1974, page 200–201) introduces this concept as "the ratio of how much the two other players would lose if a player would refuse to cooperate to how much that player would lose if it refused to cooperate".

If the negotiators consider a proposed allocative agreement (x_s, x_1, x_2) , Gately's notion of the propensity to disrupt by seller S would then be the ratio of the buyers' potential loss $x_1 + x_2 - v(B1, B2)$ to the seller's potential loss from non-cooperation, computed as $x_s - v(S)$. Hence, using $x_s + x_1 + x_2 =$

$v(N) = 9$, the seller S 's propensity to disrupt is

$$d_s(x_s, x_1, x_2) = \frac{x_1 + x_2 - v(B1, B2)}{x_s - v(S)} = \frac{x_1 + x_2}{x_s - 3} = \frac{9 - x_s}{x_s - 3} = \frac{6}{x_s - 3} - 1 > -1.$$

Similarly, we construct the propensity to disrupt for both buyers as

$$d_1(x_s, x_1, x_2) = \frac{x_s + x_2 - v(S, B2)}{x_1 - v(B1)} = \frac{x_s + x_2 - 6}{x_1} = \frac{3}{x_1} - 1 > -1$$

$$d_2(x_s, x_1, x_2) = \frac{x_s + x_1 - v(S, B1)}{x_2 - v(B2)} = \frac{x_s + x_1 - 9}{x_2} = -1$$

Gately's motivation was that, if a player would get a relatively small payout, the player's propensity to disrupt the agreement is relatively high, since the ratio of the losses due to this disruption would be substantial.⁴

Therefore, since threats to disrupt an agreement have to be taken seriously, the stated objective of Gately's proposed solution is to select an imputation that minimises the maximum propensity to disrupt at that imputation. Hence, one should select $(x_s, x_1, x_2) \geq (3, 0, 0)$ with $x_s + x_1 + x_2 = 9$ such that it solves the minimax problem

$$\min_{(x_s, x_1, x_2) \geq (3, 0, 0) : x_s + x_1 + x_2 = 9} \max \{ d_s(x_s, x_1, x_2), d_1(x_s, x_1, x_2), d_2(x_s, x_1, x_2) \}.$$

This is clearly resulting in the requirement that $d_s(x_s, x_1, x_2) = d_1(x_s, x_1, x_2)$, since $d_2(x_s, x_1, x_2) = -1$ is certainly not a maximum. This results in Gately's solution to be determined by two equations:

$$\frac{6}{x_s - 3} = \frac{3}{x_1} \quad \text{and} \quad x_s + x_1 = 9 \quad \text{with} \quad x_2 = 0.$$

Clearly, one would minimise the payout assigned to buyer B2 in order to minimise the other two propensities to disrupt. Therefore, with $x_2 = 0$, the solution is actually unique and is determined as $g(v) = (7, 2, 0)$. We remark that $g(v) \in C(v)$ is a Core allocation.

Generalising Gately's solution conception We propose a generalisation of Gately's conception. We note that it is left in the middle how much weight a player assigns to her own loss due to her disrupting the formulation of an agreement. We propose to modify the definition of a player's propensity to disrupt by imposing a weight on the denominator. Hence, we assume that a player can discount her own losses due to disruption or, conversely, assign more weight to her own losses than the losses of the other players.

Formally, we introduce a weight parameter $\beta > 0$ for the denominator in Gately's propensity to disrupt. Instead of applying this directly to the formulated propensity to disrupt d_i , $i = s, 1, 2$, itself, we apply this weight in the modified form $\rho_i = d_i + 1$. Hence, for each of the three parties in the

⁴In particular, if a player would not get any benefit in the negotiations in the sense that $x_i = v(i)$, her propensity to disrupt is infinitely large. Similarly, if the player would be proposed to receive the total generated benefit $x_i = v(N)$, her propensity to disrupt is usually negative.

trading situation considered, we introduce the β -weighted propensity to disrupt as

$$\rho_s^\beta(x_s, x_1, x_2) = \frac{6}{(x_s - 3)^\beta} > 0 \quad \rho_1^\beta(x_s, x_1, x_2) = \frac{3}{x_1^\beta} > 0 \quad \rho_2^\beta(x_s, x_1, x_2) = 0$$

For any $\beta > 0$, a generalised Gately solution would solve the following minimax problem:

$$\min_{(x_s, x_1, x_2) \geq (3, 0, 0) : x_s + x_1 + x_2 = 9} \max \left\{ \rho_s^\beta(x_s, x_1, x_2), \rho_1^\beta(x_s, x_1, x_2), \rho_2^\beta(x_s, x_1, x_2) \right\}.$$

The generalised solution for this modified Gately conception is now determined by two equations:

$$\frac{6}{(x_s - 3)^\beta} = \frac{3}{x_1^\beta} \quad \text{and} \quad x_s + x_1 = 9 \quad \text{with } x_2 = 0.$$

The equations stated above lead to the conclusion that, for every $\beta > 0$, the generalised Gately solution is given as $g(\beta) = \left(\frac{3+9\sqrt[\beta]{2}}{1+\sqrt[\beta]{2}}, \frac{6}{1+\sqrt[\beta]{2}}, 0 \right)$.

We remark that if $\beta \rightarrow \infty$, $g(\beta) \rightarrow (6, 3, 0)$ and that $\beta \downarrow 0$ implies $g(\beta) \rightarrow (9, 0, 0)$.

In general, we conclude that there is a close relationship between these generalised Gately solutions and the Core of this game in the sense that every generalised Gately solution is in the Core and that the relative interior of the Core is mapped through these generalised Gately solutions:

$$\{g(\beta) \mid \beta > 0\} = C^o(v) = \{(t, 9 - t, 0) \mid 6 < t < 9\} \subsetneq C(v).$$

This close relationship between these values and the Core refers directly to Gately (1974)'s original motivation to identify his solutions as Core selectors. This is explored further in our general discussion below.

3 Cooperative games and Gately values

We first discuss the foundational concepts of cooperative games and solution concepts. Let $N = \{1, \dots, n\}$ be an arbitrary finite set of players and let $2^N = \{S \mid S \subseteq N\}$ be the corresponding set of all (player) coalitions in N . For ease of notation we usually refer to the singleton $\{i\}$ simply as i . Furthermore, we use the simplified notation $S - i = S \setminus \{i\}$ for any $S \in 2^N$ and $i \in S$.

A *cooperative game* on N is a function $v: 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. A game assigns to every coalition a value or “worth” that this coalition can generate through the cooperation of its members. We refer to $v(S)$ as the *worth* of coalition $S \in 2^N$ in the game v . The class of all cooperative games in the player set N is denoted by

$$\mathbb{V}^N = \{v \mid v: 2^N \rightarrow \mathbb{R} \text{ such that } v(\emptyset) = 0\}.$$

For every player $i \in N$ let $v_i = v(\{i\})$ be her individually feasible worth in the game v . We refer to the game v as being *zero-normalised* if $v_i = 0$ for all $i \in N$. The collection of all zero-normalised games is denoted by $\mathbb{V}_0^N \subset \mathbb{V}^N$.

The *marginal contribution*—also known as the “utopia” value (Tijs, 1981; Branzei et al., 2008)—of

an individual player $i \in N$ in the game $v \in \mathbb{V}^N$ is defined by her marginal or “separable” contribution to the grand coalition in this game, i.e.,

$$M_i(v) = v(N) - v(N - i) \quad \text{where } N - i = N \setminus \{i\}.$$

We remark that this marginal contribution can only be considered as a “utopia” value for the following classes of cooperative games.

Definition 3.1 A cooperative game $v \in \mathbb{V}^N$ is **essential** if it holds that

$$\sum_{j \in N} v_j \leq v(N) \leq \sum_{j \in N} M_j(v) \quad (1)$$

A cooperative game $v \in \mathbb{V}^N$ is **semi-standard** if for every player $i \in N$ it holds that

$$v_i \leq M_i(v) \quad \text{or, equivalently, } v_i + v(N - i) \leq v(N) \quad (2)$$

A cooperative game $v \in \mathbb{V}^N$ is **semi-regular** if v is essential as well as semi-standard.

A cooperative game $v \in \mathbb{V}^N$ is **standard** if v is semi-standard and, additionally, for at least one player $j \in N$ it holds that $v_j < M_j(v)$, or, equivalently, $v_j + v(N - j) < v(N)$.

A cooperative game $v \in \mathbb{V}^N$ is **regular** if v is essential as well as standard. The collection of regular cooperative games is denoted by $\mathbb{V}_\star^N \subset \mathbb{V}^N$.

We emphasise that every regular game $v \in \mathbb{V}^N$ satisfies a partitional form of superadditivity in the sense that $v(N - i) + v_i \leq v(N)$ for every $i \in N$, which is aligned with the notion of a game being *weak constant-sum* as defined in [Staudacher and Anwander \(2019, Definition 5\)](#). Furthermore, [Staudacher and Anwander \(2019, Theorem 1\(a\)\)](#) is also founded on the class of regular cooperative games.

The class of regular cooperative games \mathbb{V}_\star^N is the main domain of analysis for various forms of Gately solutions and their generalisations. In particular, we denote the collection of regular zero-normalised games by $\widehat{\mathbb{V}}^N = \mathbb{V}_\star^N \cap \mathbb{V}_0^N$.

An *allocation* in the game $v \in \mathbb{V}^N$ is any point $x \in \mathbb{R}^N$ such that $x(N) = v(N)$, where we denote by $x(S) = \sum_{j \in S} x_j$ the allocated payoff to the coalition $S \in 2^N$. We denote the class of all allocations for the game $v \in \mathbb{V}^N$ by $\mathbb{A}(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N)\} \neq \emptyset$. We emphasise that allocations can assign positive as well as negative payoffs to individual players in a game.

An *imputation* in the game $v \in \mathbb{V}^N$ is an allocation $x \in \mathbb{A}(v)$ that is individually rational in the sense that $x_i \geq v_i$ for every player $i \in N$. The corresponding imputation set of $v \in \mathbb{V}^N$ is now given by $\mathbb{I}(v) = \{x \in \mathbb{A}(v) \mid x_i \geq v_i \text{ for all } i \in N\}$. We remark that $\mathbb{I}(v) \neq \emptyset$ is a polytope for any regular cooperative game $v \in \mathbb{V}_\star^N$.

Some equal surplus sharing values Let $\mathbb{V} \subseteq \mathbb{V}^N$ be some collection of TU-games on player set N . A *value* on \mathbb{V} is a map $\phi: \mathbb{V} \rightarrow \mathbb{R}^N$ such that $\phi(v) \in \mathbb{A}(v)$ for every $v \in \mathbb{V}$. We emphasise that a value satisfies the efficiency property that $\sum_{i \in N} \phi_i(v) = v(N)$ for every $v \in \mathbb{V}$.

We consider here some well-known equal surplus sharing values that consider the egalitarian or equal distribution of certain surpluses over the players in the TU-game. Most of these equal surplus sharing values are based on the worths of a certain limited number of coalitions.⁵

- The simplest of these equal sharing surplus values is the *equal division value* $E: \mathbb{V}^N \rightarrow \mathbb{R}^N$ defined by

$$E_i(v) = \frac{v(N)}{n} \quad \text{for every } i \in N \text{ and } v \in \mathbb{V}^N. \quad (3)$$

- A closely related value is the one considered by [Driessen and Funaki \(1991\)](#), referred to as the “Centre-of-gravity of the Imputation Set” value, CIS: $\mathbb{V}^N \rightarrow \mathbb{R}^N$ defined by

$$\text{CIS}_i(v) = v_i + \frac{1}{n} \left(v(N) - \sum_{j \in N} v_j \right) \quad \text{for every } i \in N \text{ and } v \in \mathbb{V}^N. \quad (4)$$

Indeed, this value assigns to every regular TU-game $v \in \mathbb{V}_\star^N$ the centre of gravity of the corresponding set of imputations $\mathbb{I}(v)$.

3.1 Gately points, Gately values and their generalisations

[Gately \(1974\)](#) seminally introduced a specific methodology to identify outcomes of a bargaining process that is different from the well-known notions of other bargaining solutions such as the Bargaining Set, the Kernel ([Davis and Maschler, 1965](#)) and the Nucleolus ([Schmeidler, 1969](#)). Gately’s approach is based on the notion of the “propensity to disrupt”.

Definition 3.2 ([Gately, 1974](#); [Littlechild and Vaidya, 1976](#))

Let $v \in \mathbb{V}^N$ be a cooperative game on N . The **propensity to disrupt** of the coalition $S \in 2^N$ at allocation $x \in \mathbb{A}(v)$ is defined by

$$d(S, x) = \frac{x(N \setminus S) - v(N \setminus S)}{x(S) - v(S)} \quad (5)$$

The **propensity to disrupt of player** $i \in N$ at allocation $x \in \mathbb{A}(v)$ is given by

$$d_i(x) = d(\{i\}, x) = \frac{x(N - i) - v(N - i)}{x_i - v_i} = \frac{M_i(v) - x_i}{x_i - v_i} = \frac{M_i(v) - v_i}{x_i - v_i} - 1 \quad (6)$$

A **Gately point** of the game $v \in \mathbb{V}^N$ is defined as an imputation $g \in \mathbb{I}(v)$ that minimises the individual propensities to disrupt, i.e., for all players $i \in N$:

$$d_i(g) \leq \min_{x \in \mathbb{I}(v)} \max_{j \in N} d_j(x) \quad (7)$$

Gately points of cooperative games have most recently been explored by [Staudacher and Anwander \(2019\)](#). They showed the following properties.⁶

⁵Most equal surplus sharing values are founded on the consideration of the grand coalition N itself and for every player $i \in N$, the singleton $\{i\}$ and its complement $N - i$.

⁶A proof of the properties collected here can be found in [Staudacher and Anwander \(2019\)](#).

Lemma 3.3 (Staudacher and Anwander, 2019)

(a) Every standard cooperative game $v \in \mathbb{V}^N$ admits a unique Gately point $g(v) \in \mathbb{I}(v)$ given by

$$g_i(v) = v_i + \frac{M_i(v) - v_i}{\sum_{j \in N} (M_j(v) - v_j)} \left(v(N) - \sum_{j \in N} v_j \right) \quad (8)$$

for every $i \in N$.

(b) For every standard zero-normalised game $v \in \mathbb{V}^N$ the unique Gately point is given by

$$g(v) = \frac{v(N)}{\sum_{j \in N} M_j(v)} M(v) \in \mathbb{I}(v). \quad (9)$$

Lemma 3.3(a) allows us to introduce the **Gately value** as the map $g: \mathbb{V}_\star^N \rightarrow \mathbb{R}^N$ on the class of regular cooperative games defined by equation (8).

We emphasise that the Gately value is only non-trivially defined on the class of regular cooperative games \mathbb{V}_\star^N , while Gately points are in principle defined for arbitrary cooperative games with the property that $M_i(v) \neq v_i$ for some $i \in N$. As pointed out by Staudacher and Anwander (2019), there might be games that admit no Gately points and other games that might admit multiple Gately points.

Generalised Gately values We generalise the notion of Gately points as seminally introduced in Gately (1974). The next definition introduces a generalised notion of the Gately value on the class of standard cooperative games.

Definition 3.4 Let $v \in \mathbb{V}^N$ be some standard cooperative game on N . For any parameter value $\alpha > 0$ we define the α -**Gately value** as the imputation $g^\alpha(v) \in \mathbb{I}(v)$ with

$$g_i^\alpha(v) = v_i + \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(v(N) - \sum_{j \in N} v_j \right) \quad \text{for every } i \in N. \quad (10)$$

We refer to $\mathcal{G} = \{g^\alpha: \mathbb{V}_\star^N \rightarrow \mathbb{R}^N \mid \alpha > 0\}$ as the family of generalised Gately values on the domain of regular cooperative games \mathbb{V}_\star^N . For any $v \in \mathbb{V}_\star^N$ the related set $\mathcal{G}(v) = \{g^\alpha(v) \mid \alpha > 0\} \subseteq \mathbb{I}(v)$ defines the **Gately set** for that particular game.

From this definition we can identify some special cases:

- We note that $g^1 = g \in \mathcal{G}$ is the original Gately value on the class of regular games \mathbb{V}_\star^N .
- Although g^α is not defined for $\alpha = 0$, note that

$$\lim_{\alpha \downarrow 0} g_i^\alpha(v) = v_i + \frac{1}{|N_0(v)|} \left(v(N) - \sum_{j \in N} v_j \right)$$

for all $i \in N_0(v)$ and $\lim_{\alpha \downarrow 0} g_i^\alpha(v) = v_i$ for $i \in N \setminus N_0(v)$, where $N_0(v) = \{i \in N \mid M_i(v) > v_i\} \neq \emptyset$. This compares to the CIS value.

Furthermore, if the game $v \in \widehat{\mathbb{V}}^N$ is additionally zero-normalised, $\lim_{\alpha \downarrow 0} g_i^\alpha(v)$ corresponds to the equal division value given by $E(v) = \frac{v(N)}{n}$ if $M_i(v) \neq 0$ for all $i \in N$.

- Finally, $\lim_{\alpha \rightarrow \infty} g_i^\alpha(v) = v_i + \frac{1}{|N_1(v)|} (v(N) - \sum_{j \in N} v_j)$ for all $i \in N_1(v)$ and $\lim_{\alpha \rightarrow \infty} g_i^\alpha(v) = v_i$ for $i \in N \setminus N_1(v)$, where $N_1(v) = \{i \in N \mid M_i(v) - v_i = \max_{j \in N} (M_j(v) - v_j)\} \neq \emptyset$. Again, this can be interpreted as a variation of the CIS value.

The next definition introduces a generalised formulation of Gately's seminal notion of the propensity to disrupt. We show that α -Gately values are closely related to optimisation problems based on this generalised notion.

Definition 3.5 *Let $v \in \mathbb{V}^N$ be some cooperative game on player set N . For every parameter $\beta > 0$ the corresponding **generalised β -propensity to disrupt of player $i \in N$ at imputation $x \in \mathbb{I}(v)$** is defined by*

$$\rho_i^\beta(x) = \frac{M_i(v) - v_i}{(x_i - v_i)^\beta} \quad (11)$$

We note here that for $\beta = 1$, this generalised propensity to disrupt corresponds exactly to the original propensity to disrupt for an individual player as introduced by Gately (1974), in the sense that $\rho_i^1(x) = \frac{M_i(v) - v_i}{x_i - v_i} = d_i(x) + 1$.

The following theorem shows the relationship between the balancing of such generalised propensities to disrupt and corresponding α -Gately values. In particular, it is shown that the α -Gately value can be interpreted as a bargaining value, like the original Gately value and the nucleolus. Furthermore, for certain values of α , the minimisation of the generated total generalised propensity to disrupt at an allocation results in the corresponding α -Gately value.

Theorem 3.6 *Let $v \in \mathbb{V}_\star^N$ be a regular cooperative game on N .*

- (a) *Let $\alpha > 0$ and define $\beta = \frac{1}{\alpha}$. Then the α -Gately value $g^\alpha(v) \in \mathbb{I}(v)$ is the unique β -Gately point in the sense that $g^\alpha(v)$ is the unique imputation that satisfies the property that*

$$\rho_i^\beta(g^\alpha(v)) \leq \min_{x \in \mathbb{I}(v)} \max_{j \in N} \rho_j^\beta(x) \quad (12)$$

for every player $i \in N$.

- (b) *Let $0 < \alpha < 1$ and define $\beta = \frac{1-\alpha}{\alpha} > 0$. Then the α -Gately value $g^\alpha(v) \in \mathbb{I}(v)$ is the unique solution to the minimisation of the total aggregated generalised β -propensity to disrupt of the game v :*

$$g^\alpha(v) = \arg \min_{x \in \mathbb{I}(v)} \sum_{j \in N} \rho_j^\beta(x) \quad (13)$$

Proof. To show assertion (a), we note that for every imputation $x \in \mathbb{I}(v)$ and every player $i \in N$: $\rho_j^\beta(x) = \frac{M_j(v) - v_j}{(x_j - v_j)^\beta} \geq 0$ from the hypothesis that $M_i(v) \geq v_i$. Furthermore, since $M_j(v) > v_j$ for at least one $j \in N$, we conclude that $r = \max_{j \in N} \rho_j^\beta(x) > 0$.

Hence, we conclude that the minimax problem $\min_{x \in \mathbb{I}(v)} \max_{j \in N} \rho_j^\beta(x)$ can be solved by identifying $r > 0$ and an imputation $x^* \in \mathbb{I}(v)$ such that $\rho_i^\beta(x^*) = r$ for all $i \in N_0 = \{i \in N \mid M_i(v) > v_i\} \neq \emptyset$. First, note that for all $j \in N \setminus N_0 = \{j \in N \mid M_j(v) = v_j\}$ we can set $x_j = v_j$.

Next, for $i \in N_0$ we can now solve for $r > 0$ as well as x_i . Rewriting $\rho_i^\beta(x) = r$, we derive that

$$x_i = \frac{(M_i(v) - v_i)^{\frac{1}{\beta}}}{r^{\frac{1}{\beta}}} + v_i$$

Note that, since $M_i(v) \geq v_i$ for all $i \in N$, it follows that $x \in \mathbb{I}(v)$. Hence,

$$\sum_{i \in N} x_i = \frac{\sum_{i \in N_0} (M_i(v) - v_i)^{\frac{1}{\beta}}}{r^{\frac{1}{\beta}}} + \sum_{i \in N} v_i \equiv v(N)$$

Since $\sum_{i \in N_0} (M_i(v) - v_i)^{\frac{1}{\beta}} = \sum_{i \in N} (M_i(v) - v_i)^{\frac{1}{\beta}}$, we conclude that

$$r = \frac{\left[\sum_{i \in N} (M_i(v) - v_i)^{\frac{1}{\beta}} \right]^\beta}{\left[v(N) - \sum_{j \in N} v_j \right]^\beta} > 0.$$

From this we conclude that the identified solution exists and is unique under the regularity conditions on the game v .

Substituting the formulated solution of r back into the formulation for the solution, we deduce that

$$x_i = v_i + \frac{(M_i(v) - v_i)^{\frac{1}{\beta}}}{\sum_{j \in N} (M_j(v) - v_j)^{\frac{1}{\beta}}} \left(v(N) - \sum_{j \in N} v_j \right) \geq v_i.$$

Recalling that $\beta = \frac{1}{\alpha}$, we indeed conclude that $x_i = g_i^\alpha(v)$, leading us to conclude that assertion (a) has been shown.

To show assertion (b), consider the minimisation problem $\min_{x \in \mathbb{I}(v)} R^\beta(x)$ as formulated, where $R^\beta(x) = \sum_{j \in N} \rho_j^\beta(x)$. Deriving the Lagrangian $L(x_1, \dots, x_n, \lambda) = \sum_{i \in N} \left[\frac{M_i - v_i}{(x_i - v_i)^\beta} \right] + \lambda (\sum_{i \in N} x_i - v(N))$, and deriving the necessary first-order conditions, we conclude that

$$\frac{M_1 - v_1}{(x_1 - v_1)^{\beta+1}} = \frac{M_2 - v_2}{(x_2 - v_2)^{\beta+1}} = \dots = \frac{M_n - v_n}{(x_n - v_n)^{\beta+1}}.$$

Thus, we arrive at $n - 1$ equations given by

$$x_k - v_k = \frac{(M_2 - v_2)^{\frac{1}{\beta+1}}}{(M_1 - v_1)^{\frac{1}{\beta+1}}} (x_1 - v_1) \quad \text{for } k = 2, \dots, n.$$

This we can rewrite as

$$v(N) - \sum_{j=3}^n x_j - x_1 - v_2 = \frac{(M_2 - v_2)^{\frac{1}{\beta+1}}}{(M_1 - v_1)^{\frac{1}{\beta+1}}} (x_1 - v_1)$$

together with

$$x_k = v_k + \frac{(M_k - v_k)^{\frac{1}{\beta+1}}}{(M_1 - v_1)^{\frac{1}{\beta+1}}} (x_1 - v_1) \quad \text{for } k = 3, \dots, n.$$

Summing up the LHSs and the RHSs, we have the following equalities:

$$v(N) - x_1 - v_2 = v_3 + \dots + v_n + \sum_{j=2}^n \frac{(M_j - v_j)^{\frac{1}{\beta+1}}}{(M_1 - v_1)^{\frac{1}{\beta+1}}} (x_1 - v_1)$$

This leads to the conclusion that

$$\begin{aligned} v(N) - \sum_{j=2}^n v_j &= x_1 + \frac{(x_1 - v_1)}{(M_1 - v_1)^{\frac{1}{\beta+1}}} \sum_{j=2}^n (M_j - v_j)^{\frac{1}{\beta+1}} \\ &= x_1 + \frac{(x_1 - v_1)}{(M_1 - v_1)^{\frac{1}{\beta+1}}} \sum_{j=1}^n (M_j - v_j)^{\frac{1}{\beta+1}} - (x_1 - v_1) \end{aligned}$$

Hence, we conclude that

$$v(N) - \sum_{j=1}^n v_j = \frac{(x_1 - v_1)}{(M_1 - v_1)^{\frac{1}{\beta+1}}} \sum_{j=1}^n (M_j - v_j)^{\frac{1}{\beta+1}},$$

or

$$\frac{(M_1 - v_1)^{\frac{1}{\beta+1}}}{\sum_{j=1}^n (M_j - v_j)^{\frac{1}{\beta+1}}} [v(N) - \sum_{j=1}^n v_j] = x_1 - v_1.$$

Remarking that $\alpha = \frac{1}{\beta+1}$ leads us immediately to the insight that the first player's allocation is actually her α -Gately value value. The resulting allocations for the other players $j = 2, \dots, n$ are derived in a similar fashion. \blacksquare

We remark that Theorem 3.6 applies to regular cost games or problems as well. Indeed, for a *cost game* $v \in \mathbb{V}^N$ satisfying $v(N) \leq \sum_{j \in N} v_j$, $M_i(v) \leq v_i \leq 0$ for all $i \in N$ and $M_j(v) < v_j \leq 0$ for some $j \in N$, both assertions of Theorem 3.6 hold. We do not consider these games here, but refer to, e.g., [Moulin \(2004\)](#) for a discussion of these cost games.

To illustrate the importance of regularity of those cooperative games for which Gately values are well-defined as imposed in Theorem 3.6(a), we consider the next example of a three-player game that exhibits non-regularities.

Example 3.7 Consider a 3-player game v on $N = \{1, 2, 3\}$ defined by $v_1 = 2$, $v_2 = 1$, $v_3 = 0$, $v(12) = v(13) = v(23) = 4$ and $v(N) = 5$.

We remark that the marginal contributions can now be computed as $M_1 = M_2 = M_3 = 1$, leading to the conclusion that $M_1 - v_1 = -1 < 0$, $M_2 = v_2$, and $M_3 - v_3 = 1 > 0$. Hence, this game is clearly neither essential nor semi-standard.⁷

⁷We also remark that this game has actually an empty Core.

We show that this game admits a continuum of Gately points, thereby illustrating that this game does not have a well-defined, unique Gately value. Note that for any proposed solution $x \in \mathbb{A}(v)$:

$$\rho_1(x) = \frac{-1}{x_1 - 2} \quad \rho_2(x) = 0 \quad \rho_3(x) = \frac{1}{x_3}$$

All Gately points are now characterised by two equations, namely $x(N) = v(N)$ and $\rho_1(x) = \rho_3(x) \geq 0$. This leads to the conclusion that the set of Gately points is a continuum given by $\{(t, 3, 2 - t) \mid 0 \leq t \leq 2\} \subset \mathbb{A}(v)$. Note that this set of Gately points includes allocations that are *not* imputations. \blacklozenge

With regard to Theorem 3.6(b) we remark that for $\beta = 1$ the minimisation of the aggregated total propensity to disrupt $R^1(x) = \sum_{i \in N} d_i(x)$ results in the $\frac{1}{2}$ -Gately value as the unique solution. Furthermore, if $\beta = 0$, the generalised propensity to disrupt for any player $i \in N$ is no longer a function of the allocation x_i , implying that the total aggregated 0-propensity to disrupt is a constant function. This implies that the minimisation problem (13) has a continuum of solutions, including all Gately points.

3.2 A constructive axiomatisation of the Gately value

It is easy to see that the Gately value on the class of regular games \mathbb{V}_\star^N is a compromise value of the individual worth vector and the net marginal contribution vector. Indeed, for any regular game $v \in \mathbb{V}_\star^N$, the individual worth vector is given by $\bar{v} = (v_1, \dots, v_n)$ and the net marginal contribution vector by $\bar{n} = M(v) - \bar{v} = (M_1(v) - v_1, \dots, M_n(v) - v_n)$. Now the Gately value $g(v)$ for game v can be written as

$$g(v) = \bar{v} + \gamma(v) \cdot \bar{n} \quad \text{with } \gamma(v) = \frac{v(N) - \sum_{i \in N} v_i}{\sum_{i \in N} (M_i(v) - v_i)}.$$

Tijs (1981) introduced the so-called τ -value, which is based on the same principles, but for a different “minimal rights” vector. The minimal right of player $i \in N$ in game $v \in \mathbb{V}^N$ is defined as

$$m_i(v) = \max_{S \subset N: i \in S} \left(v(S) - \sum_{j \in S-i} M_j(v) \right)$$

the maximal benefit to player i when she would pay the other members of a coalition their utopia marginal contribution $M_j(v)$.

Now, the τ -value is defined for any *quasi-balanced* game $v \in \mathbb{V}_Q^N = \{v \in \mathbb{V}^N \mid \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)\}$ as

$$\tau(v) = m(v) + \delta(v) \cdot \bar{n} \quad \text{where } \delta(v) = \frac{v(N) - \sum_{i \in N} m_i(v)}{\sum_{i \in N} (M_j(v) - m_j(v))} \quad (14)$$

it is clear that the construction of the Gately and the τ -values as compromise values are very similar. The Gately value seems more natural due to the natural interpretation as an individual player’s worth v_i as the natural right of that player.

Tijs (1987) devised an axiomatisation for the τ -value that is completely based on this construction method. We can replicate this axiomatisation for the Gately value to arrive at the following characterisation. In this characterisation, a variant of the compromise property and the restricted proportionality property were seminally introduced by Tijs (1987) on the class of quasi-balanced games. Here, these properties are implemented on the class of regular cooperative games.

Theorem 3.8 *The Gately value g is the unique map $f: \mathbb{V}_\star^N \rightarrow \mathbb{R}^N$ on the class of regular games \mathbb{V}_\star^N that satisfies the following three properties:*

- (i) **Efficiency:** $\sum_{i \in N} f_i(v) = v(N)$ for every $v \in \mathbb{V}_\star^N$;
- (ii) **\bar{v} -Compromise property:** For every regular game $v \in \mathbb{V}_\star^N$: $f(v) = \bar{v} + f(v - \bar{v})$, where $v - \bar{v} \in \widehat{\mathbb{V}}^N$ is the zero-normalisation of v defined by $(v - \bar{v})(S) = v(S) - \sum_{i \in S} v_i$ for every coalition $S \in 2^N$, and;
- (iii) **Restricted proportionality property:** For every zero-normalised regular cooperative game $v \in \widehat{\mathbb{V}}^N$: $f(v) = \gamma_v M(v)$ for some $\gamma_v \in \mathbb{R}$.

Proof. We first show that the Gately value $g: \mathbb{V}_\star^N \rightarrow \mathbb{R}^N$ satisfies the three stated properties. For that purpose let $v \in \mathbb{V}_\star^N$.

- (i) Obviously the Gately value $g(v)$ is efficient for v .
- (ii) Let $w = v - \bar{v} \in \mathbb{V}_0^N$ be the zero-normalisation of v . Then for every $i \in N$ we deduce that $w_i = v_i - v_i = 0$ and $M_i(w) = w(N) - w(N - i) = v(N) - v(N - i) - v_i = M_i(v) - v_i$. Hence, $M_i(w) \geq 0 = w_i$ and for those players $j \in N$ with $M_j(v) > v_j$ we deduce that $M_j(w) > 0 = w_j$.
Furthermore, $\sum_{i \in N} v_i \leq v(N) \leq \sum_{i \in N} M_i(v)$ is equivalent to $0 \leq v(N) - \sum_{i \in N} v_i = w(N) \leq \sum_{i \in N} M_i(v) - \sum_{i \in N} v_i = \sum_{i \in N} M_i(w)$, implying that $w \in \mathbb{V}_\star^N$. Therefore, $w = v - \bar{v} \in \widehat{\mathbb{V}}^N$.
Now by definition for every $i \in N$:

$$g_i(w) = \frac{M_i(w)}{\sum_{j \in N} M_j(w)} \cdot w(N) = \frac{M_i(v) - v_i}{\sum_{j \in N} (M_j(v) - v_j)} \cdot \left(v(N) - \sum_{j \in N} v_j \right) = g_i(v) - v_i.$$

This shows that $g_i(v) = v_i + g_i(v - \bar{v})$.

- (iii) Assume that $v \in \widehat{\mathbb{V}}^N$. Then for any $i \in N$: $g_i(v) = \frac{M_i(v)}{\sum_{j \in N} M_j(v)} \cdot v(N)$ showing restricted proportionality with $\gamma_v = \frac{v(N)}{\sum_{j \in N} M_j(v)}$.

Next, we show that if $f: \mathbb{V}_\star^N \rightarrow \mathbb{R}^N$ satisfies the three stated properties, it is equal to the Gately value. Take any regular game $v \in \mathbb{V}_\star^N$ and let $w = v - \bar{v} \in \widehat{\mathbb{V}}^N$ be its zero-normalisation.

Then from restricted proportionality we have that $f(w) = \gamma_w M(w) = \gamma_w (M(v) - \bar{v})$. Furthermore, from the compromise property we conclude that

$$f(v) = \bar{v} + f(v - \bar{v}) = \bar{v} + \gamma_v (M(v) - \bar{v}).$$

Using efficiency we then conclude that

$$\sum_{i \in N} f_i(v) = \sum_{i \in N} v_i + \gamma_v \left(\sum_{i \in N} M_i(v) - \sum_{i \in N} v_i \right) = v(N)$$

implying that

$$\gamma_v = \frac{v(N) - \sum_{i \in N} v_i}{\sum_{i \in N} (M_i(v) - v_i)}.$$

We immediately conclude from this that $f_i(v) = g_i(v)$, showing the assertion. ■

The axiomatisation in Theorem 3.8 is constructive in the sense that it shows basic properties satisfied by the Gately value. These three properties are independent as the next discussion shows.

The *efficiency* property is a well-established property that is used throughout the literature. It guarantees that the allocation rule selects from the set of imputations in the game rather than the broader set of allocations. We note that the allocation rule $f(v) = M(v)$ on \mathbb{V}_\star^N clearly satisfies the compromise property as well as the restricted proportionality property, but which is not efficient.

The \bar{v} -*compromise property* is a reduced form of additivity and as such decomposes the allocation rule in a translation of the allocation assigned to the zero-normalisation of the game. This property originated in Tijds (1987) as the ‘‘compromise property’’ for the minimal right vector $m(v)$ rather than the vector of individual worths \bar{v} . It is clear that the τ -value satisfies efficiency and the restricted proportionality property. It does not satisfy the \bar{v} -compromise property, but rather the compromise property based on the minimal rights vector $m(v)$.

The *restricted proportionality property* imposes zero-normalised games are assigned an allocation that is proportional to the utopia vector $M(v)$. This property originates from Tijds (1987) as well and it is satisfied by the τ -value. On the other hand, the Shapley value is a solution concept that is efficient and satisfies the \bar{v} -compromise property, but it does not satisfy restricted proportionality.

An axiomatisation of generalised Gately values From the proof of Theorem 3.8 we immediately deduce that the restricted proportionality property can be generalised and make to fit with our notion of α -Gately values. This immediately results in the following corollary.

Corollary 3.9 *Let $\alpha > 0$. The generalised α -Gately value g^α is the unique allocation rule $f: \mathbb{V}_\star^N \rightarrow \mathbb{R}^N$ on the class of regular games \mathbb{V}_\star^N that satisfies the following three properties:*

- (i) **Efficiency:** $\sum_{i \in N} f_i(v) = v(N)$ for every $v \in \mathbb{V}_\star^N$;
- (ii) **\bar{v} -Compromise property:** For every regular game $v \in \mathbb{V}_\star^N$: $f(v) = \bar{v} + f(v - \bar{v})$, where $v - \bar{v} \in \widehat{\mathbb{V}}^N$ is the zero-normalisation of v defined by $(v - \bar{v})(S) = v(S) - \sum_{i \in S} v_i$ for every coalition $S \in 2^N$, and;
- (iii) **Restricted α -proportionality property:** For every zero-normalised regular cooperative game $v \in \widehat{\mathbb{V}}^N$: $f(v) = \gamma_v \cdot (M_1(v)^\alpha, \dots, M_n(v)^\alpha)$ for some $\gamma_v \in \mathbb{R}$.

3.3 Dual Gately values

Let $v \in \mathbb{V}^N$ be a cooperative game. Then the *dual game* of v , denoted by $v^* : 2^N \rightarrow \mathbb{R}$, is defined by

$$v^*(S) = v(N) - v(N \setminus S) \quad \text{for every coalition } S \in 2^N \quad (15)$$

The dual of a game assigns to every coalition $S \subseteq N$ the worth that is lost by the grand coalition N if coalition S leaves the game. Note in particular that $v^*(\emptyset) = 0$, $v^*(N) = v(N)$ and $v_i^* = v^*({i}) = v(N) - v(N - i) = M_i(v)$ for all $i \in N$. Finally, $M_i(v^*) = v_i$ for every $i \in N$.

We investigate the “dual” of a given value, which assigns to games the value of its dual game. As an illustrative example, we note that [Driessen and Funaki \(1991\)](#) considered the dual of the CIS-value, defined as the CIS-value of the dual game. They refer to this notion as the “Egalitarian Non-Separable Contribution” value, or ENSC-value.

Formally, the *ENSC-value* is the map $\overline{\text{CIS}} : \mathbb{V}^N \rightarrow \mathbb{R}^N$ defined by $\overline{\text{CIS}}(v) = \text{CIS}(v^*)$. Hence, for every player $i \in N$ and regular game $v \in \mathbb{V}^N$ we have⁸

$$\overline{\text{CIS}}_i(v) = M_i(v) + \frac{1}{n} \left(v(N) - \sum_{j \in N} M_j(v) \right) = -v(N - i) + \frac{1}{n} \left(v(N) + \sum_{j \in N} v(N - j) \right) \quad (16)$$

The ENSC-value has been subject to analysis in contributions to the literature on equal sharing values, including [Dragan et al. \(1996\)](#), [van den Brink and Funaki \(2009\)](#), and [Driessen \(2010\)](#).⁹

The dual of α -Gately values We can apply the same procedure to the Gately value. We note first that the dual of a Gately value only can properly formulated for parameter values that are natural numbers, i.e., $\alpha \in \mathbb{N}$. This is subject to the next definition.

Definition 3.10 *Let $\alpha \in \mathbb{N}$. The **dual α -Gately value** is a map $\overline{g}^\alpha : \mathbb{V}_\star^N \rightarrow \mathbb{R}^N$ that assigns to every regular cooperative game $v \in \mathbb{V}_\star^N$ the α -Gately value of its dual game $v^* \in \mathbb{V}^N$, i.e., $\overline{g}^\alpha(v) = g^\alpha(v^*) \in \mathbb{A}(v)$.*

The next proposition considers some properties of dual α -Gately values.

Proposition 3.11 *Consider a regular cooperative game $v \in \mathbb{V}_\star^N$ and let $\alpha \in \mathbb{N}$ be a natural number. Then the following properties hold:*

- (a) *For every $\alpha \in \mathbb{N}$ the dual α -Gately value of v is well-defined and given by*

$$\overline{g}_i^\alpha(v) = M_i(v) - \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(\sum_{j \in N} M_j(v) - v(N) \right) \quad (17)$$

⁸Hence, the ENSC-value assigns the utopia payoff of that player $M_i(v)$ and an equal share of the remaining surplus in the game after the allocation of these utopia values to the players. We remark here that for all regular cooperative games it holds that $\sum_{j \in N} M_j(v) \geq v(N)$, so the surplus is really a deficit. If a player’s marginal contribution is small and the generated deficit is large, the ENSC-value assigns a negative value.

⁹[Driessen and Funaki \(1991\)](#)’s main established insight is that the ENSC-value and the (pre-)nucleolus coincide for a large class of cooperative games, including the 1-convex games ([Driessen, 1985](#)). The computational advantages of the ENSC-value over the nucleolus are significant.

for every player $i \in N$.

- (b) The dual α -Gately value of v is identical to the α -Gately value of v , i.e., $\bar{g}^\alpha(v) = g^\alpha(v)$, if and only if $\alpha = 1$ and/or $M_i(v) - v_i = M_j(v) - v_j \geq 0$ for all $i, j \in N$.

Proof. To show assertion (a), let $\alpha \in \mathbb{N}$. We compute that for every player $i \in N$:

$$\begin{aligned} \bar{g}_i^\alpha(v) &= g_i^\alpha(v^*) = v_i^* + \frac{(M_i(v^*) - v_i^*)^\alpha}{\sum_{j \in N} (M_j(v^*) - v_j^*)^\alpha} \left(v^*(N) - \sum_{j \in N} v_j^* \right) \\ &= M_i(v) + \frac{(v_i - M_i(v))^\alpha}{\sum_{j \in N} (v_j - M_j(v))^\alpha} \left(v(N) - \sum_{j \in N} M_j(v) \right) \\ &= M_i(v) - \frac{(-1)^\alpha (M_i(v) - v_i)^\alpha}{(-1)^\alpha \sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(\sum_{j \in N} M_j(v) - v(N) \right) \\ &= M_i(v) - \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(\sum_{j \in N} M_j(v) - v(N) \right) \end{aligned}$$

with the remark that $(-1)^\alpha$ attains only the values 1 and -1 due to $\alpha \in \mathbb{N}$.

Furthermore, we note that $\sum_{i \in N} \bar{g}_i^\alpha(v) = v(N)$, thereby showing that the dual α -Gately value is indeed well-defined.

To show assertion (b) let $i \in N$ and $\alpha \in \mathbb{N}$. We now note that from assertion (a) $g_i^\alpha(v) = \bar{g}_i^\alpha(v)$ if and only if

$$v_i + \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(v(N) - \sum_{j \in N} v_j \right) = M_i(v) - \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(\sum_{j \in N} M_j(v) - v(N) \right)$$

or

$$M_i(v) - v_i = \frac{(M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \left(\sum_{j \in N} M_j(v) - \sum_{j \in N} v_j \right)$$

or

$$\frac{\sum_{i \in N} (M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} = (M_i(v) - v_i)^{\alpha-1}$$

This is valid for all $i \in N$ if and only if $\alpha = 1$ and/or $M_i(v) - v_i = M_j(v) - v_j \geq 0$ for all $i, j \in N$. ■

Proposition 3.11 (b) implies immediately that the dual Gately value is the same as the Gately value on the class of regular games. This is stated in the next corollary.

Corollary 3.12 For every regular cooperative game $v \in \mathbb{V}_\star^N$, the dual Gately value of v is identical to the Gately value of v , i.e., $\bar{g}_i(v) = g_i(v)$ for all $i \in N$.

4 Gately values and the Core

In investigating the equal surplus sharing allocation rules and values, the literature has also focussed on the relationship between these allocation rules and the Core. We recall that for any cooperative game $v \in \mathbb{V}^N$, the *Core* is defined as a set of imputations $C(v) \subset \mathbb{I}(v)$ such that $x \in C(v)$ if and only for all coalitions $S \in 2^N$: $x(S) \geq v(S)$.

Gately (1974) introduced his solution concept as a Core selector within the setting of three-player games only, even though Gately did not investigate the exact conditions under which this solution is indeed in the Core. Littlechild and Vaidya (1976) point out that Gately's conception does not necessarily result in a Core selector for games with more than three players, devising a counterexample for 4 players.

In this section we first discuss the relationship between the Gately value and the Core for games with three players only. This is an exceptional case, since the worths of all coalitions in a three-player game are featured in the computation of the Gately value, in contrast to games with more than three players, in which worths of medium-sized coalitions are not considered. This is further explored in the second part of this section, which considers the relationship between the Gately value and the Core of cooperative games with an arbitrary number of players.

4.1 Gately points and the Core for 3-player games

We are able to confirm that there is a strong relationship between Gately points and the Core in three-player games. We first illustrate that there exist essential games with empty Cores for which the unique Gately point is well-defined.

Example 4.1 Consider an essential three-player game with $N = \{1, 2, 3\}$ and v given by $v_1 = 5$, $v_2 = v_3 = 0$, $v(12) = v(13) = 1$, $v(23) = 5$ and $v(N) = 6$.

First note that v is indeed essential, since $M_1(v) = 1$ and $M_2(v) = M_3(v) = 5$. On the other hand, v is not semi-standard, since $v_1 = 5 > M_1(v) = 1$.

Note that the Core of this game is empty, since for an allocation $x \in \mathbb{A}(v)$ with $x(N) = v(N) = 6$ and $x_2 + x_3 \geq v(23) = 5$ it follows that $x_1 \leq 1$. This is contradiction to the Core requirement that $x_1 \geq v_1 = 5$.

Regarding the existence of Gately points for this particular game, we note that the minimax optimisation problem can be re-stated here as the balance equation

$$\frac{M_1 - v_1}{x_1 - v_1} = \frac{M_2 - v_2}{x_2 - v_2} = \frac{M_3 - v_3}{x_3 - v_3} \quad \text{resulting into} \quad \frac{-4}{x_1 - 5} = \frac{5}{x_2} = \frac{5}{x_3}$$

which leads to a unique Gately point $g_1 = 4\frac{1}{3}$ and $g_2 = g_3 = \frac{5}{6}$. Note that this unique Gately point can also be computed by the Gately value formula stated in equation (8). \blacklozenge

The next theorem gathers some properties of three-player games regarding the relationship between the Core and the Gately points of these games. These properties generalise the insights presented through the previous two examples.

Theorem 4.2 Let $v \in \mathbb{V}^N$ be a three-player game on $N = \{1, 2, 3\}$. Then the following properties hold:

- (a) If $C(v) \neq \emptyset$, then the game v is semi-regular.
- (b) If the game v is semi-regular, then the Gately value is in its Core, $g(v) \in C(v)$.

Proof. To show assertion (a), assume that for three-player game $v \in \mathbb{V}^N$ with $N = \{1, 2, 3\}$ it holds that $C(v) \neq \emptyset$. Hence, there exists some $(x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_1 + x_2 + x_3 = v(N)$, $x_i \geq v_i$ for $i = 1, 2, 3$, and

$$x_1 + x_2 \geq v(12) \quad x_1 + x_3 \geq v(13) \quad x_2 + x_3 \geq v(23).$$

Adding the last three inequalities results in the conclusion that

$$2v(N) = 2x_1 + 2x_2 + 2x_3 \geq v(12) + v(13) + v(23),$$

which in turn leads to the conclusion that

$$M_1(v) + M_2(v) + M_3(v) = (v(N) - v(12)) + (v(N) - v(13)) + (v(N) - v(23)) \geq v(N).$$

Furthermore, from $x_i \geq v_i$ for $i = 1, 2, 3$ it follows that $v(N) = x_1 + x_2 + x_3 \geq v_1 + v_2 + v_3$.

These two inequalities leads us to the conclusion that $v_1 + v_2 + v_3 \leq v(N) \leq M_1(v) + M_2(v) + M_3(v)$, implying that v is indeed essential.

Furthermore, $v(N) = x_1 + (x_2 + x_3) \geq v_1 + v(23)$ implying that $M_1(v) = v(N) - v(23) \geq v_1$. This argument can be replicated for players 2 and 3, leading to the desired conclusion that v is indeed semi-regular.

To show assertion (b), we first consider a three-player game $v \in \mathbb{V}^N$ that is semi-regular, but not regular. Hence, $M_i = v_i$ for $i = 1, 2, 3$, implying that $v_1 + v_2 + v_3 = M_1(v) + M_2(v) + M_3(v) = v(N)$. Simple computations show that there is a unique Core imputation given by $C(v) = \{(v_1, v_2, v_3)\} = \{(M_1(v), M_2(v), M_3(v))\} \neq \emptyset$. Furthermore, it is easily established that the unique Gately point is well-defined and given by $g(v) = (v_1, v_2, v_3) \in C(v)$.

Next, we assume that v is regular in the sense that $v_1 + v_2 + v_3 \leq v(N) \leq M_1(v) + M_2(v) + M_3(v)$, $v_i \leq M_i(v)$ for all $i = 1, 2, 3$ and, without loss of generality, $v_1 < M_1(v)$. Hence, it holds that $v(12) + v(13) + v(23) \leq 2v(N)$. Furthermore, it follows that

$$3v(N) - v(12) - v(13) - v(23) - v_1 - v_2 - v_3 = \sum_j (M_j(v) - v_j) > 0.$$

Now define for every $i = 1, 2, 3$

$$\eta_i = \frac{2v(N) - v(12) - v(13) - v(23)}{3v(N) - v(12) - v(13) - v(23) - v_1 - v_2 - v_3} (M_i(v) - v_i)$$

Note that $\eta_i \geq 0$ for all $i = 1, 2, 3$ and that, in particular, $\eta_1 > 0$.

We now note the following properties of these introduced quantities:

- First, regarding their sum,

$$\begin{aligned}\eta_1 + \eta_2 + \eta_3 &= \frac{2v(N) - v(12) - v(13) - v(23)}{3v(N) - v(12) - v(13) - v(23) - \sum_i v_i} \sum_i (M_i(v) - v_i) \\ &= \sum_i M_i(v) - v(N) = 2v(N) - v(12) - v(13) - v(23).\end{aligned}$$

- Second, for every $i = 1, 2, 3$:

$$\eta_i = \frac{\sum_j M_j(v) - v(N)}{\sum_j (M_j(v) - v_j)} (M_i(v) - v_i) \leq M_i(v) - v_i.$$

- Finally, for every $i = 1, 2, 3$ we argue that $g_i(v) = M_i(v) - \eta_i$. Indeed,

$$\begin{aligned}g_i(v) &= \frac{M_i(v) - v_i}{\sum_j (M_j(v) - v_j)} \left(v(N) - \sum_j v_j \right) \\ &= \frac{\sum_j M_j(v) - v(N)}{\sum_j (M_j(v) - v_j)} v_i + \frac{v(N) - \sum_j v_j}{\sum_j (M_j(v) - v_j)} M_i(v) \\ &= M_i(v) + \frac{\sum_j M_j(v) - v(N)}{\sum_j (M_j(v) - v_j)} (v_i - M_i(v)) = M_i(v) - \eta_i.\end{aligned}$$

Using the argument of [Vorob'ev \(1977, 4.12.1\)](#), we now claim that $g(v) \in C(v)$ using the construction above. We check the conditions for $g(v)$ being a Core selector:

First, for every $i = 1, 2, 3$:

$$g_i(v) = M_i(v) - \eta_i \geq M_i(v) - (M_i(v) - v_i) = v_i.$$

Second, we can check for each 2-player coalition the Core conditions. For $\{1, 2\}$ it is easy to see that

$$\begin{aligned}g_1(v) + g_2(v) &= M_1(v) + M_2(v) - \eta_1 - \eta_2 \\ &= 2v(N) - v(23) - v(13) - \eta_1 - \eta_2 \\ &= v(12) + \eta_3 \geq v(12)\end{aligned}$$

Similar arguments show that $g_1(v) + g_3(v) \geq v(13)$ and $g_2(v) + g_3(v) \geq v(23)$.

Together with $g_1(v) + g_2(v) + g_3(v) = v(N)$, this completes the proof of assertion (b). ■

An immediate insight from Theorem 4.2 is that for every three-player game with a non-empty Core, the Gately value is a Core selector:

Corollary 4.3 *Let $v \in \mathbb{V}^N$ with $N = \{1, 2, 3\}$ be a three-player cooperative game. Then $g(v) \in C(v)$ if and only $C(v) \neq \emptyset$.*

Similar arguments as the ones used in the proof of Theorem 4.2(b) show that the Gately value of certain semi-regular three-player games is equal to the vector of marginal contributions and it is the unique Core imputation if the Core is non-empty.

Corollary 4.4 Let $v \in \mathbb{V}^N$ with $N = \{1, 2, 3\}$ be a three-player cooperative game such that v is an essential cooperative game such that $v(12) + v(13) + v(23) = 2v(N)$. Then the unique Gately point is given by the vector of marginal contributions $g(v) = (M_1(v), M_2(v), M_3(v))$. Furthermore, if $C(v) \neq \emptyset$, the Gately point is the unique Core imputation: $C(v) = \{g(v)\}$.

α -Gately values and the Core of 3-player games The analysis of the relationship between α -Gately values and the Core of a three-player game is more complex if we look beyond the standard Gately value ($\alpha = 1$).

The next example shows that there exist three-player games in which α -Gately values are in the Core for a certain closed interval of α values bounded away from zero.

Example 4.5 Consider a zero-normalised three-player game v with $N = \{1, 2, 3\}$ and $v_i = 0$ for $i = 1, 2, 3$, $v(12) = 12$, $v(13) = v(23) = 7$ and $v(N) = 16$. Clearly, this game is regular.

We easily compute that the marginal contributions are given by $M_1 = M_2 = 7$ and $M_3 = 4$. For any $\alpha > 0$ we compute the α -Gately values as

$$g_1^\alpha(v) = g_2^\alpha(v) = \frac{8 \cdot 7^\alpha}{7^\alpha + 2 \cdot 4^{\alpha-1}}$$

$$g_3^\alpha(v) = \frac{8 \cdot 4^\alpha}{7^\alpha + 2 \cdot 4^{\alpha-1}}$$

We note that there are essentially two characteristic inequalities to determine whether the α -Gately value is in the Core of v :

$$g_1^\alpha(v) + g_2^\alpha(v) \geq v(12) = 12$$

$$g_1^\alpha(v) + g_3^\alpha(v) = g_2^\alpha(v) + g_3^\alpha(v) \geq v(13) = v(23) = 7$$

The first inequality leads to the conclusion that $\alpha \geq \frac{\ln 3 - \ln 2}{\ln 7 - \ln 4} \approx 0.7245$ and the second inequality results in $\alpha \leq \frac{\ln 7 - \ln 2}{\ln 7 - \ln 4} \approx 2.2386$. Hence, the range of α values for which the α -Gately value is in the Core of this game is given by $\alpha \in \left[\frac{\ln 3 - \ln 2}{\ln 7 - \ln 4}, \frac{\ln 7 - \ln 2}{\ln 7 - \ln 4} \right]$. Note that $\alpha = 1$ is indeed in this interval, i.e., $g(v) \in C(v)$. \blacklozenge

The next example considers a game with a large set of imputations and a minimal Core, consisting of a single imputation. For this example we show that the original Gately value is the only Core selector, while all α -Gately values for $\alpha \neq 1$ are outside the Core.

Example 4.6 Consider a regular three-player game with $N = \{1, 2, 3\}$ and v given by $v_1 = v_2 = v_3 = 0$, $v(12) = 5$, $v(13) = 6$, $v(23) = 5$ and $v(N) = 9$. Note that $\mathbb{I}(v) = \{x \in \mathbb{R}_+^3 \mid \sum x_i = 9\}$ and that the Core is a singleton with $C(v) = \{(2, 3, 4)\} = \{M(v)\}$.

We note that for this game $g(v) = M(v)$ selects the unique Core imputation. However, for all $\alpha > 0$ with $\alpha \neq 1$ we have that

$$g^\alpha(v) = \frac{9}{2^\alpha + 3^\alpha + 4^\alpha} (2^\alpha, 3^\alpha, 4^\alpha) \neq (2, 3, 4).$$

Note that $g^\alpha(v) \rightarrow (3, 3, 3) = E(v)$ as $\alpha \downarrow 0$ and $g^\alpha(v) \rightarrow (0, 0, 9)$ as $\alpha \rightarrow \infty$. This convergence is not

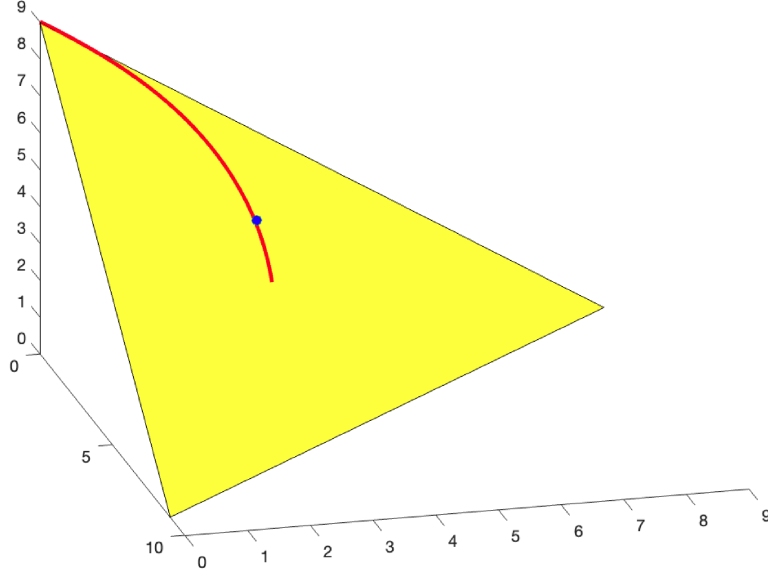


Figure 1: The Core and the trajectory of α -Gately values in Example 4.6.

monotone as one would possibly expect, since $g_2^\alpha(v)$ attains a maximal value of $g_2^{\hat{\alpha}}(v) \approx 3.0291$ at $\hat{\alpha} = \frac{1}{\ln 2} \ln \left[\frac{\ln 3 - \ln 2}{\ln 4 - \ln 3} \right] \approx 0.4951$.

The results of the analysis of this example are summarised in Figure 1. The yellow simplex represents the space of imputations $\mathbb{I}(v)$, while the Core is the unique imputation depicted as a blue point. The red curve denotes the set of α -Gately values, $\{g^\alpha(v) \mid \alpha > 0\}$. \blacklozenge

The next example discusses a game with a Core that has a non-empty relative interior.

Example 4.7 Consider a regular three-player game with $N = \{1, 2, 3\}$ and v given by $v_1 = v_2 = v_3 = 0$, $v(12) = 1$, $v(13) = 2$, $v(23) = 3$ and $v(N) = 4$. Note that $\mathbb{I}(v) = \{x \in \mathbb{R}_+^3 \mid \sum x_i = 4\}$ and the Core is given by

$$C(v) = \{x \in \mathbb{I}(v) \mid x_1 + x_2 \geq 1, x_1 + x_3 \geq 2 \text{ and } x_2 + x_3 \geq 3\} \neq \emptyset.$$

In fact, the relative interior of the Core is non-empty and contains the Gately value $g(v) = \left(\frac{2}{3}, \frac{4}{3}, 2\right)$. Indeed, $g_1 + g_2 = 2 > v(12) = 1$, $g_1 + g_3 = 2\frac{2}{3} > v(13) = 2$ and $g_2 + g_3 = 3\frac{1}{3} > v(23) = 3$.

Note that more generally the α -Gately value is computed as

$$g^\alpha(v) = \left(\frac{4}{1 + 2^\alpha + 3^\alpha}, \frac{2^{\alpha+2}}{1 + 2^\alpha + 3^\alpha}, \frac{4 \cdot 3^\alpha}{1 + 2^\alpha + 3^\alpha} \right)$$

The α -Gately value is in the Core $g^\alpha(v) \in C(v)$ for $0.44799 \leq \alpha \leq 3$.

Furthermore, we note that

$$g^\alpha(v) \rightarrow E(v) = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) \notin C(v) \quad \text{as } \alpha \downarrow 0$$

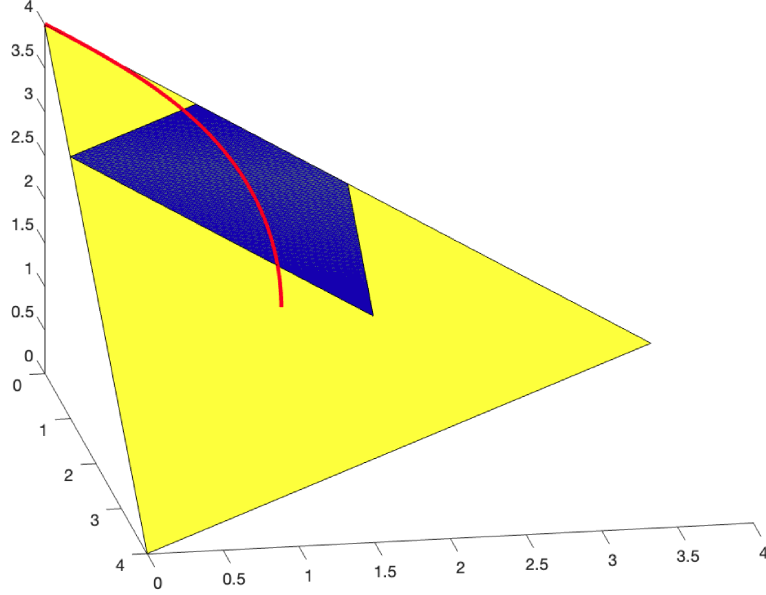


Figure 2: The Core and the trajectory of α -Gately values in Example 4.7.

and

$$g^\alpha(v) \rightarrow (0, 0, 4) \notin C(v) \quad \text{as } \alpha \rightarrow \infty.$$

This is summarised in Figure 2. The yellow simplex represents the space of imputations $\mathbb{I}(v)$, while the Core is the blue polytope. As before, the red curve denotes the Gately set $\{g^\alpha(v) \mid \alpha > 0\}$. \blacklozenge

4.2 Top dominance and α -Gately values of n -player games

The main condition for which a “symmetric” or “anonymous” cooperative game has a non-empty Core has been identified as the condition that for all coalitions $S \in 2^N$: $\frac{v(S)}{|S|} \leq \frac{v(N)}{n}$ (Shubik, 1982, page 149). This condition has been referred to as “domination by the grand coalition” by Chatterjee et al. (1993) and as “top convexity” by Jackson and van den Nouweland (2005). We generalise this condition to identify when the α -Gately value is in the Core of a regular, zero-normalised cooperative game.

Definition 4.8 Let $v \in \mathbb{V}^N$ be a semi-standard cooperative game and let $\alpha > 0$. The cooperative game v is said to be α -**top dominant** if for every coalition $S \in 2^N$

$$\left[v(S) - \sum_{j \in S} v_j \right] \cdot \sum_{j \in N} (M_j(v) - v_j)^\alpha \leq \left[v(N) - \sum_{j \in N} v_j \right] \cdot \sum_{j \in S} (M_j(v) - v_j)^\alpha. \quad (18)$$

First we remark that $\sum_{j \in S} (M_j(v) - v_j)^\alpha \geq 0$ for every semi-standard cooperative game $v \in \mathbb{V}^N$ and every $\alpha > 0$.

Furthermore, the concept of α -top dominance is akin to the notions listed above in the sense

that for a semi-standard zero-normalised game $v \in \mathbb{V}_0^N$ (18) can be rewritten as

$$\frac{v(S)}{\sum_{j \in S} M_j(v)^\alpha} \leq \frac{v(N)}{\sum_{j \in N} M_j(v)^\alpha}$$

for $\sum_{j \in N} M_j(v)^\alpha \geq \sum_{j \in S} M_j(v)^\alpha > 0$. Moreover, implementing $\alpha = 0$, the notion of α -top dominance clearly generalises the notion of top convexity, as top convexity is equivalent to 0-top dominance for zero-normalised games. Indeed, for zero-normalised game $v \in \mathbb{V}_0^N$ we straightforwardly derive $\sum_{j \in S} M_j^0(v) = |S|$, immediately leading to the conclusion that 0-top dominance is the same as top convexity.

The next definition introduces a reduced notion of superadditivity that fits with top dominance. This form of superadditivity is denoted as ‘‘partitional’’ superadditivity.

Definition 4.9 *A cooperative game $v \in \mathbb{V}^N$ is **partitionally superadditive** if for every coalition $S \subseteq N$ it holds that $v(S) + v(N \setminus S) \leq v(N)$.*

The next theorem characterises top dominant games in terms of regularity properties and the partitional superadditivity property defined above.

Theorem 4.10 *Let $v \in \mathbb{V}^N$ be a standard cooperative game. If the game v is α -top dominant for some $\alpha > 0$, then v is regular as well as partitionally superadditive.*

Proof. Let $v \in \mathbb{V}^N$ be a standard game and let $\alpha > 0$ be such that v is α -top dominant. Hence, $\sum_{j \in N} (M_v(j) - v_j)^\alpha > 0$ and $\sum_{j \in S} (M_v(j) - v_j)^\alpha \geq 0$ for every coalition $S \subset N$.

We first show that v is essential, together with the hypothesis that v is standard, implying that v is regular.

When we apply the α -top dominance property to the coalition $N - i$ for any $i \in N$ we arrive at

$$\left(v(N - i) - \sum_{j \neq i} v_j \right) \cdot \sum_{j \in N} (M_v(j) - v_j)^\alpha \leq \left(v(N) - \sum_{j \in N} v_j \right) \cdot \sum_{j \neq i} (M_v(j) - v_j)^\alpha$$

Adding these inequalities over all $i \in N$ we arrive at the conclusion that

$$\begin{aligned} \sum_{i \in N} \left(v(N - i) - \sum_{j \neq i} v_j \right) \cdot \sum_{j \in N} (M_v(j) - v_j)^\alpha &\leq \left(v(N) - \sum_{j \in N} v_j \right) \sum_{i \in N} \sum_{j \neq i} (M_v(j) - v_j)^\alpha \\ &= (n - 1) \left(v(N) - \sum_{j \in N} v_j \right) \sum_{j \in N} (M_v(j) - v_j)^\alpha \end{aligned}$$

Hence,

$$\sum_{i \in N} \left(v(N - i) - \sum_{j \neq i} v_j \right) \leq (n - 1) \left(v(N) - \sum_{j \in N} v_j \right)$$

leading to the conclusion that

$$\sum_{i \in N} [v(N) - v(N - i)] - v(N) - (n - 1) \sum_{i \in N} v_i \geq -(n - 1) \sum_{i \in N} v_i.$$

This implies that

$$\sum_{i \in N} M_i(v) \geq v(N). \quad (19)$$

Next, suppose to the contrary that $v(N) < \sum_{j \in N} v_j$. From v being a standard game, there is some $i \in N$ with $M_i(v) > v_i$. We can apply the α -top dominance property to $S = \{i\}$ and derive that

$$0 = (v_i - v_i) \cdot \sum_{j \in N} (M_j(v) - v_j)^\alpha \leq \left(v(N) - \sum_{j \in N} v_j \right) \cdot (M_i(v) - v_i) < 0$$

which is impossible. Therefore, we conclude that $v(N) \geq \sum_{j \in N} v_j$ and, together with (19), we have shown the assertion that v is essential.

Next we show that v is partitionally superadditive.

Let $S \subset N$ be some coalition. Then, from α -top dominance, it holds for S that

$$\begin{aligned} \left(v(S) - \sum_{j \in S} v_j \right) \cdot \sum_{i \in N} (M_i(v) - v_i)^\alpha &\leq \left(v(N) - \sum_{j \in N} v_j \right) \cdot \sum_{i \in S} (M_i(v) - v_i)^\alpha \\ \left(v(N \setminus S) - \sum_{j \in N \setminus S} v_j \right) \cdot \sum_{i \in N} (M_i(v) - v_i)^\alpha &\leq \left(v(N) - \sum_{j \in N} v_j \right) \cdot \sum_{i \in N \setminus S} (M_i(v) - v_i)^\alpha \end{aligned}$$

Adding these two inequalities leads to the conclusion that

$$\left(v(S) + v(N \setminus S) - \sum_{j \in N} v_j \right) \cdot \sum_{i \in N} (M_i(v) - v_i)^\alpha \leq \left(v(N) - \sum_{j \in N} v_j \right) \cdot \sum_{i \in N} (M_i(v) - v_i)^\alpha$$

Since $\sum_{i \in N} (M_i(v) - v_i)^\alpha > 0$ for any $\alpha > 0$, we have shown that

$$v(S) + v(N \setminus S) - \sum_{j \in N} v_j \leq v(N) - \sum_{j \in N} v_j$$

and, hence, $v(S) + v(N \setminus S) \leq v(N)$. We conclude that v is indeed partitionally superadditive. \blacksquare

One can ask oneself whether the condition of top dominance can be simplified or linked to other regularity properties of cooperative games. As shown in Theorem 4.10 it is clear that top dominance is closely related to the superadditivity property that is widely used in cooperative game theory. The next example shows that top dominance is actually strictly weaker than superadditivity.

Example 4.11 Consider a regular and zero-normalised three-player game with $N = \{1, 2, 3\}$ and v given by $v_i = 0$ for $i = 1, 2, 3$, $v(12) = v(13) = -1$, $v(23) = 0$ and $v(N) = 1$. We note that v is not

superadditive, since $v_1 + v_2 = 0 > v(12) = -1$.

However, for any $\alpha > 0$ we remark that

$$\begin{aligned}\frac{v(N)}{M_1(v)^\alpha + M_2(v)^\alpha + M_3(v)^\alpha} &= \frac{1}{1 + 2^{\alpha+1}} > 0 \\ \frac{v(12)}{M_1(v)^\alpha + M_2(v)^\alpha} &= \frac{-1}{1 + 2^\alpha} < 0 \\ \frac{v(13)}{M_1(v)^\alpha + M_3(v)^\alpha} &= \frac{-1}{1 + 2^\alpha} < 0 \\ \frac{v(23)}{M_2(v)^\alpha + M_3(v)^\alpha} &= 0\end{aligned}$$

Hence, we conclude that v indeed satisfies α -top dominance for every $\alpha > 0$.

Furthermore, we determine easily that $M_1(v) = 1$ and $M_2(v) = M_3(v) = 2$, leading to the conclusion that for every $\alpha > 0$ the Gately values are given as

$$g_1^\alpha = \frac{1}{1 + 2^{\alpha+1}} \quad \text{and} \quad g_2^\alpha = g_3^\alpha = \frac{2^\alpha}{1 + 2^{\alpha+1}}$$

It can also easily be checked that for every $\alpha > 0$: $g^\alpha(v) \in C(v)$. ◆

Top dominance and the Gately value as a Core selector The next theorem generalises the insights of Theorem 4.2 to games with arbitrary player sets.

Theorem 4.12 *Let $\alpha > 0$. A standard cooperative game $v \in \mathbb{V}^N$ is α -top dominant if and only if $g^\alpha(v) \in C(v)$.*

Proof. Let $v \in \mathbb{V}^N$ be standard and let $\alpha > 0$.

Now $g^\alpha(v) \in C(v)$ if and only if it holds that for every coalition $S \in 2^N$: $\sum_{j \in S} g_j^\alpha(v) \geq v(S)$. This is equivalent to the condition that for every coalition $S \in 2^N$:

$$\sum_{i \in S} v_i + \frac{\sum_{i \in S} (M_i(v) - v_i)^\alpha}{\sum_{j \in N} (M_j(v) - v_j)^\alpha} \cdot \left(v(N) - \sum_{j \in N} v_j \right) \geq v(S)$$

From v being standard, it follows that $\sum_{i \in N} (M_i(v) - v_i)^\alpha > 0$. Hence, the above is equivalent to the condition that for every coalition $S \in 2^N$:

$$\sum_{i \in S} (M_i(v) - v_i)^\alpha \cdot \left(v(N) - \sum_{j \in N} v_j \right) \geq \sum_{j \in N} (M_j(v) - v_j)^\alpha \cdot \left(v(S) - \sum_{i \in S} v_i \right)$$

This is exactly the α -top dominance property. ■

Theorems 4.10 and 4.12 now immediately imply the following corollary.

Corollary 4.13 *Let $v \in \mathbb{V}^N$ be a standard cooperative game and let $\alpha > 0$. If $g^\alpha(v) \in C(v)$, then v is regular and partitionally superadditive.*

The next example shows that the assertion of Corollary 4.13 cannot be reversed for 4-player games. Indeed, this example constructs a regular zero-normalised 4-player game in which all α -Gately values for $\alpha > 0$ are not in the Core.

Example 4.14 Consider a regular zero-normalised four-player game v with $N = \{1, 2, 3, 4\}$ and $v_i = 0$ for all $i \in N$, $v(12) = v(14) = v(23) = v(24) = v(34) = 1$, $v(13) = 4$, $v(123) = v(124) = 5$, $v(134) = v(234) = 4$, and $v(N) = 6$. Note that v is partitionally superadditive. From these worths, we derive that $M_1(v) = M_2(v) = 2$ and $M_3(v) = M_4(v) = 1$. We now compute that for every $\alpha > 0$:

$$g_1^\alpha = g_2^\alpha = \frac{3 \cdot 2^\alpha}{2^\alpha + 1} \quad \text{and} \quad g_3^\alpha = g_4^\alpha = \frac{3}{2^\alpha + 1}$$

It can easily be established that $g_1^\alpha + g_3^\alpha = 3 < v(13) = 4$, which shows that g^α is *not* in the Core of v for any $\alpha > 0$. ♦

5 Application: Measuring centrality in directed networks

The concept of *network centrality* has emerged from sociology and social network analysis (Barabási, 2016) into the field of game theory, giving rise to game theoretic methods to measure the most important and dominant nodes in a social network (Gómez et al., 2003; Pozo et al., 2011; Bloch et al., 2016; Tarkowski et al., 2018). A well-accepted approach is to formulate a cooperative game that captures features of a social network and then apply a cooperative solution concept to this associated game. Such an approach was developed for the measurement of hierarchical power in the class of directed networks by van den Brink and Gilles (1994, 2000), Borm et al. (2002), and van den Brink et al. (2008).¹⁰

The most applied solution concept to measure hierarchical power in directed social networks is the Shapley value. As such, the β -measure (van den Brink and Gilles, 1994, 2000) has been characterised as the Shapley value of two associated cooperative games to any directed social network.

Here we investigate the Gately values of these associated cooperative games. Given the particular interpretation of the Gately value—based on measuring the propensity to disrupt for each player and balancing the allocated value based on these propensities—it is clear that the Gately measure on directed networks is founded on measuring “power” in networks by players’ propensities to disrupt the functioning of these networks.

Preliminaries: Directed social networks A *directed network* on a set of players N is a map $D: N \rightarrow 2^N$ that assigns to every player $i \in N$ a set of “successors” $D(i) \subseteq N \setminus \{i\}$. We explicitly exclude that a player can succeed herself in the sense that $i \notin D(i)$. The class of all directed networks on player set N is denoted as $\mathbb{D}^N = \{D \mid D: N \rightarrow 2^N \text{ with } i \notin D(i) \text{ for all } i \in N\}$.

Inversely, in a directed network $D \in \mathbb{D}^N$, for every player $i \in N$: $D^{-1}(i) = \{j \in N \mid i \in D(j)\}$ denotes the set of her *predecessors* in D . The notions of successors and predecessors allows us to

¹⁰This research has also been summarised and discussed in Gilles (2010, Section 5.2.2).

investigate several distinct subsets of players:

$$\begin{aligned}
N_0(D) &= \{i \in N \mid D^{-1}(i) = \emptyset\} \\
N^*(D) &= \{i \in N \mid D^{-1}(i) \neq \emptyset\} = N \setminus N_0(D) \\
N_1(D) &= \{i \in N \mid \#D^{-1}(i) = 1\} \\
N_2(D) &= \{i \in N \mid \#D^{-1}(i) \geq 2\} = N^*(D) \setminus N_1(D)
\end{aligned}$$

For every directed network $D \in \mathbb{D}^N$ the player set N is partitioned in three distinct subsets of players: the set of “top” players $N_0(D)$ without predecessors; the players preceded by a single predecessor $N_1(D)$; and the players with multiple predecessors $N_2(D)$.

From the successors and predecessors of players in a directed network we can introduce the following auxiliary concepts for any coalition $S \in 2^N$:

$$\begin{aligned}
D(S) &= \cup_{i \in S} D(i) = \{j \in N^*(D) \mid D^{-1}(j) \cap S \neq \emptyset\} \\
D^*(S) &= \{j \in N^*(D) \mid D^{-1}(j) \subseteq S\} \subseteq D(S)
\end{aligned}$$

For a directed network $D \in \mathbb{D}^N$ we can now define two associated cooperative games that count the number of proper successors of a coalition in various ways.

Definition 5.1 *Let $D \in \mathbb{D}^N$ be a directed network on N .*

- (a) *The associated **successor game** to D is the cooperative game $\Gamma_D \in \mathbb{V}^N$ with for every $S \in 2^S$:*

$$\Gamma_D(S) = \#D(S) \tag{20}$$

- (b) *The associated **conservative successor game** to D is the cooperative game $\Delta_D \in \mathbb{V}^N$ with for every $S \in 2^S$:*

$$\Delta_D(S) = \#D^*(S) \tag{21}$$

The following lemma summarises the insights in the literature of the Shapley values of these games.

Lemma 5.2 *Let $D \in \mathbb{D}^N$ be a directed network on N and let φ denote the Shapley value on the class of cooperative games \mathbb{V}^N . Then for every $i \in N$ the following properties hold:*

$$\varphi(\Gamma_D) = \varphi(\Delta_D) = \beta(D) = \sum_{j \in D(i)} \frac{1}{\#D^{-1}(j)} \tag{22}$$

where β is referred to as the **β -measure** on \mathbb{D}^N .

This characterisation has been discussed in van den Brink and Gilles (2000) for successor games and by van den Brink and Borm (2002) for conservative successor games.

The Gately measure We apply the Gately value to the two successor games formulated above. We show that, similar to the β -measure, both the regular successor game and the conservative successor game result in the same Gately value, defining the *Gately measure*.

Proposition 5.3 Let $D \in \mathbb{D}^N$ be a directed network on N and let g denote the Gately value on the class of cooperative games \mathbb{V}^N . Then

$$g(\Gamma_D) = g(\Delta_D) = \xi(D) \quad (23)$$

where $\xi: \mathbb{D}^N \rightarrow \mathbb{R}^N$ is introduced as the **Gately measure** on the class of directed networks on N with

$$\xi_i(D) = \begin{cases} \#(D(i) \cap N_1(D)) + \frac{\#N_2(D)}{\sum_{j \in N_2(D)} \#D^{-1}(j)} \cdot \#(D(i) \cap N_2(D)) & \text{if } N_2(D) \neq \emptyset \\ \#(D(i) \cap N_1(D)) & \text{if } N_2(D) = \emptyset \end{cases} \quad (24)$$

for every player $i \in N$.

Furthermore, the Gately measure ξ is the unique centrality measure that balances the propensities to disrupt a network given by

$$\frac{\#(D(i) \cap N_2(D))}{\#D(i) - \xi_i(D)} = \frac{\#(D(j) \cap N_2(D))}{\#D(j) - \xi_j(D)} \quad (25)$$

over all players $i, j \in N^*(D)$.

Proof. Let $D \in \mathbb{D}^N$ be such that $N_2(D) \neq \emptyset$. Then the successor game Γ_D for $D \in \mathbb{D}^N$ is characterised for every $i \in N$ by

$$\begin{aligned} \Gamma_D(N) &= \#N^*(D) \\ \Gamma_D(i) &= \#D(i) \\ \Gamma_D(N - i) &= \#N^*(D) - \#\{j \in N_1(D) \mid D^{-1}(j) = \{i\}\} = \#N^*(D) - \#D^*(i) \end{aligned}$$

From this it follows that $M_i(\Gamma_D) = \Gamma_D(N) - \Gamma_D(N - i) = \#D^*(i)$ for every $i \in N$. Since $N_2(D) \neq \emptyset$, this implies furthermore that $\Gamma_D(i) \geq M_i(\Gamma_D)$ for every $i \in N$. Note that the Gately value can be applied to this game, since the dual Gately value applies, which is identical to the Gately value for this game.

From the previous we further derive that

$$\Gamma_D(i) - M_i(\Gamma_D) = \#\{j \in N \mid \{i\} \subsetneq D^{-1}(j)\} = \#(D(i) \cap N_2(D))$$

and that

$$\begin{aligned} \sum_{j \in N} \Gamma_D(j) - \Gamma_D(N) &= \sum_{j \in N} \#D(j) - \#N^*(D) = \sum_{h \in N} \#D^{-1}(h) - \#N^*(D) \\ &= \sum_{j \in N_2(D)} (\#D^{-1}(j) - 1) = \sum_{j \in N_2(D)} \#D^{-1}(j) - \#N_2(D) \end{aligned}$$

We now compute the Gately value of the successor game for every $i \in N$ as

$$\begin{aligned}
g_i(\Gamma_D) &= \Gamma_D(i) - \frac{\Gamma_D(i) - M_i(\Gamma_D)}{\sum_{j \in N} (\Gamma_D(j) - M_j(\Gamma_D))} \cdot \left(\sum_{j \in N} \Gamma_D(j) - \Gamma_D(N) \right) \\
&= \#D(i) - \frac{\#(D(i) \cap N_2(D))}{\sum_{j \in N} (\#(D(j) \cap N_2(D)))} \cdot \left(\sum_{j \in N_2(D)} \#D^{-1}(j) - \#N_2(D) \right) \\
&= \#D(i) - \frac{\#(D(i) \cap N_2(D))}{\sum_{j \in N_2(D)} \#D^{-1}(j)} \cdot \left(\sum_{j \in N_2(D)} \#D^{-1}(j) - \#N_2(D) \right) \\
&= \#D(i) - \#(D(i) \cap N_2(D)) + \frac{\#(D(i) \cap N_2(D))}{\sum_{j \in N_2(D)} \#D^{-1}(j)} \cdot \#N_2(D) \\
&= \#(D(i) \cap N_1(D)) + \frac{\#N_2(D)}{\sum_{j \in N_2(D)} \#D^{-1}(j)} \cdot \#(D(i) \cap N_2(D)) = \xi_i(D)
\end{aligned}$$

Similarly, the conservative successor game Δ_D for D is characterised for every $i \in N$ by

$$\begin{aligned}
\Delta_D(N) &= \#N^*(D) \\
\Delta_D(i) &= \#[D(i) \cap N_1(D)] \\
\Delta_D(N - i) &= \#N^*(D) - \#D(i)
\end{aligned}$$

For the conservative successor game Δ_D we derive from the above that $M_i(\Delta_D) = \#D(i)$, implying that $\Delta_D(i) \leq M_i(\Delta_D)$ for every $i \in N$. Also, $\Delta_D(i) < M_i(\Delta_D)$ for some $i \in N$, since $N_2(D) \neq \emptyset$. Therefore, Δ_D is regular.

Furthermore,

$$M_i(\Delta_D) - \Delta_D(i) = \#D(i) - \#[D(i) \cap N_1(D)] = \#[D(i) \cap N_2(D)]$$

and

$$\begin{aligned}
g_i(\Delta_D) &= \Delta_D(i) + \frac{M_i(\Delta_D) - \Delta_D(i)}{\sum_{j \in N} (M_j(\Delta_D) - \Delta_D(j))} \cdot \left(\Delta_D(N) - \sum_{j \in N} \Delta_D(j) \right) \\
&= \#[D(i) \cap N_1(D)] + \frac{\#[D(i) \cap N_2(D)]}{\sum_{j \in N} \#[D(j) \cap N_2(D)]} \cdot \#N_2(D) \\
&= \#[D(i) \cap N_1(D)] + \frac{\#N_2(D)}{\sum_{j \in N_2(D)} \#D^{-1}(j)} \cdot \#[D(i) \cap N_2(D)] = \xi_i(D)
\end{aligned}$$

This shows the assertion.

Next, let $D \in \mathbb{D}^N$ be such that $N_2(D) = \emptyset$. Then $\#D^{-1}(j) = 1$ for all $j \in N^*(D)$. This implies that for every $i \in N$: $M_i(\Gamma_D) = \Gamma_D(i) = \#D(i)$. Hence,

$$g_i(\Gamma_D) = \Gamma_D(i) = \#D(i) = \#[D(i) \cap N_1(D)] = \xi_i(D).$$

Furthermore, for every $i \in N$: $M_i(\Delta_D) = \#D(i) = \#D^*(i) = \Delta_D(i)$. Hence,

$$g_i(\Delta_D) = \Delta_D(i) = \#D(i) = \#(D(i) \cap N_1(D)) = \xi_i(D).$$

Combined with the previous case, this shows the first assertion of the proposition.

Finally, the final assertion of the proposition follows immediately from identifying the propensity to disrupt in the score game Γ_D for some imputation $m \in \mathbb{I}(\Gamma_D)$ as

$$\frac{M_i(\Gamma_D) - \Gamma_D(i)}{m_i - \Gamma_D(i)} = \frac{\#D^*(i) - \#D(i)}{m_i - \#D(i)} = \frac{\#(D(i) \cap N_2(D))}{\#D(i) - m_i}.$$

Using the definition of a Gately point and noting that $\xi_i = \Gamma_D(i) = \#D(i) = 0$ for every $i \in N_0(D)$, the final assertion is confirmed. ■

The nature of the Gately measure in comparison with the β -measure for any network $D \in \mathbb{D}^N$ is that the β -measure is based on the localised measurement of how power is shared over the predecessors of nodes in $N_2(D)$, while the Gately measure has more holistic an approach to sharing this power. This difference vanishes for networks $D \in \mathbb{D}^N$ with $N_2(D) = \emptyset$. The next corollary of Proposition 5.3 summarises this insight.

Corollary 5.4 *Let $\beta: \mathbb{D}^N \rightarrow \mathbb{R}^N$ and $\xi: \mathbb{D}^N \rightarrow \mathbb{R}^N$ denote the β -measure and the Gately measure, respectively. For any directed network $D \in \mathbb{D}^N$ with $N_2(D) \neq \emptyset$ and any player $i \in N$:*

$$\beta_i(D) - \xi_i(D) = \sum_{j \in N_2(D): j \in D(i)} \frac{1}{\#D^{-1}(j)} - \frac{\#N_2(D)}{\sum_{j \in N_2(D)} \#D^{-1}(j)} \cdot \#(D(i) \cap N_2(D)) \quad (26)$$

Furthermore, for any directed network $D \in \mathbb{D}^N$ with $N_2(D) = \emptyset$: $\beta(D) = \xi(D)$.

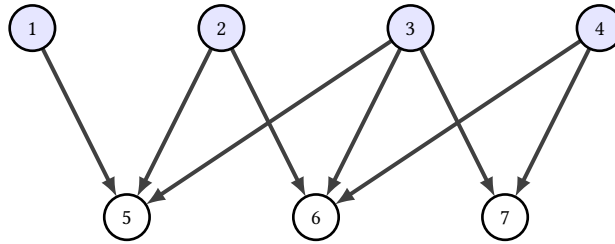


Figure 3: The directed network considered in Example 5.5

We also illustrate the difference between these two measures with the use of the following two examples.

Example 5.5 Consider the player set $N = \{1, 2, 3, 4, 5, 6, 7\}$ and the directed network on N depicted in Figure 3. This depicts a matching network. The hierarchically higher ranked players are 1–4, while the lower ranked players are 5–7.

For this network both the β -measure as well as the Gately measure assign a zero to the lower ranked players 5–7. Therefore, we only turn to the computation of the two measure for the higher ranked

players 1–4.

The β -measure is based on the addition of locally determined values for each of the lower ranked players. This results in the determination that

$$\beta(D) = (\beta_1, \beta_2, \beta_3, \beta_4) = \left(\frac{1}{3}, \frac{2}{3}, 1\frac{1}{6}, \frac{5}{6}\right)$$

Player 4 is higher ranked than player 2 based on their respective β -measures, since player 4 dominates player 7 instead of player 5, where player 7 has 2 superiors versus player 5 having 3 superiors. In the computed β -measures this difference—which is a local phenomenon—is taken as an indication that player 4 has more power in her relationship with player 7 than player 2 in his relationship with player 5.

The Gately measure is founded on more holistic considerations. Indeed, the lower ranked players 5–7 are assigned an equal weight given by $w = \frac{\#N_2(D)}{\sum_{j \in N_2(D)} \#D^{-1}(j)} = \frac{3}{8}$ and for the higher ranked players this weight is multiplied by the number of players in $N_2(D)$ that each higher ranked player dominates. So, each shared domination relationship is weighted equally.

Now, $N_2(D) = \{5, 6, 7\}$ and $\#D^{-1}(5) = \#D^{-1}(6) = 3$ and $\#D^{-1}(7) = 2$. Hence, $w = \frac{3}{8}$ and

$$\xi(D) = (\xi_1, \xi_2, \xi_3, \xi_4) = \left(\frac{3}{8}, \frac{3}{4}, 1\frac{1}{8}, \frac{3}{4}\right)$$

Note that in the Gately measure, players 2 and 4 are ranked equal. The localised difference between dominating player 5 versus dominating player 7 is not taken into account for the Gately measure. ♦

The previous example shows the mechanism that underlies the Gately measure in comparison with the β -measure on directed networks. The network depicted in Figure 3, however, does not result in a drastic change in the ranking of players induced by these two measures. The next example shows that the two measures can lead to very different player rankings for relatively simple matching networks.

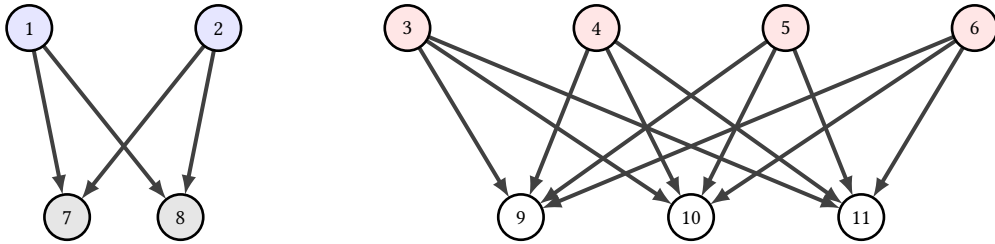


Figure 4: The directed network considered in Example 5.6

Example 5.6 Consider the directed network D on the player set $N = \{1, \dots, 11\}$ as depicted in Figure 4. The network is fully characterised by $D(1) = D(2) = \{7, 8\}$ and $D(3) = D(4) = D(5) = D(6) = \{9, 10, 11\}$.

Note that $N_0(D) = \{1, \dots, 6\}$, $N_1(D) = \emptyset$ and $N_2(D) = \{7, \dots, 11\}$. From this it is easy to compute

that

$$\begin{aligned}\beta_1(D) &= \beta_2(D) = 1 \\ \beta_3(D) &= \beta_4(D) = \beta_5(D) = \beta_6(D) = \frac{3}{4} < 1 \\ \xi_1(D) &= \xi_2(D) = \frac{5}{8} \\ \xi_3(D) &= \xi_4(D) = \xi_5(D) = \xi_6(D) = \frac{15}{16} > \frac{5}{8}\end{aligned}$$

where we use that $\#N_2(D) = 5$ and $\sum_{j \in N_2(D)} \#D^{-1}(j) = 16$.

This example illustrates that, in general, the Gately measure is different from the β -measure for measuring power in directed networks, possibly reversing the power ranking of players. In the example discussed, Players 2 and 3 are ranked higher than Players 3, 4, 5 and 6 for the β -measure. However, for the Gately measure this reverses and Players 3, 4, 5 and 6 are ranked higher than Players 1 and 2. \blacklozenge

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