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## On the Integration of Shapley-Scarf Housing Markets

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# On the Integration of Shapley–Scarf Housing Markets

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## Abstract

We study the welfare consequences of merging disjoint Shapley–Scarf housing markets. We obtain tight bounds on the number of agents harmed by integration and on the size of their losses. We show that, in the worst-case scenario, market integration may harm the vast majority of agents, and that the average rank of an agent’s house can decrease (asymptotically) by 50% of the length of their preference list.

We also obtain average-case results. We exactly compute the expected gains from integration in random markets, where each of the preference profiles is chosen uniformly at random. We show that, on average, market integration benefits all agents, particularly those in smaller markets. Using the expected number of cycles in the top trading cycles algorithm, we bound the expected number of agents harmed by integration. In particular, the expected fraction of agents harmed by integration is less than 50% if each market has the same size and this is below 26 (independent of the number of markets that merge). We conclude by providing a preference domain that ensures that those harmed by market integration are a minority.

*Keywords:* Shapley–Scarf markets, gains from integration, random markets.

*JEL Codes:* C78.

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## 1. Introduction

Shapley–Scarf markets, in which agents own one house each which they can exchange among themselves without using monetary transfers, have been helpful to analyse several real-life allocation problems, such as kidney exchanges involving donor-patient pairs that are incompatible with each other (Roth et al., 2004). A common complication in these allocation problems is that a big market is fragmented into several small and disjoint ones, causing inefficiencies. For example, Agarwal et al. (2019) document that most kidney exchanges in the US are conducted locally, despite the existence of centralized clearinghouses, which if used could increase the number of transplants by up to 63 percent. However, despite these potential large gains from integration, some American hospitals refrain from participating in centralized allocation schemes because they expect to match more of their donor-patient pairs internally (Ashlagi and Roth, 2014).

Motivated by this observation, we investigate theoretically the effects of integrating disjoint Shapley–Scarf markets, where before and after integration occurs the unique core allocation is implemented. In our model, there are  $k$  Shapley–Scarf markets with  $n_j$  agents each ( $n_j$  is potentially different for each market) and  $n$  agents in total. Before integration occurs, each agent obtains their core allocation within their initial market. However, after integration occurs, agents obtain their core allocation in the market where all agents are allowed to trade their initial endowments.

Our first result (Proposition 1) states that up to, but not more than,  $n - k$  agents may be harmed by integration, i.e. they receive a house they prefer more when trade is only allowed within their own disjoint markets. This upper bound holds for any choice of  $n$  and  $k$ . It shows that Shapley–Scarf markets may fail to integrate because doing so could generate more losers than winners. Consequently, if market integration would be decided by a majority vote, our result implies that integration may not be achieved.

Our second result (Proposition 2) concerns the size of the gains from integration in terms of house rank. For example, if an agent receives her 3rd best house before integration, but her 1st best after integration, the size of her gains from integration is  $3 - 1 = 2$ . Even if most agents are harmed by the merge of disjoint markets, integration may still be justified if the size of the gains from integration experienced by a few is substantially larger than the size of the losses from many. We show that, in the worst-case scenario, the size of the average gains from integration may be down to, but not less than,  $\frac{-n^2+n+k^2+k}{2n^2}$ . This lower bound can be achieved for any choice of  $n$  and  $k$ , and shows that, asymptotically, integration may reduce the average assigned house rank by 50% of the size of agents' preference lists.

Taken together, our first two results show that there are real obstacles to the integration of Shapley–Scarf markets. For example, if we have three small markets that merge into one with 60, 30 and 10 agents respectively, up to 97 agents may obtain a worse house after integration occurs, and on average (across all agents) each agent may receive a house 50 positions down on her preference list, equivalent from going from her top choice to her 51st choice.

However, these results are obtained in worst-case scenarios, which occur only when preferences are very specific. Consequently, studying the expected gains from integration across all possible preference profiles may be more informative. Therefore, our third result studies the size of the expected gains from integration in random Shapley–Scarf markets, in which agents’ preferences over houses are drawn uniformly and independently.

In our third result (Proposition 3), we compute the exact expected gains from integration, which equal  $\frac{(n+1)[(n_j+1)H_{n_j}-n_j]}{n_j(n_j+1)n} - \frac{(n+1)H_n-n}{n^2}$ ; ( $H_n$  is the  $n$ -th harmonic number). This result immediately shows that the expected welfare gains from integration are positive for all agents, and larger for agents belonging to smaller markets. Going back to our example of three markets integrating with 60, 30 and 10 agents, the agents of the market with size ten go up 16 positions in their expected house rank, whereas those in the market of size sixty also increase their expected allocated house rank, but only by 2 rank positions. Our third result, which we confirm with the use of simulations, puts some context to our first two propositions, and shows that on average we should expect an overall positive effect from integration in Shapley–Scarf markets for agents from all disjoint markets.

Our fourth result (Proposition 4) establishes a connection between the number of trade cycles that occur in the top trading cycles algorithm before integration and the expected number of agents harmed by integration. We use this connection to show that the expected number of agents harmed by integration in each economy is less than  $n_j - \sqrt{2\pi n_j} + O(\log n_j)$ , and consequently the expected number of agents harmed by integration in the grand economy is smaller than  $n - \sqrt{2\pi}(\sum_{j=1}^k \sqrt{n_j}) + O(\log \prod_{j=1}^k n_j)$ . In our example regarding the integration of markets with size 60, 30 and 10, our result implies that the expected number of agents harmed by integration is less than 59 in the entire economy, and furthermore less than 41, 16, and 2 for each respective market. A direct consequence of our result is that, when all markets are of the same size, the expected fraction of agents harmed by integration is less than 50% whenever each market has less than  $8\pi \approx 25.13$  agents, independent of the number of markets integrating.

A different approach to ensure that integration does not harm a majority

of agents is to focus on specific preference domains. We find a preference domain that achieves this purpose, called sequential dual dictatorship, which enforces a high correlation among agents' preferences. When preferences satisfy this particular property, we can guarantee that no more than 50% of agents in any individual market are harmed by integration (which implies that no more than 50% of the total agents in the grand economy are harmed, see Proposition 5). The sequential dual dictator property is equivalent to assigning the title of dictator to two agents at each step of the top trading cycle algorithm, therefore bounding the length of cycles that can occur.

We conclude the paper by showing that, although implementing the core allocation for the grand economy cannot guarantee not harming any agent (a property which we call integration monotonicity), there is no other matching rule that is clearly better. In particular, there is no matching rule that is integration monotonic, Pareto optimal and strategy-proof (Proposition 6).

*Structure of the paper.* Section 2 discusses the relevant literature. Section 3 introduces our model. Section 4 presents worst-case results. Section 5 presents average-case results. Section 6 discusses preference domains. Section 7 concludes.

## 2. Related Literature

A few other papers study the effects of integration on variations of Shapley–Scarf markets. For example, Ashlagi and Roth (2014) study the incentives for hospitals to fully reveal their patient–donor pairs to a centralized clearinghouse by constructing a model in which the set of agents is partitioned into sets of hospitals. In their model, agents do not have preferences but only dichotomous compatibility restrictions, whereas hospitals have preferences over the number of exchanges involving agents affiliated to them. Thus, welfare is measured by the size of the matching rather than by ranks, and therefore our proof techniques are different and unrelated. They obtain worst- and average-case results that have a similar flavour to ours: the average-case cost for hospitals to fully integrate into a centralized clearinghouse is small, but the worst-case cost is high. In the same framework as them, Toulis and Parkes (2015) propose a mechanism that is efficient and asymptotically individually rational for hospitals.

Another paper that relates to the integration of Shapley–Scarf markets is by Ehlers et al. (2002). They focus on slightly different markets in which agents may own several houses. In this setup, they study allocation rules that are population monotonic, i.e. those in which reducing the set of agents has a positive effect for all agents. Although population monotonicity is a hard

property to satisfy, they characterize the set of rules satisfying population monotonicity, efficiency and strategy-proofness.

Our paper is also related to a series of recent articles that have studied the integration of other types of markets without money, in particular for Gale–Shapley one-to-one matching markets (Ortega, 2018, 2019), Gale–Shapley many-to-one matching markets with applications to school choice (Manjunath and Turhan, 2016; Doğan and Yenmez, 2019; Ekmekci and Yenmez, 2019; Turhan, 2019; Aue et al., 2020), exchange economies (Chambers and Hayashi, 2017, 2020) and networking markets (Gersbach and Haller, 2019; Ghelfi, 2019; Bykhovskaya, 2020). Among these, the closest to ours are Ortega (2018, 2019). He shows that, in Gale–Shapley marriage markets, market integration never harms more agents than it benefits, even though the average rank of an agent’s spouse can decrease by 37.5% of the length of agents’ preference list. He also provides an approximation for the gains from integration in random markets. Although some of our results parallel his for Gale–Shapley marriage markets, ours are substantially better as: i) they provide tight bounds on the welfare losses, ii) they apply to the integration of markets of different sizes, thus providing useful comparative statics and iii) in the case of the gains from integration in random markets, our results are exact rather than approximations.

Our average-case results rely on two seminal papers from the computer science literature regarding random Shapley–Scarf markets with uniform and independent preferences. The first of these, by Frieze and Pittel (1995), computes the expected number of iterations that the top trading cycles algorithm takes to find the unique core allocation and the number of cycles created in the process. The second paper, by Knuth (1996), finds the expected sum of ranks of obtained houses and establishes the equivalence between the core allocation obtained from random endowments and the random serial dictatorship mechanism with no property rights.<sup>1</sup> Che and Tercieux (2019) use a similar random market approach to show that, in a related two-sided model, the top trading cycles algorithm achieves efficiency and stability asymptotically when agents’ preferences are independent, and propose a variation of this algorithm that achieves both properties even when preferences are correlated.

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<sup>1</sup>The latter result was also independently discovered by Abdulkadiroğlu and Sönmez (1998).

### 3. Model

*Preliminary definitions.* We study the housing markets proposed by [Shapley and Scarf \(1974\)](#), where there are  $n$  agents, each of them owning an indivisible good (say a house). The agents have strict ordinal preferences over all houses, including their own, and no agent has any use for more than one house.<sup>2</sup>

Formally, let  $N := \{1, \dots, n\}$  be the set of agents and let  $\omega := \{\omega_1, \dots, \omega_n\}$  be the initial endowment of the market. Let  $\succ_i$  denote the strict preference of agent  $i$  and let  $\succ := (\succ_i)_{i \in N}$ . As usual, the weak preference corresponding to  $\succ_i$  is denoted by  $\succsim_i$ . A *housing market (HM)* is a pair  $(N, \succ)$ . An *allocation*  $x = \{x_1, \dots, x_n\}$  is any permutation of the initial endowment. Allocation  $x$  *dominates* allocation  $y$  if  $\exists S \subseteq N$  such that

1.  $\{x_i : i \in S\} = \{\omega_i : i \in S\}$
2.  $x_i \succ_i y_i \quad \forall i \in S$

The first condition says that coalition  $S$  is *effective* for the allocation  $x$ , and the second condition says that every member of coalition  $S$  strictly prefers  $x$  to  $y$ . A *core allocation* is one that is undominated. An allocation  $x$  is *individually rational* if  $x_i \succsim_i \omega_i$  for all  $i \in N$ . An allocation  $x$  is *Pareto optimal* if, for every alternative allocation  $x'$  such that  $x'_i \succ x_i$  for some  $i \in N$ , there exists some  $j \in N$  for which  $x_j \succ_j x'_j$ . A *matching mechanism*  $\mathcal{M}$  is a map from HMs to allocations, and is said to be a *core one* (resp. *individually rational, Pareto optimal*) if it produces a *core* (resp. *individually rational, Pareto optimal*) allocation for every HM. The mechanism  $\mathcal{M}$  is *strategy-proof* if, for every  $i, \succ'_i, \succ, \mathcal{M}_i(N, \succ) \succsim_i \mathcal{M}_i(N, (\succ'_i, \succ_{-i}))$ .

For any housing market, there is always a unique core allocation that can be found with an algorithm known as *top trading cycles (TTC)* ([Shapley and Scarf, 1974](#); [Roth and Postlewaite, 1977](#)), which works by repeating the following two steps until all agents have been assigned a house.

1. Construct a graph with one vertex per agent. Each agent points to the owner of his top-ranked house among the remaining ones. At least one cycle must exist and no two cycles overlap. Select the cycles in this graph.

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<sup>2</sup>We only consider the case where agents have strict preferences; for an analysis of housing markets with weak preferences, see [Quint and Wako \(2004\)](#); [Alcalde-Unzu and Molis \(2011\)](#); [Aziz and De Keijzer \(2012\)](#); [Jaramillo and Manjunath \(2012\)](#) and [Saban and Sethuraman \(2013\)](#).



2. Permanently assign to each agent in a cycle the object owned by the agent he points to. Remove all agents and objects involved in a cycle from the problem.

TTC is the only mechanism satisfying individual rationality, Pareto-efficiency and strategy-proofness on the strict preference domain (Ma, 1994).

*New definitions.* We study *extended housing markets* (EHM), which consist of a HM and a partition of the set of agents into  $k$  disjoint communities  $C_1, \dots, C_k$ . This is, an EHM is a triple  $(N, \succ, C)$ , where  $C := \{C_1, \dots, C_k\}$ . An *integrated allocation* is any allocation for the HM  $(N, \succ)$ , whereas a *segregated allocation* is an allocation for  $(N, \succ)$  in which every agent receives a house owned by an agent in her own community. This is, a segregated allocation  $x$  is such that  $\{x_i : i \in S\} = \{\omega_i : i \in S\} \forall S \in C$ . A matching scheme  $\sigma$  is a map from EHMs into an integrated and a segregated allocation, denoted by  $\sigma(\cdot, C)$  and  $\sigma(\cdot, \Omega)$ , respectively.<sup>3</sup>

For agent's  $i \in C_j$  preference  $\succ_i$ , we denote its restriction to  $C_j$  by  $\tilde{\succ}_i$ . In other words,  $\tilde{\succ}_i$  is the strict ranking of agent  $i$  on all the houses belonging to agents in community  $C_j$  (including his own) that is consistent with  $\succ_i$ . The matching scheme  $\sigma^*$  is the *core matching scheme* if  $\sigma^*(\cdot, \Omega)$  is the core matching for the HM  $(N, \succ)$  and, for every community  $C_j$ ,  $\sigma^*(\cdot, C_j)$  is the core matching for the HM  $(C_j, \tilde{\succ}_{C_j})$ , where  $\tilde{\succ}_{C_j} := (\tilde{\succ}_i)_{i \in C_j}$ . Example 1 presents the core matching scheme for a EHM.

The *rank* of house  $\omega_h$  in the preference order of agent  $i$  is defined by  $\text{rk}_i(\omega_h) := |\{j \in N : \omega_j \succ_i \omega_h\}|$ . The *gains from integration* for agent  $i$  under the matching scheme  $\sigma$  are defined as  $\gamma_i(\sigma) := \text{rk}_i(\sigma(i, C)) - \text{rk}_i(\sigma(i, \Omega))$ . The *total gains from integration* are given by  $\Gamma(\sigma) := \sum_{i \in N} \gamma_i$ . If these are negative, we speak of the total losses from integration. The *average percentile gains from integration* are denoted by  $\bar{\Gamma}(\sigma) := \frac{\Gamma(\sigma)}{n^2}$ . We divide by  $n^2$  to account for both the number of agents ( $n$ ) and the length of preference list (which is also  $n$ ). Thus,  $\bar{\Gamma}(\sigma) \in (-1, 1)$ , where  $\bar{\Gamma}(\sigma) = -1$  means that everybody was harmed by integration and moved from their best possible house to the worst possible one.

We use  $N^+(\sigma) := \{i \in N : \sigma(i, \Omega) \succ_i \sigma(i, C)\}$  to denote the set of agents who benefit from integration. Similarly,  $N^0(\sigma) := \{i \in N : \sigma(i, \Omega) = \sigma(i, C)\}$  and  $N^-(\sigma) := \{i \in N : \sigma(i, C) \succ_i \sigma(i, \Omega)\}$  denote the set of agents that are unaffected and harmed by integration, respectively. For all  $j \in \{1, \dots, k\}$ , we define  $N_{C_j}^+(\sigma) := \{i \in C_j : \sigma(i, \Omega) \succ_i \sigma(i, C)\}$  to be the set of agents in

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<sup>3</sup>Matching schemes are similar to the concept of assignment schemes in cooperative game theory (Sprumont, 1990).

Example 1: An EHM with  $N = 7$ ,  $k = 2$ ,  $C_1 = \{a, b, c\}$  and  $C_2 = \{d, e, f, g\}$ .  
The integrated (resp. segregated) core allocation appears in a diamond (resp. circle).

a	b	c	d	e	f	g
◊ d	○ c	○ a	◊ a	○ f	○ g	○ d
○ b	a	d	○ e	a	a	a
⋮	d	◊ c	⋮	d	d	c
	g	⋮		b	b	◊ g
	◊ b			c	c	⋮
	⋮			g	◊ f	
				◊ e	e	

community  $C_j$  who benefit from integration. The sets  $N_{C_j}^0(\sigma)$  and  $N_{C_j}^-(\sigma)$  are defined analogously.

Henceforth we focus on  $\sigma^*$ , i.e. we study the gains from integration that occur when the allocation obtained before and after integration occurs is the unique core allocation.

#### 4. Results

Unfortunately, the integration of housing markets may harm the vast majority of agents. In the worst-case scenario, up to  $n - k$  agents are harmed by integration, and this upper bound is tight.

**Proposition 1.** *For any pair  $(n, k)$ , there exists an EHM in which  $|N^-(\sigma^*)| = n - k$ ; whereas there is no EHM in which  $|N^-(\sigma^*)| > n - k$ .*

*Proof.* The EHM in Example 1 illustrates an EHM showing that the  $n - k$  bound is attainable. In such example, there are two communities with three and four agents each, such that one agent from each community (in this case  $a$  and  $d$ ) is assigned to their second best house in the segregated core allocation, whereas all remaining agents are assigned to their most preferred house. However, when both communities integrate,  $a$  and  $d$  exchange their houses, each obtaining their most preferred house, and provoking that all other agents are assigned to their own house (which they prefer less than the segregated core allocation). Keeping this structure, we can extend the construction of this example to arbitrary values of  $n$  and  $k$  to guarantee

that  $n - k$  agents strictly prefer the core segregated allocation to the core integrated one.

To see that the  $n - k$  upper bound is tight, assume by contradiction that more than  $n - k$  agents are harmed by integration, which implies that there is one community in which all agents are harmed by integration, say  $C_j$ . But then  $\sigma^*(\cdot, \Omega)$  is not a core allocation for  $(N, \succ)$ , because any alternative allocation  $x$  such that  $x_i = \sigma^*(i, C) \forall i \in C_j$  dominates it (since  $C_j$  is effective for allocation  $x$  and every agent in  $C_j$  prefers the segregated over the integrated allocation). That the integrated core allocation is not a core allocation is a contradiction, which terminates the proof.  $\square$

Proposition 1 implies that, if market integration is decided via majority voting (where agents who benefit from integration vote in favor of it, and those who are harmed by integration vote against it), integration may not be achieved in Shapley–Scarf housing markets. This is a striking observation, since the integration of Gale–Shapley marriage markets (in which two sets of agents are matched to each other) always benefits more agents than those it harms, and thus it can be implemented via majority voting (see Proposition 2 in Ortega (2018), also Gale and Shapley (1962); Gärdenfors (1975)).

A key property of the EHM in Example 1 that allows us to find  $n - k$  agents who are harmed by integration is that, when computing the segregated core allocation using TTC, there is only one trade cycle in each community. This is an interesting observation to which we will come back later on when studying random markets.

Given the negative result in Proposition 1, we may think that integration can still be justified if the size of the welfare gains experienced by a minority are much larger than the size of the welfare losses suffered by a majority. Unfortunately, in the worst-case scenario, the size of the losses from integration is much larger than the size of the gains from integration. In particular, we show below that the agents' average welfare loss may be negative and asymptotically equivalent to an increase in ranking of 50% of the length of agents' preference list. We provide a tight lower bound on the size of agents' average welfare loss.

**Proposition 2.** *For any pair  $(n, k)$ , there exists an EHM in which  $\bar{\Gamma}(\sigma^*) = \frac{-n^2+n+k^2+k}{2n^2}$ ; whereas there is no EHM in which  $\bar{\Gamma}(\sigma^*) < \frac{-n^2+n+k^2+k}{2n^2}$*

*Proof.* Example 1 shows that our lower bound for  $\bar{\Gamma}(\sigma^*)$  is attainable. The agents who experience welfare gains (namely  $a$  and  $d$ ) go from their second to their first best after integration occurs, obtaining a rank gain of +1. However, agent  $c$  goes from his first to his third best (a change of -2 in rank), agent  $g$  goes from his first to his fourth best (a change of -3 in rank), and so on,

until agent  $e$  who goes from his best to his worst option (a change of -6 in rank). When we add the total welfare losses (+1 + 1 - 2 - 3 - 4 - 5 - 6), we obtain  $-\frac{1}{2}(n^2 - n - k^2 - k) = -18$ . Dividing -18 by  $n^2 = 49$ , we obtain an average welfare reduction of 36.7% of the length of agents' preferences.

We constructed the EHM in Example 1 in such a way that the minimum possible number of agents gain from integration (i.e.  $k$ , per Proposition 1), and that the size of such gains is as small as possible (+1). On the other side, the welfare losses of the remaining  $n - k$  individuals go from -2 to  $-n + 1$  (the largest possible welfare loss). We can replicate such construction for EHMs with arbitrary values of  $n$  and  $k$  to obtain:

$$\bar{\Gamma}(\sigma^*) = \frac{1}{n^2} \left( k * 1 - \sum_{i=1}^{n-k} n - i \right) \quad (1)$$

$$= -\frac{1}{2n^2}(n^2 - n - k^2 - k) \quad (2)$$

Which establishes that our lower bound can be attained for arbitrary values of  $n$  and  $k$ . It is interesting that our lower bound does not depend on the size of each community relative to the size of the whole society. Note that when  $n$  grows and  $k$  remains constant,  $\bar{\Gamma}(\sigma^*) \sim -1/2$ .

We now show that our lower bound for  $\bar{\Gamma}(\sigma^*)$  is tight, with the help of some additional definitions and two auxiliary lemmas. Given a core allocation  $x^*$  for a HM  $(N, \succ)$  and an integer  $r$  such that  $1 \leq r \leq n$ , let  $m(r, x^*) := |\{i \in N : rk_i(x_i^*) = r\}|$ . Similarly, let  $M(r, x^*) := |\{i \in N : rk_i(x_i^*) \geq r\}|$ .

**Lemma 1.** *In any core allocation  $x^*$ ,  $rk_i(x_i^*) \leq rk_i(\omega_i)$ .*

*Proof.* This is a well-known fact due to any core allocation being individually rational.  $\square$

**Lemma 2.** *In any core allocation  $x^*$ ,  $m(r, x^*) \leq n - r + 1$ .*

*Proof.* For  $r = n$ , our lemma says  $m(n, x^*) \leq 1$ . Note that if  $rk_i(x_i^*) = n$ , then  $x_i^* = \omega_i$  because of Lemma 1. Therefore, we cannot have  $m(n, x^*) > 1$ , as otherwise two agents are assigned their own house but they would like to exchange their house with each other, and thus  $x^*$  is not a core allocation.

For  $r = n - 1$ , suppose by contradiction that  $m(n - 1, x^*) > 2$ . Then there exists three agents  $j, l, h$  for which  $rk_i(x_i^*) = n - 1$  for all  $i \in \{j, l, h\}$ . But for each of those agents, there exists a house  $\omega'_i \in \{\omega_j, \omega_l, \omega_h\}$  such that  $\omega'_i \succ_i x_i^*$  and  $\omega'_i \succ_i \omega_i$  for all  $i \in \{j, l, h\}$ . Therefore,  $x^*$  is not a core allocation, since there is a reallocation of houses among  $j, l, h$  that is effective for such coalition and that is strictly preferred.

The same argument applies for any other values of  $r < n - 1$ . Suppose by contradiction that there exist some  $r' \leq n - 1$  such that  $m(r', x^*) > n - r' + 2$ . Then there are  $n - r' + 2$  agents for which  $\text{rk}(x_i^*) = r'$ . But for each of these agents  $i$ , there exists a house  $\omega_j$  belonging to one of these  $n - r' + 2$  agents such that  $\omega_j \succ_i x_i^*$  and  $\omega_j \succ_i \omega_i$ . Therefore,  $x^*$  is not a core allocation, since there is a reallocation of houses among those  $n - r' + 2$  agents that is effective for such coalition and that is strictly preferred. Hence, the argument holds for all  $r$ .  $\square$

**Lemma 3.** *In any core allocation  $x^*$ ,  $M(r, x^*) \leq n - r + 1$ .*

*Proof.* For  $r = n$ , the statement in Lemma 3 is the same as in Lemma 2. For  $r = n - 1$ , assume by contradiction that  $M(n - 1, x^*) > 2$ . By Lemma 2 we cannot have that two agents are allocated a house ranked  $n$  for both, or that three agents are allocated a house ranked  $n - 1$ . Thus, it must be that one agent gets a house ranked  $n$  (agent  $j$ ) and two agents get a house ranked  $n - 1$  (agents  $h$  and  $l$ ). Then we have  $x_j = \omega_j$  by Lemma 1. Furthermore, for  $i \in \{h, l\}$ , there are two houses  $x'_i, x''_i \in \{\omega_j, \omega_h, \omega_l\}$  such that  $x'_i \succ_i x_i$  and  $x_i \succ_i x''_i$ , where  $x'_i \neq \omega_i$  per Lemma 1. If, for either agent  $h$  or  $l$ ,  $x'_i = \omega_j$ , then  $j$  and such agent would like to exchange their endowments and would be strictly better off, and thus  $\text{rk}_h(\omega_j) = \text{rk}_l(\omega_j) = n$ . But because  $\text{rk}_h(x_h) = \text{rk}_l(x_l) = n - 1$ , they must be getting their own houses, i.e.  $x_h = \omega_h$  and  $x_l = \omega_l$ . But then, agents  $h$  and  $l$  are better off by trading their endowments, and thus  $x^*$  is not a core allocation, a contradiction. The same argument applies for all other values of  $r < n - 1$ .  $\square$

Armed with these three auxiliary lemmas, we are ready to prove that  $\bar{\Gamma}(\sigma^*) \geq \frac{-n^2 + n + k^2 + k}{2n^2}$ . By Proposition 1, at most  $n - k$  people may experience negative gains from integration. These are defined, for each agent  $i$ , as  $\gamma_i(\sigma^*) := \text{rk}_i(\sigma_i^*(i, C)) - \text{rk}_i(\sigma_i^*(i, \Omega))$ . To make  $\gamma_i(\sigma_i^*)$  as small as possible, we need to fix  $\sigma_i^*(i, C) = 1$  and make  $\sigma_i^*(i, \Omega)$  as large as possible. But Lemma 3 shows that  $\sigma_i^*(i, C) = n$  for at most one agent,  $\sigma_i^*(i, C) \geq n - 1$  for at most two agents, and so on. Thus, in the worst case scenario, the sum of the welfare gains from integration among those  $n - k$  agents equals

$$-\sum_{i=1}^{n-k} (n - i) = \frac{-n^2 + n + k^2 - k}{2} \quad (3)$$

Similarly, the smallest positive gains from integration for the remaining  $k$  agents (which must exist by Proposition 1) are equal to 1. Thus, the

smallest possible value for  $\bar{\Gamma}(\sigma^*)$  is

$$\bar{\Gamma}(\sigma^*) = -\frac{1}{2n^2}(n^2 - n - k^2 - k) \quad (4)$$

□

Proposition 2 can be compared to an analogous result in Gale–Shapley marriage markets. The average welfare gains may also be negative in Gale–Shapley markets, but only up to 37.5% of the length of preference lists (Ortega, 2019). Taken together, Propositions 1 and 2 show that the integration of Shapley–Scarf housing markets can be hard to achieve, and in particular is more difficult to obtain (in the worst-case scenario) than in Gale–Shapley marriage markets.

## 5. Random Housing Markets

In the previous section we found out two negative results regarding the integration of housing markets; however both results are about worst-case scenarios. Whereas these results are interesting, one may argue that these are knife-edge scenarios, and wonder whether the integration of housing markets would generate welfare gains on average.

To answer this question, we study *random housing markets* (RHM). Given a set of agents, a RHM is generated by drawing a complete preference list for each agent independently and uniformly at random. Similarly, a *random extended housing market* (REHM) is a RHM where the set of agents is partitioned into disjoint communities  $C_1, \dots, C_k$ , each of size  $n_1, \dots, n_k$  (where  $n = n_1 + \dots + n_k$ ). Random housing markets were first studied by Frieze and Pittel (1995) and Knuth (1996). The latter proved the following seminal result.

**Lemma 4 (Knuth, 1996).** *In a RHM,  $\mathbb{E}(\sum_{i=1}^n rk_i(x_i^*)) = (n+1)H_n - n$ , where  $H_n$  is the  $n$ -th harmonic number, i.e.  $H_n := \sum_{i=1}^n \frac{1}{i}$ .*

We can use Knuth’s theorem to find the expected size of the average welfare gains in REHMs. Let us define the *total gains from integration for community  $C_j$*  as  $\Gamma_{C_j}(\sigma) := \sum_{i \in C_j} \gamma_i$ . The *average percentile gains from integration for community  $C_j$*  are denoted by  $\bar{\Gamma}_{C_j}(\sigma) := \frac{\Gamma_{C_j}(\sigma)}{n n_j}$ . We divide by  $n_j$  to take the average across all agents in community  $C_j$ , and by  $n$  to normalize by the length of agents’ preference lists. Equipped with these new definitions, we can show that the average gains from integration are positive for agents belonging to any community.

**Proposition 3.**  $\mathbb{E}[\bar{\Gamma}_{C_j}(\sigma^*)] = \frac{(n+1)[(n_j+1)H_{n_j}-n_j]}{n_j(n_j+1)n} - \frac{(n+1)H_n-n}{n^2}$ .

*Proof.* For any  $i \in C_j$  and any community  $C_j$ , define the *relative rank* of house  $\omega_h$  in the preference order of agent  $i$  by  $\hat{\text{rk}}_i(\omega_h) := |\{l \in C_j : \omega_l \succ_i \omega_h\}|$ . This is, the relative rank indicates the position of a house in an agent's preference ranking compared *only* to houses owned by other agents belonging to the same community. Knuth's result directly implies that

$$\mathbb{E}\left[\sum_{i=1}^n \text{rk}_i(\sigma^*(i, \Omega))\right] = (n+1)H_n - n, \text{ and} \quad (5)$$

$$\mathbb{E}\left[\sum_{i=1}^{n_j} \hat{\text{rk}}_i(\sigma^*(i, C_j))\right] = (n_j+1)H_{n_j} - n_j, \forall j \in \{1, \dots, k\} \quad (6)$$

So that before integration, agents are assigned to a house relatively ranked  $(n_j+1)H_{n_j} - n_j$ . To complete the proof, we need to figure out in which position is such house are in the absolute rank of all houses (i.e. convert the relative rank into the full rank). To do so, suppose that a house assigned to an agent in a segregated allocation has a relative rank  $q$ . A randomly chosen house, belonging to an agent from another community, could be better ranked than house 1, between houses 1 and 2, ..., between houses  $q-1$  and  $q$ , and so on. Therefore, a random house belonging to another agent is in any of those gaps with probability  $\frac{1}{n_j+1}$  and thus has  $\frac{q}{n_j+1}$  chances of being more highly ranked than the house with relative ranking  $q$ . There are  $(n - n_j)$  houses from other communities. On average,  $\frac{q(n-n_j)}{n_j+1}$  houses will be better ranked. Furthermore, there were already  $q$  houses in his own community ranked better than it. This implies that its expected ranking is  $q + \frac{q(n-n_j)}{n_j+1} = \frac{q(n+1)}{n_j+1}$ . Substituting  $q$  for the expression obtained in equation (6), we obtain

$$\mathbb{E}[\bar{\Gamma}_{C_j}(\sigma^*)] = \frac{(n+1)[(n_j+1)H_{n_j}-n_j]}{n_j(n_j+1)n} - \frac{(n+1)H_n-n}{n^2} \quad (7)$$

□

Simulation exercises confirm the correctness of Proposition 3, which has an interesting implication.

**Corollary 1.** *The expected welfare gains from integration are positive for all agents, and higher for agents in smaller communities.*

For example, if we merge three Shapley–Scarf markets of size 60, 30 and 10, the corresponding welfare gains in terms of house rank are 1.9828, 5.9525

and 16.1615, i.e. agents from the market with only 10 agents improve the ranking of their assigned house by 16 positions, whereas those in the market with 60 agents only improve theirs by 2 positions. In percentile terms, agents from the smallest market improve the rank of their assigned house by 16% of the length of their preference list, whereas agents from the largest market increase their corresponding rank only by 2% of the length of their preference list.

Now we study the size of the set of agents who are harmed by integration in each community, i.e.  $N_{C_j}^-(\sigma^*)$ . To do so, we first note that the number of agents harmed by integration crucially depends on the number of trading cycles generated by the TTC algorithm when computing the segregated core allocation. Then, we use a result by [Frieze and Pittel \(1995\)](#) that establishes the expected number of trading cycles generated by TTC in random markets.

**Proposition 4.**  $\mathbb{E}[N_{C_j}^-(\sigma^*)] \leq n_j - \sqrt{2\pi n_j} + O(\log n_j)$ .

*Proof.* To start, we prove an auxiliary lemma relating the number of trading cycles in TTC for the segregated markets to the number of agents harmed by integration. For any community  $C_j$ , let  $t_j$  be the number of cycles obtained by TTC when computing the segregated core allocation  $\sigma^*(\cdot, C_j)$ , and let  $t := \sum_{i=1}^k t_j$ .

**Lemma 5.** *In any EHM,  $|N_{C_j}^-(\sigma^*)| \leq n_j - t_j$ .*

*Proof.* In any cycle obtained by TTC when computing the segregated core allocation  $\sigma^*(\cdot, C_j)$ , we must either have that all agents in the cycle are in  $N_{C_j}^0(\sigma^*)$  or that at least one agent is in  $N_{C_j}^+(\sigma^*)$ . Otherwise there is a cycle (involving a set of agents  $S$ ) with at least one agent in  $N_{C_j}^-(\sigma^*)$  and with no agent in  $N_{C_j}^+(\sigma^*)$ . Such a combination cannot occur. If all agents in the cycle are in  $N_{C_j}^-(\sigma^*)$ , then those agents are clearly a blocking coalition to the integrated core allocation. If some agents are in  $N_{C_j}^-(\sigma^*)$  and some in  $N_{C_j}^0(\sigma^*)$ , then when we run TTC to find the integrated core allocation, there is an agent  $i \in N_{C_j}^-(\sigma^*)$  who is pointed by an agent  $h \in N_{C_j}^0(\sigma^*)$ , i.e.  $h$ 's assignment does not change (it is  $\omega_i$  before and after integration) but the one of  $i$  becomes worse. But when we run TTC,  $i$  points to the agent owning the best house available. Now, if  $\sigma^*(i, C)$  is no longer available, it means that its owner exited in an earlier cycle during TTC, and thus she must have received a better house, and thus there is an agent in  $N_{C_j}^+(\sigma^*)$ , a contradiction.  $\square$

Then, we present Theorem 2 in [Frieze and Pittel \(1995\)](#) regarding the expected number of cycles in random housing markets. Let  $t'$  denote the



number of cycles formed during the execution of TTC in a RHM with  $n'$  agents.

**Lemma 6** (Frieze and Pittel, 1995).  $\mathbb{E}[t'] = \sqrt{2\pi n'} + O(\log n')$ .

Frieze and Pittel's result directly implies that, in a REHM

$$\mathbb{E}[t_j] = \sqrt{2\pi n_j} + O(\log n_j) \quad (8)$$

Putting Lemma 5 and equation (8) together, we directly obtain the proof of our result.  $\square$

A direct implication of Proposition 4 is a bound on the expected total number of agents harmed by integration.

**Corollary 2.**  $\mathbb{E}[N^-(\sigma^*)] \leq n - \sqrt{2\pi}(\sum_{j=1}^k \sqrt{n_j}) + O(\log \prod_{j=1}^k n_j)$ .

Proposition 4 is our only bound that is not tight, but is nevertheless informative. Returning to our example of a EHM divided into three communities of sizes 60, 30 and 10, Proposition 4 tells us that, on average, the TTC algorithm generates around 25 trade cycles when computing the segregated core allocation. Moreover, in each of those cycles, at least one person is not harmed by integration. Consequently, at most 59 agents can be harmed by integration. But in fact Proposition 4 says more: it tells us the distribution of agents harmed by integration across communities. Thus, in the market of size 60, the expected number of agents harmed by integration is smaller than 41. Similarly, for the markets of size 30 and 10, the expected number of agents harmed by integration is smaller than 16 and 2, respectively.

A corollary of Proposition 4 is that, whenever all communities have the same number of agents  $n_1$ , market integration never harms more than half of the total population if  $n_1 < 26$ . This is an interesting implication because it holds irrespective of the number of markets that merge.

**Corollary 3.** *If  $n_1 = \dots = n_k$ , then  $\mathbb{E}[N^-(\sigma^*)] \leq \frac{n}{2}$  if  $n_1 < 8\pi \approx 25.13$ .*

*Proof.*

$$\mathbb{E}[N^-(\sigma^*)] = kn_1 - \sqrt{2\pi}(\sqrt{n_1} + \dots + \sqrt{n_1}) \quad (9)$$

$$= k(n_1 - \sqrt{2\pi n_1}) \quad (10)$$

$\mathbb{E}[N^-(\sigma^*)]$  is weakly less than  $n/2$  if

$$k(n_1 - \sqrt{2\pi n_1}) < \frac{kn_1}{2} \quad (11)$$

$$\implies n_1 < 8\pi \quad (12)$$

□

## 6. Specific Preference Domains

Although uniform and independent preferences are the most natural and simple preferences to consider in random markets, it is well known that in real-life applications such as kidney exchange, agents' preferences are strongly correlated, with some "houses" being particularly desired by most agents. In this section, we show that if we impose a particular type of correlation structure in agents' preferences, we can guarantee that no more than half of the total population of agents is harmed by integration.

To do so, let  $q(r, \tilde{\succ}_{C_j})$  be the set of agents in community  $C_j$  placed at rank  $r$  by any agent in their own community (including themselves) in preference profile  $\tilde{\succ}_{C_j}$ . This is, for any positive integer  $r$  and any  $j \in \{1, \dots, k\}$ ,  $q(r, \tilde{\succ}_{C_j}) := \{i \in C_j : \exists h \in C_j : \text{rk}_j(\omega_i) = r\}$ . Similarly, let  $Q(r, \tilde{\succ}_{C_j}) := \sum_{t=1}^r q(t, \tilde{\succ}_{C_j})$  be the set of agents in community  $j$  placed at rank  $r$  and above.

Now we introduce the property that will ensure market integration, which we call *sequential dual dictator* (the name choice will become obvious later on).<sup>4</sup>

**Definition 1** (Sequential dual dictator property). A preference profile  $\succ$  satisfies the sequential dual dictator property if, for any positive integer  $r$  and  $\forall j \in \{1, \dots, k\}$ , each of their corresponding preference restriction  $\succ_{C_j}$  satisfies

$$|Q(r, \tilde{\succ}_{C_j})| \leq r + 1$$

In Example 2 we show that the preference profile in Example 1 does not satisfy this property and provide a preference profile that does. In Example 1,  $|Q(1, \tilde{\succ}_{C_1})| = |\{b, c, a\}| > 2$ , violating the sequential dual dictator property. Similarly,  $|Q(1, \tilde{\succ}_{C_2})| = |\{e, f, g, d\}| > 2$ . In contrast, in the example on the right,  $|Q(1, \tilde{\succ}_{C_1})| = |\{c, a\}| \leq 2$ ,  $|Q(1, \tilde{\succ}_{C_2})| = |\{e, f\}| \leq 2$  and  $|Q(2, \tilde{\succ}_{C_2})| = |\{e, f, d\}| \leq 3$ . Whenever preferences satisfy the sequential dual dictator property, we can guarantee that no more than half of the agents in each community are harmed by integration. Note that, unlike the result in Proposition 3, here we bound the number of agents harmed by integration in every EHM, instead of the expected number of agents harmed by integration across all REHMs.

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<sup>4</sup>Troyan (2019) introduce this property in a two-sided extension of a Shapley–Scarf market. He imposes an acicliety condition for houses' priorities which he shows is equivalent

Example 2: The preference profile on the right satisfies the sequential dual dictator property, unlike the one on the left.

a	b	c		d	e	f	g
b	c	a		e	f	g	d
⋮	⋮	⋮		⋮	⋮	⋮	⋮

a	b	c		d	e	f	g
c	c	a		e	f	f	e
b	b	c		f	d	e	d
a	c	b		d	g	g	f
				g	e	d	g

**Proposition 5.** *If  $\succ$  satisfies the sequential dual dictator property, then  $|N_{C_j}^-(\sigma^*)| \leq \frac{n_j}{2}$ .*

*Proof.* To complete the proof, we examine the number and length of trading cycles generated by the TTC algorithm when computing the segregated core allocation  $\sigma^*(\cdot, C_j)$  for community  $C_j$ . At the first iteration, all agents point to the owner of their most preferred house, and if the sequential dual dictator property is satisfied, there are only two vertices with a positive in-degree. A trade cycle is created, either of those agents pointing to themselves or pointing at each other, and therefore each cycle created in the first iteration of TTC has length at most 2. In the second iteration, at most two agents have positive in-degree (because at least one agent was removed in the first iteration). Either one or two cycles are formed in iteration 2, and they have length of at most 2. The argument repeats for each iteration: each trade cycle has length at most 2.

Now we invoke an argument that we used in the proof of Lemma 5, showing that in any cycle, we must either have that all agents are in  $N^0(\sigma^*)$  or that at least one agent is in  $N_{C_j}^+(\sigma^*)$ . We have showed that there are at least  $n_j/2$  cycles in each community. Therefore,  $|N_{C_j}^-(\sigma^*)| \leq \frac{n_j}{2}$ .  $\square$

Note that one particular case of preference profiles satisfying the sequential dual dictator property are those in which all agents have the same preferences. Such preferences has been extensively studied in Gale–Shapley marriage markets because they guarantee the uniqueness of the core allocation and ensure that truth-telling is a Nash equilibrium of the revelation game induced by any stable mechanism (Gusfield and Irving, 1989; Eeckhout, 2000; Consuegra et al., 2013).

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to a sequential dual dictatorship. He uses the dual dictatorship property to characterize the obvious strategy-proof implementation of TTC.

The sequential dual dictator property bears some resemblance to the serial dictatorship allocation rule (or more generally, hierarchical exchange rules as in Pápai (2000); Ehlers et al. (2002)), where there exists a hierarchy of the agents specified a priori and agents choose their preferred object according such hierarchy. When preference profiles satisfy the sequential dual dictator property, we choose one or two agents among the remaining agents at each step of the procedure (as opposed to always choosing one agent in serial dictatorships), and allocate the favourite remaining object to the agent if he is chosen alone, and to one or both agents if chosen in a pair. In the latter case, if one agent does not receive his first-ranked house among the remaining ones (which means that it is the other chosen agent’s favourite object as well, and it has been allocated to him), then he receives his second-ranked object. In other words, while we have a single dictator at each stage of the procedure for serial dictatorships, the preference profiles that we have described allow the choice of dual dictators as well as ordinary (single) dictators at any given stage of the procedure. Of course, in our case the dictators are determined endogenously by agents’ preferences and cannot be chosen arbitrarily.

## 7. Concluding Comments

Market integration can be a difficult goal to achieve in practice. Here, we have showed that market integration is also challenging to achieve theoretically by showing that two natural welfare measures can substantially decrease after integration occurs in Shapley–Scarf markets. Nevertheless, our average-case results show that losses from integration are rare, particularly when the size of the individual markets that integrate is small or preferences exhibit a particular correlation structure.

Throughout this paper, we have focused on the welfare effects of integration assuming that the core allocation is implemented before and after integration occurs. As we have seen, choosing the core allocation in the integrated market may harm a substantial number of agents, and thus implementing a different allocation that harms no one may be a sensible alternative. We conclude the paper with a description of one mechanism that achieves such purpose efficiently.

The *double TTC* mechanism implements the allocation obtained by TTC in each segregated market, which then uses as the agents’ endowment to run TTC on the global market. Although the double-TTC mechanism is Pareto optimal<sup>5</sup> and assigns to each agent a house weakly better than the one they obtain in the segregated core allocation (a property which we call *integration*

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<sup>5</sup>To see that double TTC is Pareto optimal, recall that Abdulkadiroğlu and Sönmez

*monotonicity*), the double-TTC has an important drawback: it fails to be strategy-proof, as we show in Example 3. Unfortunately, Ma’s seminal result implies that there is no mechanism that satisfies integration monotonicity, Pareto optimality and strategy-proofness, since integration monotonicity implies individual rationality. Thus, even if the core allocation rule is not perfect to achieve integration, no better alternative is in sight. We summarize our final findings in Proposition 6.

**Proposition 6.** *The double TTC mechanism is integration monotonic and Pareto optimal, but not strategy-proof. There is no matching mechanism that is integration monotonic, Pareto optimal and strategy-proof.*

Example 3: In the EHM on the left, agent  $c$  has an incentive to manipulate the double TTC mechanism by pretending to have the preferences on the right (the double TTC allocation appears in circles).

a	b	c
ⓐ	ⓐ	a
b	c	ⓑ
a	b	c

a	b	c
ⓐ	a	ⓐ
b	c	b
a	ⓑ	c

A possible way to relax this impossibility result is focusing on smaller preference domains, where several mechanisms other than TTC satisfy Pareto optimality, individual rationality and strategy-proofness (Bade, 2019). We leave this conjecture for future research.

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(1998) show that the core from random endowments is ex-post Pareto optimal. The allocation derived by running TTC in each community can be considered as of one of such random endowments, and therefore double TTC is Pareto optimal.

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