

Mataramvura, Sure

**Article**

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Cogent Economics & Finance

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Taylor & Francis Group

*Suggested Citation:* Mataramvura, Sure (2019) : An optimal reinsurance management and dividend payout strategy when the insurer's reserve is an Ito-Levy process, Cogent Economics & Finance, ISSN 2332-2039, Taylor & Francis, Abingdon, Vol. 7, Iss. 1, pp. 1-11, <https://doi.org/10.1080/23322039.2019.1698939>

This Version is available at:

<https://hdl.handle.net/10419/270705>

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To cite this article: Sure Mataramvura | (2019) An optimal reinsurance management and dividend payout strategy when the insurer's reserve is an Ito–Levy process, Cogent Economics & Finance, 7:1, 1698939, DOI: [10.1080/23322039.2019.1698939](https://doi.org/10.1080/23322039.2019.1698939)

To link to this article: <https://doi.org/10.1080/23322039.2019.1698939>



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Published online: 10 Dec 2019.



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Received: 18 February 2019  
Accepted: 21 November 2019  
First Published: 30 November 2019

\*Corresponding author: Sure Mataramvura, Division of Actuarial Science, University of Cape Town, South Africa  
E-mail: [sure.mataramvura@uct.ac.za](mailto:sure.mataramvura@uct.ac.za)

Reviewing editor:  
David McMillan, University of Stirling, UK

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## FINANCIAL ECONOMICS | RESEARCH ARTICLE

# An optimal reinsurance management and dividend payout strategy when the insurer's reserve is an Ito–Levy process

Sure Mataramvura\*

**Abstract:** We solve the problem of an insurer who decides to optimally allocate a proportion  $(1-a(t))$  of premiums to a re-insurance company (thereby retaining a proportion  $a(t)$  of premiums) and who also has to optimally pay dividends  $c(t)$  at any time  $t$  to shareholders. If the insurer's reserve  $x(t)$  is a given Ito–Levy process and the shareholders' preferences are given by a general constant relative risk aversion utility function, we set the problem as a stochastic control problem and solve the resulting HJB equation. Results are discussed in detail and show that it is optimal that premiums and dividends be directly proportional to the reserve. High premium lead to high reserves which in turn leads to high premium payouts. This paper contributes to the rich area of stochastic control and also it helps insurers whose reserves can be modelled, to make technical decisions of how to charge premiums, reinsure liabilities and pay dividends to shareholders.

**Subjects:** Statistics & Probability; Economics; Finance

**Keywords:** optimal premium; HJB equation; reinsurance; optimal dividend  
**MR Subject classifications:** 60G51; 60HXX; 60J65; 49J20; 62P05

### ABOUT THE AUTHOR

The author, Professor Sure Mataramvura, is an Associate Professor in the Department of Actuarial Science at the University of Cape Town in South Africa. He is also the current Deputy Dean of Research and Internationalization in the Commerce Faculty. With a Ph.D in Mathematics of Finance, his main research interests focusses on risk minimization, Mathematics of insurance and general topics of applications of stochastic calculus to finance which include derivative pricing and hedging.

### PUBLIC INTEREST STATEMENT

This paper investigates how an insurer can charge premiums, reinsure some of its liability and pay dividends to investors. The insurer, Ås reserve process whose model is prescribed, depends on premiums and dividend payments. Results show that the risk retention level (optimal premiums) and dividend payouts depend linearly on the reserve process. Therefore, an increase in premiums results in an increase in reserves which in turn increases the dividend payouts. The problem has been solved using stochastic control methods applicable to cases like these where the reward function is continuous. The risk retention level and dividend payout also depend on the investor, Ås level of risk aversion and bank interest rates. It turns out that the choice of constant relative risk aversion (CRRA) utility type indicates that a low relative risk aversion and high interest rates imply demands for high dividend payouts from investors.

## 1. Introduction

The motivation of this research stems from De Finetti (1957) where the feeling among researchers is that insurance companies can maximize dividend payouts to shareholders through reinsurance. This then has to be done in an optimal way. We solve a problem of an insurance company with known reserve process and which can control the proportion  $a(t)$  of its premiums (and claims) while the proportion  $1 - a(t)$  is held by a reinsurance company. The idea of constructing the reserve of the insurance company stems from Cramér—Lundberg discrete time model with current extensions to continuous time models like we have. The emphasis then was on ruin probability arguments. This paper considers a reserve process which is a Lévy- Itô process with a safety loading on the premium. A similar model with no jumps exists in literature (see, eg, Yong and Zhou (1999) or Marufu (2014)). The insurance company is also responsible for the payment of a dividend  $c(t)$  to its shareholders who have a CRRA utility function. The choice of the utility function together with the addition of the jumps makes this research different from previous work. An optimal premium and optimal dividend payout are found which maximize the shareholder’s expected cumulative utility of dividend payment up to bankruptcy or insolvency. In Højgaard and Taskar (1999) the authors consider a Brownian motion with drift as model for reserve process and a completely different utility function (linear in their case) while in Taksar and Zhou (1998) the reserve process is similar to ours on drift and diffusion part while the dividend payout structure and utility functions are different plus there are no jumps. Our utility function is of a generalized Constant Absolute Risk Aversion (CARA) which in part is a power utility function and, on the other hand, a logarithmic utility function. The optimal risk allocation (premium sharing with the reinsurance company) and dividend payout are obtained by use of dynamic programming which is preferred to the other common maximum principle since our control is Markovian. One surprising result is that the optimal premium and dividend payout are independent of time  $t$ , but both are proportional to the reserve.

Our contribution in this paper is threefold, first, to the best of our knowledge this is the first time the reserve process of the type used has been used for a study of this type, second, the consideration of the investor’s general CRRA utility type is the first time it has been applied to a reinsurance-dividend payout problem and third we have managed to prove that the value function in our stochastic control setup is independent of time, an important contribution to any future studies. Other important results discussed at the end are also new results for this study. Our paper is structured as follows: the next section looks at the model framework after which we solve the HJB equations in the next section which has the main results of this paper. Each result is discussed with respect to implications to insurance-reinsurance strategy. Finally, we conclude.

## 2. The model

Consider a time horizon  $T \in [0, \infty)$  and  $t \in [0, T]$  be a starting time of  $x(s)$ , ( $s \in [t, T]$ ), where  $x(s)$ , the insurer’s reserve is given by

$$dx(s) = (a(s)\mu - c(s))ds + a(s)dB(s) + \int_{\mathbb{R}} \gamma a(s)z\tilde{N}(ds, dz) \tag{2.1}$$

$$x(t) = x > 0$$

Here  $0 \leq a(s) \leq 1$  is the insurer’s premium, such that  $1 - a(s)$  is the reinsurance’s portion of premiums and claims,  $\mu$  is the “safety loading” and  $c(s)$  is the dividend payout to shareholders of the company. Also  $B(s)$  is a standard Brownian motion and  $\tilde{N}(ds, dz)$  is a Poisson random measure independent of  $B(s)$  (see Øksendal & Sulem, 2007 for more treatment of Lévy processes). We consider  $\nu$  to be the Lévy measure of  $x(s)$  satisfying  $\int_{\mathbb{R} \setminus \{0\}} (1 \wedge z^2)\nu(dz) < \infty$ , ie,  $x(\cdot)$  has paths of bounded quadratic variation and such that  $\int_{|z| \geq 1} |z|\nu(dz) < \infty$  (meaning  $x(s)$  has on average a finite number of jumps of size greater than 1 by unit time). Observe that the threshold of 1 is just arbitrary. Any other threshold, eg,  $R > 0$  could be chosen.

Consider a probability space  $(\Omega, \mathcal{F}, P, \mathcal{F}_s)$  where  $\mathcal{F}_s$  is the natural filtration of  $x(s)$ . We assume that both  $a(s)$  and  $c(s)$  are non-anticipative. Define the control  $\pi(s) = (a(s), c(s))$  and let  $\mathcal{A}(t)$  be the set of all admissible controls  $\pi$  such that (2.1) has a unique strong solution  $x(s)$  (in reality we are happy with Markov controls which are enough for our problem).

We assume that the shareholder's utility function over dividend payouts is  $U(c) = \begin{cases} \frac{c^\beta}{\beta}, & \text{if } 0 < \beta < 1 \\ \ln(c) & \text{if } \beta = 0 \end{cases}$

Note that, for  $\beta \neq 0$ , the case  $\beta = 1 - \ln|\varepsilon|$  ( $\varepsilon < 0.1$ ) was studied in Kravych (1992). In this case,  $\varepsilon$  is defined as the maximal accepted level of the insurer's ruin probability as demanded by the regulatory authority. This would restrict our case to  $0 < \beta < 1$ . Our choice of utility function is to have a utility function that exhibits constant relative risk aversion (CRRA) for all  $0 \leq \beta < 1$ .

Define  $\tau = \inf \{s \geq t : x(s) = 0\}$  as the bankruptcy time where  $\tau = \infty$  is equivalent to saying that almost surely the company never being bankrupt. Note that  $(\tau, x(\tau)) \in \partial\mathcal{D}^*$  where  $\mathcal{D}^* = [0, T] \times \mathbb{R}^+$ . In that case, the boundary of  $\mathcal{D}^*$  is  $\partial\mathcal{D}^* = ([0, T] \times \partial\mathbb{R}^+) \cup (\{T\} \times \mathbb{R}^+)$  so either the insurer will become bankrupt before terminal time in which case  $\tau$  is the first exit time of  $x(s)$  from  $\mathbb{R}^+$  or the insurer will remain almost surely solvent till maturity in which case  $x(s) > 0$  for all  $s \in [t, T]$ .

$$J(t, x, \pi) = E \left[ \int_t^T e^{-rs} U(c(s)) ds \right] \text{ and} \tag{2.2}$$

$$V(t, x) = \sup_{\pi \in \mathcal{A}(t)} J(t, x, \pi) \tag{2.3}$$

Here  $J(\cdot, \cdot, \cdot)$  is the reward function which gives the shareholder's discounted expected cumulative dividends payout up to bankruptcy and  $V(t, x)$  is the value function which gives optimal reward. The problem is to find an optimal control  $\pi^*(\cdot) = (a^*(\cdot), c^*(\cdot))$  and optimal reward  $V(t, x)$  such that  $V(t, x) = J(t, x, \pi^*)$ .

**Proposition 2.1**  $V(t, x)$  defined on (2.3) is a concave function in  $x$ .

PROOF.

We follow the same argument as in Taksar and Zhou (1998) albeit in a jump case. Let  $\pi_1(\cdot) = (a_1(\cdot), c_1(\cdot))$  and  $\pi_2(\cdot) = (a_2(\cdot), c_2(\cdot))$  be two policies admissible at  $x_1(\cdot)$  and  $x_2(\cdot)$  respectively where  $x_1(s)$  and  $x_2(s)$  are two solutions of (2.1). Let also  $\lambda \in \mathbb{R}$  be such that  $0 \leq \lambda \leq 1$ . Fix  $t \in [s, \tau]$  and define

$$x_3(\cdot) = \lambda x_1(\cdot) + (1 - \lambda)x_2(\cdot)$$

and

$$\pi_3(\cdot) = (a_3(\cdot), c_3(\cdot)) = \lambda(a_1(\cdot), c_1(\cdot)) + (1 - \lambda)(a_2(\cdot), c_2(\cdot))$$

Then

$\pi_3(\cdot) \in \mathcal{A}(\cdot)$ . Let also  $\tau_1, \tau_2$  and  $\tau_3$  be respectively bankruptcy times for  $x_1(\cdot), x_2(\cdot)$  and  $x_3(\cdot)$ , then (2.1) implies that

$J(t, x_3, a_3(\cdot), c_3(\cdot)) = \lambda J(t, x_1, a_1(\cdot), c_1(\cdot)) + (1 - \lambda)J(t, x_2, a_2(\cdot), c_2(\cdot))$ . Taking supremum both sides, we get

$$V(t, \lambda x_1 + (1 - \lambda)x_2) \geq \lambda V(t, x_1) + (1 - \lambda)V(t, x_2).$$

Note also that  $\tau_3 = \tau_1 \wedge \tau_2$  □

### 3. The HJB equations and main results

We now setup the HJB equations. For a more generalized setting, see Øksendal and Sulem (2007). For our case, we define the domain

$\mathcal{D} = \{(s, x) : t \leq s \leq \tau; 0 < x \leq x_0\}$  as the “continuation region”, for some  $x_0$  to be determined. From Øksendal and Sulem (2007), and for  $\phi(t, x) \in C^{1,2}(\mathcal{D})$  using the convention

$$D_x \phi(t, x) = \frac{d\phi(t, x)}{dx} = \phi_x(t, x) \text{ and } D_{xx} \phi(t, x) = \frac{d^2 \phi(t, x)}{dx^2} = \phi_{xx}(t, x) \text{ we get that the generator of the controlled process } y(t) = \begin{pmatrix} s+t \\ x(t) \end{pmatrix} \text{ with } y(0) = \begin{pmatrix} t \\ x \end{pmatrix} \text{ is}$$

$$A^\pi \phi(t, x) = \phi_t(t, x) + (a\mu - c)\phi_x(t, x) + \frac{1}{2} a^2 \phi_{xx}(t, x) + \int_{\mathbb{R}} \{\phi(t, x + \gamma az) - \phi(t, x) - \gamma az \phi_x(t, x)\} \nu(dz).$$

The HJB equation is (see Øksendal & Sulem, 2007 for this specific control problem)

$$\sup_{\pi \in \mathcal{A}} \{V_t(t, x) + \mathcal{H}(t, x, \pi, D_x V(t, x), D_{xx} V(t, x))\} = 0 \tag{3.1}$$

where

$$\mathcal{H}(t, x, \pi, D_x V(t, x), D_{xx} V(t, x)) = (a\mu - c)V_x(t, x) + \frac{1}{2} a^2 V_{xx}(t, x) + \int_{\mathbb{R}} \{V(t, x + \gamma az) - V(t, x) - \gamma az V_x(t, x)\} \nu(dz) + e^{-rt} U(c)$$

is the Hamiltonian for this control problem.

**Case 1:**  $\nu = 0$  and  $\beta \neq 0$

Note that the utility function  $U(c) = \frac{c^\beta}{\beta}$  is homothetic, and hence one can easily prove that  $V(t, x) = V(t, 1)x^\beta$ .

It is thus natural to assume a reward function of the form  $V(t, x) = e^{-rt}K(t)x^\beta$  for some function  $K(t)$  to be determined.

**Proposition 3.1** Let  $\nu = 0$  and  $0 < \beta < 1$ . Then there cannot be  $K(t)$  such that  $V(t, x) = e^{-rt}K(t)x^\beta$ .

PROOF. The optimal portfolio is found directly from the HJB Equation (3.1) whereby the value function  $V(t, x)$  is expected to be of the form

$V(t, x) = e^{-rt}K(t)\frac{x^\beta}{\beta}$  for some  $K(t)$  to be determined. Note that a similar result can be obtained with  $V(t, x) = e^{-rt}K(t)x^\beta$  so the above assumption is without loss of generality. However, substituting this into the HJB equation leads to a Bernoulli differential equation

$$K'(t) + \left( \frac{\mu^2 \beta}{2(1-\beta)} - r \right) K(t) + (1-\beta) (K(t))^{\frac{\beta}{\beta-1}} = 0.$$

Let  $\theta = \frac{1}{1-\beta} \left( \frac{\mu^2 \beta}{2(1-\beta)} - r \right)$  then the natural condition  $V(t, 0) = 0$  gives a solution to the Bernoulli equation as  $K(t) = \left[ \frac{1}{\theta} (e^{-\theta t} - 1) \right]^{1-\beta}$ , which immediately gives  $c^*(t) = \frac{\theta x}{e^{-\theta t} - 1}$ .

Note that if  $\theta > 0$  then  $e^{-\theta t} - 1 < 0$  for all  $t > 0$ . Equally if  $\theta < 0$  then  $e^{-\theta t} - 1 > 0$  for all  $t > 0$ . Therefore there is no  $\theta$  such that  $K(t)$  is well defined and such that  $c^*(t) \geq 0$  □

**Proposition 3.2** Let  $\nu = 0$ ,  $0 < \beta < 1$ ,  $0 < \mu < \sqrt{\frac{2r(1-\beta)}{\beta}}$  and define  $x_0 := \frac{1-\beta}{\mu}$ . Then the optimal control  $\pi^* = (a^*, c^*)$  is given by

$a^* = \frac{\mu x}{1-\beta}$  and  $c^*$  is independent of time and given by  $c^* = \frac{1}{1-\beta} \left( r - \frac{\mu^2 \beta}{2(1-\beta)} \right) x$ . Moreover, if  $x = x_0$  then  $a^* = 1 \Rightarrow 1 - a^* = 0$

The value function for this stochastic control problem is

$$V(t, x) = e^{-rt} \left( \frac{1}{1-\beta} \left( r - \frac{\mu^2 \beta}{2(1-\beta)} \right) \right)^{\beta-1} \frac{x^\beta}{\beta}, x \geq 0$$

PROOF.

Due to (3.1), we know that  $\pi^*$  cannot be time dependent for this case.

Now, assume that  $V(t, x)$  is of the form  $V(t, x) = e^{-rt} K \frac{x^\beta}{\beta}$  for some constant  $K$  to be determined. Then substitution in the HJB equation yields

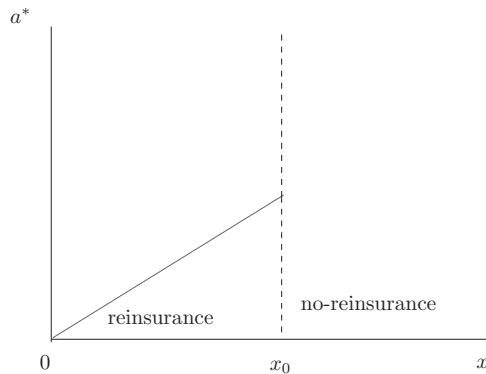
$$K = \left( \frac{1}{1-\beta} \left( r - \frac{\mu^2 \beta}{2(1-\beta)} \right) \right)^{\beta-1}. \text{ Now the condition } c^* = K^{\frac{1}{\beta-1}} x \text{ then gives the result for both } c^* \text{ and } V(t, x)$$

The condition  $0 < \mu < \sqrt{\frac{2r(1-\beta)}{\beta}}$  is necessary for  $K$  to be defined for all  $0 < \beta < 1$ , otherwise the value function will be negative or undefined for some values of  $r$  and  $\beta$ . Clearly  $V(t, x) \in C^{1,2}(\mathcal{D})$  and a Verification Theorem (see Øksendal, 2003 for theorem) applied to  $V(t, x)$  shows that it is really the optimal reward.  $\square$

**Remark 3.3**

We draw a lot of insights from Proposition 3.2.

- (1) First, the portion of premium  $a^*$  due to the insurance company is directly proportional to the reserve up to the threshold  $x_0 = \frac{1-\beta}{\mu}$  after which the insurer would take all premiums and pay all liabilities, ie,  $a^* = 1$  for  $x = x_0$ . Note that if  $x > x_0$ , then  $a^* > 1$  which is not possible. We thus view  $x_0$  as the insurance vs no-reinsurance threshold as indicated on the graph below.



- (2) The threshold  $x_0$  is inversely proportional to the safety loading  $\mu$ . The smaller the safety loading the greater the threshold meaning that if the insurer wants to quickly “go it solo” without the aid of reinsurance, then the insurer should raise its safety loading. Alternatively, if the insurance company charges less safety loading, they would need more money in reserve before switching to the no-reinsurance region. But the insurance company cannot afford not to charge the safety loading, ie, the case  $\mu = 0$  is not possible. In limit (since  $x_0$  will not be defined) it would result in  $a^* = 0$  meaning the insurance company will have to reinsure all the liability which does not make sense. It is as good as not insuring the liability at all. Actually, the condition  $0 < \mu < \sqrt{\frac{2r(1-\beta)}{\beta}}$  makes it impossible to consider the case of no safety loading.

(3) The optimal dividend payout is also directly proportional to the reserve for the case  $\beta \neq 0$ . The constant of proportionality is time independent, a result we proved before. For the case  $0 < \mu < 1$ , then for fixed  $\beta$ , then  $c^*$  is an increasing function of  $\mu$ , that is, the more safety loading, the more dividend payout. On the other hand, for  $\mu > 1$ , then  $c^*$  is a decreasing function of  $\mu$ , ie, the more safety loading, the less dividend payout.  $\square$

**Case 2  $\nu \neq 0$  and  $\beta \neq 0$**

As in the other case, we look for  $\pi(\cdot) = (a(\cdot), c(\cdot))$  that maximizes

$$h_0(a, c) = V_t(t, x) + (a\mu - c)V_x(t, x) + \frac{1}{2}a^2V_{xx}(t, x) + \int_{\mathbb{R}} \{V(t, x + \gamma az) - V(t, x) - \gamma azV_x(t, x)\} \nu(dz) + e^{-rt}U(c)$$

and such that

$h_0(a^*, c^*) = 0$ . The HJB equation from this is an integral-differential equation

$$\sup_{\pi \in \mathcal{A}(x)} [V_t(t, x) + (a\mu - c)V_x(t, x) + \frac{1}{2}a^2V_{xx}(t, x) + \int_{\mathbb{R}} \{V(t, x + \gamma az) - V(t, x) - \gamma azV_x(t, x)\} \nu(dz) + e^{-rt}U(c)] = 0 \tag{3.2}$$

$$\frac{1}{2}a^2V_{xx}(t, x) + \int_{\mathbb{R}} \{V(t, x + \gamma az) - V(t, x) - \gamma azV_x(t, x)\} \nu(dz) + e^{-rt}U(c) = 0$$

By “guessing” the value function as in the previous case (due to the homothetic property), we get  $c^* = K^{\frac{1}{\beta-1}}x$  for some K to be determined and  $a^*$ , if it exists, is a solution of the polynomial equation

$$\mu - \frac{a(1-\beta)}{x} + \int_{\mathbb{R}} \left\{ \left( 1 + \frac{\gamma az}{x} \right)^{\beta-1} \gamma z - \gamma z \right\} \nu(dz) = 0.$$

It is not clear whether this equation always gives a unique solution  $0 \leq a \leq 1$ . However, we can show that in special circumstances, a unique optimal control  $\pi^*$  can be found. To see this, we assume as before that  $V(t, x) = e^{-rt}K \frac{x^\beta}{\beta}$ , so that maximizing  $h_0(a, c)$  implies:

$$\mu x^{\beta-1} - (1-\beta)ax^{\beta-2} + \int_{\mathbb{R}} \left\{ \gamma z (x + \gamma az)^{\beta-1} - \gamma z x^{\beta-1} \right\} \nu(dz) = 0 \tag{3.3}$$

$$c^* = K^{\frac{1}{\beta-1}}x \tag{3.4}$$

Let

$$f(a) = \mu x^{\beta-1} - (1-\beta)ax^{\beta-2} + \int_{\mathbb{R}} \left\{ \gamma z (x + \gamma az)^{\beta-1} - \gamma z x^{\beta-1} \right\} \nu(dz)$$

then

- $f(0) = \mu x^{\beta-1} > 0$
- $\lim_{a \rightarrow \infty} f(a) = -\infty$
- $f'(a) = -(1-\beta) \left[ x^{\beta-2} + \int_{\mathbb{R}} \gamma z (x + \gamma az)^{\beta-2} \nu(dz) \right] < 0$ , so that  $f(a)$  is a decreasing function for all  $0 < a < 1$ .

Therefore, there exists a unique solution  $a^*$  to Equation (3.3). Given numerical values of parameters, one can easily solve numerically for  $a^*$  where algebraic methods are not sufficient.

Substituting  $a^*$  and  $c^*$  in (3.4) into the HJB equation, and assuming that

$$\mu > \frac{x}{\beta a^*} \left[ \int_{\mathbb{R}} \left\{ \left( 1 + \frac{\gamma a^* z}{x} \right)^\beta - 1 - \frac{\gamma \beta a^* z}{x} \right\} \nu(dz) - \frac{(a^*)^2 \beta (1-\beta)}{2x^2} - r \right],$$



$$K = \left( \frac{1}{1-\beta} \left( r + \frac{\mu\beta\alpha^*}{x} + \frac{(\alpha^*)^2\beta(1-\beta)}{2x^2} - \int_{\mathbb{R}} \left\{ \left( 1 + \frac{\gamma\alpha^*z}{x} \right)^\beta - 1 - \frac{\gamma\beta\alpha^*z}{x} \right\} \nu(dz) \right) \right)^{\beta-1}.$$

Note that, the results for the case  $\nu = 0$  in Case 1 shows that  $\alpha^*$  is of the form  $\alpha^* = g(\mu)x$ , for some function  $g$  of the safety loading  $\mu$ . In that case,  $K$  will be constant and equal to

$$K = \left( \frac{1}{1-\beta} \left( r + \mu\beta g(\mu) + \frac{1}{2}\beta(1-\beta)g^2(\mu) - \int_{\mathbb{R}} \left\{ (1 + \gamma g(\mu)z)^\beta - 1 - \gamma\beta g(\mu)z \right\} \nu(dz) \right) \right)^{\beta-1}$$

and finally, the value function is  $V(t, x) = e^{-rt}K\frac{x^\beta}{\beta}$ . This proves the following general result for the case  $\beta \neq 0$ :

**Proposition 3.4** Suppose that the reserve of the insurance company is given by the Lévy-Itô process (2.1). If the shareholder's utility function of dividend yield is  $U(c) = \frac{c^\beta}{\beta}$  then the optimal control  $\pi^* = (\alpha^*, c^*)$  that solves (2.3) is given by

$$\alpha^* = g(\mu, \beta)x \tag{3.5}$$

$$c^* = (K(\mu, r, \beta))^{\frac{1}{\beta-1}}x \tag{3.6}$$

where both  $g(\dots)$  and  $K(\dots)$  are constants which depend on the safety loading  $\mu$ , the interest rate  $r$  and the risk-reversion index  $\beta$ . The value function is  $V(t, x) = e^{-rt}K(\mu, r, \beta)\frac{x^\beta}{\beta}$ . In particular, for the case  $x > x_0 = \frac{1}{g(\mu)}$ , then the insurance company will take all risk and does not re-insure the liability.

For the case  $\nu = 0$  then

$$g(u) = \frac{\mu}{1-\beta} \text{ while } K = \left( \frac{1}{1-\beta} \left( r - \frac{\mu^2\beta}{2(1-\beta)} \right) \right)^{\beta-1}$$

Surely  $V(t, x) \in C^{1,2}(\mathcal{D})$  and a Verification Theorem (see Øksendal, 2003 for the case  $\nu = 0$ ; Øksendal & Sulem, 2007 for the case  $\nu \neq 0$ ), shows that  $\pi^*$  is really optimal and that  $V(t, x)$  is the value function. The verification theorem proof will be standard and will be omitted here.

**Example 3.5**

We look at a special example of the above for some special approximation:

Assume that  $(1 + \frac{\gamma\alpha z}{x})^{\beta-1} \approx 1 + (\beta-1)\frac{\gamma\alpha z}{x}$  by Taylor approximation and assuming that

$$0 < \mu < \sqrt{\frac{2r(1-\beta)(1 + \int_{\mathbb{R}} \gamma^2 z^2 \nu(dz))}{\beta}}$$

then we get

$$\alpha^* = \frac{\mu x}{(1-\beta) \left( 1 + \int_{\mathbb{R}} (\gamma z)^2 \nu(dz) \right)} \text{ and the optimal dividend payout for this case is } c^* = K^{\frac{1}{\beta-1}}x$$

$$\text{where } K = \left( \frac{1}{1-\beta} \left( r - \frac{\mu^2\beta}{2(1-\beta)(1 + \int_{\mathbb{R}} \gamma^2 z^2 \nu(dz))} \right) \right)^{\beta-1}.$$

The corresponding threshold for reinsurance is now

$$x_0 = \frac{(1-\beta)}{\mu} \left( 1 + \int_{\mathbb{R}} (\gamma z)^2 \nu(dz) \right).$$

Note that since  $1 + \int_{\mathbb{R}} (\gamma z)^2 \nu(dz) > 0$  for all  $z$ , then the threshold for the special case with jumps is always more than the threshold for the case without jumps. We then conclude for this case that in the presence of jumps the insurance company would need more in reserve before insuring all liability than in the case with no jumps. The jumps bring more risk. All the other observations for  $a^*$  raised in Case 1 also apply for this case.

In this case, we can retrieve the same value for  $K$  as in Case 1 by simply letting  $\nu = 0$ .

Substituting, we get  $c^* = \frac{1}{1-\beta} \left( r - \frac{\mu^2 \beta}{2(1-\beta)(1 + \int_{\mathbb{R}} \gamma^2 z^2 \nu(dz))} \right) x$

and note that the dividend payment for the case with jumps is less than that for the case with no jumps because of the existence of the factor  $1 + \int_{\mathbb{R}} (\gamma z)^2 \nu(dz)$  on the denominator.

The reward function is

$$V(t, x) = e^{-rt} \left( \frac{1}{1-\beta} \left( r - \frac{\mu^2 \beta}{2(1-\beta)(1 + \int_{\mathbb{R}} \gamma^2 z^2 \nu(dz))} \right) \right)^{\beta-1} \frac{x^\beta}{\beta}$$

**Case 3:  $\beta = 0$**

**Proposition 3.6** Let  $\beta = 0$ , and without loss of generality, let's consider the approximation in the example above so that  $\lambda = 1 + \int_{\mathbb{R}} \gamma^2 z^2 \nu(dz)$ . Then the optimal control  $\pi^* = (a^*, c^*)$  is given by

$a^* = \mu x$  and  $c^* = rx$  where the optimal payoff or reward function is

$$V(t, x) = \begin{cases} e^{-rt} \left( \frac{\frac{1}{2}u^2 - r + r \ln r}{r^2} + \frac{1}{r} \ln x \right) & \text{if } \nu = 0 \\ e^{-rt} \left( \frac{\frac{1}{2}u^2 \left( \frac{1}{\lambda} - \frac{1}{\lambda^2} \right) - r + r \ln r}{r^2} + \frac{1}{r} \ln x \right) & \text{if } \nu \neq 0 \end{cases}$$

where for the case  $\nu \neq 0$  we have ignored higher order powers of  $a$  in the Taylor expansion of  $V(t, x + \gamma az)$ .

PROOF. □

In this case we have

$$a^* = \begin{cases} -\mu \frac{V_x}{V_{xx}} & \text{if } \nu = 0 \\ -\mu \frac{V_x}{V_{xx\lambda}} & \text{if } \nu \neq 0 \end{cases}$$

and

$c^* = \frac{e^{-rt}}{V_x}$  and the HJB equation becomes

$$V_t - \frac{\mu^2}{2} \frac{V_x^2}{V_{xx}} - e^{-rt}(1 + rt) - e^{-rt} \ln V_x = 0 \text{ if } \nu = 0 \tag{3.7}$$

$$V_t - \frac{\mu^2}{2} \left( \frac{1}{\lambda} - \frac{1}{\lambda^2} \right) \frac{V_x^2}{V_{xx}} - e^{-rt}(1 + rt) - e^{-rt} \ln V_x = 0 \text{ if } \nu \neq 0 \tag{3.8}$$

whose solution is

$$V(t, x) = e^{-rt} \left( \frac{\frac{1}{2}u^2 - r + r \ln r}{r^2} + \frac{1}{r} \ln x \right) \text{ for } \nu = 0 \text{ and}$$

$$V(t, x) = e^{-rt} \left( \frac{\frac{1}{2}u^2 \left( \frac{1}{\lambda} - \frac{1}{2\lambda^2} \right) - r + r \ln r}{r^2} + \frac{1}{r} \ln x \right) \text{ for } \nu \neq 0. \quad \square$$

The following observations hold for this case

- (1) The first observation of Remark 3.3 is confirmed through separate computations for the case  $U(c) = \ln(c)$  (ie, the case  $\beta = 0$ ). As a result, the threshold is now  $x_0 = \frac{1}{\mu}$  and the first two remarks above hold for the proportion  $\sigma^*$  in this case.
- (2) The dividend payout is directly proportional to the reserve with the bank interest rate being the constant of proportionality. The more the reserve or the higher the interest rates or both, the more dividend to be paid out.

Just as in the case  $\beta \neq 0$ , we generalized the result for the case  $\nu \neq 0$ , as follows:

**Proposition 3.7** *Suppose that the reserve of the insurance company is given by the Lévy -Itô process (2.1) . If the shareholder's utility function of dividend yield is  $U(c) = \ln c$  then the optimal control  $\pi^* = (\sigma^*, c^*)$  that solves (2.3) is given by*

$$\sigma^* = g(\mu)x \tag{3.9}$$

$$c^* = f(r)x \tag{3.10}$$

where both  $g(\cdot)$  and  $f(\cdot)$  are constants which depend on the safety loading  $\mu$  and the interest rate  $r$  respectively. The value function is  $V(t, x) = e^{-rt} (K(\mu, r) + \frac{1}{r} \ln x)$ . In particular, for the case  $x > x_0 = \frac{1}{g(u)}$ , then the insurance company will take all risk and does not re-insure the liability. In particular, for the case  $\nu = 0$  then

$$g(u) = \mu, \quad f(r) = r \text{ while } K = \frac{\frac{1}{2}u^2 - r + r \ln r}{r^2}$$

**Remark 3.8**

The stopping time  $\tau$ , which is the insolvency time, is not a control variable since the insurance company cannot decide when to be insolvent. As a result it is not part of the solution. It turns out that in both cases

$$\tau = \inf \{t \geq 0 : x(t) = x_0\} \text{ where } x_0 \text{ has been determined as "threshold" time.}$$

**4. Conclusion**

We have solved an insurance-reinsurance problem of maximizing the dividend payout to shareholders through reinsurance. The shareholder is deemed to have CRRA utility function with results showing that the optimal dividend payout and the premium must be proportional to the reserve with constant coefficients of proportionality which depend on the safety loading, interest rate and coefficient of risk aversion where applicable. The constants of proportionality will never be time dependent in all cases. An extension of this work is to consider the case when dividend payouts are not continuous but depend on the status of the reserve. This will lead to regime-switching arguments which can still be solved on condition that the controlled process  $x(\cdot)$  is proved to have a unique strong solution and that an optimal control will exist. The results help insurers in decision-making. The interplay between paying dividends and deciding on reinsuring liabilities is complex as demonstrated. Results show that retention of risk and dividend payments are expected to increase as the insurer's reserve increases and both depend on the investor's level of risk aversion. If this problem involved decisions on investment, the type of utility function, ie, one that exhibits constant relative risk aversion would have resulted in a constant allocation of the proportion of wealth allocated to risky investments. Our results offer a different scenario. Take the case of power utility function. If  $\beta$  is high, then  $1 - \beta$  is low and both  $\sigma^*$  and  $c^*$  increase. The optimal reward also increases for the investor. This is more evident

when there are no jumps. Therefore, if investors have low relative risk aversion then insurers should pay them more dividends because the premiums will be high resulting in increased reserve. This results in increased payoff. The reverse is also true. Of interest is also the effect of interest rates on dividend payouts. We note that an increase in interest rates results in an increase in dividend payouts. This is independent of the presence of jumps. The economic significance of this is not certain suffice to say that the value, in present value terms, of cashflows decreases with high interest rates. Therefore, increasing dividend payout works for investors who are averse to interest rate risk, who may demand more in dividends in line with the liquidity preference theory, with time replaced by interest rates. Finally, we posit that this paper does not solve the problems of minimum cash requirements as proposed by Solvency I, II or II regulations because it only looks at one aspect of the insurance company, ie, its reserves. It will be interesting to include this aspect in future discussions.

#### Funding

This work was supported by the University of Cape Town [408320].

#### Author details

Sure Mataramvura

E-mail: [sure.mataramvura@uct.ac.za](mailto:sure.mataramvura@uct.ac.za)

ORCID ID: <http://orcid.org/0000-0002-8073-2070>

Division of Actuarial Science, School of Management Studies, University of Cape Town, Rondebosch, Cape Town 7701, South Africa.

#### Citation information

Cite this article as: An optimal reinsurance management and dividend payout strategy when the insurer's reserve is an Ito-Levy process, Sure Mataramvura, *Cogent Economics & Finance* (2019), 7: 1698939.

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