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Article

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Suggested Citation: Benjlijel, Bacem; Mansali, Hatem (2021) : The expected sharpe ratio of efficient portfolios under estimation errors, Cogent Economics & Finance, ISSN 2332-2039, Taylor & Francis, Abingdon, Vol. 9, Iss. 1, pp. 1-16, https://doi.org/10.1080/23322039.2021.1943910

This Version is available at:
https://hdl.handle.net/10419/270114

Cogent Economics & Finance

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/oaef20

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To cite this article: Bacem Benjlijel & Hatem Mansali | (2021) The expected sharpe ratio of efficient portfolios under estimation errors, Cogent Economics & Finance, 9:1, 1943910, DOI: 10.1080/23322039.2021.1943910

To link to this article: https://doi.org/10.1080/23322039.2021.1943910

Published online: 19 Jul 2021.

Benjlijel & Mansali, *Cogent Economics & Finance* (2021), 9: 1943910
https://doi.org/10.1080/23322039.2021.1943910

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FINANCIAL ECONOMICS | RESEARCH ARTICLE

The expected sharpe ratio of efficient portfolios under estimation errors

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Received: 17 October 2020
Accepted: 10 June 2021

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Reviewing editor:
David McMillan, University of Stirling, Stirling UK.

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Abstract: This paper aims to develop a feasible estimator of the Sharpe ratio that the investor would expect from estimated efficient portfolios. Based on the analytical expression of the expected Sharpe ratio, we construct an estimator that captures all the errors involved in the estimated efficient portfolios. We conduct a simulation study and find that our estimator delivers the lowest mean square error with comparison to existing estimators. Our result is robust to sample size, to number of assets and to non-normality. It works well, particularly, with short sample sizes. The superior performance of the proposed estimator is confirmed through empirical analysis. The ex-ante method developed in this work allows the investor to assess the value of efficient portfolios before investing capital.

Subjects: Econometrics; Mathematical Economics; Finance

Keywords: portfolio performance; mean-variance analysis; estimation errors; Sharpe ratio; estimator performance

1. Introduction

Since the seminal work of Markowitz (1952, 1959), the mean-variance analysis has become the most dominant approach used in practice.[1] Investors often use the mean-variance model to build their portfolios and evaluate the performance of their investments based on the Sharpe ratio,[2] despite the existence of several alternatives to this measure.

Bacem Benjlijel

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PUBLIC INTEREST STATEMENT

Investors in the stock market often seek efficient portfolios with higher returns and lower risks. A prominent portfolio choice framework that meets these requirements is the mean-variance model established by Markowitz in 1952. This model promises to investors to hold efficient portfolios that offer maximum performance. The practical implementation of this model requires the estimation of the optimal portfolio weights invested in the risky assets. However, these estimated weights suffer from estimation errors, which affect portfolio performance. The aim of this article is to quantify the performance that we expect from estimated efficient portfolios using a popular performance measure: the Sharpe ratio. We propose an improved method that helps the investor to assess the value of efficient portfolios before investing capital.

The Sharpe ratio, introduced by Sharpe (1966, 1994), measures the expected excess return per unit of risk. Many researchers find that alternative performance measures generate identical rank ordering as the Sharpe ratio, despite the significant deviations from the normal distribution (e.g., see Eling and Schuhmacher (2007), Eling (2008), and Auer (2015)). Thus, investors can use simple metrics such as the Sharpe ratio instead of more complex ones.

This performance metric is derived under the mean-variance framework that promises to investors to hold efficient portfolios that offer the maximum Sharpe ratio. To achieve this performance, mean-variance analysis assumes that the investor knows, with certainty, the parameters of the portfolio selection model namely the vector of expected returns on individual assets and the corresponding covariance matrix. However, in the real world, investors need to estimate these parameters using samples of historical return data. Replacing true parameters with estimated parameters often generates errors. These estimation errors negatively affect the performance of efficient portfolios (e.g., see DeMiguel et al. (2009), Tu and Zhou (2011), and Kan et al. (2016)). Therefore, the Sharpe ratio promised by portfolio theory is never achieved in practice.

A natural question arises: what is the Sharpe ratio the investor would expect from estimated efficient portfolios? To answer this question, previous researches rely on ex-post approach to estimate the Sharpe ratio (e.g., see DeMiguel et al. (2009), Chen and Yuan (2016), and Shi et al. (2020)). With this method, portfolio performance is measured after an investment decision has been made. Thus, the portfolio manager is not able to know if he made the right decision only after the end of the investment period. While this method is helpful in quantifying the ex-post results, we need an ex-ant method to assess the out-of-sample Sharpe ratio that can help the investor at the time of investment.

Relatively few studies have investigated the problem from this perspective. For example, Kourtis (2016) evaluates the out-of-sample performance of estimated efficient portfolios using an estimator based on analytical expressions of the expected squared Sharpe ratio and assuming that the covariance matrix is known by the investor. Nevertheless, the squared Sharpe ratio is not the appropriate measure of performance in this context since it is not founded theoretically as a performance measure and not compatible with the mean-variance framework. In addition, errors on the covariance matrix can have a large impact on portfolio performance, especially when the estimation length is small compared to the number of assets, as Kan and Zhou (2007), Ledoit and Wolf (2012) show. Thus, ignoring these errors when constructing the estimator of the Sharpe ratio of an estimated efficient portfolio leads to inaccurate results.

Recently, Paulsen and Söhl (2020) derived an unbiased estimator adjusting for both sources of bias: noise fit and estimation error. The authors show how to use their adjusted Sharpe ratio as a model selection criterion analogously to the Akaike Information Criterion. However, their result also depends on knowledge of the true covariance matrix of the efficient portfolio. Furthermore, to compute the proposed estimator, the authors utilize the same sample return data used to construct the efficient portfolio. Practically, portfolios are constructed with short samples of historical returns. Although their estimator is unbiased, it is not clear how it behaves for small sample sizes.

The aim of this paper is to derive a feasible estimator of the Sharpe ratio we expect from estimated efficient portfolios. Based on the analytical expression of the expected Sharpe ratio, we construct an estimator that captures all the errors involved in the estimated efficient portfolios and improves upon existing estimators under quadratic loss. It works well, particularly, for short sample lengths.

The remainder of the paper is organized as follows: Section 2 outlines the portfolio choice problem. Section 3 provides an analytical expression of the expected Sharpe ratio of estimated mean-variance portfolios. Section 4 develops a method to assess the Sharpe ratio of estimated

efficient portfolios. We conduct a simulation study and empirical analysis in Section 5 and 6, respectively, to evaluate the performance of the suggested estimator by comparison to existing competitors. Section 7 concludes. The proofs are moved to the Appendix.

2. The portfolio choice problem

In this section, we describe the portfolio choice problem for a mean-variance investor who chooses an optimal portfolio among N risky assets and a risk-free asset.

2.1. Under parameter certainty

Denote the returns of the N risky assets and the riskless asset at time t by R_t and R_t^f, respectively. Let then $r_t = R_t - R_t^f \ell$ be the vector of excess returns, where ℓ is an $N \times 1$ vector of ones. We assume that r_t is independent and follows a multivariate normal distribution with mean μ and covariance matrix Σ. Under the mean-variance framework, the objective of the investor is to maximize the following utility function:

$$U(w) = w'\mu - \frac{\gamma}{2} w'\Sigma w, \tag{1}$$

where w is the vector of weights of the N risky assets and γ is the investor's relative risk aversion. $w'\mu$ and $w'\Sigma w$ are the portfolio mean excess return and variance, respectively. Under parameter certainty, i.e., μ and Σ are known to the investor, it is easy to show that the optimal weights allocated to the risky assets are

$$w^* = \frac{\Sigma^{-1}\mu}{\gamma}, \tag{2}$$

and thus $1 - \ell' w^*$ is invested in the risk-free asset.[3] This optimal portfolio delivers the highest Sharpe ratio

$$\theta = \sqrt{\mu'\Sigma^{-1}\mu} \tag{3}$$

2.2. Under parameter uncertainty

In reality, μ and Σ are unknown to investors and need to be estimated. To compute the portfolio weights, the investor is constrained to estimate the expected returns of risky assets, the variances and covariances from a set of T periods of historical return data.

The sample estimates of the vector of expected returns μ and covariance Σ matrix are given, respectively, by

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} t \tag{4}$$

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}(r_t - \hat{\mu})(r_t - \hat{\mu})' \tag{5}$$

The simplest way to estimate w^* is to replace μ and Σ in (2) by $\hat{\mu}$ and $\hat{\Sigma}$ to obtain $\hat{w} = \frac{1}{\gamma}\hat{\Sigma}^{-1}\hat{\mu}$, which is the maximum likelihood estimator of w^*. Then, the Sharpe ratio of the estimated efficient portfolio is

$$\hat{S} = \frac{\hat{w}'\mu}{\sqrt{\hat{w}'\Sigma\hat{w}}}. \tag{6}$$

We notice from equation (6) that the Sharpe ratio of estimated efficient portfolios is a nonlinear function of a random variable, $\hat{w}$, which itself depends on the precision degree of parameter estimation. Due to estimation errors in $\hat{w}$, this will generally be lower than θ. With the stochastic representation of $\hat{S}$, we can obtain the expression of the expected Sharpe ratio.

3. The expected Sharpe ratio

In this section, we, first, derive analytical formula of the expected Sharpe ratio, i.e., the average out-of-sample performance the investor would realize if he estimates his efficient portfolio by a set of historical return data. Second, we study its accuracy. Finally, we analyze the determinants and the properties of this analytical formula of the expected Sharpe ratio.

3.1. Analytical expression

The following proposition derives an approximation of the expected Sharpe ratio that delivers an efficient portfolio where the elements of the vector of expected returns and the covariance matrix used to compute this portfolio are estimated from historical data.

Proposition 1 *For $T>N+4$, the expected Sharpe ratio for the estimated efficient portfolios can be expressed as follows*:

$$E\left(\hat{S}\right) := \sqrt{\frac{(T-N-1)(T-N-4)}{(T-N-2)(T-2)\left(\frac{N}{T\theta^2}+1\right)}}\theta \tag{7}$$

3.2. Accuracy of the analytical expression

The previous formula of the expected Sharpe ratio is derived under a Taylor series expansion and assuming independent and identically normally distributed returns. To study the accuracy of this analytical expression and the validity of normality assumption on return data, we conduct a simulation experiment.

The simulation design attributes to the covariance matrix a Toeplitz structure, i.e., $\Sigma = \Sigma_{ij} = \delta^{|i-j|}$ for $i,j = 1,\ldots,N$. The vector of expected returns is proportional to a constant, i.e., $\mu = kl$ We set $N = 10, 30, 40$ and 50. For each number of assets, N, we compute the analytical expression of the expected Sharpe ratio, $E\left(\hat{S}\right)$ using four different estimation lengths ($T = 120, 240, 360$ and 480). To study its accuracy, we compare it to the true values ζ and ξ computed using simulated returns generated from normal and student distributions with 8 degrees of freedom, respectively. Setting $\delta = 0.5$ and $k = 0.05$, the results from this experiment are reported in Table 1.

First, comparing the true values of the expected Sharpe ratio under different distribution assumptions, we find that they are close to each other since the difference is too small. This suggests that the assumption of normality is irrelevant in evaluating portfolio performance in terms of the expected Sharpe ratio for a mean-variance investor. Tu and Zhou (2004) find similar result for the certainty-equivalent performance measure.

Second, comparing the analytical formula of the expected Sharpe ratio to the true values, we can see that the error is relatively small for the different number of assets and decreases with the

Table 1. Accuracy of the approximation of the expected Sharpe ratio

N	θ	T	$E(\dot{S})$	ζ	ξ
10	0.100	120	0.031	0.031	0.031
		240	0.043	0.043	0.043
		360	0.051	0.051	0.051
		480	0.056	0.056	0.056
30	0.163	120	0.044	0.044	0.046
		240	0.064	0.064	0.065
		360	0.077	0.077	0.078
		480	0.086	0.086	0.087
50	0.208	120	0.048	0.049	0.052
		240	0.077	0.077	0.079
		360	0.094	0.094	0.096
		480	0.107	0.107	0.108

Note: This table reports the values of the expected Sharpe ratio from holding an efficient portfolio of N risky assets and a riskless asset estimated using T periods of historical returns. $E(\dot{S})$ is computed using the closed form expression derived in equation (7). The true values ζ and ξ are computed using 5,000 simulated returns from normal and student distributions, respectively.

sample length. As a result, we can conclude that the analytical formula is an accurate approximation of the expected Sharpe ratio.

3.3. Determinants and properties

Equation (7) in proposition one affirms, first, that the expected Sharpe ratio is a function of three determinants: the number of assets N, the sample size, T, and the maximum Sharpe ratio θ. It increases with the estimation length and the maximum Sharpe ratio and decreases with the number of assets. Intuitively, as the estimation length increases, $\hat{\mu}$ and $\hat{\Sigma}$ become more accurate estimators of μ and Σ, so the out of sample Sharpe ratio increases. On the other hand, the greater the number of assets, the greater the number of elements of μ and Σ that must be estimated. This leads to more errors in estimating the portfolio weights, which drives down the performance. Furthermore, the higher the maximum Sharpe ratio (i.e., the more the portfolios are diversified), the larger the expected Sharpe ratio of estimated portfolios is.

Second, for a given number of assets, when T goes to infinity, we note that the expected Sharpe ratio converges to its maximum value. This implies that the investor needs a huge quantity of historical data to estimate parameters with precision and thus reach the maximum performance as predicted by the theory. While in practice he possesses a limited quantity of historical information, attaining such performance remains a difficult task.

4. Estimating the Sharpe ratio

The analytical formula of the expected Sharpe ratio developed in the previous section depends on an unknown parameter, θ^2, which makes it unfeasible tool to compute the out-of-sample performance of estimated efficient portfolios. In this section, we show how to construct a feasible estimator. This is achieved by replacing θ^2 by an improved estimator.

4.1. Estimation design

Technically, we estimate, first, θ^2 based on the available data. Then, we calculate the estimated expected Sharpe ratio of the efficient portfolio using the expression (7) in proposition one by replacing θ^2 by its estimate. A natural estimator of θ^2 is its sample counterpart,

$$\hat{\theta}^2 = \hat{\mu}'\hat{\Sigma}^{-1}\hat{\mu}. \tag{8}$$

However, this estimator can be heavily biased when the sample length, T, is small (see, for instance, Kan and Zhou (2007)); which is a common practical situation that the investor faces. Indeed, the investor has access to a limited quantity of historical information available for the estimation. To alleviate the problem, we can replace θ^2 by its unbiased estimator (see expression (49) of Kan and Zhou (2007))

$$\hat{\theta}_u^2 = \frac{(T-N-2)\hat{\theta}^2 - N}{T}. \tag{9}$$

Despite the fact that this is an accurate estimator of θ^2 for large sample size, it can take negative values for short sample size. Thus, it is also an undesirable estimator of θ^2.

4.2. Adjusted estimator

For our application, we use an adjusted estimator of θ^2.

$$\tilde{\theta}^2 = max\left(\frac{(T-N-2)\hat{\theta}^2 - N}{T}, \frac{2(T-N-2)}{T(N+2)}\hat{\theta}^2\right), \tag{10}$$

where $T>N+2$ and $\hat{\theta}^2$ is given by equation (8). This estimator is due to the fact that $T\hat{\theta}^2$ follows a non-central F-distribution with N and $T-N$ degrees of freedom and a non-centrality parameter $T\theta^2$. We express $\tilde{\theta}^2$ as a function of the unbiased estimator $\hat{\theta}_u^2$ to get:

$$\tilde{\theta}^2 = max\left(\hat{\theta}_u^2, \frac{2(T\hat{\theta}_u^2 + N)}{T(N+2)}\right). \tag{11}$$

From the last equation, we see that if $\hat{\theta}_u^2$ is large (i.e., $\hat{\theta}_u^2 \geq \frac{2}{T}$) then $\tilde{\theta}^2$ is equal to $\hat{\theta}_u^2$ and behaves exactly like the unbiased estimator. On the other hand, if $\hat{\theta}_u^2$ is small (i.e., $\hat{\theta}_u^2 < \frac{2}{T}$) then $\tilde{\theta}^2$ is greater than $\hat{\theta}_u^2$. Although in this case the estimator presents some bias, it has some desirable features. First, it dominates the unbiased estimator in terms of minimizing the quadratic loss (see Theorem 3.1 of Kubokawa et al. (1993)). Second, this dominance persists even if the estimator is computed with a short estimation length. The reason is that the condition of small $\hat{\theta}_u^2$ is most likely to be verified when T is short. Third, with this design we prevent our estimator from taking negative values.

Substituting $\tilde{\theta}^2$ for θ^2 in (7), we can gauge the Sharpe ratio of estimated efficient portfolios. For $T>N+4$, the resulting estimator is as follows:

$$\bar{S} = \sqrt{\frac{(T-N-1)(T-N-4)}{(T-N-2)(T-2)\left(\frac{N}{T\tilde{\theta}^2}+1\right)}}\tilde{\theta}. \tag{12}$$

5. Simulation study

In this section, we examine and compare the performance of our estimator with that of alternative estimators. We begin with a presentation of the simulation set-up. Then, we provide a description of alternative competitors. Finally, we report and discuss our simulation results.

5.1. Set-up

We use the same design for the covariance matrix, Σ, and the vector of expected returns, μ, as in section 3. We perform two simulation experiments.

The first simulation aims to study the behavior of the mean square error (MSE) of our estimator and the competitor estimators as a function of the estimation length. For $N = 10$, we simulate a random sample of returns of length T ($T = 60, 120, \ldots, 600$), generated from a multivariate normal distribution $N_N(\mu, \Sigma)$. We then compute the estimators of the out of sample Sharpe ratio for the specified sample length. We iterate the design 5000 times. From the iterations, we compute the MSE for each sample length.

The goal of the second simulation is to examine the behavior of the MSE as a function of the number of assets. We set $T = 240$. For each number of assets N ($N = 10, 20, 30, 40, 50$), we draw a random sample of returns from a multivariate normal distribution. Then, we compute the estimators. We repeat the simulation 5000 times and we get the MSE delivered by our estimator and the alternative estimators for each number of assets. We repeat these two simulation experiences using data generated by a heavy-tailed distribution.

5.2. Competitors

We consider two competitors to our proposed estimator. The first one is derived by Kourtis (2016) used to assess the value of the expected squared Sharpe ratio. The author derives two approximations of the squared Sharpe ratio estimator using first order and second-order Taylor series expansions. We use the second-order approximation as it is more precise by construction. It is defined by the following expression:

$$\bar{\zeta}^2 = \bar{\theta}^2 - \frac{(N-1)\bar{\theta}^2}{N + T\bar{\theta}^2} - \frac{2(N-1)T\bar{\theta}^4}{\left(N + T\bar{\theta}^2\right)^3}. \tag{13}$$

$$\text{Where } \bar{\theta}^2 = \max\left(\ddot{\theta}\frac{N}{T}, \frac{2}{N+2}\ddot{\theta}\right) \tag{14}$$

$$\text{and } \ddot{\theta} = \hat{\mu}'\Sigma^{-1}\hat{\mu} \tag{15}$$

To make a fair comparison between our estimator and the Kourtis estimator, we use the formula of the Sharpe ratio estimator, $\bar{\zeta}$, instead of the squared Sharpe ratio estimator. Moreover, we replace the unknown covariance matrix in equation (15) by its sample counterpart in order to ensure a practical estimation framework.

The second competitor is an unbiased estimator proposed by Paulsen and Söhl (2020) and has the following expression:

$$\tau' = \ddot{\theta}\frac{N}{T\ddot{\theta}} \tag{16}$$

Where $\ddot{\theta}$ is given by equation (15). As in the Kourtis estimator case, we replace the unknown covariance matrix by its sample counterpart.

5.3. Results

In Figures 1 and Figures 2, we present the simulation results for normally distributed data. Figure 1 presents the global behavior of the estimators for different estimation lengths T. While Figure 1 shows the properties of the studied estimators for a fixed portfolio dimension ($N = 10$), Figure 2 depicts the general behavior for different dimensions.

To start with the results of Figure 1, we notice that the MSE is a decreasing function of the estimation length. Clearly, we see our proposed estimator shows the best performance since it delivers the lowest MSE among all others regardless of the sample size used to construct it. Our method works well especially with small sample estimation lengths. To give a specific example, with $T = 60$, our estimator error is 0.011, which is approximately three times less than the Kourtis estimator error (0.031) and more than four times less than the Paulsen and Söhl estimator error (0.049).

From Figure 2, we can see that the MSEs of the estimators increase with the number of assets. The Paulsen and Söhl estimator has the worst performance since it involves much more errors as the portfolio dimension becomes larger. The Kourtis estimator behaves much better for small portfolio dimensions. However, the errors grow up rapidly for large portfolio dimensions and reach similar badly results as in the case of the Paulsen and Söhl estimator. Our estimator outperforms all the competitors since it has the lowest MSE regardless the size of the portfolio. Furthermore, we notice that our proposed estimator is less sensitive to the dimension of the portfolio since the MSE evolves slowly with high number of assets.

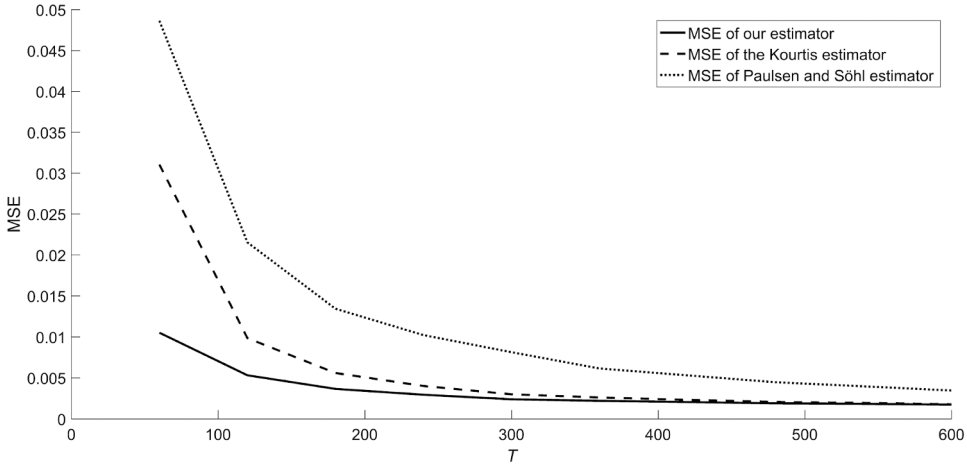

Figure 1. MSE of the expected Sharpe ratio estimators as a function of the estimation length.

Note: This figure plots the MSE of our estimator of the expected Sharpe ratio derived in equation (12), the Kourtis estimator and the Paulsen and Söhl estimator as a function of the estimation length T. The number of assets N is equal to 10. For each sample of returns of length T, simulated from a multivariate normal distribution, we compute the estimators of the out of sample Sharpe ratio. Then, we iterate the design 5000 times to compute the MSE.

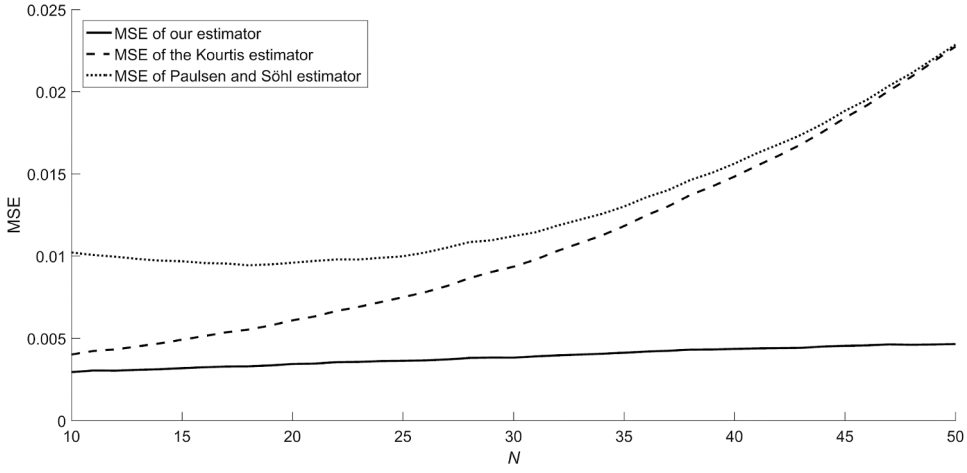

Figure 2. MSE of the expected Sharpe ratio estimators as a function of the portfolio dimension.

Note: This figure plots the MSE of our estimator of the expected Sharpe ratio derived in equation (12), the Kourtis estimator and the Paulsen and Söhl estimator as a function of the number of assets N. The estimation length T is equal to 240. For each number of assets N, we simulate a sample of returns of length T from a multivariate normal distribution. Then, we compute the estimators of the out of sample Sharpe ratio. We iterate the design 5000 times to compute the MSE.

In Figures 3 and Figures 4, we present the results for the student distributed asset returns with 8 degrees of freedom. Figure 3 illustrates the relationship between the MSE and the estimation length, while Figure 4 plots the MSE as a function of the number of assets in the portfolio.

The structure of the comparison study is the same as in the case of the normally distributes data. In general, the behavior observed in Figures 3 and Figures 4 does not differ significantly from that obtained from the normal distribution. The best estimator is, as usual, our proposed estimator since it clearly dominates the other competitors for all estimation lengths and portfolio dimensions.

To sum up, our estimator of the expected Sharpe ratio of estimated efficient portfolios shows superior performance compared with existing competitors. It has the lowest MSE for all sample sizes and portfolio dimensions regardless of the probability distribution of asset returns.

This outperformance is due, first, to the fact that our estimator captures all the errors involved in the estimated efficient portfolios. While Kourtis (2016) and Paulsen and Söhl (2020) neglect to incorporate the impact of the errors from estimating the covariance matrix when constructing their estimators by assuming a known covariance matrix, we take into account this feature. In fact, estimation errors in the covariance matrix affect the expected Sharpe ratio, especially, when the estimation length is small by comparison to the number of assets as demonstrated by Kan and Zhou (2007). This explain, particularly, the outperformance of our estimator when computed with small estimation lengths.

Second, we accept some bias when constructing the estimator, which leads to a great decrease of the overall error. This conception allows beating the non-biased estimator of Paulsen and Söhl and the biased estimator of Kourtis.

6. Empirical study

In this section, we illustrate empirically the performance of our estimator of the expected Sharpe ratio and its competitors. We, first, describe the data used in this analysis. Then, we explain the

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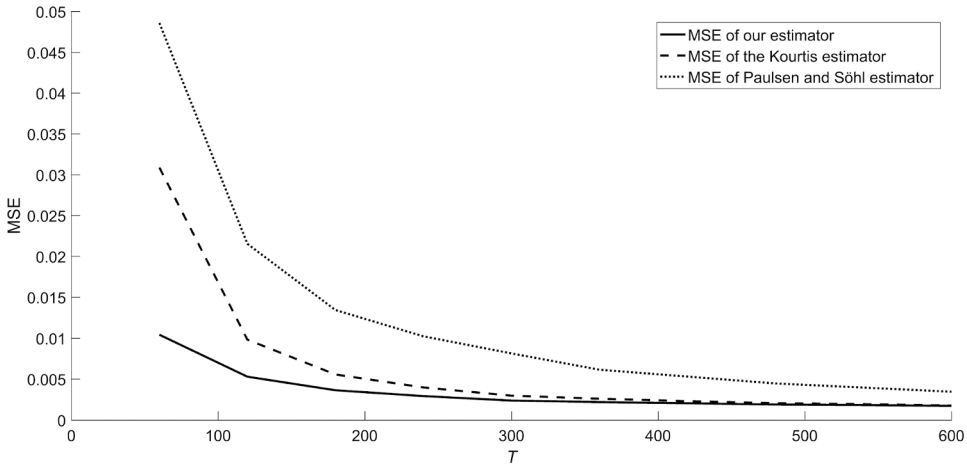

Figure 3. MSE of the expected Sharpe ratio estimators as a function of the estimation length.

Note: This figure plots the MSE of our estimator of the expected Sharpe ratio derived in equation (12), the Kourtis estimator and the Paulsen and Söhl estimator as a function of the estimation length T. The number of assets N is equal to 10. For each sample of returns of length T, simulated from a multivariate student distribution with 8 degrees of freedom, we compute the estimators of the out of sample Sharpe ratio. Then, we iterate the design 5000 times to compute the MSE.

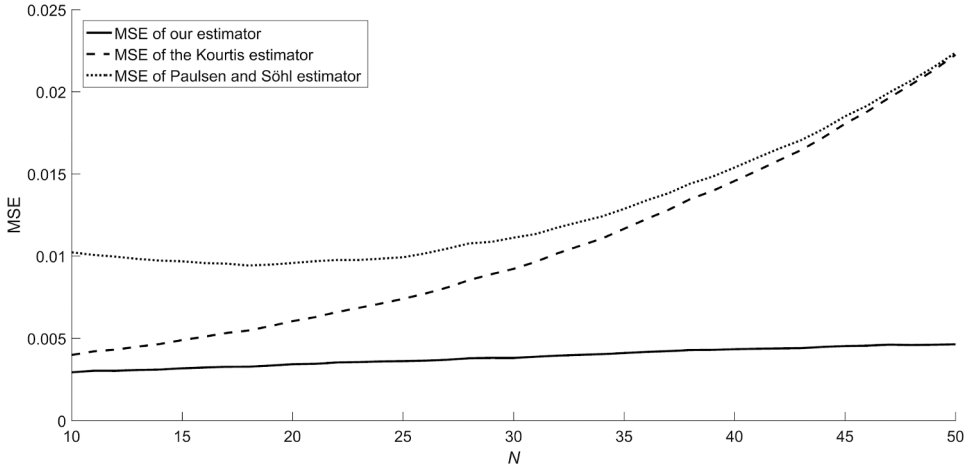

Figure 4. MSE of the expected Sharpe ratio estimators as a function of the portfolio dimension.

Note: This figure plots the MSE of our estimator of the expected Sharpe ratio derived in equation (12), the Kourtis estimator and the Paulsen and Söhl estimator as a function of the number of assets N. The estimation length T is equal to 240. For each number of assets N, we simulate a sample of returns of length T from a multivariate student distribution with 8 degrees of freedom. Then, we compute the estimators of the out of sample Sharpe ratio. We iterate the design 5000 times to compute the MSE.

methodology to evaluate the performance of these estimators. Finally, we present the results of the experiment.

6.1. Data

We use five empirical data sets of monthly returns outlined in Table 2. The data sets considered are from Ken French's website. These are i) monthly excess returns of the standard 10 industry portfolios: Consumer-Discretionary, Consumer-Staples, Manufacturing, Energy, High-Tech,

Telecommunication, Wholesale and Retail, Health, Utilities, and Others; ii) monthly excess returns of 10 portfolios sorted on momentum; iii) monthly excess returns of 25 portfolios formed on Size- and book-to-market; iv) monthly excess returns of 25 portfolios sorted by size and momentum; and v) monthly excess returns of 100 portfolios sorted by size and book-to-market. All the data sets cover the period from January 1927 to December 2017 except for the last data set which is available from January 1972 to December 2017.

6.2. Methodology of performance evaluation

In each data set, we study how the estimators of the out-of-sample Sharpe ratio perform using a rolling window approach. Specifically, given a H-month-long data set of asset returns, we choose an estimation window of length T. Using return data corresponding to the chosen window, we compute our estimator ($\bar{S}$), the Kourtis estimator ($\bar{\zeta}$) and the Paulsen and Söhl estimator ($\bar{\tau}$). We repeat this after rolling the window forward. This means estimation is repeated after adding data for the next month and dropping data for the earliest month. This is continued until all the available data is used. As a result, $H-T$ estimations of the out-of-sample Sharpe ratio of each estimator are produced.

We, also, compute the true expected Sharpe ratio as follows: First, we calculate the true parameters (i.e., μ and Σ) using the entire data set. Then, for each estimation window of length T, the Sharpe ratio of estimated efficient portfolios is computed using equation (6). This process is continued by adding data for the next month and dropping the earliest data until the end of the data set is reached. Finally, we average on the resulting $H-T$ Sharpe ratio outcome. Given the $H-T$ series of the out-of-sample Sharpe ratio and the true expected Sharpe ratio, we compute the MSE of the estimators.

6.3. Results

Table 3 reports the MSE of the estimators of the out-of-sample Sharpe ratio for empirical data using an estimation window of 240 months. Table 3 clearly shows that our estimator performs the best among all data sets since it delivers the lowest MSE compared to its competitors. Even with a short estimation window (i.e., T=120), Table 4 confirms the superiority of the suggested estimator of the Sharpe ratio. For example, for the SBM100 data set, the MSE of the competitors is about 10

Table 2. List of data sets

No.	Data set	N	Time period	Abbreviation
1	Ten industry portfolios	10	01/1927-12/2017	IND10
2	Ten momentum portfolios	10	01/1927-12/2017	MOM10
3	Twenty five portfolios formed on Size- and book-to-market	25	01/1927-12/2017	SBM25
4	Twenty five portfolios formed on size and momentum	25	01/1927-12/2017	SMOM25
5	One hundred portfolios formed on Size- and Book-to-Market	100	01/1972-12/2017	SBM100

Note: This table lists the various data sets analyzed, the number of assets N contained in each data set, the time period and the abbreviation used to identify each data set. Each data set contains monthly excess returns over the one month nominal US T-bill (from Ken French's Web site).

Table 3. MSE of the out-of-sample Sharpe ratio estimators for empirical data ($T = 240$)

	IND10	MOM10	SBM25	SMOM25	SBM100
$\bar{s}$	0.0067	0.0175	0.0405	0.0770	0.0178
$\bar{\zeta}$	0.0068	0.0229	0.0654	0.1233	0.1710
$\bar{\tau}$	0.0119	0.0224	0.0648	0.1221	0.1702

Note: For each of the empirical data sets listed in Table 2, this table reports the MSE of the estimators of the out-of-sample Sharpe ratio computed using an estimation window of $T = 240$ months.

Table 4. MSE of the out-of-sample Sharpe ratio estimators for empirical data ($T = 120$)

	IND10	MOM10	SBM25	SMOM25	SBM100
$\bar{s}$	0.0097	0.0281	0.0492	0.0944	0.0158
$\bar{\zeta}$	0.0144	0.0430	0.1315	0.2354	6.9795
$\bar{\tau}$	0.0370	0.0492	0.1311	0.2336	6.9738

Note: For each of the empirical data sets listed in Table 2, this table reports the MSE of the estimators of the out-of-sample Sharpe ratio computed using an estimation window of $T = 120$ months.

times compared to our estimator when $T = 240$. However, when $T = 120$, the MSEs of the same competitors grows up to more than 400 times.

7. Conclusion

Efficient mean-variance portfolios are subject to estimation errors that affect their future performance. To evaluate the out-of-sample performance, this article derives a feasible estimator of the expected Sharpe ratio that captures all the errors involved in these portfolios.

In a simulation study, we find that our estimator dominates the existing competitors in the literature since it delivers the lowest MSE regardless off the size of the estimation length, the number of assets and the probability distribution of asset returns. Our estimation method works well, particularly, in practical investment settings (i.e., short estimation lengths). The empirical study confirms these results.

Our analysis suggests a number of potentially fruitful avenues for future research. It would be of interest to further improve the precision of the proposed estimator. Two techniques could be adopted: the address of a limitation of this work (i.e., the expected Sharpe ratio formula developed under first order Taylor approximation) by incorporating higher-order approximations and the development of accurate estimators of the unknown maximum Sharpe ratio.

Furthermore, future work could use the methods in this work to develop estimators of the Sharpe ratio for sample-based mean-variance rules that have been proposed in the literature to deal with the problem of estimation errors.

8. Appendix: Proofs

We state three lemmas that will be used along the proofs:

Lemma 1 *For a bivariate function $f(x,y)$, a first-order Taylor series approximation to f around the point $\left(\mu_x = E(x), \mu_y = E(y)\right)$ is*

$$f(x,y) \approx f\left(\mu_x,\mu_y\right) + f'_x\left(\mu_x,\mu_y\right)(x-\mu_x) + f'_y\left(\mu_x,\mu_y\right)\left(y-\mu_y\right).$$

The expected value of $f(x,y)$ is

$$E(f(x,y)) = f\left(\mu_x, \mu_y\right).$$

When $f(x,y) = \frac{x}{\sqrt{y}}$, then the expectation is

$$E\left(\frac{x}{\sqrt{y}}\right) = E(f(x,y)) = f\left(\mu_x, \mu_y\right) = \frac{\mu_x}{\sqrt{\mu_y}}. \tag{17}$$

Lemma 2 *Under the multivariate normality assumption on* r_t, $\hat{\mu}$ *and* $\hat{\Sigma}$ *are independent and they have the following exact distributions*:

$$\hat{\mu}\ N_N(\mu, \Sigma/T), \tag{18}$$

$$\hat{\Sigma}\ W_N(T-1, \Sigma/T), \tag{19}$$

where $W_N(T-1, \Sigma/T)$ is a Wishart distribution with $T-1$ degrees of freedom and covariance matrix Σ/T.

The proof of the exact distributions of $\hat{\mu}$ and $\hat{\Sigma}$ is a well-known result in econometrics. See, for instance, Muirhead (1982), theorem 3.1.2.

Lemma 3 *Let* $W = \Sigma^{-\frac{1}{2}}\hat{\Sigma}\Sigma^{-\frac{1}{2}}$, *then* $W\tilde{W}_N(T-1, I_N/T)$, *where* I_N *is the identity matrix. The inverse moments of W are as follows (see, e.g.,* Haff (1979)):

$$E\left(W^{-1}\right) = \left(\frac{T}{T-N-2}\right)I_N, \tag{20}$$

$$E\left(W^{-2}\right) = \left[\frac{T^2(T-2)}{(T-N-1)(T-N-2)(T-N-4)}\right]I_N, \tag{21}$$

where $T>N+4$.

9. Proof of proposition 1

If μ and Σ are unknown, then the estimated portfolio is $\hat{w} = \frac{1}{\gamma}\hat{\Sigma}^{-1}\hat{\mu}$. The expected Sharpe ratio of this portfolio is

$$E\left(\hat{S}\right) = E\left(\frac{\mu'\hat{w}}{\sqrt{\hat{w}'\Sigma\hat{w}}}\right)$$

$$= E\left(\frac{\mu'\hat{\Sigma}^{-1}\hat{\mu}}{\sqrt{\hat{\mu}'\hat{\Sigma}^{-1}\Sigma\hat{\Sigma}^{-1}\hat{\mu}}}\right)$$

$$= \frac{E\left(\mu'\hat{\Sigma}^{-1}\hat{\mu}\right)}{\sqrt{E\left(\hat{\mu}'\hat{\Sigma}^{-1}\Sigma\hat{\Sigma}^{-1}\hat{\mu}\right)}}$$

$$= \frac{E\left(\mu'\Sigma^{-\frac{1}{2}}W^{-1}\Sigma^{-\frac{1}{2}}\hat{\mu}\right)}{\sqrt{E\left(\hat{\mu}'\Sigma^{-\frac{1}{2}}W^{-2}\Sigma^{-\frac{1}{2}}\hat{\mu}\right)}}$$

$$= \frac{E\left[E\left(\mu'\Sigma^{-\frac{1}{2}}W^{-1}\Sigma^{-\frac{1}{2}}\hat{\mu}\backslash\hat{\mu}\right)\right]}{\sqrt{E\left[E\left(\hat{\mu}'\Sigma^{-\frac{1}{2}}W^{-2}\Sigma^{-\frac{1}{2}}\hat{\mu}\backslash\hat{\mu}\right)\right]}}$$

$$= \frac{E\left(\mu'\Sigma^{-\frac{1}{2}}E\left(W^{-1}\right)\Sigma^{-\frac{1}{2}}\hat{\mu}\right)}{\sqrt{E\left(\hat{\mu}'\Sigma^{-\frac{1}{2}}E(W^{-2})\Sigma^{-\frac{1}{2}}\hat{\mu}\right)}}$$

$$= \frac{\frac{T}{T-N-2}\mu'\Sigma^{-1}\mu}{\sqrt{\frac{T^2(T-2)}{(T-N-1)(T-N-2)(T-N-4)}\left(\frac{N+T\mu'\Sigma^{-1}\mu}{T}\right)}}$$

$$= \sqrt{\frac{(T-N-1)(T-N-4)}{(T-N-2)(T-2)\left(\frac{N}{T\theta^2}+1\right)}}\theta \tag{22}$$

10. Proof of (10)

By expression (49) of Kan and Zhou (2007), we have $\hat{\theta}^2$ follows a non-central F distribution with non-centrality parameter $\lambda = T\theta^2$:

$$\hat{\theta}^2\left(\frac{N}{T-N}\right)F_{N,T-N}\left(T\theta^2\right).$$

Theorem 3.1 of Kubokawa et al. (1993) states that if a random variable $wF_{p,n}(\lambda)$, then the unbiased estimator of λ is $(n-2)w-p$. Under quadratic loss, this estimator is dominated by

$$\tilde{\lambda} = max\left((n-2)w-p, \frac{2(n-2)}{p+2}w\right).$$

Our adjusted estimator $\tilde{\theta}^2$ is then obtained by letting $\tilde{\lambda} = T\tilde{\theta}^2$, $\lambda = T\theta^2$, $w = \hat{\theta}^2$, $p = N$ and $n = T-N$

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: The expected sharpe ratio of efficient portfolios under estimation errors, Bacem Benjlijel & Hatem Mansali, *Cogent Economics & Finance* (2021), 9: 1943910.

Notes

1. Fabozzi et al. (2007), in a survey of managers at 38 asset management firms managing a total of $4.3 trillion in equities, find that the mean-variance optimization is by far the most used method in practice.
2. Amenc et al. (2003) find that the majority of the distributors of hedge funds measure the performance of their investments using the Sharpe ratio.
3. If $\ell'w^* = 1$, then the investor chooses the tangency portfolio composed of risky assets only.

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***Cogent Economics & Finance* (ISSN: 2332-2039) is published by Cogent OA, part of Taylor & Francis Group.**