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Suggested Citation: Becker, Denis M. (2021) : The difference between Modigliani-Miller and Miles-Ezzell and its consequences for the valuation of annuities, Cogent Economics & Finance, ISSN 2332-2039, Taylor & Francis, Abingdon, Vol. 9, Iss. 1, pp. 1-24,
<https://doi.org/10.1080/23322039.2020.1862446>

This Version is available at:

<https://hdl.handle.net/10419/270030>

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To cite this article: Denis M. Becker | (2021) The difference between Modigliani–Miller and Miles–Ezzell and its consequences for the valuation of annuities, Cogent Economics & Finance, 9:1, 1862446, DOI: [10.1080/23322039.2020.1862446](https://doi.org/10.1080/23322039.2020.1862446)

To link to this article: <https://doi.org/10.1080/23322039.2020.1862446>



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Received: 30 September 2020
Accepted: 05 December 2020

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Reviewing editor:
David McMillan, University of Stirling, Stirling, UK

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FINANCIAL ECONOMICS | RESEARCH ARTICLE

The difference between Modigliani–Miller and Miles–Ezzell and its consequences for the valuation of annuities

Denis M. Becker^{1*}

Abstract: This paper addresses the differences between the Modigliani–Miller [M&M] model (1958, 1963) and the Miles–Ezzell [M&E] model (1980, 1985). The main difference between these two models concerns the stochasticity of the free cash flows. While M&M assumes a strictly stationary process, M&E's process is a martingale. However, this subtle difference has not been fully exposed, and previous literature has produced partly erroneous statements or inconsistent valuation models. Therefore, the main objective of this paper is to illustrate and accentuate the effect of these two mutually exclusive stochastic processes on the timely behavior of cash flows, discount rates, and values of the firm, equity, debt, and tax shield. For this purpose, we perform a numerical experiment that allows the determination of values and discount rates by means of the risk-neutral approach. We show that in the M&E model, all cash flows and values are path-dependent, while they are not in M&M's world. Furthermore, in M&E's model, all discount rates are time-invariant, except for the discount rate applied to tax shields, which depends on the lifetime of the cash flows. Contrarily, in the M&M setup, all discount rates change across time, except for the constant discount rate of the tax shield. This has consequences for the applicability of the well-known present-value formula for annuities and for building consistent valuation models for both finite and perpetual cash flows.

ABOUT THE AUTHOR

Denis M. Becker is an associate professor at the NTNU Business School, Norwegian University of Science and Technology (NTNU). Denis holds a PhD in managerial economics and operations research from the NTNU and an MBA from the Chemnitz University of Technology (Germany). His research comprises different areas of finance, economics and operations research. This study is a result of his long-standing interest in the theory and application of discounted-cash-flow methods.

PUBLIC INTEREST STATEMENT

The valuation of firms and the effect of the financing structure on the value of the firm are important topics in corporate finance. One category of approaches applied to firm valuation is based on discounting expected future cash flows. There are several methods that fall into this category. The formulas involved in these methods rely on several assumptions. One of these assumptions concerns the type of the stochastic behavior of the cash flows. Specifically, two mutually exclusive stochastic processes have been assumed in the previous literature. However, the subtle difference between these two alternatives has not been fully exposed, and previous literature has produced partly erroneous statements or inconsistent valuation models. Therefore, the main objective of this paper is to illustrate and accentuate the effect of these two stochastic processes on the timely behavior of cash flows, discount rates, and value of the firm.

Keywords: Capital structure; Leverage; firm valuation; discounted-cash-flow methods; value of tax shield

1. Introduction

In 1958 and 1963, Modigliani and Miller (in the following abbreviated as M&M) published their seminal papers about the effect of financial leverage on the value of a firm and the costs of capital. Since then, the valuation of cash-flow streams, the development of discounted-cash-flow (DCF) models, the determination of the appropriate discount factors in these models, and the effect of financial leverage on both the DCF values and discount rates (required returns) have received considerable attention among researchers, academics, and practitioners.

During the course of past research several DCF models have been developed (for 10 such models, see Fernández, 2007). The most prominent models are the equity method, the free cash-flow method (FCF method; sometimes also referred to as weighted-average-costs-of-capital [WACC] method), the adjusted-present-value (APV) method (developed by Myers, 1974), and the capital-cash-flow (CCF) method (see, e.g. McConnell & Sandberg, 1975; Nantell & Carlson, 1975; Ruback, 2002). An important and well-understood claim (e.g. Booth, 2007; Fernández & Magni, 2007) is that these different models, if applied to the same set of assumptions (like the assumptions of M&M) need to give the same values of the unlevered firm, levered equity, debt, and interest tax shield. If they would not, the practical usefulness would be greatly impaired because different firm or project values may result in different decisions concerning project acceptance or resource allocation.

Previous literature has produced partly confusing and erroneous statements or inconsistent valuation models. Detailed examples for this will be given in Section 2. One of the reasons for these disagreements or misspecifications of DCF models is that one essential assumption has not been paid enough attention to. It concerns the type of the stochastic process of the cash-flows. Two types of cash-flow stochasticity have been implicitly or explicitly used in the valuation literature. The first specification was originally used by M&M who model the free cash flow as a strictly stationary process. A second and different specification was applied by Miles and Ezzell (1980, 1985), in the following abbreviated as M&E who presume a martingale process. *Ceteris paribus*, replacing an M&M type of cash flow by an M&E type of cash flow does not only affect the timely behavior of discount rates and values but also changes the way in which consistent valuation can be carried out.

Thus, fully understanding the difference between the stochasticity of M&M and M&E cash flows is essential for building consistent DCF models both in the case of perpetual cash flows as well as cash flows of finite life.

The main purpose of this paper is therefore to illustrate and accentuate the effect of these two mutually exclusive stochastic processes on the timely behavior of cash flows, discount rates, and values. For this purpose, we introduce a numerical experiment. More particularly, we simulate a simple scenario-tree of cash flows, one based on M&M and another one based on M&E. By means of the risk-neutral approach we will then calculate the values of the unlevered firm, levered firm, levered equity, debt, and tax shield. The observations made in this experiment will then support general statements about the timely behavior of discount rates and values. More particularly, we will show that for M&E all discount rates are time-invariant, except for the discount rate applied to tax shields, which is increasing with the remaining lifetime of the cash flows. Contrarily, in the M&M setup, all discount rates depend on the remaining lifetime of the cash flow, except for the discount rate of the tax shield, which is constant. This knowledge is important when valuing cash flows that behave like annuities or perpetuities.

This paper will be structured as follows. In section 2 we will look more closely into the previous literature on this topic and give examples for statements that, in our opinion, are misleading or incorrect. Section 3 shows the set of assumptions applied by Modigliani and Miller (1963) and Miles

and Ezzell (1980, 1985). In section 4, the four most common DCF models are formalized. Section 5 introduces the evaluation of cash flows via a backward-iteration process and risk-neutral probability approach. This approach is then applied to a numerical example. Section 6 summarizes the observations with respect to all cash flows, values, and discount rates involved in the evaluation process. Section 7 highlights the consequences for the evaluation of perpetuities and annuities. Finally, section 8 summarizes the findings and gives short indications for further research.

2. Disagreements, Confusion, or Misspecifications in the Previous Literature

In this section, we want to pinpoint the disagreements, confusion, or misspecifications in the previous literature. We will start with looking at the translation (mathematical) formulae between the required return on the unlevered equity r_U and the discount rate in the FCF method r_{FCF} that play a central role in the consistency of different DCF methods. Modigliani and Miller (1963, based on their formula 31.c) find the following relationship:

$$r_{FCF} = r_U - r_U \cdot \tau \cdot (1 - q) \quad (1)$$

Here τ is the corporate tax rate, and q represents the equity-to-firm-value ratio. Contrary to (1), Miles and Ezzell (1980) show that:

$$r_{FCF} = r_U - \tau \cdot (1 - q) \cdot r_f \cdot \frac{1 + r_U}{1 + r_f} \quad (2)$$

with r_f being the risk-free rate, which equals the required return of the debt holders. Cooper and Nyborg (2006) apply yet a different translation, which is:

$$r_{FCF} = r_U - r_f \cdot \tau \cdot (1 - q) \quad (3)$$

Intrinsically tied to this translation is the determination of the required return on the interest-tax shield r_{TS} , which is required in the APV method. Concerning this rate, Modigliani and Miller (1963) suggest the risk-free rate $r_{TS} = r_f$. Harris and Pringle (1985), on the other hand, suggest the application of the required return on unlevered equity as the appropriate discount rate for the interest tax shield across all periods: $r_{TS} = r_U$. Likewise, Booth (2007, p. 38), in his CCF formula (formula enumerated with 20), exclusively applies the required return on unlevered equity to the tax shield. As we will show later, this cannot be true for neither M&M nor M&E. From the tax-shield-value calculation by Arzac and Glosten (2005, formula 13, but without growth) or Barbi (2012, formula 15) yet a third discount rate can be derived, namely: $r_{TS} = r_U \cdot \frac{1+r_f}{1+r_U}$.

Let us now turn to some discrepancies concerning the constant leverage and constant level of debt. Arzac and Glosten (2005) state that “Modigliani and Miller (1963) show that when the firm maintains a constant level of debt D and pays a tax rate t , the value of the tax shield is $VTS = \tau \cdot D$. How to value the tax shield in the more interesting case in which the firm maintains a constant leverage ratio is a matter of contention.” While it is true that M&M maintains a constant level of debt, the second sentence of this statement suggests that a constant level of debt excludes constant leverage, or vice versa. This is incorrect with respect to M&M where the constant-debt assumption implies constant leverage, which again is a consequence of the type of stochasticity of M&M’s cash flow.

Later in their paper, Arzac and Glosten (2005) say that “Fernandez [referring to Fernandez (2004)] arrives at (14) by making use of the Modigliani-Miller’s expression $VTS = \tau \cdot D$, which [...] does not apply under a constant leverage policy in the presence of systematic risk.” However, in the world of M&M the firm has both constant leverage and systematic risk in the free cash flow (as well as the flow to the equity holders).

Cooper and Nyborg (2006) state that *“The difference is that [M&E] assume that the amount of debt is adjusted to maintain a fixed market value leverage ratio, whereas [M&M] assume that the amount of debt in each future period is set initially and not revised in light of subsequent developments.”* Although true, this statement is somewhat misleading, because both M&E and M&M maintain a fixed market-value-leverage ratio. However, the cash-flow process according to M&E requires the amount of debt to change, while the cash-flow process of M&M does not require the debt to be revised. In other words, if we assume constant leverage in the first place then revising or not revising the amount of debt is not an assumption, it is an implication of the stochastic behavior of the cash flows. Cooper and Nyborg (2006) further claim that the: *“[M&M model ...] generates a tax saving from interest that does not vary as the value of the firm varies.”* However, the firm value in M&M cannot vary because of the strict stationarity of the future free cash flow.

Sabal (2007) states that *“If cash flows were not perpetuities and the firm wished to keep a constant debt ratio, the level of debt must be adjusted every period to reflect the changing present values of both tangible assets and the tax shield.”* This statement seems to imply that if cash flows were perpetuities, then the value of the firm, equity, debt, and tax shield would not have to be adjusted at all. This, however, is invalid with respect to the M&E setup, in which all of these values are almost surely adjusted in all states in the future.

Beside the aforementioned disagreements concerning perpetual free cash flows, previous research has also produced different opinions concerning finite-life annuities. Miles and Ezzell (1980, p. 722, assumption 2) assume that the required return on unlevered equity is constant throughout time. They furthermore conclude (p. 726) that the discount rate in the FCF method is independent of the magnitude and timing of the FCF. Booth (2007), however, comes to the conclusion that the discount rate in the FCF method (the weighted average costs of capital) increases over time for cash flows with a finite lifetime. As we will show below, this statement is not valid for M&E type of cash flows.

Arditti and Levi (1977) studied the validity of the weighted average costs of capital for both perpetual and finite-life annuities. Their formula (26) represents the FCF method, where they assume a constant discount rate. As we will discuss later, this formula is correct only for cash flows that fulfill the stochastic properties according to M&M. Their formula (27), which corresponds to the CCF method, assumes both constant interest payments and constant discount rates. As we will show later, both these conditions are inconsistent with cash flows according to M&M and M&E. Arditti and Levi (1977) also generate an inconsistency by applying the equity-to-firm-value ratio to the investment outlay (see formula (15) and (27)) instead of the firm value when calculating interest payments.

Brusov et al. (2011) attempted to develop a valuation formula for the M&M style constant finite-life annuity. In their analysis (more precisely their formula 14), they neglect that the continuing value of debt (like all of the other values) decreases over time. This fact was already pointed out by Boudreaux and Long (1979, p. 8). If the debt decreases, then tax savings will also decrease. Hence, the annuity formula applied in their analysis (see Brusov et al., 2011, formula 14) cannot be valid. Brusov et al. (2011) furthermore presume that the unlevered return on equity and the discount rate in the FCF method remain constant over time. However, as we will show below, these rates can be subject to change (more precisely they increase) throughout the lifetime of the FCF. If discount rates change throughout time, the traditional annuity formula (as in their analysis enumerated with 20) is inapplicable.

Before we can evaluate the statements in the previous literature, it is necessary to look carefully at the assumptions and construction of DCF methods contained in the aforementioned literature and in the following analysis. The next two sections are devoted to setting out these assumptions and specifying the four valuation methods mentioned above.

3. Definition of Stochastic Cash Flows According to Miles/Ezzell and Modigliani/Miller

Before stating the one and only decisive difference between Modigliani and Miller (1963) and Miles and Ezzell (1980), the set of assumptions that are equal in both of these frameworks will be introduced. As Qi (2010) points out, one can argue against a theory (here the theory of M&M or M&E) by attacking its assumptions. The purpose of this paper, however, is not to question these assumptions, but to ask for results (discount rates and translation formulae) that are consistent with these assumptions. The assumptions are the following:

(1)	The earnings before interest and taxes (EBIT) represent a stochastic infinite-life annuity (perpetuity): more precisely, we assume that $\{EBIT_t\}$ with $t = 1, \dots, T$ is a stochastic process with constant unconditional expectation $\mathbb{E}[EBIT_t] = EBIT_B$ for all $t = 1, \dots, T$. $EBIT_B$ denotes this constant, and we will occasionally refer to it as base <i>EBIT</i> . $EBIT_B$ is furthermore positive to receive a positive value for the firm. Schematically, the first four points in time of such an annuity are illustrated as a scenario tree in Figure 1. In this figure, the nodes of the tree are enumerated from $n = 1, \dots, 15$. The branches (edges) of this tree indicate the existence of transition probabilities p_n from one state in t to another state in $t + 1$.
(2)	The EBIT equals the operating cash flow. This means that depreciations, investments or any changes in working capital cancel each other out.
(3)	We apply only corporate taxation, no personal taxation. The corporate tax rate is furthermore deterministic (non-stochastic), time invariant, and does not depend on the size of the <i>EBIT</i> .
(4)	The flow to debt consists of interest payments and changes in the principal of debt only. This means that there do not exist additional fees, discounts, etc.
(5)	No transaction or information costs appear when leveraging or delevering the firm.
(6)	The risk-free rate r_f is deterministic (non-stochastic) and constant throughout time.
(7)	Debt is risk free. The debt holders receive the negotiated nominal amount of debt and interest. Early research like Modigliani and Miller (1958, 1963), Myers (1974), Miles and Ezzell (1980), and Harris and Pringle (1985) explicitly or implicitly assume risk-free debt financing. Others, who applied other than the risk-free rate to debt still treated debt deterministically (for example Ruback (2002) or Cooper and Nyborg (2008)). The modeling of risky debt, particularly in finite-life projects, requires additional assumptions and complicates the computations. Throughout this paper it will therefore be convenient to assume debt as risk-free.
(8)	The value of debt DV_t equals the nominal (contractual) amount of debt DN_t : $DV_t = DN_t$ for all t . This implies that the nominal (contractual) interest rate equals the risk-free rate. Because the risk-free rate is deterministic and constant over time, the nominal interest rate is also deterministic and time invariant.
(9)	In the case of negative income before taxes ($EBIT_t - I_t < 0$, with I_t representing the interest payment in point in time t), there is a tax transfer to the firm (reverse taxation). This means, for example, that negative income cannot be carried over to another point in time.
(10)	We assume a constant equity-to-firm value ratio q throughout the lifetime of the <i>EBIT</i> . Accordingly, the debt-to-firm value ratio will be $1 - q$.
	When it comes to the only difference between Modigliani and Miller (1963) and Miles and Ezzell (1980) we have the following mutually exclusive assumptions. Modigliani and Miller (1963) assume:
(11a)	The conditional expected EBIT are constant (time invariant): $\mathbb{E}[EBIT_t EBIT_{t-1}] = EBIT_B$ with $EBIT_B$ being a constant.
(11b)	The conditional expected <i>EBIT</i> at point in time t equals the realized <i>EBIT</i> in the previous point in time ($t - 1$): $\mathbb{E}[EBIT_t EBIT_{t-1}] = EBIT_{t-1}$

Figure 2 and Figure 3 exemplify the difference between these two settings. The numerical values in these figures are generated as follows: assume a base-cash flow of $EBIT_B = 50$. With respect to the M&M setup in Figure 2 this means, from any given state in point of time t there will be a 50% transition probability to a state in $t + 1$ with $EBIT_{t+1} = EBIT_B \cdot u$, and with the same probability we move into the state with $EBIT_{t+1} = EBIT_B \cdot d$. The factors u (like up) and d (like down) are chosen as: $u = 1.1$ and $d = 0.9$. Hence, being in some state (node) in point of time t , the next period's EBIT will be either: $50 \cdot 1.1 = 55$ or $50 \cdot 0.9 = 45$. This means that both the unconditional and conditional expected EBIT equals the base $EBIT_B$ of 50. The probabilities shown in brackets represent the risk-neutral probabilities that will be applied during the valuation of the firm below.

Figure 1. Schematic illustration of a stochastic annuity as a scenario tree.

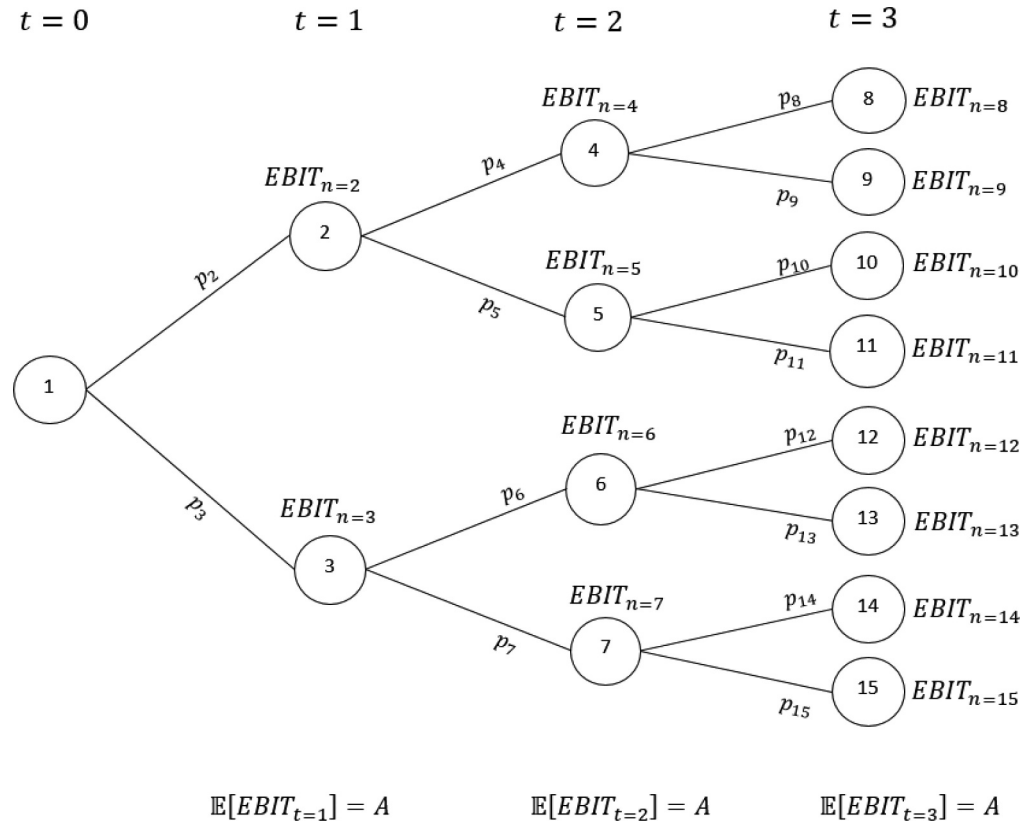


Figure 2. The Modigliani–Miller scenario tree.

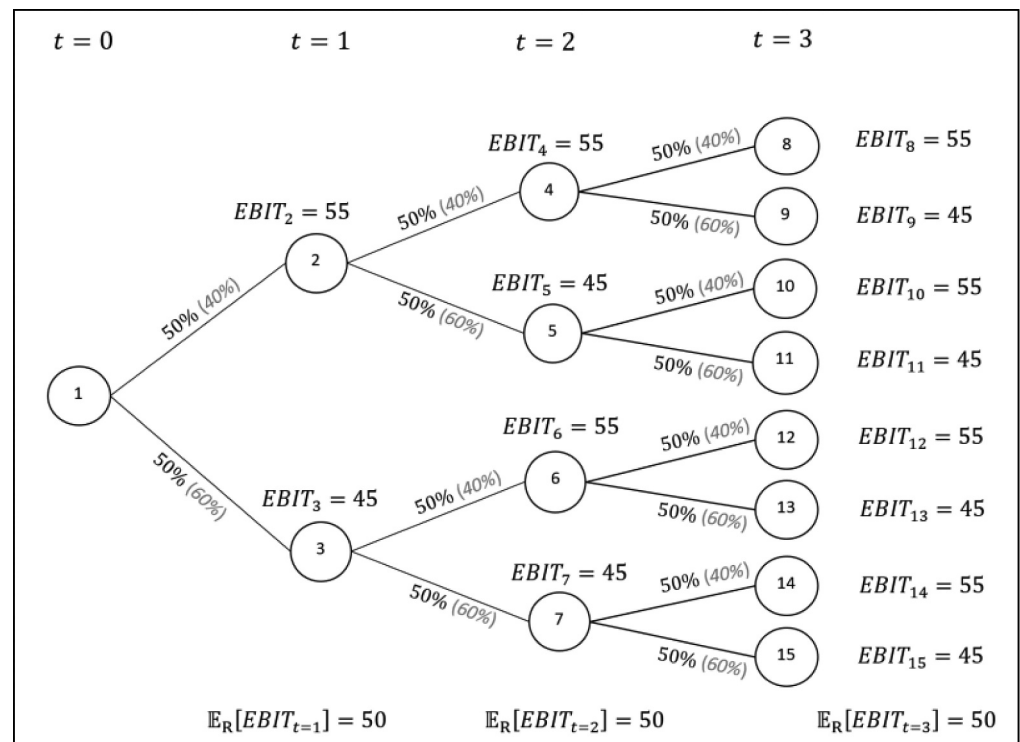
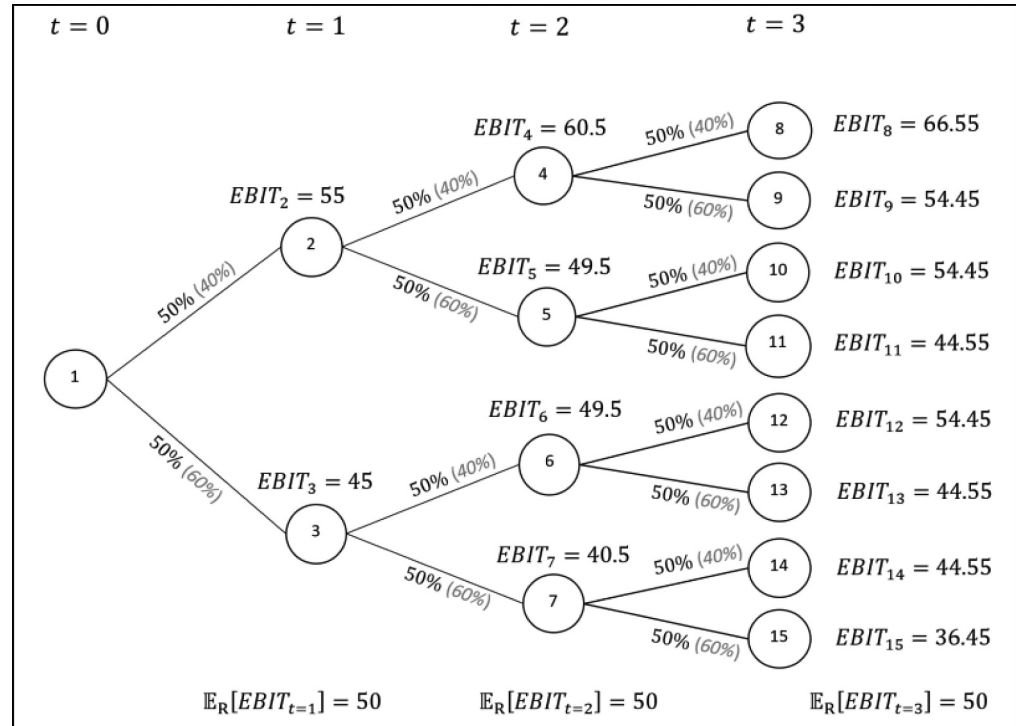


Figure 3. The Miles–Ezzell scenario tree.



With respect to the M&E setup in Figure 3 this means: from any given state in point of time t there can be a transition to either a state with $EBIT_{t+1} = EBIT_t \cdot u$ or to a state with $EBIT_{t+1} = EBIT_t \cdot d$. For the calculation of the EBIT in $t = 1$ we use the base EBIT. The real probabilities, the risk-neutral probabilities, and the factors u and d have the same values as in the M&M setup. While the unconditional expected $EBIT_t$ is equal to the base $EBIT_B$ for all points in time t , the conditional expected EBIT is now path dependent. For example, having moved on the path from node 1 to node 4 through intermediary node 2, the conditional expected $EBIT_{t=3|n=4}$ will be $50\% \cdot 66.55 + 50\% \cdot 54.45 = 60.5$.

In section 5, we will evaluate these cash flows by means of the risk-neutral probability approach. For both M&M and M&E, we will apply the same risk-neutral probabilities and the risk-free rate. In the following section, we will specify the four methods of valuation addressed in this paper.

4. Four Discounted Cash Flow Methods

To prevent any confusion about the construction of valuation methods addressed in this analysis, we will quickly outline these models here. The notation applied in these models is given as follows:

FV_L : Value of the levered firm

FV_U : Value of the unlevered firm, respectively, the unlevered equity

EV_L : Value of the levered equity

DV : Value of debt

ΔDV : Change of debt because of down payments or issue of new debt

TSV : Value of the interest tax shield

$EBIT$: Earnings before interest and taxes

I : Interest payments

τ : Corporate tax rate

q : Equity-to-firm value ratio

r_{EL} : Required return on levered equity

r_U : Required return on unlevered equity or unlevered firm

r_D : Required return on debt

r_{TS} : Required discount rate for the interest tax shields

r_{FCF} : Required discount rate in the free-cash-flow method

The equity method: In this method the value of the levered equity is calculated directly by discounting the flow to the equity holders by means of the required return on levered equity r_{EL} .

Backward iteration formula:	Perpetuity formula:
$EV_{L,t} = \frac{(EBIT_{t+1} - I_{t+1}) \cdot (1 - \tau) - \Delta DV_{t+1} + EV_{L,t+1}}{1 + r_{EL,t}}$	$EV_L = \frac{(EBIT - I) \cdot (1 - \tau)}{r_{EL}}$

The change in debt is determined as: $\Delta DV_{t+1} = DV_{t+1} - DV_t$. In the case of an M&M perpetuity, the change in debt is always zero. In the case of an M&E perpetuity, there will be debt adjustments in the single states of the world (nodes). However, both the unconditional and conditional expectation of the change in debt will be zero. The value of the levered firm can be calculated by adding the debt value to the equity value ($FV_{L,t} = EV_{L,t} + DV$) or by dividing the firm value by the constant equity-to-firm-value ratio ($FV_{L,t} = EV_{L,t}/q$).

The FCF method: In this method the value of the levered firm is calculated by discounting the free cash flow (the flow to the unlevered firm) by means of a corresponding discount rate $r_{FCF} = q \cdot r_{EL} + (1 - q) \cdot (1 - \tau) \cdot r_D$.

Backward iteration formula:	Perpetuity formula:
$FV_{L,t} = \frac{EBIT_{t+1} \cdot (1 - \tau) + FV_{L,t+1}}{1 + q_t \cdot r_{EL,t} + (1 - q_t) \cdot (1 - \tau) \cdot r_{D,t}}$	$FV_L = \frac{EBIT \cdot (1 - \tau)}{q \cdot r_{EL} + (1 - q) \cdot (1 - \tau) \cdot r_D}$

The discount rate in this formula is often referred to as the after-tax weighted average costs of capital (see, e.g., Harris & Pringle, 1985, p. 237; McConnell & Sandberg, 1975, p. 885). The FCF method evolves from the equity method, and this is a well-known relationship. Nevertheless, the derivation is outlined in Appendix 1.

The APV method: In this method the value of the levered firm is determined as the sum of the value of the unlevered firm plus the value of the interest tax shield. The value of the unlevered firm is computed by discounting the free cash flow (the flow to the unlevered equity) with the required return on unlevered equity r_U . The value of the tax shield needs to be computed by discounting the interest-tax shield with the corresponding discount rate r_{TS} .

Backward iteration formula:	Perpetuity formula:
$FV_{L,t} = FV_{U,t} + TSV_t$	$FV_L = FV_U + TSV$
$FV_{U,t} = \frac{EBIT_{t+1} \cdot (1-\tau) + FV_{U,t+1}}{1+r_{U,t}}$	$FV_U = \frac{EBIT \cdot (1-\tau)}{r_U}$
$TSV_t = \frac{TS_{t+1} + TSV_{t+1}}{1+r_{TS,t}}$	$TSV = \frac{TS}{r_{TS}}$

The CCF method: In this method the cash flow to the capital holders is discounted with the corresponding weighted average costs of capital $r_{CCF} = q \cdot r_{EL} + (1-q) \cdot r_D$. The flow to the capital holders consists of the flow to both the equity and debt holders after corporate taxation.

Backward iteration formula:	Perpetuity formula:
$FV_{L,t} = \frac{(EBIT_{t+1} - I_{t+1}) \cdot (1-\tau) + I_{t+1} + FV_{L,t+1}}{1+q_t \cdot r_{EL,t} + (1-q_t) \cdot r_{D,t}}$	$FV_L = \frac{(EBIT - I) \cdot (1-\tau) + I}{q \cdot r_{EL} + (1-q) \cdot r_D}$

This method also evolves directly from the equity method together with the valuation of debt. The derivation is shown in Appendix 2.

5. Backward Iteration Process with Risk-Neutral Probabilities

In what follows, we will describe the procedure for evaluating all of the aforementioned values: EV_L (value of levered equity); FV_L (value of the levered firm); FV_U (value of the levered firm); DV (value of debt); and TSV (value of the interest tax shield). For this purpose, we will apply the risk-neutral probability

Step 0:	Initialization: Start with the next-to-last period $t = T - 1$
Step 1:	Let N_t be the set of all nodes at point in time t . Let $F(n)$ be the set of all the offspring-nodes that evolve from node n . For each node $n \in N_t$, determine the value of the stochastic cash flow $CF_{s \in F(n)}$ and continuing value $V_{s \in F(n)}$ as follows: $V_n = \frac{\mathbb{E}_{s \in F(n)}^{\text{RN}}[CF_s + V_s]}{1+r_f}$ for all $n \in N_t$. Here $\mathbb{E}_{s \in F(n)}^{\text{RN}}$ is the conditional risk-neutral expectation of the cash flow and continuing values.
Step 2:	If $t = 0$, the valuation is complete. Otherwise, go back one time period, i.e., $t - 1 \rightarrow t$ and continue with step 1.

approach, based on the numerical values provided in Figure 2 and Figure 3. Generally, the valuation of a stochastic cash flow needs to follow a backward iteration process with the following steps:

This procedure can be simplified if the cash flows, continuing values, probabilities, and risk-free rate are path-independent. We can then apply a deterministic backward iteration process described by:

$$V_t = \frac{\mathbb{E}_{s \in N_t}^{\text{RN}}[CF_{t+1} + V_{t+1}]}{1+r_f} \text{ for all } t = 0, \dots, T-1 \quad (4)$$

As we notice for the M&M setup, the cash flows are state independent. The risk-free rate and the set of transition probabilities are also constant. This implies that all of the computed values will be state independent. Hence, in the M&M world, we can apply deterministic backward induction (3). For the M&E setup, this is not the case.

More precisely, the equations in the backward iteration process are the following:

Value of levered equity:

$$EV_{L,n} = \frac{\mathbb{E}_{s \in F(n)}^{\text{RN}} [(EBIT_s - I_s) \cdot (1 - \tau) - \Delta DV_s + EV_{L,s}]}{1 + r_f} \quad (5)$$

Value of debt:

$$DV_n = \frac{\mathbb{E}_{s \in F(n)}^{\text{RN}} [I_s + \Delta DV_s + DV_s]}{1 + r_f} \quad (6)$$

Value of unlevered equity:

$$EV_{U,n} = \frac{\mathbb{E}_{s \in F(n)}^{\text{RN}} [EBIT_s \cdot (1 - \tau) + EV_{U,s}]}{1 + r_f} \quad (7)$$

Value of interest tax shield:

$$TSV_n = \frac{\mathbb{E}_{s \in F(n)}^{\text{RN}} [I_s \cdot (1 - \tau) + TSV_s]}{1 + r_f} \quad (8)$$

To calculate the interest payments, we will introduce the set $P(s)$ that denotes the parent node of which node s springs off.

Calculation of interest payment:

$$I_s = DN_{P(s)} \cdot r_{\text{nom}} \quad (9)$$

Calculation of down payment:

$$\Delta DN_s = DN_{P(s)} - DN_s \quad (10)$$

By assumption (7) we impose $DN_n = DV_n$, and by assumption (10) we also fix a target equity-to-firm-value ratio:

$$EV_{L,n} = FV_{L,n} \cdot q \text{ and } DV_n = FV_{L,n} \cdot (1 - q) \quad (11)$$

After adding (8) into (4), (5), and (7), and after adding (9) into (4) and (5), we are left with the following equation system for each state of the world:

$$EV_{L,n} = \frac{\mathbb{E}_{s \in F(n)}^{\text{RN}} [EBIT_s \cdot (1 - \tau) - DV_{P(s)} \cdot (1 + r_f \cdot (1 - \tau)) + DV_s + EV_{L,s}]}{1 + r_f} \quad (12)$$

$$FV_{U,n} = \frac{\mathbb{E}_{s \in F(n)}^{\text{RN}} [EBIT_s \cdot (1 - \tau) + FV_{U,s}]}{1 + r_f} \quad (13)$$

$$TSV_n = \frac{\mathbb{E}_{s \in F(n)}^{\text{RN}} [DV_{P(s)} \cdot r_f \cdot (1 - \tau) + TSV_s]}{1 + r_f} \quad (14)$$

$$EV_{L,n} = FV_{L,n} \cdot q \text{ and } DV_n = FV_{L,n} \cdot (1 - q) \quad (15)$$

For all nodes n in the last period of time we assume: $DV_n = 0$, $EV_{L,n} = 0$, $EV_{U,n} = 0$ and $TSV_n = 0$. Expression (12), the value of the unlevered firm, can undergo the backward induction directly, that is, it is possible to determine the continuing value of the unlevered firm for all nodes in $t = T - 1$. Once these values are known, it is possible to determine the values in all preceding nodes (states) in $t = T - 2$, and so forth. The value of the levered equity, debt, and tax shield cannot be calculated directly. For this reason, we solve equation (11) together with (14) for the levered firm value. This leads us to the FCF method as follows:

$$FV_{L,n} = \frac{\mathbb{E}_{S \in F(n)}^{\text{RN}} [EBIT_s \cdot (1 - \tau) + FV_{L,s}]}{1 + q \cdot r_f + (1 - q) \cdot r_f \cdot (1 - \tau)} \quad (16)$$

By means of backward induction, it is now possible to compute the value of the levered firm in all nodes of the tree (from $t = T - 1$ to $t = 0$). Once the values of the levered and unlevered firm are known, both the discount rates r_U and r_{FCF} can be computed. The value of the levered equity and debt can be deduced by applying relationships (14). The value of the tax shield can also be determined as the difference between the levered and unlevered value of the firm. After the value of debt becomes known, the interest payments and the changes in debt can be calculated. Once these cash flows are available, the tax shield and the flow to levered equity can also be computed. Once all of the values and cash flows are calculated, it is possible to compute the remaining discount rates: r_{CCF} , r_{TS} , r_D , and r_{EL} .

Let us now turn back to our numerical example. Assume the following additional information:

Tax Rate: $\tau = 30\%$

Equity-to-firm-value ratio: $q = 40\%$

Risk-free interest rate: $r_f = 5\%$

For the M&M setup, the results are represented in Figure 4 and Figure 5. The results corresponding to the M&E setup are shown in Figure 6 and Figure 7. In these figures, $FtE_{L,n}$ (FtD_n , TS_n) refers to

Figure 4. The Modigliani–Miller flows to stakeholders.

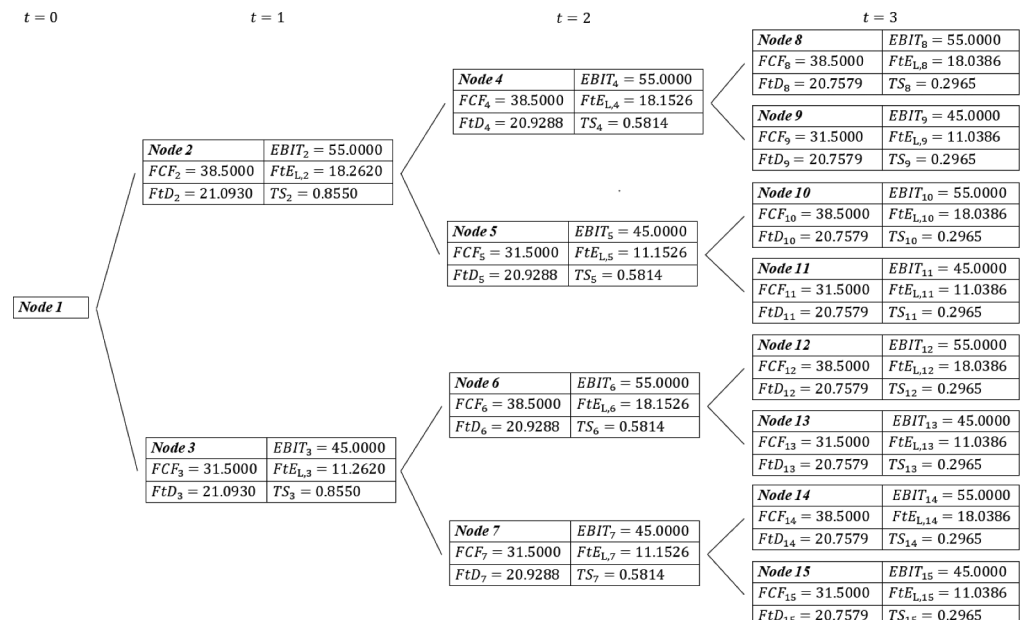


Figure 5. The Modigliani–Miller values and discount rates.

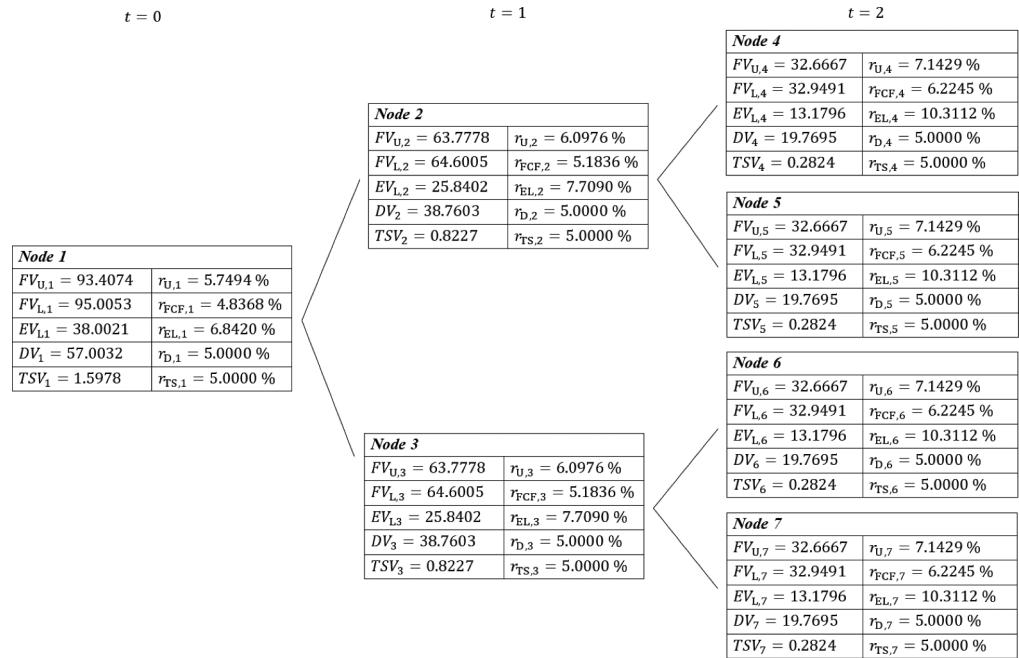
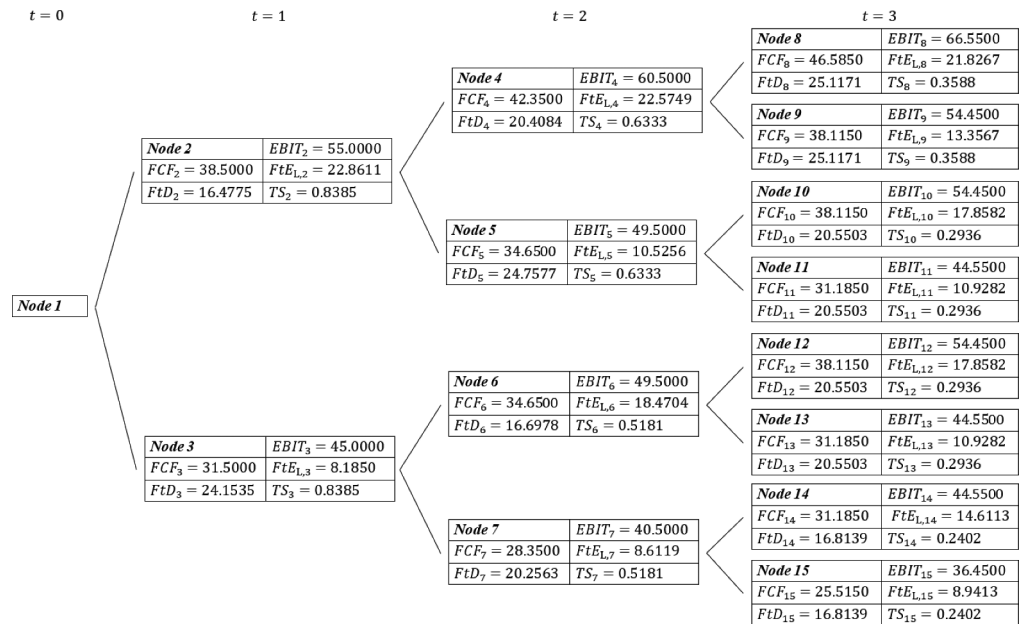


Figure 6. The Miles–Ezzell flows to stakeholders.



the flow to levered equity (flow to debt, tax shield) in node n of the tree. Some of these calculations are given in more detail in Appendix 3.

In the following section, we will summarize our observations and discuss the results with respect to the literature mentioned in sections 1 and 2.

Figure 7. The Miles–Ezzell values and discount rates.

$t = 0$		$t = 1$		$t = 2$	
Node 1 $FV_{U,1} = 91.6119$ $r_{U,1} = 7.1429\%$ $FV_{L,1} = 93.1682$ $r_{FCF,1} = 6.2245\%$ $EV_{L,1} = 37.2673$ $r_{EL,1} = 10.3112\%$ $DV_1 = 55.9009$ $r_{D,1} = 5.0000\%$ $TSV_1 = 1.5563$ $r_{TS,1} = 6.0433\%$		Node 2 $FV_{U,2} = 69.4711$ $r_{U,2} = 7.1429\%$ $FV_{L,2} = 70.3642$ $r_{FCF,2} = 6.2245\%$ $EV_{L,2} = 28.1457$ $r_{EL,2} = 10.3112\%$ $DV_2 = 42.2185$ $r_{D,2} = 5.0000\%$ $TSV_2 = 0.8931$ $r_{TS,2} = 5.6957\%$		Node 4 $FV_{U,4} = 39.5267$ $r_{U,4} = 7.1429\%$ $FV_{L,4} = 39.8684$ $r_{FCF,4} = 6.2245\%$ $EV_{L,4} = 15.9474$ $r_{EL,4} = 10.3112\%$ $DV_4 = 23.9210$ $r_{D,4} = 5.0000\%$ $TSV_4 = 0.3417$ $r_{TS,4} = 5.0000\%$	
		Node 3 $FV_{U,3} = 56.8400$ $r_{U,3} = 7.1429\%$ $FV_{L,3} = 57.5707$ $r_{FCF,3} = 6.2245\%$ $EV_{L,3} = 23.0283$ $r_{EL,3} = 10.3112\%$ $DV_3 = 34.5424$ $r_{D,3} = 5.0000\%$ $TSV_3 = 0.7307$ $r_{TS,3} = 5.6957\%$		Node 5 $FV_{U,5} = 32.3400$ $r_{U,5} = 7.1429\%$ $FV_{L,5} = 32.6196$ $r_{FCF,5} = 6.2245\%$ $EV_{L,5} = 13.0478$ $r_{EL,5} = 10.3112\%$ $DV_5 = 19.5718$ $r_{D,5} = 5.0000\%$ $TSV_5 = 0.2796$ $r_{TS,5} = 5.0000\%$	
				Node 6 $FV_{U,6} = 32.3400$ $r_{U,6} = 7.1429\%$ $FV_{L,6} = 32.6196$ $r_{FCF,6} = 6.2245\%$ $EV_{L,6} = 13.0478$ $r_{EL,6} = 10.3112\%$ $DV_6 = 19.5718$ $r_{D,6} = 5.0000\%$ $TSV_6 = 0.2796$ $r_{TS,6} = 5.0000\%$	
				Node 7 $FV_{U,7} = 26.4600$ $r_{U,7} = 7.1429\%$ $FV_{L,7} = 26.6888$ $r_{FCF,7} = 6.2245\%$ $EV_{L,7} = 10.6755$ $r_{EL,7} = 10.3112\%$ $DV_7 = 16.0133$ $r_{D,7} = 5.0000\%$ $TSV_7 = 0.2288$ $r_{TS,7} = 5.0000\%$	

6. Observations

For the numerical experiment conducted in section 5, we can summarize the observations below. Let us begin with the findings regarding all of the discount rates.

- (1) **Discount rates in last period:** in period $t = 2$ (the last period for which we have calculated values) the M&M returns coincide with the M&E returns.
- (2) **Discount rates in different states of the same period:** for both valuation settings, we notice that the discount rates in all nodes of the same time period are always the same. In other words, the discount rates are path independent.
- (3) **Discount rate for debt ($r_{D,s}$):** we have treated debt as risk-free. Therefore, it can be confirmed that the return on debt is the same in both trees across all nodes (states of the world).
- (4) **Discount rate for tax shield ($r_{TS,s}$):** in the M&M tree, the tax shield is discounted with the risk-free rate at all times, while in the M&E tree it is only risk-free in the last period. The further backward we go in time (from future to present) the higher the discount rate of the tax shield (including the continuing value) becomes in the M&E tree.
- (5) **Return on the unlevered firm:** in the M&E tree, the required return on the unlevered firm is constant for all nodes and time periods. In the M&M setup, this return changes, and more particularly, it decreases from the future to the present.
- (6) **Return on levered equity (r_{EL}):** in the M&E tree, the required return on the levered equity is constant for all nodes and time periods. In the M&M setup, this return changes, and more particularly, it decreases from the future to the present.
- (7) **Discount rate applied in the FCF method, $r_{FCF} = q \cdot r_{EL} + (1 - q) \cdot r_f \cdot (1 - \tau)$:** in the M&E tree, this discount rate is constant because all of its constituents are constant. In the M&M setup, this return decreases from the future to the present because the required return on levered equity decreases.

Let us now also look at the cash-flow behavior:

- (8) The non-conditional expected EBIT, as well as the non-conditional expected FCF were assumed to be constant (see assumption 1).
- (9) For both the M&M and the M&E setting, the nonconditional expected flow to the debt holders is not constant (i.e. not an annuity). More specifically for this experiment, the cash flow to the debt holders increases from the present to the future.
- (10) For both the M&M and M&E setting, the nonconditional expected flow to levered equity is not constant (i.e. not an annuity). More specifically for this experiment, it decreases from the present to the future.
- (11) For both the M&M and M&E setting, the nonconditional expected tax shield is not constant (i.e. not an annuity). More specifically for this experiment, it decreases from the present to the future.

These observations have the following immediate consequences:

- (a) In both cash-flow scenarios, the discount rates are path independent. This makes it possible to discount the unconditional expectations of the cash flows and values. Hence, it is possible to use the deterministic backward induction scheme:

$$V_{i,t} = \frac{\mathbb{E}[CF_{i,t+1} + V_{i,t+1}]}{1 + r_i} \text{ for all } t = 0, \dots, T - 1 \quad (17)$$

where \mathbb{E} now represents the expectation under real probabilities, and r_i is the risk-adjusted discount rate for $\mathbb{E}[CF_{i,t+1} + V_{i,t+1}]$.

- (b) For M&M, none of the values in $t = 0$ can be computed directly using the formula for the present value of an annuity of the form: Present Value = Annuity \times Annuity Factor with the annuity factor commonly being defined as:

$$\text{Annuity Factor} = \frac{(1 + r)^T - 1}{(1 + r)^T \cdot r}$$

where r is the discount rate, and T represents the lifetime of the annuity. This is because either the expected cash flows, the discount rates, or both are not constant.

- (c) For M&E, the FCF method can be carried out by means of the formula: *Present Value = Annuity \times Annuity Factor*. The same applies to the value of the unlevered firm. For these valuations, both the FCF and the corresponding discount rates are constant across time. The value of levered equity, debt, and tax-shield cannot be computed by means of an annuity valuation formula. Here a backward induction of the form (17) is required. Note, however, that these values can always be deduced from the levered and unlevered firm value as follows:

$$EV_L = FV_L \cdot q, \quad DV = FV_L \cdot (1 - q), \quad TSV = FV_L - FV_U \quad (18)$$

7. Consequences for Valuation of Annuities and Perpetuities

Based on the implications above, we will now numerically analyze the behavior and convergence of the discount rates.

7.1 The Modigliani/Miller Annuity

The discount rates for the M&M cash flows cannot be derived by applying an approach based on the annuity formula, but we can, however, apply the deterministic backward induction process (17). Let us start with the equity approach, restated in the following, where we apply the remaining

lifetime $R = T, \dots, 1$ of the annuity (instead of time index $t = 1, \dots, T$):

$$EV_{L,R} = \frac{(EBIT_B - I_R) \cdot (1 - \tau) - \Delta DV_{R-1} + EV_{L,R-1}}{1 + r_{EL,R}}$$

Note that we replace the time index on EBIT because we are now looking at a stochastic annuity with constant expectation $EBIT_B$. Let us rewrite this approach as shown in the following frame:

$$\begin{aligned} & FV_{L,R} \cdot (1 - q) \cdot r_f \quad FV_{L,R} \cdot (1 - q) - FV_{L,R-1} (1 - q) \\ & FV_{L,R} \cdot q \quad DV_R \cdot r_f \quad DV_R - DV_{R-1} \quad FV_{L,R-1} \cdot q \end{aligned} \quad (19)$$

$$EV_{L,R} = \frac{EBIT_B \cdot (1 - \tau) - I_{R-1} \cdot (1 - \tau) - DV_{R-1} + EV_{L,R-1}}{1 + r_{EL,R}}$$

This brings us to:

$$FV_{L,R} \cdot q = \frac{EBIT_B \cdot (1 - \tau) - FV_{L,R} \cdot (1 - q) \cdot r_f \cdot (1 - \tau) - FV_{L,R} \cdot (1 - q) + FV_{L,R-1}}{1 + r_{EL,R}}$$

We recognize that $r_{EL,t}$ is a compound rate because the numerator consists of the stochastic EBIT with expectation $EBIT_B$. The remaining terms of the numerator are deterministic and can be discounted with the risk-free rate:

$$FV_{L,R} \cdot q = \frac{EBIT_B \cdot (1 - \tau)}{1 + r_{EBIT}} + \frac{FV_{L,R-1}}{1 + r_f} - \frac{FV_{L,R} \cdot (1 - q) \cdot [r_f \cdot (1 - \tau) + 1]}{1 + r_f}$$

By solving this expression for the value of the levered firm, we obtain:

$$FV_{L,R} = \frac{\frac{EBIT_B \cdot (1 - \tau)}{1 + r_{EBIT}} \cdot (1 + r_f) + FV_{L,R-1}}{1 + q \cdot r_f + (1 - q) \cdot r_f \cdot (1 - \tau)} \quad (20)$$

For given r_{EBIT} (the required return on unlevered equity at point in time T) and $EBIT_B$ we are able to compute the value of the levered firm for all points in time by means of backward induction. Subsequently, we can determine the discount rate $r_{FCF,R}$ in the FCF method for a given remaining lifetime as follows:

$$FV_{L,R} = \frac{EBIT_B \cdot (1 - \tau) + FV_{L,R-1}}{1 + r_{FCF,R}} \rightarrow r_{FCF,R} = \frac{EBIT_B \cdot (1 - \tau) + FV_{L,R-1}}{FV_{L,R}} - 1 \quad (21)$$

The backward iteration for the unlevered firm value can be written as:

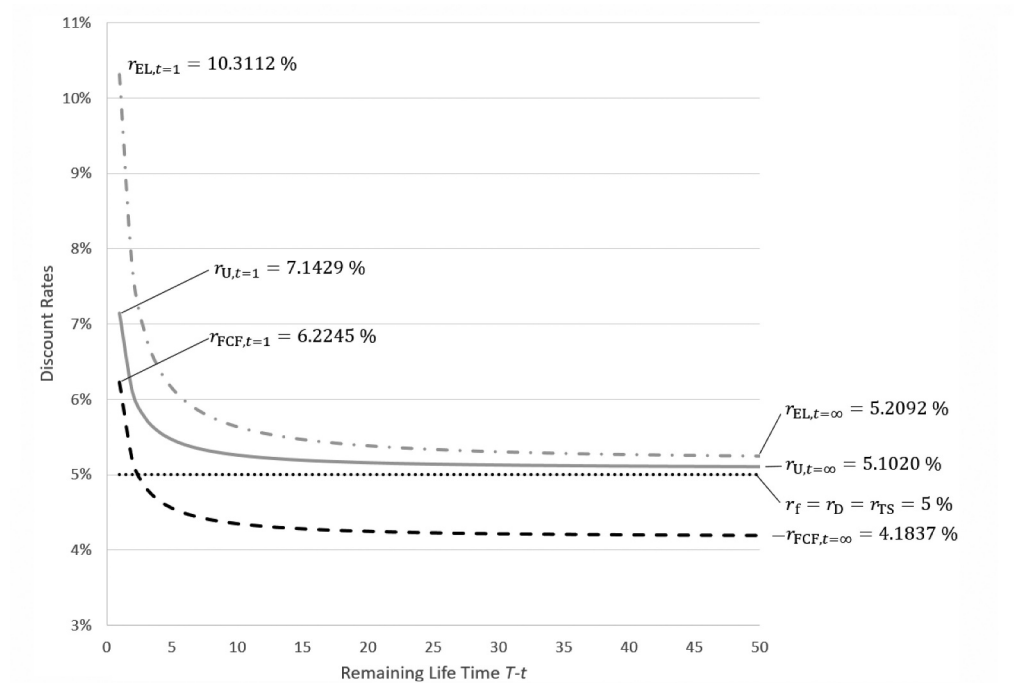
$$FV_{U,R} = \frac{EBIT_B (1 - \tau) + FV_{U,t+1}}{1 + r_{U,t}} = \frac{EBIT_B (1 - \tau)}{1 + r_{EBIT}} + \frac{FV_{U,t+1}}{1 + r_f}$$

This value can be used to deduce the required return on the unlevered firm:

$$r_{U,R} = \frac{EBIT_B (1 - \tau) + FV_{U,R-1}}{FV_{U,R}} - 1$$

Once the values for the levered and unlevered firm are known, the remaining values can be deduced according to (18). Based on these values and the corresponding cash flows, we are

Figure 8. Behavior of discount rates (required returns) in the Modigliani–Miller model.



then able to determine all of the discount rates. Figure 8 illustrates these rates depending on the remaining lifetime of the M&M annuity for the numerical values given above.

7.2 The Miles–Ezzell Annuity

In the M&E setup, all the discount rates are constant except for the discount rate applied to the tax shield. This discount rate can be determined by the following considerations. Again, let $R = T, \dots, 1$ represent the remaining lifetime of the annuity $EBIT_B$.

The discount rate for the tax shield is defined as:

$$TSV_R = \frac{DV_R \cdot r_f \cdot \tau + TSV_{R-1}}{1 + r_{TS,R}} \rightarrow r_{TS,R} = \frac{DV_R \cdot r_f \cdot \tau + TSV_{R-1}}{TSV_R} - 1 \quad (22)$$

The value of the tax shield is the difference between the levered and unlevered firm value.

$$TSV_R = FV_{L,R} - FV_{U,R} \text{ and } TSV_{R-1} = FV_{L,R-1} - FV_{U,R-1} \quad (23)$$

As concluded above, the values of the levered and unlevered firm can be calculated by means of the annuity valuation formula as follows:

$$FV_{L,R} = EBIT_B \cdot (1 - \tau) \cdot \frac{(1 + r_{FCF})^R - 1}{(1 + r_{FCF})^R \cdot r_{FCF}}$$

$$FV_{L,R-1} = EBIT_B \cdot (1 - \tau) \cdot \frac{(1 + r_{FCF})^{R-1} - 1}{(1 + r_{FCF})^{R-1} \cdot r_{FCF}} \quad (24)$$

$$FV_{U,R} = EBIT_B \cdot (1 - \tau) \cdot \frac{(1 + r_U)^R - 1}{(1 + r_U)^R \cdot r_U}$$

$$FV_{U,R-1} = EBIT_B \cdot (1 - \tau) \cdot \frac{(1 + r_U)^{R-1} - 1}{(1 + r_U)^{R-1} \cdot r_U} \quad (25)$$

The debt value can be calculated by means of the debt-to-firm-value ratio.

$$DV_R = (1 - q) \cdot FV_{L,R} \quad (26)$$

Substituting (22) to (25) into (21) yields the following expression for the discount rate of the tax shield:

$$r_{TS,R} = \frac{(1 - q) r_f \tau \varphi_{FCF,R} - (1 + r_{FCF})^{-R} + (1 + r_U)^{-R}}{\varphi_{FCF,R} - \varphi_{U,R}}$$

where $\varphi_{FCF,R}$ and $\varphi_{U,R}$ represent the annuity factors:

$$\varphi_{FCF,R} = \frac{(1 + r_{FCF})^R - 1}{(1 + r_{FCF})^R r_{FCF}} \text{ and } \varphi_{U,R} = \frac{(1 + r_U)^R - 1}{(1 + r_U)^R r_U}$$

Figure 9 illustrates the behavior of the discount rates for an M&E annuity depending on the remaining lifetime and for the numerical values chosen in our example.

7.3. The Modigliani/Miller Perpetuity

At the end of this section, we will quickly compare the numerical results with the considerations in the literature for the case of perpetual cash flows. For M&M, the necessary conditions for the consistency of the four valuation models are well established. These are:

- (1) The tax shield is discounted by means of the risk-free rate (or the rate of debt):

$$r_{TS} = r_f$$

- (2) The translation from the unlevered return on equity to the discount rate in the FCF method is (Modigliani & Miller, 1963, p. 438):

$$r_{FCF} = r_U [1 - \tau (1 - q)] \quad (27)$$

$$r_{FCF} = 5.102041\% [1 - 30\% (1 - 40\%)] = 4.1837\%$$

- (3) The discount rate in the FCF method is also defined as (see FCF method in Section 4 or Appendix 1):

$$r_{FCF} = q r_{EL} + (1 - q) (1 - \tau) r_D \quad (28)$$

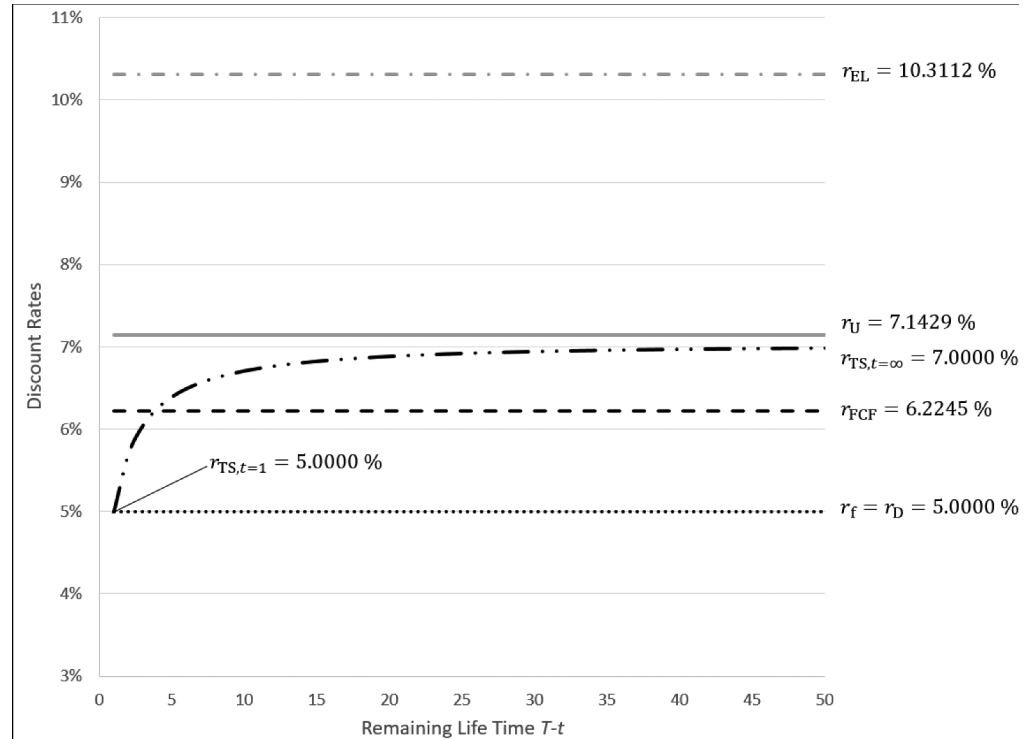
$$r_{FCF} = 40\% 5.2092\% + (1 - 40\%) (1 - 30\%) 5\% = 4.1837\%$$

- (4) The translation from the unlevered return on equity to the levered return on equity is given as follows (Modigliani & Miller, 1963, p. 439):

$$r_{EL} = r_U + (1 - \tau) (r_U - r_f) \frac{(1 - q)}{q} \quad (29)$$

$$r_{EL} = 5.102041\% + (1 - 30\%) (5.102041\% - 5\%) \frac{(1 - 40\%)}{40\%} = 5.2092\%$$

Figure 9. Behavior of discount rates (required returns) in the Miles–Ezzell model.



(5) The discount rate in the CCF method is (see CCF method in Section 4 or Appendix 2):

$$r_{CCF} = q \cdot r_{EL} + (1 - q) \cdot r_f \quad (30)$$

(5) From (29) and (30), we can also derive the translation from the unlevered return on equity to the discount rate in the CCF method as shown by Nantell and Carlson (1975, p. 1348):

$$r_{CCF} = r_U + (r_f - r_U) \tau (1 - q) \quad (31)$$

$$r_{CCF} = 5.102041\% + (5\% - 5.102041\%) 30\% (1 - 40\%) = 5.0837\%$$

Taking (31) the other way around we obtain:

$$r_U = \frac{r_{CCF} - r_f \tau (1 - q)}{1 - \tau (1 - q)} = \frac{r_{FCF}}{1 - \tau (1 - q)}$$

Contrary to this result, Brealey et al. (2017) suggest unlevering by means of $r_U = r_{FCF}$ and at the same time leveraging by means of (29), which generates an inconsistency.

7.4. The Miles–Ezzell Perpetuity

For the M&E perpetuity our analysis suggests the following:

- (1) Both Arzac and Glosten (2005, equation 13 with growth rate $g = 0$) and Barbi (2012, equation 15) derived the correct formula for the value of the tax shield. Their formula directly induces the discount rate for the tax shield as follows:

$$r_{TS} = r_U \frac{1 + r_f}{1 + r_U}$$

$$r_{TS} = 7.14286\% \frac{1 + 5\%}{1 + 7.1429\%} = 7.000\% \quad (32)$$

We therefore reject the opinion of Harris and Pringle (1985), who claim to discount the tax-shield with the required return of the unlevered firm.

(2) The analysis in this paper confirms the results of Miles and Ezzell (1980, p. 726, formula 20) concerning the translation of r_{FCF} and r_U which is:

$$r_{FCF} = r_U - r_f \tau (1 - q) \frac{1 + r_U}{1 + r_f} \quad (33)$$

$$r_{FCF} = 7.14286\% - 5\% 30\% (1 - 40\%) \frac{1 + 7.14286\%}{1 + 5\%} = 6.2245\%$$

(3) As in the case of M&M, the discount rate in the FCF method can also be represented by formula (28).

(4) From the r_U -to- r_{FCF} translation we can directly deduce the relationship between the unlevered and levered return on equity:

$$r_{EL} = \frac{r_U - r_f \cdot (1 - q) \cdot \left[\frac{\tau \cdot (r_U - r_f)}{1 + r_f} + 1 \right]}{q} \quad (34)$$

$$r_{EL} = \frac{7.14286\% - 5\% (1 - 40\%) \left[\frac{30\% (7.14286\% - 5\%)}{1 + 5\%} + 1 \right]}{40\%} = 10.3112\%$$

In a world with corporate taxes, we can therefore reject the formula of Taggart (1991, p. 14) and Cooper and Nyborg (2006) who stated:

$$r_{EL} = \frac{r_U - r_f \cdot (1 - q)}{q} \quad (35)$$

However, it is interesting to recognize that formula (29), in the absence of taxes, reduces to (35).

(5) The discount rate in the CCF method is the same as for M&M (see formula (30)).

(6) From the expressions (30) and (34), we can derive the translation from the unlevered return on equity to the discount rate in the CCF method.

$$r_{CCF} = r_U - r_f (1 - q) \frac{\tau (r_U - r_f)}{1 + r_f} \quad (36)$$

$$r_{CCF} = 7.14286\% - 5\% (1 - 40\%) \frac{30\% (7.14286\% - 5\%)}{1 + 5\%} = 7.1245\%$$

8. Summary and Outlook

In this paper, we have addressed the differences between the Modigliani–Miller (M&M) model from 1958 and 1963 and the Miles–Ezzell (M&E) model from 1980. The only different assumption between these two frameworks is related to the stochasticity of the free cash flow: while M&M assumes a strictly stationary process, M&E departs from a process with the martingale property. This difference forces three implications for perpetuities: first, in the M&M setting, the level of debt and equity is state independent. In the M&E setting, however, the value of debt and equity are state dependent. For the latter, this means that the outstanding amount of debt needs to be adjusted throughout time in different states of the world. The second implication concerns the discount rate on the tax shield: in the M&M setting, this rate equals the risk-free rate, while in the M&E setting it represents a compound of the risk-free rate and the required return on the unlevered firm (see equation (32) above). The third implication concerns the translation formulae between r_U , r_{EL} , r_{FCF} , and r_{CCF} . As shown in the previous section, the formulae for a strictly stationary process (M&M) are different from the martingale process (M&E). When it comes to annuities, these implications are also valid. In addition, we observe that the

difference in the cash-flow stochasticity affects the timely behavior of the discount rates. In the M&E framework, all of the discount rates except the discount rate for the tax shield remain constant. In other words, the discount rate of the tax shield depends on the remaining maturity. Contrary to this, in the M&M annuity, the discount rate of debt and the tax shield remain constant over time; they are equal to the risk-free rate. The discount rates r_U , r_{EL} , r_{FCF} , and r_{CCF} , however, depend on the remaining maturity.

This has consequences for the calculation of the value of the unlevered firm, the levered firm, equity, debt, and the tax shield when free cash flows are annuities: annuity formulae of the form *Present Value = Annuity × Annuity Factor* need to be applied with care. More precisely, this straightforward formula cannot be applied to M&M type cash flows. In the case of M&E, this approach is exclusively applicable to the FCF approach and the valuation of the unlevered firm.

Finally, we will suggest some obvious directions for further research. The simplest extension would be to address growth in the cash flows (see, e.g., Dempsey, 2013; Fernandez, 2004). In the M&M model, we would then propose that $\mathbb{E}[EBIT_t|EBIT_{t-1}] = EBIT_B \cdot (1+g)^{t-1}$, and in the M&E setup, we would claim that $\mathbb{E}[EBIT_t|EBIT_{t-1}] = EBIT_{t-1} \cdot (1+g)$. In the latter model, this can be forced by changing the factors u and d or by skewing the real probabilities (see Section 3). Other immediate suggestions are the inclusion of personal taxes (see Cooper & Nyborg, 2008; Miller, 1977; Stapleton, 1972; Taggart, 1991), and risky debt, both with and without the possibility of bankruptcy.

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Citation information

Cite this article as: The difference between Modigliani-Miller and Miles-Ezzell and its consequences for the valuation of annuities, Denis M. Becker, *Cogent Economics & Finance* (2021), 9: 1862446.

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Appendix 1. Derivation of the FCF Method from the Equity Method

In this appendix, we briefly outline the derivation of the free-cash-flow (FCF) method. We start from the equity method:

$$EV_{L,t} = \frac{(EBIT_{t+1} - I_{t+1})(1 - \tau) - \Delta DV_{t+1} + EV_{L,t+1}}{1 + r_{EL,t}}$$

We replace the following:

$$I_{t+1} = DN_t \cdot r_{nom} = DV_t \cdot r_f = FV_{L,t} \cdot (1 - q) \cdot r_f$$

$$\Delta DV_{t+1} = DN_t - DN_{t+1} = DV_t - DV_{t+1} = FV_{L,t} \cdot (1 - q) - FV_{L,t+1} \cdot (1 - q)$$

$$EV_{L,t} = q \cdot FV_{L,t}$$

$$EV_{L,t+1} = q \cdot FV_{L,t+1}$$

After adding these expressions into the equity method formula, it is possible to solve for the levered firm value $FV_{L,t+1}$, which results in the FCF method presented in Section 4.

Appendix 2. Derivation of the Capital Cash Flow Method

In this appendix we briefly show the derivation of the capital-cash-flow (CCF) method. The equity method can be stated as:

$$EV_{L,t}(1 + r_{EL,t}) = (EBIT_{t+1} - I_{t+1})(1 - \tau) - \Delta DV_{t+1} + EV_{L,t+1}$$

From the valuation of debt we know:

$$DV_t(1 + r_f) = I_{t+1} + \Delta DV_{t+1} + DV_{t+1}$$

Adding these two equations together,

we obtain:

$$EV_{L,t} \cdot (1 + r_{EL,t}) + DV_t \cdot (1 + r_f) = (EBIT_{t+1} - I_{t+1}) \cdot (1 - \tau) + EV_{L,t+1} + I_{t+1} + DV_{t+1}$$

On the left side of this expression, we replace $EV_{L,t} = FV_{L,t} \cdot q$ and $DV_t = FV_{L,t} \cdot (1 - q)$. On the right side of this equation we see that $EV_{L,t+1} + DV_{t+1} = FV_{L,t+1}$.

Hence, we obtain:

$$FV_{L,t} \cdot q \cdot (1 + r_{EL,t}) + FV_{L,t} \cdot (1 - q)(1 + r_f) = (EBIT_{t+1} - I_{t+1}) \cdot (1 - \tau) + I_{t+1} + FV_{L,t+1}$$

On the left side, we factor out $FV_{L,t}$. Then we divide by the remainder, which brings us to:

$$FV_{L,t} = \frac{(EBIT_{t+1} - I_{t+1}) \cdot (1 - \tau) + I_{t+1} + FV_{L,t+1}}{q \cdot (1 + r_{EL,t}) + (1 - q) \cdot (1 + r_f)}$$

We can finally rearrange the denominator (discount factor), which brings us to the formula presented in Section 4.

Appendix 3 Calculations in the Numerical Example

In this appendix, we will show a selection of the detailed numerical calculations of the flows, values, and discount rates shown in Figure 2 through Figure 7. Because the calculations are conceptually equal for both the M&M and M&E models, we will here only show the latter. Please note that possible decimals beyond the fourth decimal are suppressed. The development of the EBIT in case of M&E is determined as follows:

$EBIT_2 = EBIT_8 \cdot u = 50 \cdot 1.1 = 55$	$EBIT_3 = EBIT_8 \cdot d = 50 \cdot 0.9 = 45$
$EBIT_4 = EBIT_2 \cdot u = 55 \cdot 1.1 = 60.5$	$EBIT_5 = EBIT_2 \cdot d = 55 \cdot 0.9 = 49.5$
$EBIT_6 = EBIT_3 \cdot u = 45 \cdot 1.1 = 49.5$	$EBIT_7 = EBIT_3 \cdot d = 45 \cdot 0.9 = 40.5$
...	...

All of the free cash flows are calculated as:

$$\begin{aligned} FCF_n &= EBIT_n (1 - \tau) \\ &= EBIT_n (1 - 30\%) \text{ for all } n = 2, \dots, 15 \end{aligned}$$

By backward iteration, the value of the levered and unlevered firm can be determined. We start calculating the values in nodes 4 to 7. For example, for node 4 the precise calculations are as follows:

Levered firm value in node 4	Unlevered firm value in node 4
$\begin{aligned} FV_{L,4} &= \frac{p_u^N (FCF_8 + FV_{L,8}) + p_d^N (FCF_9 + FV_{L,9})}{1 + q \cdot r_f + (1 - q) \cdot (1 - \tau) \cdot r_f} \\ &= \frac{40\% \cdot (46.5850 + 0) + 60\% \cdot (38.115 + 0)}{1 + 40\% \cdot 5\% + (1 - 40\%) \cdot (1 - 30\%) \cdot 5\%} \\ &= 39.8684 \end{aligned}$	$\begin{aligned} FV_{U,4} &= \frac{p_u^N (FCF_8 + FV_{U,8}) + p_d^N (FCF_9 + FV_{U,9})}{1 + r_f} \\ &= \frac{40\% \cdot (46.5850 + 0) + 60\% \cdot (38.115 + 0)}{1 + 5\%} \\ &= 39.5267 \end{aligned}$
$\begin{aligned} r_{FCF,4} &= \frac{p_u^B (FCF_8 + FV_{L,8}) + p_d^B (FCF_9 + FV_{L,9})}{FV_{L,4}} - 1 \\ &= \frac{50\% \cdot (46.5850 + 0) + 50\% \cdot (38.115 + 0)}{39.8684} - 1 \\ &= 6.2245\% \end{aligned}$	$\begin{aligned} r_{U,4} &= \frac{p_u^B (FCF_8 + FV_{U,8}) + p_d^B (FCF_9 + FV_{U,9})}{FV_{U,4}} - 1 \\ &= \frac{50\% \cdot (46.5850 + 0) + 50\% \cdot (38.115 + 0)}{39.5267} - 1 \\ &= 7.1429\% \end{aligned}$

After the continuing values for the nodes in $t = T - 1$ are calculated, we proceed with the nodes in $t = T - 2$. In our example, these are nodes $n = 2$ and $n = 3$. For node $n = 2$, the detailed calculations are:

Levered firm value in node 2	Unlevered firm value in node 2
$FV_{L,2} = \frac{p_u^N \cdot (FCF_4 + FV_{L,4}) + p_d^N \cdot (FCF_5 + FV_{L,5})}{1 + q \cdot r_f + (1 - q) \cdot (1 - \tau) \cdot r_f}$ $= \frac{40\% \cdot (42.35 + 39.8684) + 60\% \cdot (34.65 + 32.6196)}{1 + 40\% \cdot 5\% + (1 - 40\%) \cdot (1 - 30\%) \cdot 5\%}$ $= 70.3642$ $r_{FCF,2} = \frac{p_u^R \cdot (FCF_4 + FV_{L,4}) + p_d^R \cdot (FCF_5 + FV_{L,5})}{FV_{L,2}} - 1$ $= \frac{50\% \cdot (42.35 + 39.8684) + 50\% \cdot (34.65 + 32.6196)}{70.3642} - 1 = 6.2245\%$	$FV_{U,2} = \frac{p_u^N \cdot (FCF_4 + FV_{U,4}) + p_d^N \cdot (FCF_5 + FV_{U,5})}{1 + r_f}$ $= \frac{40\% \cdot (42.35 + 39.5267) + 60\% \cdot (34.65 + 32.34)}{1 + 5\%}$ $= 26.4600$ $r_{U,2} = \frac{p_u^R \cdot (FCF_4 + FV_{U,4}) + p_d^R \cdot (FCF_5 + FV_{U,5})}{FV_{U,2}} - 1$ $= 7.1429\%$

We now proceed to period $t = T - 3$. In our case, this is the present point of time, $t = 0$, where we find the root node of the tree.

Levered firm value in node 1	Unlevered firm value in node 1
$FV_{L,1} = \frac{p_u^N \cdot (FCF_2 + FV_{L,2}) + p_d^N \cdot (FCF_3 + FV_{L,3})}{1 + q \cdot r_f + (1 - q) \cdot (1 - \tau) \cdot r_f}$ $= \frac{40\% \cdot (38.5 + 70.3642) + 60\% \cdot (31.5 + 57.5707)}{1 + 40\% \cdot 5\% + (1 - 40\%) \cdot (1 - 30\%) \cdot 5\%}$ $= 93.1682$ $r_{FCF,1} = \frac{p_u^R \cdot (FCF_2 + FV_{L,2}) + p_d^R \cdot (FCF_3 + FV_{L,3})}{FV_{L,1}} - 1$ $= \frac{50\% \cdot (38.5 + 70.3642) + 50\% \cdot (31.5 + 57.5707)}{93.1682} - 1 = 6.2245\%$	$FV_{U,1} = \frac{p_u^N \cdot (FCF_2 + FV_{U,2}) + p_d^N \cdot (FCF_3 + FV_{U,3})}{1 + r_f}$ $= \frac{40\% \cdot (38.5 + 69.4711) + 60\% \cdot (31.5 + 56.84)}{1 + 5\%}$ $= 91.6119$ $r_{U,1} = \frac{p_u^R \cdot (FCF_2 + FV_{U,2}) + p_d^R \cdot (FCF_3 + FV_{U,3})}{FV_{U,1}} - 1$ $= \frac{50\% \cdot (38.5 + 69.4711) + 50\% \cdot (31.5 + 56.84)}{91.6119} - 1 = 7.1429\%$

From the values above, we can now deduce the value of levered equity, debt, and the tax shield. The precise calculations for nodes 1, 2, and 4 are the following.

Levered equity value and debt	Tax shield value
$EV_{L,1} = q \cdot FV_{L,1} = 40\% \cdot 93.1682 = 37.2673$ $DV_1 = (1 - q) \cdot FV_{L,1} = (1 - 40\%) \cdot 93.1682 = 55.9009$	$VTS_1 = FV_{L,1} - FV_{U,1} = 93.1682 - 91.6119 = 1.5563$

Levered equity value and debt	Tax shield value
$EV_{L,2} = q \cdot FV_{L,2} = 40\% \cdot 70.3642 = 28.1457$ $DV_2 = (1 - q) \cdot FV_{L,2} = (1 - 40\%) \cdot 70.3642 = 42.2185$	$VTS_2 = FV_{L,2} - FV_{U,2} = 70.3642 - 69.4711 = 0.8931$

Levered equity value and debt	Tax shield value
$EV_{L,4} = q \cdot FV_{L,4} = 40\% \cdot 39.8684 = 15.9474$ $DV_4 = (1 - q) \cdot FV_{L,4} = (1 - 40\%) \cdot 39.8684 = 23.9210$	$VTS_4 = FV_{L,4} - FV_{U,4} = 39.8684 - 39.5267 = 0.3417$

Once, we know the value of debt, we can determine the interest payments, the interest tax shield, and the change in debt. Again the calculations are shown for nodes 2, 4, and 8.

	Interest payment	Interest tax shield
Node 2	$I_2 = DV_1 \cdot r_f = 55.9009 \cdot 5\% = 2.7950$	$TS_2 = DV_1 \cdot r_f \cdot \tau = 55.9009 \cdot 5\% \cdot 30\% = 0.8385$
Node 4	$I_4 = DV_2 \cdot r_f = 42.2185 \cdot 5\% = 2.1109$	$TS_4 = DV_2 \cdot r_f \cdot \tau = 42.2185 \cdot 5\% \cdot 30\% = 0.6333$
Node 8	$I_8 = DV_4 \cdot r_f = 23.9210 \cdot 5\% = 1.1961$	$TS_8 = DV_4 \cdot r_f \cdot \tau = 23.9210 \cdot 5\% \cdot 30\% = 0.3588$

	Change in debt
Node 2	$\Delta DV_2 = DV_1 - DV_2 = 55.9009 - 42.2185 = 13.6824$
Node 4	$\Delta DV_4 = DV_2 - DV_4 = 42.2185 - 23.9210 = 18.2975$
Node 8	$\Delta DV_8 = DV_4 - DV_8 = 23.9210 - 0 = 23.9210$

Once we know the interest payments and the change in the outstanding debt, it is possible to determine the flow to equity. For nodes 2, 4, and 8 the calculations are:

	Flow to equity
Node 2	$FtE_{L,2} = (EBIT_2 - I_2)(1 - \tau) - DV_2 = (55 - 2.7950)(1 - 30\%) - 13.6824 = 22.8611$
Node 4	$FtE_{L,4} = (EBIT_4 - I_4)(1 - \tau) - DV_4 = (60.5 - 2.1109)(1 - 30\%) - 18.2975 = 22.5749$
Node 8	$FtE_{L,8} = (EBIT_8 - I_8)(1 - \tau) - DV_8 = (66.55 - 1.1961)(1 - 30\%) - 23.9210 = 21.8267$

Once all of the cash flows and continuing values are determined, it is possible to deduce all of the required returns or discount rates. We will illustrate this for nodes 2 and 4.

	Return on levered equity:
Node 2	$r_{EV,2} = \frac{p_u^8 \times (FtE_{L,4} + EV_{L,4}) + p_D^8 \times (FtE_{L,5} + EV_{L,5})}{EV_{L,2}} - 1 = 10.3112\%$
Node 4	$r_{EV,4} = \frac{p_u^8 \times (FtE_{L,8} + EV_{L,8}) + p_D^8 \times (FtE_{L,9} + EV_{L,9})}{EV_{L,4}} - 1 = 10.3112\%$



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