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Solution Strategies of Dynamic Stochastic General Equilibrium (DSGE) models

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Abstract

DSGE models are the main tool for analysing various questions in problems of monetary, business cycle theory and fiscal policy problems, growth and other fields in international macroeconomics and macroeconomics. Many macroeconomic publications use the DSGE framework. A consensus has been reached on the methodology for using such kind of model. The resolution of DSGE models remains an area of ongoing interest. This paper provides an overview of the available solution techniques. Linear approximation methods and perturbation methods have been explored in detail. Solving strategies such as the eigenvalue auto-decomposition of Blanchard and Kahn (1980) or the method of indefinite coefficients are explained. A Bayesian estimate is drawn shortly. The evaluation methods are briefly described. Finally, the paper provides some useful resources for practical implementation.

Key Words: DSGE models, solution strategies, Blanchard-Kahn conditions, perturbation methods, practical implementation

JEL-Classification: C3, C63, C68, C88

1 Introduction

There has been a tremendous and rapid change in macroeconomic thinking and modeling during the last 20 to 30 years, which may well be compared to a jump “from the Wright brothers to an Airbus 380 in one generation.”¹ This “quantum leap”² of macroeconomic theory started with the seminal paper by Kydland and Prescott (1982) modeling the U.S. economy and introducing the Real Business Cycle model (RBC model). Despite many opponents, this new methodological approach of dynamic stochastic general equilibrium models (DSGE) became more and more accepted over the years, and now represents a “significant share of publications in macroeconomics.”³

These days, DSGE models are a standard tool in various fields of economics, especially in macroeconomics and international economics, to deal with aggregate economic problems in business cycles, growth, monetary and fiscal stabilization policy. According to Blanchard (2008), among others, after a long debate about the appropriate macroeconomic model during the past several decades,⁴ a consensus on the methodological approach to macroeconomic questions has been reached.

During the last few years there has been extensive research concerning monetary policy within the framework of New Keynesian models,⁵ and not surprisingly, these models have become very popular at central banks.⁶ They even build their own models such as the European Central Bank (NAWM),⁷ the Bank of Canada (ToTEM),⁸ the Bank of England (BEQM),⁹ the Central Bank of Chile (MAS),¹⁰ the Norges Bank (NEMO),¹¹ the Sveriges Riksbank

¹Fernández-Villaverde (2009a), p. 1.

²Fernández-Villaverde (2009a), p. 1.

³An and Schorfheide (2007), p.113.

⁴For an extensive overview of the history of the theories of business cycles see Landmann (2007).

⁵Extensive textbook treatments of NK models for monetary policy analysis are given by Woodford (2003) and Galí (2008).

⁶Despite the fact that the baseline sticky price model does not generate time series for important macroeconomic variables as can be observed in the data. For instance, one has to make the assumption of implausible levels of nominal rigidities to generate the persistence in inflation, output and real wages. Rabanal and Rubio-Ramirez (2005) give some examples and compare the outcome of three extended New Keynesian models with the baseline model with Calvo (1983) price setting. Using a Bayesian approach on the estimation side they find that adding price indexation to the baseline model results in higher inflation persistence than in the baseline model.

⁷Christoffel et al. (2008) build a micro-founded open-economy model of the euro area (the New Area-Wide Model (NAWM)), which is used by the ECB/Eurosystem staff to perform macroeconomic projection exercises. The model builds on the one developed by Smets and Wouters (2003) and succeeds the Area-Wide Model (Fagan et al., 2001), a traditional macroeconomic model of the euro area which has been used by the ECB for several years.

⁸Terms-of-Trade Economic Model, Murchison and Rennison (2006).

⁹Bank of England Quartely Model, Harrison et al. (2005).

¹⁰Model of Analysis and Simulation, Medina and Soto (2007).

¹¹Norwegian Economy Model, Brubakk et al. (2006).

(RAMSES)¹² or the US Federal Reserve (SIGMA).¹³ Moreover there are DSGE models of supranational institutions like the IMF (GEM)¹⁴.

Apparently, the main features of this class of models are well summarized by its denotation: DSGE models are dynamic, referring to the models' focus on the time paths of the variables, in contrast to one-time variables. General alludes to the attempt to explaining the "whole economy" and not just partial markets.¹⁵ **Equilibrium** relates to the fact that the model is based on an economic theory where market mechanisms create a balance between demand and supply in the different markets of the economy. Finally, some kind of **stochastic** disturbance is introduced by assumption.

Within the broad field of macroeconomic models there are two types of DSGE models which can be distinguished, Real Business Cycle Models (RBC models) or, falling in the same class, New Keynesian Models (NK models) depending on the mix of techniques included and the choice of sources of stochastic shocks on the one side, and Overlapping Generations Models (OLG) on the other side.

The former are a combination of the stochastic neoclassical growth model (the core model) and some kind of real shocks to study business cycle fluctuations in an environment with flexible prices. The seminal paper by Kydland and Prescott (1982) started the RBC theory in particular and DSGE modeling in general.¹⁶

As the name suggests, RBC models focus on the real economy and hence abstract from monetary issues. As the importance of money has been documented by empirical studies, NK models introduce money into the framework as well as monetary authorities and in addition, they abandon the assumption of perfect competition in the markets for goods and services or for the labor market, and private sector consumption and investment decisions are characterized by rigidities.¹⁷ Nominal rigidities are introduced via sticky prices or sticky wages or both implying that monetary policy affects real variables.

Solving DSGE models remains a wide area of research. Both theoretical and computational issues are explored when setting up a DSGE model and applying it for macroeconomic analysis. They are not only of interest to the research field because of theoretical assumptions, but

¹²Riksbanks Aggregated Macro model for Studies of the Economy in Sweden. Adolfson et al. (2007) give a detailed description of the theoretical side of the model, whereas Adolfson et al. (2008) is a detailed empirical evaluation of the model.

¹³Erceg et al. (2006) set up a multicountry open economy model which is based on the seminal open economy modeling framework by Obstfeld and Rogoff (1995).

¹⁴Global Economic Model, Tchakarov et al. (2004).

¹⁵But this does not necessarily mean that the model is able to reproduce every important part of an economy, as for example very often parts of an economy are not modeled in a sophisticated way (fiscal policy, financial markets).

¹⁶Good surveys for this branch of literature can be found in Cooley (1995).

¹⁷NK models are RBC models with a spanner thrown in the works.

from a mathematical perspective as well, as new assumptions might require different solution techniques. But in general, solving DSGE models follows the same solution procedure no matter what kind of model is analyzed. After having identified the model's assumptions, the first-order equilibrium conditions have to be derived. Together with the structural equations, they build a system of stochastic difference equations. Usually, this system is non-linear, so in the next step, approximation methods lead to a linear system whose solution approximates the solution of interest. Finally, in order to analyze the performance of the model, impulse response functions or second moments are computed.

Due to the importance and the widespread use of this methodological approach in macroeconomics, this paper serves as a survey of the solution methods of DSGE models. It is supposed to be a general introduction which does not focus on a special model. The aim of the paper is rather to shed some light on the various methods which are in practical use and to serve as a starting point and overview for individual research for those wanting to build their own DSGE model. Each of the necessary steps in this procedure is described in general. Furthermore, this paper provides literature to work with the methodology in detail. There are publications and even books on this field, but they focus on very different parts of the procedure of working with DSGE models, or they demonstrate the method using a specified model. To name a few, Canova (2007) is an extensive introduction to macroeconometrics, whereas Fernández-Villaverde (2009a) introduces Bayesian estimation methods. Woodford (2003) or Galí (2008) focus on one type of model, the New Keynesian framework. Tovar (2009) highlights the application of DSGE models at central banks.

The remainder of the paper is organized as follows. The next section briefly describes benefits and weaknesses of this class of models. In section 3 we will focus on the basic setup of the model. The main part of this paper will be section 4, focusing on various solution methods of DSGE models with an emphasis on linear approximation (section 4.1), solving linear difference equations in 4.2 and perturbation methods in section 4.3. Section 5 briefly introduces estimation of DSGE models. Practical implementation is discussed in 6. The final section concludes.

2 Advantages and Disadvantages of DSGE Models

As laid out in the introduction, a consensus in methodology has been reached for using DSGE models. Nevertheless, many macroeconomists point out the limitations of DSGE models. The models come under criticism particularly with regard to their assumptions, their solution strategies, the complexity of their solutions, and for their overall implementation in the decision making process at political institutions.

With respect to the simplifying assumptions imposed in DSGE models (or in particular with regard to RBC models), there are serious concerns about the "complete markets paradigm", efficiency of markets, and the focus on representative households with rational expectations.¹⁸ But for the last few years, many papers have focused on DSGE models with more realistic assumptions, e.g., rule-of-thumb behavior of agents, frictions in financial markets, or heterogeneous agents¹⁹.

Regarding the solution strategy, several authors complain that the procedure of linearizing the model, or as Buiter (2009) calls it, the "castration of the macroeconomic models", leads to trivializing the problem under consideration, which is highly non-linear. Approximating optimality conditions leads to long and complicated systems of equations with forward-looking expectations in a messy notation which is hard to understand in detail. Sometimes it is not easy to provide an economic interpretation of the equations to the public, and it is hard to explain transmission mechanism or identify the transmission channels of a shock or a policy decision in this economy. In most cases, analytical solutions cannot be computed, thus an additional challenge when working with these models is to find reliable numerical solution algorithms. Due to their complex nature, there might be limited acceptance among policy makers because models and solutions are hard to communicate to the public.

To conclude, in order to understand the set-up and the workings of models the researcher or policy maker must be a well-trained macroeconomist with a modeling culture and strong statistical and programming skills. However, macroeconomic modeling is a mandatory class in many PhD programs, and publications and papers provide a useful guide to the topic.

Though researchers or policy makers should be aware of all caveats of DSGE models mentioned above, they can rely on this methodology, as its benefits are manifold. The first advantage of these models compared to models with *ad hoc* assumptions, i.e., models which do not model individual behavior in-depth, is the explicit microeconomic foundation of the underlying assumptions with optimizing agents under rational expectations. This approach explicitly specifies the objectives and constraints of the private and the public sectors, followed by the determination of prices and allocations. Effects of policy decisions on optimal decisions such as the labor supply schedule and the consumption demand by households or the labor demand and price decisions by firms can be analyzed directly. This optimizing behavior also means that individuals and firms base their conduct on the best possible forecasts of the future that they are able to produce. It is assumed that they have rational expectations in contrast to the rule-of-thumb behavior of the models studied previously. Moreover, micro-founded models give an exact relationship of reduced-form parameters to deeper structural parameters of the model, which are less likely to change in response to changes in the policy

¹⁸Buiter (2009) even excoriates DSGE models for their assumptions and the failure to predict the current financial crisis.

¹⁹For an example, see Krusell et al. (1998).

regime. Thus, these models are robust to the Lucas (1976) critique.

Furthermore, approximating the utility function of the representative agent of the economy results in a welfare criterion which is consistent within the model. In this way a DSGE model is a powerful tool that provides a coherent framework for policy discussion and analysis, as with the help of the welfare criterion alternative policy decisions can easily be assessed and are consistent with the model's assumptions.

DSGE models are not only attractive from a theoretical perspective, they have also emerged as a useful tool for forecasting and quantitative policy analysis in macroeconomics. Along with the advancements in theoretical modeling, a tremendous research effort is spent on the development of taking these models to the data. Owing to improved time series fit, these models are gaining credibility in policy-making institutions such as central banks (An and Schorfheide, 2007). They are competitive with VARs in terms of forecasting power (Edge et al., 2009). Furthermore, quick progress is made for estimation these models (Fernández-Villaverde, 2009a).

All in all, DSGE models provide a fully integrated framework for policy analysis. Nevertheless, as models always are a description of reality but not a one-to-one-mapping, policy implications must be looked at through the lens of common sense, keeping in mind that simplified models cannot capture all important aspects of reality. Furthermore, because of the complex nature of DSGE models and though many central banks try to build new or improve already existing DSGE models which are in use to justify monetary policy decisions undertaken by central bankers, it is generally acknowledged that DSGE models are "unlikely to displace expert judgment".²⁰ Policy makers should take into account "anecdotal and extra-model information".²¹

3 From the assumptions to the solution - an overview

The "recipe" for building a DSGE model, solving it and comparing it to actual data includes several steps which can be summarized as follows. Figure 1 gives a visual overview of all steps. This procedure is commonly applied when analyzing infinite horizon DSGE models with representative agents.

- a) Set up of the economic model.
- b) Derivation of the first-order equilibrium conditions. Together with the structural equations, these build a system of non-linear stochastic difference equations.

²⁰Bernanke (2007).

²¹Bernanke (2007).

- c) As this system usually does not have a closed analytical solution, the solution is approximated in the neighborhood of a given point, in most cases the non-stochastic steady state which is determined in step 3.
- d)
 - Either (log-)linear approximation around the steady-state leading to a system of linear difference equations in state-space form and solution of this system with the help of the usual procedures,
 - or second (or higher) order approximation around the steady-state.
- e) Calibration of the parameters or estimation of parameters or both.
- f) Calculation of variances and conducting a variance decomposition of the underlying shocks and impulse response functions of the variables of interest.
- g) Evaluation of the model by looking at measures of fit to the data.

The remainder of the paper describes these stages successively.

The specification of the model assumptions should suit the problems under investigation. It must be clarified whether normative or positive questions should be discussed. The model should help to find consistent answers to questions such as which shocks play a role in the economy, what is the transmission mechanism of these shocks, and how should policy be conducted to react to these shocks?

In a general equilibrium model every sector of the economy is built independently. Typical parts of such a model are the private and the public sectors. The former consists of consumers and firms. Normally the consumer is modeled as a representative household choosing optimal consumption and leisure over their lifetime according to personal preferences. Firms, on the other hand, maximize profits, given some restriction on technology. Expectations (e.g., rational or rule-of-thumb behavior) have to be specified. The public sector consists of fiscal and monetary policy makers. Governments are in charge of fiscal policy, which is in most cases restricted to Ricardian equivalence. The central bank sets monetary policy. Assumptions should be set for how policy is conducted, i.e., whether the public sector follows some rules, which budget constraints they have to consider or whether they follow an optimal welfare maximizing policy. Moreover, market mechanisms and the institutional constraints under which agents interact have to be specified, i.e., whether and which markets are complete or incomplete, or whether policy makers have to set policy restricted to some policy rules. Finally, shocks have to be identified and modeled.

All the above assumptions must be converted to mathematical formulae assuming properties of functions which make the problem solvable (e.g., convex and concave preferences and technology, non-Ponzi game conditions, etc.).

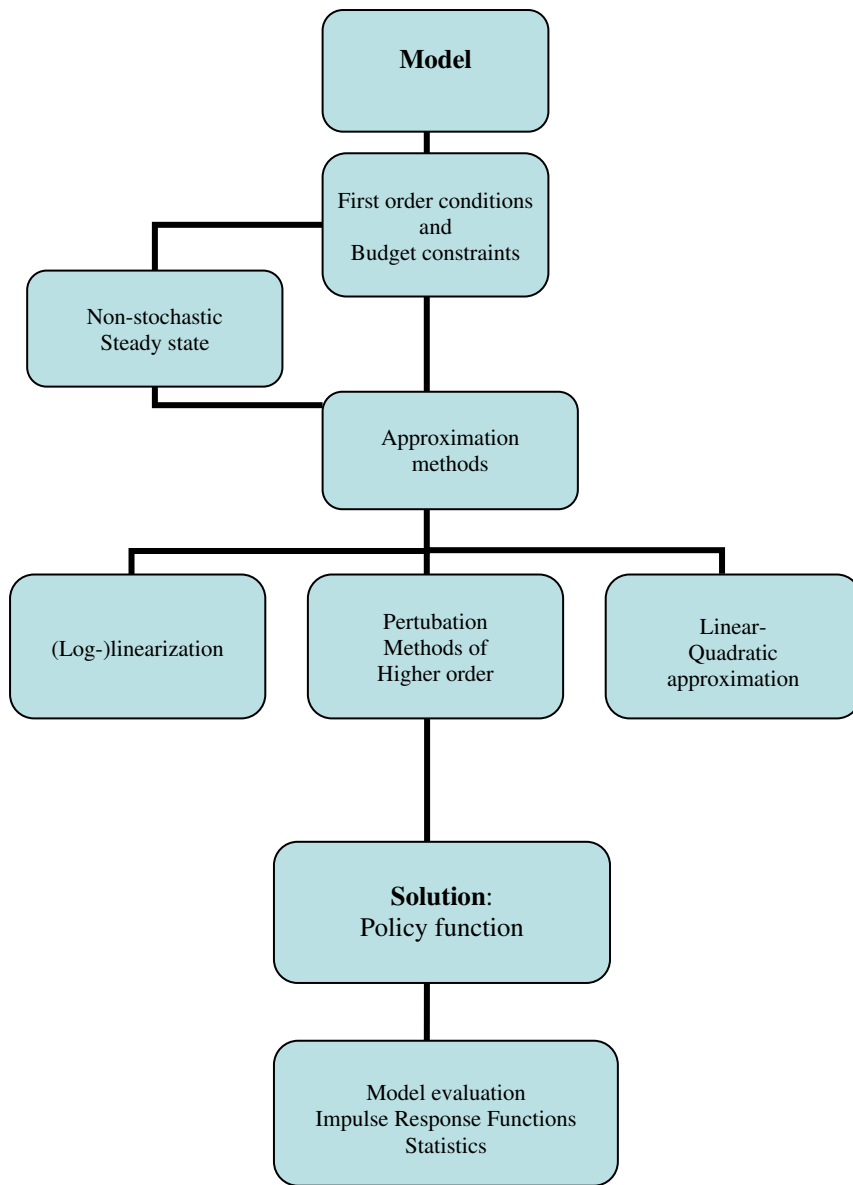


Figure 1: Overview of Solution Methods to DSGE models

The next step is to derive all optimization-based first order optimality conditions and constraints of households, firms and the public sector of an economy. These equations usually form a non-linear rational expectations system with current and/or backward and/or forward looking elements. An exact solution to this non-linear system can be computed for just a few simple examples. For instances where there are no closed analytical solutions, various numerical methods or procedures based on approximation to solve these systems are available and are discussed in the next section. Approximation methods are local methods and in most cases the reference point is the non-stochastic steady state of the model. So, in order to proceed with the next step the deterministic stationary point of the model has to be derived. If no such point exists, the model assumptions have to be modified to meet this requirement.

So far, stages 1 to 3 of the solution agenda have been performed. The next section presents the fourth step of the solution procedure.

4 Solution Methods

DSGE models do not have "paper and pencil" solutions aside from very few exceptions as the stochastic problems considered usually are non-linear. In addition, solving these systems involves expectations of future endogenous variables.

The most direct approach to solve dynamic models (such as the neoclassical growth model) is to apply value function iteration (dynamic programming). Though this method is quite safe and reliable and has good convergence properties, it has several drawbacks: it is slow and suffers from the curse of dimensionality. Its disadvantages have led to the development of new solution procedures, such as perturbation methods, and projection algorithms, which are much faster than the value function iteration, while maintaining good convergence properties.

The focus of this paper is not on projection methods and the value function iteration procedure. For information on projection methods the reader is referred to Aruoba et al. (2006) and the literature cited therein. For (linear quadratic) dynamic programming, or, in general, for recursive methods in macroeconomics, the books by Lucas and Stokey (1989) and Ljungqvist and Sargent (2004) provide a good introduction and overview.

Moreover, global methods (e.g. Chebyshev-polynomial method and finite-elements) are beyond the scope of this paper. For an extensive introduction see the textbook by Judd (1998).

Instead, approximation methods are the major component of the paper. The idea is that the non-linear system of optimality conditions and resource constraints are converted to a linear system of equations via a linear approximation of all equations in the neighborhood of the non-stochastic steady state. Then a solution to the linear system is derived, which is an

approximation to the solution of the underlying problem in the near of the stationary point. The core of solving DSGE models consists of finding the solution of these linear stochastic difference equations.

In this section approximation methods of first (part 4.1) and perturbation techniques of higher order (section 4.3) will briefly be discussed, and the solution methods of the resulting linear stochastic difference equations (section 4.2) will be demonstrated.

4.1 Linear approximation

As the optimality conditions and the budget constraints of the model form a non-linear system of difference equations, this system hardly has a closed analytical solution. In most cases it is much easier and more convenient to approximate the model's equation around a given point (which is a non-stochastic steady state) and find a solution to the resulting linear system using well-developed methods. The conversion of the nonlinear model leads to a sufficiently good linear approximation whose solutions are helpful in understanding the behavior of the underlying nonlinear system.

This section summarizes (log-)linear approximation methods currently used in the literature. The focus is on local approximation methods, i.e., procedures which are valid in the neighborhood of a particular point. Though it might be interesting to analyze problems which are far away from the steady state or involve switches of steady states, this is beyond the scope of this section.²²

Following King et al. (2002), a now-standard approach is linear approximation, with the choice of either loglinearization or linearization in levels. In both cases the approximation terms are the first order terms in a Taylor series approximation around a steady state. To illustrate, assume that for a smooth non-linear function f ,

$$y = f(x)$$

where y is a non-predetermined variable in a dynamic model and x is a predetermined variable.²³ Further let the deterministic steady state of x be given by \bar{x} , so that $\bar{y} = f(\bar{x})$. A

²²In these cases one has to approximate the conditions globally. A useful method is the one of parametrizing expectations as proposed by Marcet and Lorenzoni (1999).

²³Following Klein (2000), a variable X_t is a *predetermined* variable if and only if it has exogenous one-period-ahead forecast errors, i.e. $X_{t+1} - E_t X_{t+1} = C\varepsilon_{t+1}$, is exogenous. Thus, a predetermined variable is determined by lagged variables and contemporaneous exogenous shocks. Exogenous variables are predetermined variables which only depend on lagged values of themselves and contemporaneous exogenous shocks. A variable that is not a predetermined variable is a *non-predetermined* variable. These are the forward looking variables.

second-order Taylor series approximation²⁴ of f around \bar{x} yields

$$y \approx f(\bar{x}) + f_x(\bar{x})(x - \bar{x}) + \frac{1}{2}f_{xx}(\bar{x})(x - \bar{x})^2 + O^3 \quad (1)$$

where the first term on the right hand side is of zero order, the second term of first order, the third term of second order and the notation at the end signifies that the error in the approximation is of third or higher order. Here, f_x respectively f_{xx} denotes the first respectively second derivative of f with respect to x .

Up to first order (i.e. neglecting the terms of higher order) this equation can be rearranged to give a *linear approximation in levels*:

$$\underbrace{\frac{y - \bar{y}}{\bar{y}}}_{\text{percentage deviation from } \bar{y}} \approx \underbrace{\frac{f_x(\bar{x})\bar{x}}{\bar{y}}}_{\text{coefficient}} \underbrace{\frac{(x - \bar{x})}{\bar{x}}}_{\text{percentage deviation from } \bar{x}}$$

The case of *log-linearizing* includes the transformation of variables $y = e^{\ln y} = f(e^{\ln x})$. Taking the logarithm on both sides of this equation and denoting with $Y = \ln y$ and $X = \ln x$ we get $Y = \ln f(e^X)$. Taking a linear Taylor series approximation around the non-stochastic steady state and rearranging yields the result:

$$\underbrace{\ln\left(\frac{y}{\bar{y}}\right)}_{\text{percentage deviation in log-terms from } \bar{y}} \approx \underbrace{\frac{f_x(\bar{x})\bar{x}}{\bar{y}}}_{\text{coefficient}} \underbrace{\ln\left(\frac{x}{\bar{x}}\right)}_{\text{percentage deviation in log terms from } \bar{x}}$$

The underlying model determines which approximation method (either logs or levels) one might find convenient to choose. Comparing the results one has to keep in mind that in the case of log-linear approximation the deviation from the steady state is given in logarithmic terms. But keeping in mind that the percentage deviation from the steady state can also be expressed as

$$\hat{x} = \frac{x - \bar{x}}{\bar{x}} = \frac{x}{\bar{x}} - 1 \approx \ln\left(\frac{x}{\bar{x}}\right) = \ln x - \ln \bar{x}$$

both methods lead to similar results.

²⁴The Taylor series approximation of a smooth function f around \bar{x} is given by

$$y \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(\bar{x})}{n!} (x - \bar{x})^n$$

where $f^{(n)}$ denotes the n -th derivative of f with respect to x .

Uhlig (1999) proposes a simpler method for finding log-linear approximations, one which does not require taking explicit derivatives, if some simple rules are followed.²⁵ The result is the same as in the above method, except that the linear model is expressed in terms of log differences of the variables.

The main idea is to substitute a variable x by $xe^{\hat{x}}$ where $\hat{x} = \ln \frac{x}{\bar{x}}$ represents the log-deviation with respect to the steady state. Then the equation is linearized with respect to \hat{x} . Using the above-mentioned rules simplifies the process, as the substitutions are direct and mostly mechanical.

Some noteworthy conclusions about linear approximation can be drawn and are easily derived from equation (1):

- a) The expected value of y is equal to the steady state value \bar{y} , when evaluated up to a first order. This implies that uncertainty (variances or higher order moments) does not play a role.
- b) When evaluated up to a higher order the expected value of y depends on the curvature of the function and the variance of x .
- c) In order to characterize impulse response and second moments of variables it suffices to analyze first order properties of the model. For example, in order to calculate the variance of y , (1) yields

$$\text{var}(y) = E(y - Ey)^2 = f_x^2 \text{var}(x)$$

In many cases first-order linear approximation are adequate, but there is evidence that higher-order leads to better results. First-order accurate approximations to the solution to a DSGE model are insufficient in many economic examples, in particular for comparing welfare across policies which do not have any first-order effects on the model's non-stochastic steady state. An extensive discussion of these two points occurs in section 4.3 on perturbation methods.

Linear approximation methods lead to a state-space representation of the model, which is quite convenient for further empirical work, such as estimation, Kalman filter, forecasting, etc. Moreover, many solution techniques are available to solve these systems of linear difference

²⁵These rules are for some constant a and if x and y are close to zero

$$e^{x+ay} \approx 1 + x + ay$$

$$xy \approx 0$$

$$E_t(ae^{x_{t+1}}) \approx E_t(ax_{t+1}) \quad \text{up to a constant.}$$

equations. Applying some of them to systems under rational expectations will be discussed in the next section.

4.2 Linear stochastic difference equations

The non-linear system of optimality conditions and structural equations has been linearized, resulting in a system of linear stochastic difference equations under rational expectations, which can be written in state-space form

$$A_0 E_t Y_{t+1} = A_1 Y_t + B_0 \varepsilon_{t+1} \quad (2)$$

where A_0 , A_1 and B_0 are matrices of coefficients of the linearized model and Y_t denotes a vector of all endogenous and ε_t a vector of all exogenous variables.²⁶ These linearized equations form a dynamic system that determines the path of the variables of the model. Usually this system contains backward and forward-looking elements. Depending on the parameterization of the model, either no stable rational expectations solution exists, or the model yields determinacy, i.e., the stable solution is unique. In case of indeterminacy, multiple stable solutions exist. So the parameter space has to be restricted in order to focus on the stable case.

Many different methods for finding the rational expectations solution of the system (2) are explored in the literature. This solution is a feedback rule relating the current endogenous variables to the state variables of the model. The key reference is still Blanchard and Kahn (1980), which has been studied by many authors, resulting in many published surveys.²⁷ The main differences between all methods are the way they characterize the stable solution manifold, how the solution strategy treats expectations in the model, how much non-linearity is kept when applying the numerical solution algorithm, and the distinction between predetermined and non-predetermined variables.²⁸

The standard method of eigenvector-eigenvalue decomposition is to decouple the stable and unstable variables of the system and solve the easier "decoupled" system. A second method is one of undetermined coefficients. This section focuses on these two techniques.

The first method of separating the system (2) into stable and unstable variables was originally

²⁶Either ε_t can be a vector of i.i.d. shocks with zero mean, variance of $E(\varepsilon_t \varepsilon_t') = \Sigma$, and $E(\varepsilon_s \varepsilon_t') = 0$ for $s \neq t$, or of AR(1) processes which depend on i.i.d. exogenous shocks.

²⁷For an overview of solving nonlinear rational expectations models in the special case of nonlinear stochastic growth models see, for example the special issue of Vol. 8 of the *Journal of Business and Economic Statistics* (1990). The special issue of vol. 20 of the *Computational Economics* (2002) publishes several papers on the different methods of solving linear rational expectations models. Another survey is provided by the collection of all papers in the book of Marimon and Scott (1999) or by Binder and Pesaran (1995).

²⁸Focusing on backward looking variables Meyer-Gohde (2007) provides a fast algorithm which can help looking at models with sticky information like Mankiw and Reis (2002).

developed by Blanchard and Kahn (1980). The decoupled equations are much easier to solve. There are two main advantages of this method. First it provides conditions both for the existence and uniqueness of the solution. Moreover, it delivers a solution relatively fast. Therefore, the computational aspects have been developed further by many authors, among them King and Watson (1998), King and Watson (2002), Anderson (2000), Anderson and Moore (1985), Sims (2002), Klein (2000) and Soderlind (1999).

Because of its great importance, the method of Blanchard and Kahn (1980) is briefly sketched. The whole procedure can be divided into the following steps. First the model is partitioned into predetermined and non-predetermined variables, then a Jordan decomposition of the coefficient matrix is computed. As a next step the problem is rearranged into a transformed problem of decoupled equations. The solutions of the unstable and stable equations are combined yielding the final solution to the beginning problem.

The underlying assumption of the Blanchard-Kahn method is that the coefficient matrix A_0 is invertible. Applying A_0^{-1} to both sides of (2) and letting $A_0^{-1}A_1 = A$ and $A_0^{-1}B_0 = B$, the multivariate system of linear difference equations under rational expectations changes to

$$E_t Y_{t+1} = AY_t + B\varepsilon_{t+1} \quad (3)$$

The vector Y_t is partitioned into predetermined variables k_t and non-predetermined variables y_t . With this partitioning of the model (3) becomes

$$\begin{bmatrix} k_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} k_t \\ y_t \end{bmatrix} + B\varepsilon_{t+1}$$

A Jordan decomposition of the matrix A is given by $A = P\Lambda P^{-1}$, where P is a matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues.

At this point, it can be explored whether the model has a solution or not. This condition is known as the Blanchard-Kahn condition for determinacy: The solution of the rational expectations model exists and is unique if and only if the number of unstable eigenvectors (i.e., the number of eigenvalues outside the unit circle) is exactly equal to the number of non-predetermined variables. In this case the system has saddle path stability. In other words, the Blanchard-Kahn condition is not satisfied either if the number of unstable eigenvectors (roots) exceeds the number of non-predetermined variables, resulting in no solution or explosive paths as the transversality condition is violated, or if the number of unstable roots is less than the number of non-predetermined variables yielding multiple equilibria.

So, rewriting the last equation and multiplying both sides by P^{-1} , and denoting $P^{-1}B \equiv R$

yields

$$P^{-1} \begin{bmatrix} k_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \Lambda P^{-1} \begin{bmatrix} k_t \\ y_t \end{bmatrix} + R \varepsilon_{t+1}$$

with Λ_1 denoting the stable eigenvalues and Λ_2 denoting the unstable ones,

$$\Lambda = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix}$$

Defining

$$P^{-1} \begin{bmatrix} k_t \\ y_t \end{bmatrix} \equiv \begin{bmatrix} \tilde{k}_t \\ \tilde{y}_t \end{bmatrix} \quad \text{and} \quad P^{-1} \begin{bmatrix} k_{t+1} \\ E_t y_{t+1} \end{bmatrix} \equiv \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{y}_{t+1} \end{bmatrix}$$

where the matrix P^{-1} is partitioned as

$$P^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

leads to the transformed problem

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \varepsilon_{t+1}$$

resulting in the decoupled system of equations

$$\tilde{k}_{t+1} = \Lambda_1 \tilde{k}_t + R_1 \varepsilon_{t+1} \tag{4}$$

$$\tilde{y}_{t+1} = \Lambda_2 \tilde{y}_t + R_2 \varepsilon_{t+1} \tag{5}$$

where (4) is the stable equation and (5) is the unstable equation.

These decoupled equations can be solved separately. First, the unstable equation (5) is solved forward to time $t + j$ to yield

$$\tilde{y}_{t+j} = (\Lambda_2)^j \tilde{y}_t$$

As $|\Lambda_2| > 1$, the only stable solution is given by $\tilde{y}_{t+j} = 0$ for all t . From the partition of the matrix P it follows from the transformed problem that $P_{21}k_t + P_{22}y_t = \tilde{y}_t = 0$. Hence

$$y_t = -P_{22}^{-1} P_{21} k_t. \tag{6}$$

So, forward-looking variables are a function of the predetermined variables.

Second, the stable equation (4) is solved forward to time $t + j$ to yield

$$\tilde{k}_{t+j} = (\Lambda_1)^j \tilde{k}_t$$

As $|\Lambda_1| < 1$, there are no instability problems. Insert (6) into $P_{11}k_t + P_{12}y_t = \tilde{k}_t$ to get $\tilde{k}_t = (P_{11} - P_{12}P_{22}^{-1}P_{21})k_t$ for all t . Substitute this into (4) on both sides to derive

$$k_{t+1} = (P_{11} - P_{12}P_{22}^{-1}P_{21})^{-1}\Lambda_1(P_{11} - P_{12}P_{22}^{-1}P_{21})k_t + (P_{11} - P_{12}P_{22}^{-1}P_{21})^{-1}R_1\varepsilon_{t+1} \quad (7)$$

As a result, future predetermined variables are a function of current backward-looking variables. As a final step, this recursive formulation (7) can be used to derive solutions for k_t and y_t for all t . Starting from a steady-state value $k_0 = 0$, and drawing shocks ε_t from a normal distribution, the k_t are simulated from the shocks recursively with (7). Finally y_t are calculated from k_t with the solution formula (6).

This method works well in the case of a regular coefficient matrix A in (2). Others have systematized the methodology. King and Watson (1998) and Klein (2000) provide algorithms for the case when the matrix A of (2) is not invertible, i.e., in the case of singular systems. Whereas the former use a decomposition in canonical forms, the latter implements a Schur decomposition, which is useful when (generalized) eigenvalues are not distinct. When all eigenvectors are linearly independent, the method by Blanchard and Kahn (1980) leads to the same results. But the Schur decomposition is not necessarily unique. On the other hand, Sims (2002) discusses the case where the distinction between states and controls is unclear. All authors provide software to implement their solution strategies (see section 6).

A second method which is widespread in the literature is the method of the undetermined coefficient.²⁹ The advantage of this method is that it is quick when feasible. On the other hand, however, this procedure presumes that a solution exists. Moreover, the method is applicable only when no redundant variables are included, i.e., the state space is of minimal size. Otherwise, zero eigenvalues lead to bubble solutions.³⁰

The main idea is to conjecture a solution of the model, i.e., to assume that the endogenous variables are linear functions of state variables. This guess is not completely random. Solving the model implies finding the unknown (undetermined) coefficients of the conjectured solution by substituting the conjectured solution into the structural equations, finding a system of equations relating the coefficients and solving this linear system. The general solution is built on solving matrix quadratic equations.

From a computational aspect, both this method and the method of Blanchard and Kahn

²⁹The key reference here is McCallum (1983). Other significant contributions are Binder and Pesaran (1995), Uhlig (1999) and Christiano (2002) to name a few.

³⁰Canova (2007), p. 57.

require the solution of an eigenvalue problem including finding the eigenvectors. It can be shown that the vector of the unknown coefficients which are to be determined are equal to the expression of $-P_{22}^{-1}P_{21}$ of (6). For a detailed exposition of the single steps refer to Uhlig (1999).

4.3 Perturbation Methods

First order approximations may be inaccurate if considering welfare measures. In order to evaluate expected utility, a second order approximation of utility and a second order approximation of the economic equations have to be taken. Otherwise some relevant second-order terms are left out, as the equivalence result no longer holds, as has been shown by Kim and Kim (2003) and Schmitt-Grohe and Uribe (2004), among others. To illustrate this issue assume that a welfare measure is given by $EU(y)$. The evaluation of expected utility of the agents is based on this welfare measure and on the distribution of x . A second order approximation of $U(y)$ is given by

$$U(y) \approx U(\bar{y}) + U_y(\bar{y})(y - \bar{y}) + \frac{1}{2}U_{yy}(\bar{y})(y - \bar{y})^2 + O^3 \quad (8)$$

Taking expectations on both sides yields

$$EU(y) \approx U(\bar{y}) + U_y(\bar{y})E(y - \bar{y}) + \frac{1}{2}U_{yy}E(\bar{y})(y - \bar{y})^2 + O^3$$

In order to evaluate expected utility the first order approximation of y from (1) should not be used. This would be tempting but would lead to the wrong result:

$$EU(y) \approx U(\bar{y}) + \frac{1}{2}U_{yy}(\bar{y})f_x^2 var(x) + O^3$$

Why is this wrong? Taking (1) and evaluating up to a second order results in

$$Ey \approx \bar{y} + \frac{1}{2}f_{xx}(\bar{x})var(x) + O^3$$

Plugging this into the welfare equation this yields the correct approximation

$$EU(y) \approx U(\bar{y}) + \frac{1}{2} [U_{yy}(\bar{y})f_x^2 + U_y(\bar{y})f_{xx}(\bar{x})] var(x) + O^3 \quad (9)$$

To conclude, both a second order approximation of utility and a second order approximation of the economic model are necessary. Otherwise, important second order terms are left out. In other words, the solution of interesting problems requires perturbation methods.

In a nutshell, the main idea of perturbation methods is to replace the original difficult problem with a simpler one which is much easier to handle and solve. Then the solution of the simpler model is used to approximate the solution of the problem of interest. To be more precise, in

practice this method refers to finding Taylor expansion of the policy function describing the dynamics of the variables of the model around the deterministic steady state. Judd (1998) refers to solution methods including the calculation of smooth Taylor series approximation to the solution of the equations as "perturbation methods".

First-order perturbation corresponds to the solution obtained by standard linearization of first-order conditions. One important advantage of higher-order approximation methods is the deviation from the certainty equivalence principle, i.e., higher-order moments of the distribution of the shocks (volatility) of the models are taken into account, as illustrated above.³¹ Second, through the quadratic term of the expansion the curvature of the true decision rule is handled better. This is important when the curvature of utility is high. Besides these benefits – for example, in the case of welfare evaluation – quadratic approximation techniques may be better in approximating the system when one is a bit farther from the steady state.

For the last few years, many algorithms have been developed to take a second-order expansion of the equilibrium conditions of DSGE models: see, for instance, Judd (1998), Collard and Juillard (2001), Judd and Jin (2002), Schmitt-Grohe and Uribe (2004), Kim et al. (2005), and Swanson et al. (2006). All these methods approximate the policy functions locally.

Judd (1998) is the pioneer in this field presenting the general method for deterministic discrete-time models (perturbation methods have been applied to continuous time before). Judd (1998) describes which adjustments have to be made when applying the method to discrete-time models.

Judd and Jin (2002) describe very extensively how to compute approximations of arbitrary order in discrete-time rational expectations models. They analyze regularity conditions which justify the local approximations and they provide methods for checking the validity of the approximations. Judd and Jin (2002) argue that deriving perturbation solutions is no more difficult than deriving the linear approximation solutions, but they also discuss some cases when linear approximation is better than higher-order approximation.

Accuracy criteria of higher order perturbation methods are provided by Collard and Juillard (2001). They compare the solutions of perturbation to closed form solution of an asset pricing model. They find that in comparison to linearization second-order and fourth-order approximation performs better.

Swanson et al. (2006) focus on Taylor series expansions of n th-order to approximate solutions to dynamic, discrete-time rational expectations models. These higher order approximations have the advantage (compared to the first- or second-order approximated solutions) that they are valid everywhere within the domain of convergence of the Taylor series, i.e., they are valid globally (in a very rigorous sense).

³¹This is of particular interest when studying risk premia, or the effect of uncertainty on welfare which is an interesting question in the recent literature on monetary and fiscal stabilization policy.

An algorithm for computing second order approximation, forecasts and impulse response functions in dynamic models is given by Kim et al. (2005).

The relative performance of two main solution methods, such as perturbation methods (first, second, and fifth order) and projection algorithms (finite elements method, Chebyshev-polynomials), is analyzed by Aruoba et al. (2006), when computing the solution of the canonical stochastic neoclassical growth model with leisure choice. They compare these methods with regard to computing time, implementation complexity, and accuracy applied to a canonical stochastic neoclassical growth model with leisure choice.

The main idea of the perturbation method can be summarized by the following three steps:

- a) Transform the problem, rewriting it in terms of a small perturbation parameter in such a way that the zeroth-order approximation has an analytical solution.
- b) Solve the new problem for a particular choice of the perturbation parameter, i.e., find the analytical solution of step 1.
- c) Use the previous solution to approximate the solution of the original problem. This step requires finding an algorithm. Solving for the first iteration involves a nonlinear equation. All further iterations only require solving a linear equation in one unknown.

To illustrate the method an intuitive example from elementary calculus is given.³² The problem is to find the negative root of the cubic equation³³

$$x^3 - 4.1x + 0.2 = 0$$

The first step includes the transformation of the problem into a perturbed problem indexed by a small perturbation parameter ε . There are several ways to perform this step. In this case the most suitable way is to consider the following equation:

$$x^3 - (4 + \varepsilon)x + 2\varepsilon = 0$$

where $\varepsilon \equiv 0.1$. Figure 2 shows the polynomial for some values of ε .

For solving the new problem in step 2 it is assumed that the solutions x of the problem can be written as a function of the perturbation parameter, i.e. $x = g(\varepsilon)$:

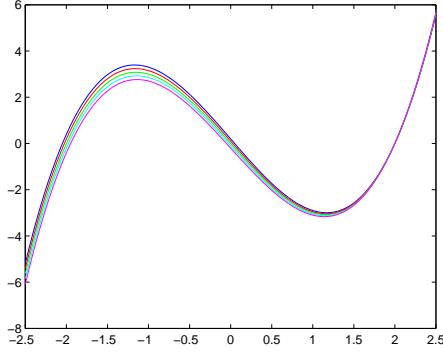
$$g(\varepsilon)^3 - (4 + \varepsilon)g(\varepsilon) + 2\varepsilon = 0$$

The solution for a particular choice of one of the perturbation parameters, namely for $\varepsilon = 0$, is quite easy to obtain. In this case the equation $x^3 - 4x = 0$ has to be solved, implying the roots given by -2, 0, 2. As a result, $g(0) = -2$.

³²Example follows closely Fernández-Villaverde (2009b).

³³The root is given by -2.04880884817015 (up to 14 decimal places).

Figure 2: The Polynom for various values of ε



In the third step this derived solution is used to approximate the original solution. A Taylor's series approximation is given by

$$x = g(\varepsilon)|_{\varepsilon=0} = g(0) + \sum_{n=1}^{\infty} \frac{g^{(n)}(0)}{n!} \varepsilon^n$$

The main idea is to find an algorithm to recover the coefficients $g(0)$ and $\frac{g^{(n)}(0)}{n!}$ for $n = 1, \dots, n$ in an iterative way with the starting point of $g(0) = -2$ derived in the previous step. Taking the derivative of

$$g(\varepsilon)^3 - (4 + \varepsilon)g(\varepsilon) + 2\varepsilon = 0$$

with respect to ε yields

$$3g(\varepsilon)^2 g'(\varepsilon) - g(\varepsilon) - (4 + \varepsilon)g'(\varepsilon) + 2 = 0. \quad (10)$$

Setting $\varepsilon = 0$ and recalling $g(0) = -2$ results in $g'(0) = -\frac{1}{2}$. The next step in the algorithm requires taking the derivative of (10) with respect to ε , set $\varepsilon = 0$ and recalling the previous results $g(0) = -2$ and $g'(0) = -\frac{1}{2}$ to solve for

$$g''(0) = \frac{1}{4}$$

Table 1 summarizes the results up to second order.

After this simple example the rest of the section will describe the second order perturbation method following Schmitt-Grohe and Uribe (2004) in case of a DSGE rational expectations model, which is in general given by:

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0 \quad (11)$$

where f is a non-linear function $f : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \longrightarrow \mathbb{R}^n$. The vector y_t denotes non-predetermined variables of size $n_y \times 1$, and x_t denotes the vector of predetermined variables of

Table 1: The problem of the cubic equation

order	Taylor approximation	$\varepsilon \equiv 0.1$	$x^3 - 4.1x + 0.2$
0th	$x \simeq -2$	$x = -2$	0.4
1st	$x \simeq -2 - \frac{1}{2}\varepsilon$	$x = -2.05$	-0.010125
2nd	$x \simeq -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2$	$x = -2.04875$	0.00049976758...

size $n_x \times 1$. Define $n = n_x + n_y$. E_t is the mathematical expectations operator conditional on information available at time t . Assume that the state vector x_t can be partitioned as a vector of endogenous predetermined variables x_t^1 and a vector of exogenous state variables x_t^2 which follows an exogenous stochastic process given by $x_{t+1}^2 = \tilde{h}(x_t^2, \sigma) + \tilde{\eta}\sigma\varepsilon_{t+1}$. The innovation ε_t is independently and identically distributed, with mean zero and variance/covariance matrix I . A broad class of DSGE models falls into this category.

The general solution of models like (11) is given by the policy functions

$$y_t = \hat{g}(x_t), \quad \text{and} \quad x_{t+1} = \hat{h}(x_t) + \eta\sigma\varepsilon_{t+1}.$$

Usually, the policy functions will depend on the amount of uncertainty in the model. Thus, the key idea of the perturbation method is to introduce the parameter σ as the perturbation parameter of the problem to scale the amount of uncertainty. The solution of the model can now be interpreted to depend on this scaling factor and the predetermined variables x_t thus the general solutions to this model are the following policy functions:

$$y_t = g(x_t, \sigma) \tag{12}$$

$$x_{t+1} = h(x_t, \sigma) + \eta\sigma\varepsilon_{t+1}, \tag{13}$$

where g maps $\mathbb{R}^{n_x} \times \mathbb{R}^+$ into \mathbb{R}^{n_y} , and $h : \mathbb{R}^{n_x} \times \mathbb{R}^+ \rightarrow \mathbb{R}^{n_x}$.

Usually it is hard to find an exact formula for these two policy functions $g(x_t, \sigma)$ and $h(x_t, \sigma)$, so an approximation around some specific point is taken. Finding the solution proceeds in three steps. First, the point around which the solution should be approximated is derived; second, the first-order perturbation of the solution is computed; and finally, the second-order approximation is computed.

Discussing the first step of the program, the point of interest is the non-stochastic steady state of the model, which is defined by the condition

$$f(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0$$

Secondly, to get the first order approximation, use a linear Taylor series expansion of (12) and (13) around the steady state:³⁴

$$y_t \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma$$

$$y_t \approx h(\bar{x}, 0) + h_x(\bar{x}, 0)(x - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma$$

where g_x , and h_x denote the partial derivative of the functions with respect to x and g_σ and h_σ the derivatives with respect to σ . Obviously, $\bar{y} = g(\bar{x}, 0)$ $\bar{x} = h(\bar{x}, 0)$.

The technically demanding step of the procedure is now to calculate the coefficients g_x , h_x , g_σ , and h_σ . The coefficients are obtained by substituting the solutions (12) and (13) in (11) and noting that for any initial x and σ

$$F(x, \sigma) \equiv E_t(g(h(x, \sigma) + \eta\sigma\varepsilon'), g(x), h(x) + \eta\sigma\varepsilon', x) = 0$$

so that all derivatives $F_{x^i\sigma^j} = 0$. First, setting $F_x = 0$, leads to the expressions g_x and h_x by solving for the roots of the resulting first order dynamic system. This requires applying an eigenvalue-eigenvector or alternately the Schur decomposition of a matrix problem as discussed in section 4.2. Second, setting $F_\sigma = 0$ leads to a homogeneous equation in g_σ and h_σ . For all x and σ the only solution is given by $g_\sigma = h_\sigma = 0$. This is an important theoretical result implying that up to first order, the size of the variance of the shocks does not matter for the policy function, in other words the expected values of x_t and y_t are equal to their non-stochastic steady state values \bar{x} and \bar{y} in a first-order approximation. So in this case the principle of certainty equivalence holds.

To perform the third and last step of the algorithm, i.e., the second order approximation, take a second order Taylor series approximation of (12) and (13). The results are substituted into the conditions $F_{xx} = 0$ to determine the coefficients g_{xx} and h_{xx} , in $F_{x\sigma} = 0$, to determine $g_{x\sigma}$ and $h_{x\sigma}$, and finally in $F_{\sigma\sigma} = 0$ to determine the coefficients $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$. Though the algebra is quite cumbersome, this step is much easier than the second step of the procedure due to the fact that it does not involve the solving of an eigenvalue problem. The resulting systems are all linear in the unknown coefficients, and in addition, similar to the above problem, $g_{x\sigma} = h_{x\sigma} = 0$.

4.4 Calibration

The main methods to evaluate the models are either calibration or estimation of the DSGE models. Whereas the latter is currently more popular, the usual approach in the literature to

³⁴To keep notation simple, assume from now on that $n_x = n_y = 1$. For the general case, refer to Schmitt-Grohe and Uribe (2004).

take a DSGE model to the data has been calibration, which is a useful tool for analyzing the dynamic properties of DSGE models.³⁵

Calibration means that values of structural parameters are taken from different microeconomic studies which have estimated these parameters on a microeconomic basis.

4.5 Variances and Impulse Response Functions

There are two ways to evaluate the performance of a model.³⁶ One way is to compare standard errors and correlations of the endogenous variables with those from the data. The model is doing a good job if statistics are close to those of the economy. Unconditional moments of the simulated economic series are computed and compared with the actual data series. Evaluation of the model is performed by looking at the distance between these two sets of moments

Another way is to analyze dynamic properties of the model by studying impulse response functions. In this case, the starting point of the analysis is the steady state with all shocks to stochastic processes set to zero (and if the model is in log differences from steady state with all variables set to zero). Then a one-change, an impulse, is applied to one of the model's exogenous variables and the resulting time-paths of the dynamic (endogenous) variables are the responses of the model to impulse. The model performs well if the impulse response functions resemble those of the economy.

To be more precise assume that the system describing the economy can be cast in the form of (3). Then impulse response functions calculate the effect of a shock from the matrix ε_t on the y_t and k_t , simulating y_t and k_t recursively using (7) and (6). To derive a mathematical expression for the impulse response functions, rearrange (3) to

$$(I - AL)Y_{t+1} = B\varepsilon_{t+1} \quad (14)$$

where L denotes the lag operator such that $LY_{t+1} = Y_t$. Applying $(I - AL)^{-1}$ on both sides and observing that $(I - AL)^{-1} = (I + AL + A^2L^2 + \dots)$ results in

$$Y_{t+1} = \sum_{j=0}^{\infty} A^j B \varepsilon_{t+j}.$$

The function $\Gamma_t^j \equiv A^j B$ is the impulse response function of the shock.

This analysis can be complemented by a variance decomposition to analyze the relative importance of the various shocks in explaining the conditional and unconditional variances of the variables.

³⁵Kydland and Prescott (1996) discuss advantages of calibration in greater detail.

³⁶These procedures are generally applied when evaluating RBC or NK models.

5 Estimation of DSGE Models

There is extensive research literature on how to take DSGE models to the data, i.e., how to work with these models empirically. The focus is on estimation and evaluation methods of these models. Whereas at the beginning of this macroeconometric research field classical estimation techniques prevailed, there has been a trend towards advanced econometric methods (Bayesian estimation) for the last several years due to improved computational skills. Bayesian estimation is now en-vogue when working with DSGE models empirically.

However, the classical approach (meaning non-Bayesian techniques) has been elaborated extensively. Surveys of these methods can be found in works such as Kim and Pagan (1995) or Canova (2007). The latter also provides an introduction to Bayesian estimation. Another good introduction and overview is provided by An and Schorfheide (2007), reviewing this estimation technique and evaluation techniques applying these methods to a DSGE model. Ruge-Murcia (2007) introduces and compares four methods (GMM, ML with Bayesian priors, SMM and Indirect Inference). A very extensive and detailed discussion and overview of Bayesian estimation is laid out in Fernández-Villaverde (2009a). The difference between all methods has to do with the amount of information each method is able to handle. Methodological discussion of various estimation and model evaluation techniques can be found, for example, in Sims (1996) and Kydland and Prescott (1996).

Following Ruge-Murcia (2007), several advantages of fully-fledged econometric estimation (vs. calibration) are noteworthy. First, parameter values are obtained using the model of interest. Parameter values taken from some other study might be inconsistent with the model's assumptions. Furthermore, one can estimate all relevant parameters, even those where there is no microeconomic study so far. Second, in order to take care of parameter uncertainty, confidence intervals for model's response to a shock can be constructed more easily. Finally, it is easier to evaluate the models.

Furthermore, the use of Bayesian estimation yields the following benefits. An and Schorfheide (2007) point out that Bayesian estimation takes advantage of the general equilibrium approach; in contrast, GMM estimation is based on equilibrium relationships, so it fits the model to a vector of aggregate time series which are based on the likelihood function generated by the model. Last, prior distributions can be used to incorporate additional information into the parameter estimation. In addition, Bayesian estimation outperforms the techniques of general method of moments (GMM) and maximum likelihood (MLH) in small samples. It does not rely on the identification scheme of the VAR but follows the likelihood principle (see Tovar (2009) for a more detailed discussion). Moreover, in cases of mis-specified models, Bayesian estimation and model comparison are consistent (Rabanal and Rubio-Ramirez, 2005).

Nevertheless, taking models to the data might be quite tricky. Tovar (2009), among others, provides a discussion of the weaknesses of estimation techniques. To name but a few, the researcher should be aware which data sets are implemented (especially when dealing with emerging and underdeveloped countries), the right estimation methods, misspecification issues (invalid cross-equation restrictions, stochastic singularity) and identification issues.

6 Practical Implementation

This overview of solution methods of DSGE models ends with a short note on the practical implementation of these solution methods. Analyzing the above described steps, it is obvious that a computer implements some algorithm to solve the above-mentioned problems practically. It is not necessary to write one's own code to implement one of the discussed solution strategies, as there are many different algorithms available on the web nowadays. See table 2 for an overview.³⁷

Anderson (2008) provides an overview and comparison of some of the main established solution strategies to solve linear rational expectations models. In his paper the author compares the solution algorithms provided by Sims (2002), Anderson and Moore (1985), Binder and Pesaran (1995), King and Watson (1998), Klein (2000), and Uhlig (1999)³⁸ with regard to their practical implementation, functionality, efficiency, and accuracy. By and large, the differences are negligible. All codes are easy to implement and user-friendly. The algorithms provide equivalent results for models meeting the Blanchard and Kahn conditions.

An alternative to the above-mentioned algorithms is DYNARE, which is a platform for the solution, simulation, and estimation of DSGE models in economics developed by Michel Juillard and collaborators. This program is more like a very user-friendly "blackbox approach", but the codes are easy to access.

³⁷Usually the codes are programmed using MATLAB, GAUSS, FORTRAN, among others.

³⁸For differences related to the theoretical background of the rational expectations model refer to section 4.2.

Table 2: Overview of selected codes for solving DSGE models

Anderson (2000)	http://www.federalreserve.gov/pubs/oss/oss4/aimindex.html
Uhlig (1999)	http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm
Sims (2002)	http://www.princeton.edu/~sims/
Klein (2000)	http://economics.uwo.ca/faculty/klein/personal/
Soderlind (1999)	http://home.datacomm.ch/paulsoderlind/
Juillard and collaborators (2009)	http://www.cepremap.cnrs.fr/dynare
Meyer-Gohde (2007)	http://www.macroeconomics.tu-berlin.de/Meyer-Gohde/papers/LinLagEx.htm
Binder and Pesaran (1995)	http://dge.repec.org/codes/binder/handbook/
Schmitt-Grohe and Uribe (2004)	http://www.columbia.edu/~ss3501/
International Network for DSGE modeling	http://www.dsge.net/

7 Conclusion

Nowadays, DSGE models are the main tool for analyzing various questions in business cycle theory, monetary and fiscal policy problems, growth or other fields in macroeconomics, or international macroeconomics. Many macroeconomic publications use a DSGE framework. Despite some disadvantages of this methodology, this approach is still attractive both in academics and central banks.

Solving DSGE models remains an area of intense interest. Many techniques are available. Among them, linear approximation methods and perturbation methods have been explored in detail in this paper. The non-linear equations of the models are transformed into a system of linear difference equations under rational expectations. Solution strategies such as the eigenvalue-eigenvector decomposition of Blanchard and Kahn (1980) or the method of undetermined coefficients are explained. Bayesian estimation is briefly described. Evaluation methods are described briefly. Finally, the paper provides some useful sources for practical implementation.

References

- Adolfson, M., S. Lasen, J. Lind, and M. Villani (2007, July). Bayesian estimation of an open economy dsge model with incomplete pass-through. *Journal of International Economics* 72(2), 481–511.
- Adolfson, M., S. Lasen, J. Lind, and M. Villani (2008). Evaluating an estimated new keynesian small open economy model. *Journal of Economic Dynamics and Control* 32(8), 2690 – 2721.
- An, S. and F. Schorfheide (2007). Bayesian analysis of dsge models. *Econometric Reviews* 26(2), 113–172.
- Anderson, G. (2008, March). Solving linear rational expectations models: A horse race. *Computational Economics* 31(2), 95–113.
- Anderson, G. and G. Moore (1985). A linear algebraic procedure for solving linear perfect foresight models. *Economics Letters* 17(3), 247–252.
- Anderson, G. S. (2000). A reliable and computationally efficient algorithm for imposing the saddle point property in dynamic models. Occasional Staffs Studies 4, Federal Reserve Board, Washington D.C.
- Aruoba, S. B., J. Fernández-Villaverde, and J. F. Rubio-Ramírez (2006). Comparing solution methods for dynamic equilibrium economies. *Journal of Economic Dynamics and Control* 30(12), 2477 – 2508.
- Bernanke, B. S. (2007, July). Inflation expectations and inflation forecasting. Speech at the Monetary Economics Workshop of the National Bureau of Economic Research Summer Institute.
- Binder, M. and M. H. Pesaran (1995). Multivariate rational expectations models: a review and some new results. In M. H. Pesaran and M. Wickens (Eds.), *Handbook of Applied Econometrics*, pp. 139–187. Basil Blackwell, Oxford.
- Blanchard, O. and C. Kahn (1980). The solution of linear difference models under rational expectations. *Econometrica* 48(5), 1305–1311.
- Blanchard, O. J. (2008, August). The state of macro. Working Paper 14259, National Bureau of Economic Research.
- Brubakk, L., T. A. Husebø, J. Maih, K. Olsen, and M. Østnor (2006). Finding nemo: Documentation of the norwegian economy model. Staff Memo 2006/6, Norges Bank.
- Buiter, W. (2009, March). The unfortunate uselessness of most "state of the art" academic monetary economics. Financial Times, Willem Buiter's Maverecon.

- Calvo, G. (1983). Staggered prices in a utility maximizing framework. *Journal of Monetary Economics* (12), 383–398.
- Canova, F. (2007). *Methods for Applied Macroeconomic Research*. Princeton University Press.
- Christiano, L. J. (2002). Solving dynamic equilibrium models by a method of undetermined coefficients. *Computational Economics* 20(1-2), 21–55.
- Christoffel, K., G. Coenen, and A. Warne (2008, September). The new area-wide model of the euro area - a micro-founded open-economy model for forecasting and policy analysis. Working Paper Series 944, European Central Bank.
- Collard, F. and M. Juillard (2001, June). Accuracy of stochastic perturbation methods: The case of asset pricing models. *Journal of Economic Dynamics and Control* 25(6-7), 979–999.
- Cooley, T. F. (Ed.) (1995). *Frontiers of Business Cycle Research*. Princeton University Press.
- Edge, R. M., M. T. Kiley, and J.-P. Laforte (2009). A comparison of forecast performance between federal reserve staff forecasts, simple reduced-form models, and a dsge model. Technical report.
- Erceg, C. J., L. Guerrieri, and C. Gust (2006, March). Sigma: A new open economy model for policy analysis. *International Journal of Central Banking* 2(1).
- Fagan, G., J. Henry, and R. Mestre (2001, January). An area-wide model (awm) for the euro area. Working Paper Series 42, European Central Bank.
- Fernández-Villaverde, J. (2009a, January). The econometrics of dsge models. Working Paper 14677, National Bureau of Economic Research.
- Fernández-Villaverde, J. (2009b). Methods in macroeconomic dynamics - lecture notes 3: Solving dynamic equilibrium models. <http://www.econ.upenn.edu/jesusfv/teaching.html>.
- Galí, J. (2008). *Monetary policy, inflation, and the business cycle: an introduction to the New Keynesian framework*. Princeton University Press.
- Harrison, R., K. Nikolov, M. Quinn, G. Ramsey, A. Scott, and R. Thomas (2005). The bank of england quarterly model. Technical report, Bank of England.
- Judd, K. L. (1998). *Numerical Methods in Economics*. MIT Press.
- Judd, K. L. and H. Jin (2002). Perturbation methods for general dynamic stochastic models. Technical report, Manuscript, Hoover Institute.
- Juillard, M. and collaborators (2009). Dynare. <http://www.ceprenap.cnrs.fr/dynare>.

- Kim, H., J. Kim, E. Schaumburg, and C. A. Sims (2005). Calculating and using second order accurate solutions of discrete time dynamic equilibrium models. Technical report.
- Kim, J. and S. H. Kim (2003, August). Spurious welfare reversals in international business cycle models. *Journal of International Economics* 60(2), 471–500.
- Kim, K. and A. Pagan (1995). Econometric analysis of calibrated macroeconomic models. In H. Pesaran and M. Wickens (Eds.), *Handbook of Applied Econometrics: Macroeconomics*, Oxford, pp. 356–390. Basil Blackwell.
- King, R. G., C. I. Plosser, and S. T. Rebelo (2002, October). Production, growth and business cycles: Technical appendix. *Computational Economics* 20(1-2), 87–116.
- King, R. G. and M. W. Watson (1998, November). The solution of singular linear difference systems under rational expectations. *International Economic Review* 39(4), 1015–26.
- King, R. G. and M. W. Watson (2002, October). System reduction and solution algorithms for singular linear difference systems under rational expectations. *Computational Economics* 20(1-2), 57–86.
- Klein, P. (2000, September). Using the generalized schur form to solve a multivariate linear rational expectations model. *Journal of Economic Dynamics and Control* 24(10), 1405–1423.
- Krusell, P., A. A. Smith, and Jr. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy* 106(5), 867–896.
- Kydland, F. E. and E. C. Prescott (1982, November). Time to build and aggregate fluctuations. *Econometrica* 50(6), 1345–70.
- Kydland, F. E. and E. C. Prescott (1996, Winter). The computational experiment: An econometric tool. *Journal of Economic Perspectives* 10(1), 69–85.
- Landmann, O. (2007). Die entwicklung der konjunkturtheorie. In *Jahresbericht 2007*, Volume 386, pp. 74–89. Kommission für Konjunkturfragen.
- Ljungqvist, L. and T. Sargent (2004). *Recursive Macroeconomic Theory* (2nd ed.). MIT Press.
- Lucas, R. E. and N. L. Stokey (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.
- Lucas, R. J. (1976, January). Econometric policy evaluation: A critique. *Carnegie-Rochester Conference Series on Public Policy* 1(1), 19–46.

- Mankiw, N. G. and R. Reis (2002, November). Sticky information versus sticky prices: A proposal to replace the new keynesian phillips curve. *The Quarterly Journal of Economics* 117(4), 1295–1328.
- Marcet, A. and G. Lorenzoni (1999). The parameterized expectations approach: Some practical issues. In R. Marimon and A. Scott (Eds.), *Computational Methods for the Study of Dynamic Economics*, pp. 143–171. Oxford University Press.
- Marimon, R. and A. Scott (Eds.) (1999). *Computational Methods for the Study of Dynamic Economics*. Oxford University Press.
- McCallum, B. T. (1983). On non-uniqueness in rational expectations models : An attempt at perspective. *Journal of Monetary Economics* 11(2), 139–168.
- Medina, J. P. and C. Soto (2007, December). The chilean business cycles through the lens of a stochastic general equilibrium model. Working Papers Central Bank of Chile 457, Central Bank of Chile.
- Meyer-Gohde, A. (2007, December). Solving linear rational expectations models with lagged expectations quickly and easily. SFB 649 Discussion Papers SFB649DP2007-069, Sonderforschungsbereich 649, Humboldt University, Berlin, Germany.
- Murchison, S. and A. Rennison (2006). Totem: The bank of canada’s new quarterly projection model. Technical report.
- Obstfeld, M. and K. Rogoff (1995, June). Exchange rate dynamics redux. *Journal of Political Economy* 103(3), 624–60.
- Rabanal, P. and J. F. Rubio-Ramirez (2005, September). Comparing new keynesian models of the business cycle: A bayesian approach. *Journal of Monetary Economics* 52(6), 1151–1166.
- Ruge-Murcia, F. J. (2007, August). Methods to estimate dynamic stochastic general equilibrium models. *Journal of Economic Dynamics and Control* 31(8), 2599–2636.
- Schmitt-Grohe, S. and M. Uribe (2004, January). Solving dynamic general equilibrium models using a second-order approximation to the policy function. *Journal of Economic Dynamics and Control* 28(4), 755–775.
- Sims, C. A. (1996, Winter). Macroeconomics and methodology. *Journal of Economic Perspectives* 10(1), 105–20.
- Sims, C. A. (2002, October). Solving linear rational expectations models. *Computational Economics* 20(1-2), 1–20.

- Smets, F. and R. Wouters (2003, 09). An estimated dynamic stochastic general equilibrium model of the euro area. *Journal of the European Economic Association* 1(5), 1123–1175.
- Soderlind, P. (1999, April). Solution and estimation of re macromodels with optimal policy. *European Economic Review* 43(4-6), 813–823.
- Swanson, E., G. Anderson, and A. Levin (2006). Higher-order perturbation solutions to dynamic, discrete-time rational expectations models. Working Paper Series 2006-01, Federal Reserve Bank of San Francisco.
- Tchakarov, I., P. D. Karam, T. Bayoumi, H. Faruqee, B. Hunt, D. Laxton, J. Lee, and A. Rebucci (2004, November). Gem: A new international macroeconomic model. IMF Occasional Papers 239, International Monetary Fund.
- Tovar, C. E. (2009). Dsge models and central banks. *Economics: The Open-Access, Open-Assessment E-Journal* 3(2009-16).
- Uhlig, H. (1999). A toolkit for analyzing nonlinear dynamic stochastic models easily. In R. Marimon and A. Scott (Eds.), *Computational Methods for the Study of Dynamic Economics*, pp. 30–61. Oxford University Press.
- Woodford, M. (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

A Appendix

A.1 Log-linearizing the Neoclassical Growth Model

The social planner maximizes the utility of the representative agent

$$\max_{(C_t, K_t)_{t=0}^{\infty}} E \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\eta} - 1}{1-\eta} \right]$$

subject to the following constraints

$$Y_t = C_t + K_t$$

$$K_t = I_t + (1 - \delta)K_{t-1}$$

$$Y_t = Z_t K_{t-1}^p$$

$$\log Z_t = (1 - \psi) \log \bar{Z} + \psi \log Z_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim i.i.d.N(0, \sigma^2)$$