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Hellwig, Martin

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Incentive provision with hidden characteristics: A unified approach

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Martin Hellwig

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Max Planck Institute for Research on Collective Goods, Kurt-Schumacher-Str. 10, D-53113 Bonn
http://www.coll.mpg.de

Incentive Provision with Hidden Characteristics: A Unified Approach

Martin F. Hellwig
Max Planck Institute for Research on Collective Goods
Kurt-Schumacher-Str. 10, D - 53113 Bonn, Germany
hellwig@coll.mpg.de

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Abstract

The paper develops a unified approach for studying incentive problems with unidimensional hidden characteristics. Previous work had used separate approaches for problems with finitely many types and for problems with type distributions having continuous densities. The approach developed here applies to both these specifications, as well as the general case of type distributions that have mass points and an atomless part. The paper confirms the proposition that optimal incentive schemes induce no distortion "at the top" and downward distortions "below the top." Mass points are pooled with immediately neighbouring higher types.

Key Words: Hidden Characteristics, with General Type Distributions, Principal-Agent Problem, Incentive Schemes

JEL Classification: C61, D82, D86

1 Introduction

For incentive problems with hidden characteristics, a standard result asserts that, if the set of possible types is well ordered and, relative to this ordering of types, the relevant utility function satisfies a single-crossing condition, then an optimal incentive scheme involves no distortion "at the top" of the type set and downward distortions "below the top." This result is derived from a comparison of the losses that are entailed by a departure from efficiency with the distributive gains that can be obtained because a downward distortion for any type alleviates incentive constraints for adjacent higher types, and makes room for worsening the terms that are provided to these

higher types. At the top of the type set, these distributive gains are zero because there are no higher types; a departure from efficiency is therefore not warranted. Below the top, the distributive gains are significant; a departure from efficiency is warranted because, starting from an efficient outcome, the efficiency loss from introducing a small distortion is of the second order of smalls and is dominated by the distributive gains.

In the literature, this argument has been given for models in which the type set is finite and for models in which the type set is a continuum and the type distribution has a continuous density. These models permit the application of known optimization techniques, which are, however, quite different for the two specifications.

What about type distributions that have mass points *and* a continuous part at the same time? For such distributions, the above argument runs into difficulties. For suppose that a type below the top has positive probability mass and, on an interval just above this mass point, the type distribution has a positive density. Then any one the neighbouring types just above the mass point has zero probability mass. The efficiency loss from introducing a small distortion for the type with positive mass would therefore seem to outweigh the gains from alleviating incentive constraints for adjacent higher types. To be sure, in comparison to the distortion, the efficiency loss is of the second order of smalls. However, the mass point is "infinitely" more important than any of the neighbouring higher types. Could it be that, in this case, a distortion is undesirable even though the mass point in question lies "below the top" of the type distribution?

Incentive problems involving hidden characteristics with mixed type distributions have not been much studied. These problems do not immediately lend themselves to the application of known optimization techniques. This paper develops a new technique for analysing them.

The new technique can be used for arbitrary models with hidden characteristics when the type set is well ordered, regardless of wh$\ll\ll$at the type distribution is. A change of variables serves to redefine the notion of "type" in such a way that the original incentive problem is transformed into a new one, where the distribution of the "redefined types" has a density; this density need not be continuous.

The new technique is used to extend the standard result about no distortion "at the top" and downward distortions "below the top" to models with arbitrary type distributions. Whereas the result itself is standard, the argument given is not. For mass points below the top of the type set, the puzzle that has been posed above is resolved by showing that any such point is *necessarily* pooled with the adjacent higher types. Without pooling, the

above argument would imply that the outcome for the type with positive mass must be efficient and the outcomes for the neighbouring higher types must be distorted downwards from efficiency. But then, the relation between outcomes and types could not be monotonic. Monotonicity, however, is a necessary property of any solution for an optimal incentive compatible arrangement. To have monotonicity in the relation of outcomes to types, any mass point must be pooled with immediately adjacent higher types if such types exist.

The analysis is presented for a principal-agent problem with unidimensional hidden characteristics. The simplicity of this problem facilitates the presentation. The new technique can, however, be applied to *any* incentive problem with unidimensional hidden characteristics.

2 An Agency Problem with Hidden Characteristics

2.1 Statement of the Problem

Consider the following agency problem with hidden characteristics. A principal wants an agent to produce some output $y \geq 0$ in return for a wage payment $w \geq 0$. The payoffs from the pair (w, y) are $y - w$ for the principal and $u(w, y, t)$ for the agent, where $t \in \Re$ is a productivity parameter. The function u is assumed to be twice continuously differentiable, as well as increasing in w, decreasing in y, and strictly quasi-concave in w and y jointly. The agent's utility function also satisfies $u(0, 0, t) = 0$, $u_y(w, 0, t) = 0$, and $\lim_{y\to\infty} u_y(w, y, t) = -\infty$ for all w and t, as well as

$$u_{yt}(w, y, t) > 0 \quad \text{and} \quad u_{wt}(w, y, t) \leq 0 \tag{2.1}$$

for all $w, y > 0$, and t. Condition (2.1) implies that u also satisfies the strict single-crossing condition

$$\frac{\partial}{\partial t} \frac{|u_y(w, y, t)|}{u_w(w, y, t)} < 0 \tag{2.2}$$

for all $w, y > 0$, and t. Since $u(0, 0, t) = 0$ for all t implies $u_t(0, 0, t) = 0$ for all t, condition (2.1) also implies

$$u_t(w, y, t) > 0 \tag{2.3}$$

for all $w, y > 0$, and t.

The principal is assumed to have all the bargaining power. If he offers the agent a contract (w, y), the agent can only accept or reject this offer. The agent's payoff from rejecting the principal's offer is assumed to be zero. Under complete information, the principal would be able to hire the agent at a wage that just compensates him for the disutility from working, without letting the agent participate in the surplus from production. He would therefore propose the contract $(w^*(t,0), y^*(t,0))$, where, for any v, $(w^*(t,v), y^*(t,v))$ is defined as the pair that maximizes the surplus $y - w$ under the constraint that $u(w, y, t) \geq 0$.

However, the principal has incomplete information. Whereas the agent knows t, the principal only knows that t is the realization of a random variable $\tilde{t}$, which has a probability distribution F. The support T of the distribution F is assumed to be compact, with minimum t_0 and maximum t_1.

Given his lack of information, the principal offers a menu of contracts, leaving the agent to choose whichever contract in the menu he likes best, or to reject the principal's offer altogether. The agent's choice typically depends on t. A *contract menu* is a pair $(w(\cdot), y(\cdot))$ of functions on T such that, for any $t \in T$, $(w(t), y(t))$ is the contract, i.e., the wage/output combination that is chosen by the agent when his productivity parameter is t. The principal's problem is to choose the contract menu $(w(\cdot), y(\cdot))$ so that his expected net payoff,

$$\int [y(t) - w(t)] dF(t), \tag{2.4}$$

is maximized subject to the incentive compatibility condition that

$$u(w(t), y(t), t) \geq u(w(t'), y(t'), t) \tag{2.5}$$

for all t and t' in T, and subject to the individual-rationality condition that

$$u(w(t), y(t), t) \geq 0 \tag{2.6}$$

for all t in T.[1] A contract menu that satisfies the incentive compatibility conditions (2.5) and the individual-rationality conditions (2.6) is said to

[1] Condition (2.6) presumes that the principal does not want to make an offer which, for some t, the agent wants to reject. Because $u(0, 0, t) = 0$ for all t, this is without loss of generality. For any t, the contract $(0, 0)$ provides the principal and the agent with the same payoffs as a rejection of the principal's offer by the agent. A contract menu $(w(\cdot), y(\cdot))$ with the property that, for some $t \in T$, the agent rejects the principal's offer is therefore payoff-equivalent to the contract menu that is obtained if the contract offers $(w(t), y(t))$ for the rejecting types are replaced by $(0, 0)$; this latter contract menu satisfies the individual-rationality condition (2.6) for all t.

be *admissible*. A contract menu that maximizes the principal's expected net payoff (2.4) under the incentive compatibility conditions (2.5) and the individual-rationality conditions (2.6) is said to be *optimal*.

2.2 The Main Theorem

Conceptually, the principal's problem is a standard incentive problem with hidden characteristics. Fudenberg and Tirole (1991), Mas-Colell, Whinston, Green (1994), or Laffont and Martimort (2002) provide textbook treatments under special assumptions about the type set T and the distribution F. By contrast, I only assume that T is compact.

Let $(w(\cdot), y(\cdot))$ be an optimal contract menu and let

$$v(\cdot) := u(w(\cdot), y(\cdot), \cdot) \tag{2.7}$$

be the associated indirect utility function for the agent. In those cases that have been treated in the literature, $(w(\cdot), y(\cdot))$ and $v(\cdot)$ have been shown to exhibit the following properties.

Property A There is no distortion at the top:

If $F(\{t_1\}) > 0$, then

$$(w(t_1), y(t_1)) = (w^*(t_1, v(t_1)), y^*(t_1, v(t_1))); \tag{2.8}$$

if $F(\{t_1\}) = 0$, then

$$\lim_{k\to\infty} (w(t^k), y(t^k)) = (w^*(t_1, v(t_1)), y^*(t_1, v(t_1))) \tag{2.9}$$

for any sequence $\{t^k\}$ in T that converges to t_1 from below.

Property B There are downward distortions below the top:

For $t \in [t_0, t_1)$,

$$(w(t), y(t)) \ll (w^*(t, v(t)), y^*(t, v(t))). \tag{2.10}$$

The following theorem shows that these properties continue to hold under the more general assumptions of this paper. The statement of the theorem involves the notion of an equivalent contract menu. Two contract menus are said *equivalent* if, up to modifications on a null set in terms of the measure F, they are the same.The choice between equivalent incentive-compatible, individually rational contract menus does not affect the joint distribution of wage/output combinations and types or the value of the principal's objective function and is therefore arbitrary.

Theorem 2.1 *For any solution $(w(\cdot), y(\cdot))$ to the principal's problem, there exists an equivalent solution $(\bar{w}(\cdot), \bar{y}(\cdot))$ that exhibits Properties A and B, as well as*

Property C If $F(\{t\}) > 0$ for some $t \in [t_0, t_1)$, then, for some $\bar{t} \in (t, t_1]$, the functions $\bar{w}(\cdot)$, $\bar{y}(\cdot)$, and $\bar{y}(\cdot) - \bar{w}(\cdot)$ are constant on $[t, \bar{t})$. If $F((t, \bar{t})) > 0$, all types in $[t, \bar{t})$ are pooled.

Property C adds the new feature that any mass point of the type distribution must be pooled with some immediately adjacent higher types. The economic rationale has been sketched in the introduction: If the contract $(\bar{w}(t), \bar{y}(t))$ that is designed for a type t with positive mass were considered in isolation, then this type would be deemed to have so much weight that a distortion away from efficiency would seem to be undesirable. For the immediate right-hand neighbours of such a type, standard arguments imply that downward distortions are desirable. The resulting contract menu would not be monotonic and, by the single-crossing condition (2.2), would violate incentive compatibility.

2.3 A Preliminary Result

In the remainder of the paper, I provide a proof for Theorem 2.1. I begin with a preliminary result showing that the incentive compatibility and individual-rationality constraints can be replaced by a differential equation and a boundary condition for the indirect utility function in combination with a monotonicity condition for outcomes. This is basically the result of Mirrlees (1976). However, I need to allow for the possibility that the domain T of the contract menu is not an interval. A contract menu with domain X is said to be incentive-compatible on X if condition (2.5) holds for all t and t' in X.

Proposition 2.2 *A contract menu is incentive-compatible on T and individually rational if and only if there exists an extension $(w(\cdot), y(\cdot))$ of the contract menu from T to the interval $[t_0, t_1]$ such that $y(\cdot)$ is nondecreasing and the induced indirect utility function $v(\cdot)$ is absolutely continuous, with a derivative satisfying*

$$v'(t) = u_t(w(t), y(t), t) \tag{2.11}$$

and initial value

$$v(t_0) \geq 0. \tag{2.12}$$

Proof. By Lemma A.1 in the Appendix, a contract menu is incentive-compatible on T if and only if it admits an extension $(w(\cdot), y(\cdot))$ to the interval $[t_0, t_1]$ that is incentive-compatible on $[t_0, t_1]$. By the result of Mirrlees (1976), a contract menu $(w(\cdot), y(\cdot))$ is incentive-compatible on $[t_0, t_1]$ if and only if $y(\cdot)$ is nondecreasing and the indirect utility function is absolutely continuous, with a derivative satisfying (2.11) almost surely. Because (2.3) and (2.11) imply that v is nondecreasing, it follows that an incentive-compatible contract menu with domain $[t_0, t_1]$ is individually rational if and only if v satisfies (2.12). ■

3 A Reformulation of the Principal's Problem

Proposition 2.2 implies that the problem of choosing an optimal contract menu is equivalent to the problem of choosing $w(\cdot), y(\cdot)$, and $v(\cdot)$ so as to maximize (2.4) under the constraints that $y(\cdot)$ must be nondecreasing and that $v(\cdot) = u(w(\cdot), y(\cdot), \cdot)$ must satisfy the differential equation (2.11) and the boundary condition (2.12). I will refer to this problem as *the principal's problem.*

If it was not for the monotonicity requirement on $y(\cdot)$ and the lack of any structure on F, the principal's problem would be a standard problem of optimal control. In this section, I will reformulate this problem so that control-theoretic methods can be used anyway.

For this purpose, I change the variable of integration in (2.4), using a new variable x, rather than t, as the argument of the functions that are to be chosen. In a sense, this amounts to a redefinition of the notion of "type". The new variable x corresponds to the sum of t and $F(t)$, which is strictly increasing in t. This change of variables will ensure that the relevant distribution function has a density. The density need not be continuous.

Proceeding formally, for any $t \in [t_0, t_1]$, let

$$\xi(t) := t + F(t). \tag{3.1}$$

Being strictly increasing, the function ξ has an inverse $\tau = \xi^{-1}$. The inverse is defined on the range of ξ, a subset of the interval $[x_0, x_1] := [t_0, t_1 + 1]$. Using the fact that the function $\xi(\cdot)$ is right-continuous, as well as increasing, one can extend its inverse to the entire interval $[x_0, x_1]$ by setting

$$\tau(x) = t \quad \text{if} \quad x \in [\lim_{k \to \infty} \xi(t^k), \xi(t)] \tag{3.2}$$

for $x \in [x_0, x_1]$, where $\{t^k\}$ is any sequence that converges to t from below.

If one sets $\tilde{x} = \xi(\tilde{t})$, one has $\tilde{t} = \tau(\tilde{x})$. The distribution of $\tilde{x}$ is $G := F \circ \xi^{-1}$, and the distribution F of $\tilde{t}$ satisfies $F = G \circ \tau^{-1} = G \circ \xi$. By the change-of-variable formula, it follows that, for any function h on $[t_0, t_1]$, one has

$$\int_{t_0}^{t_1} h(t)dF(t) = \int_{x_0}^{x_1} h(\tau(x))dG(x). \tag{3.3}$$

The following lemma shows that G has a density so that (3.3) can actually be written in the form

$$\int_{t_0}^{t_1} h(t)dF(t) = \int_{x_0}^{x_1} h(\tau(x))g(x)dx. \tag{3.4}$$

Lemma 3.1 *The function $\tau(\cdot)$ that is defined by (3.1) and (3.2) is absolutely continuous. Its derivative $\tau'(\cdot)$ satisfies*

$$\tau'(x) = \frac{1}{1 + f(\tau(x))} \tag{3.5}$$

if, at $t = \tau(x)$, the derivative $F'(t) = f(t)$ is well defined, and

$$\tau'(x) = 0 \tag{3.6}$$

otherwise. The distribution function $G = F \circ \xi^{-1}$ is also absolutely continuous. Its density $g(x) = G'(x)$ satisfies

$$g(x) = 1 - \tau'(x) \tag{3.7}$$

for all x.

Proof. From (3.1), (3.2), and the definition of G, one has

$$x = \tau(x) + F(\tau(x)) = \tau(x) + G(x) \tag{3.8}$$

for all $x \in [x_0, x_1]$. Since $\tau(\cdot)$ and $G(\cdot)$ are both nondecreasing, it follows that both are Lipschitz continuous, hence absolutely continuous. Moreover, their slopes must add to one.

For any x, $\tau(x + \Delta) > \tau(x - \Delta)$ for all $\Delta > 0$ implies

$$1 = \frac{\tau(x + \Delta) - \tau(x - \Delta)}{2\Delta} + \frac{F(\tau(x + \Delta)) - F(\tau(x - \Delta))}{2\Delta},$$

hence

$$\lim_{\Delta \to 0} \frac{\tau(x + \Delta) - \tau(x - \Delta)}{2\Delta} = \frac{1}{1 + \lim_{\Delta \to 0} \frac{F(\tau(x+\Delta)) - F(\tau(x-\Delta))}{\tau(x+\Delta) - \tau(x-\Delta)}},$$

which yields (3.5) if

$$\lim_{\Delta \to 0} \frac{F(\tau(x+\Delta)) - F(\tau(x-\Delta))}{\tau(x+\Delta) - \tau(x-\Delta)} = f(\tau(x))$$

is well defined and (3.6) if

$$\lim_{\Delta \to 0} \frac{F(\tau(x+\Delta)) - F(\tau(x-\Delta))}{\tau(x+\Delta) - \tau(x-\Delta)} = \infty.$$

Trivially, (3.6) holds also if $\tau(x+\Delta) = \tau(x-\Delta)$ for some $\Delta > 0$. For this case, (3.2) indicates that ξ and F are discontinuous at $t = \tau(x+\Delta) = \tau(x-\Delta)$. The derivative $F'(t)$ is then not well defined at t. ■

For any contract menu $(w(\cdot), y(\cdot))$, (3.2) and (3.4) imply that the principal's payoff (2.4) can be rewritten as

$$\int_{x_0}^{x_1} [y(\tau(x)) - w(\tau(x))] g(x) dx. \tag{3.9}$$

Moreover, if $\hat{u}$ is defined so that

$$\hat{u}(w, y, x) := u(w, y, \tau(x)) \tag{3.10}$$

for all w, y, x, (2.7), (2.11), and (2.12) are equivalent to the conditions

$$v(\tau(x)) = \hat{u}(w(\tau(x)), y(\tau(x)), x), \tag{3.11}$$

$$v'(\tau(x))\tau'(x) = \hat{u}_x(w(\tau(x)), y(\tau(x)), x), \tag{3.12}$$

and

$$v(\tau(t_0)) \geq 0, \tag{3.13}$$

If one replaces the functions $w(\cdot), y(\cdot)$, and $v(\cdot)$ by

$$\hat{w} := w \circ \tau, \hat{y} := y \circ \tau, \text{ and } \hat{v} := v \circ \tau, \tag{3.14}$$

one can rewrite (3.9) - (3.13) as

$$\int_{x_0}^{x_1} [\hat{y}(x) - \hat{w}(x)] g(x) dx, \tag{3.15}$$

$$\hat{v}(x) = \hat{u}(\hat{w}(x), \hat{y}(x), x), \tag{3.16}$$

$$\hat{v}'(x) = \hat{u}_x(\hat{w}(x), \hat{y}(x), x), \tag{3.17}$$

and

$$\hat{v}(t_0) \geq 0. \tag{3.18}$$

The principal's problem, i.e., the problem of maximizing (2.4) under the constraints that $y(\cdot)$ must be nondecreasing. and that $v(\cdot)$ must satisfy (2.7), (2.11), and (2.12) is thus equivalent to the problem of choosing functions $\hat{w}(\cdot), \hat{y}(\cdot)$, and $\hat{v}(\cdot)$ to maximize (3.15) subject to the constraints that $\hat{y}(\cdot)$ be nondecreasing, that $\hat{v}(\cdot)$ satisfy (3.16) - (3.18), and that $\hat{w}(\cdot), \hat{y}(\cdot)$, and $\hat{v}(\cdot)$ can be represented in the form (3.14).

Fortunately, the requirement that $\hat{w}(\cdot), \hat{y}(\cdot)$, and $\hat{v}(\cdot)$ can be represented in the form (3.14) can be neglected. The following lemma shows that, up to modifications on null sets, any solution to the problem of maximizing (3.15) subject to monotonicity of $\hat{y}(\cdot)$ and (3.16) - (3.18) can *automatically* be represented in the form (3.14).

Lemma 3.2 *If $\hat{w}, \hat{y}$, and $\hat{v}$ maximizes (3.15) under to the constraints that $\hat{y}$ be nondecreasing and that $\hat{v}$ satisfy (3.16) - (3.18), then there exist functions w, y, and v, such that, up to modifications on null sets, the triples $\hat{w}, \hat{y}, \hat{v}$ and w, y, v satisfy (3.14).*

Proof. To prove this lemma, it is enough to show that, for almost all x_1 and x_2, $\tau(x_1) = \tau(x_2)$ implies $\hat{w}(x_1) = \hat{w}(x_2)$, $\hat{y}(x_1) = \hat{y}(x_2)$, and $\hat{v}(x_1) = \hat{v}(x_2)$. From (3.17) and (3.10), one has

$$\hat{v}'(x) = u_t(\hat{w}(x), \hat{y}(x), \tau(x))\ \tau'(x).$$

Since $\tau(x_1) = \tau(x_2)$ implies $\tau'(x) = 0$ for almost all $x \in [x_1, x_2]$, it follows that $\tau(x_1) = \tau(x_2)$ implies $\hat{v}(x_1) = \hat{v}(x_2)$. By standard arguments,[2] it follows that there exists a function v such that $\hat{v}(x) = v(\tau(x))$ for all x.

Next, consider the function w^* such that

$$w^*(t) = E[\hat{w}(\tilde{x})|\tau(\tilde{x}) = t] \tag{3.19}$$

for all t, where $\tilde{x}$ is distributed as G. By the definition of the conditional expectation, one has

$$\int_{x_0}^{x_1} w^*(\tau(x))dG(x) = \int_{x_0}^{x_1} \hat{w}(x)dG(x). \tag{3.20}$$

Given w^*, consider also the function y^* such that

$$u(w^*(t), y^*(t), t) = v(t) \tag{3.21}$$

[2] See, e.g., Result (8), p. 43, in Hildenbrand (1974).

for all t. By the definition of v, one also has

$$u(\hat{w}(x), \hat{y}(x), \tau(x)) = v(\tau(x)) \tag{3.22}$$

for all x. By the strict quasi-concavity of u, (3.21), (3.19) and (3.22) imply

$$y^*(t) \geq E[\hat{y}(\tilde{x})|\tau(\tilde{x}) = t] \tag{3.23}$$

for all t, hence

$$\int_{x_0}^{x_1} y^*(\tau(x))dG(x) \geq \int_{x_0}^{x_1} \hat{y}(x)dG(x). \tag{3.24}$$

Moreover, the inequality in (3.24) is strict unless one has $w^*(\tau(x)) = \hat{w}(x)$ for G-almost all x.

If the inequality in (3.24) is strict, one has

$$\int_{x_0}^{x_1} [y^*(\tau(x)) - w^*(\tau(x))]dG(x) > \int_{x_0}^{x_1} [\hat{y}(x) - \hat{w}(x)]dG(x). \tag{3.25}$$

Given that, trivially, the triple $w^* \circ \tau, y^* \circ \tau, \hat{v}$ has $y^* \circ \tau$ nondecreasing and $\hat{v} = v \circ \tau$ satisfying (3.16) - (3.18), (3.25) is incompatible with the assumption that $\hat{w}, \hat{y}, \hat{v}$ maximizes (3.15) subject to the constraints that $\hat{y}$ be nondecreasing and that $\hat{v}$ satisfy (3.16) - (3.18). Therefore, the inequality in (3.24) cannot be strict. It follows that $w^*(\tau(x)) = \hat{w}(x)$ and, by (3.21), $y^*(\tau(x)) = \hat{y}(x)$ for G-almost all x, as claimed in the lemma. ■

The argument for Lemma 3.2 is illustrated in Figure 1. If contracts are conditioned on x, rather than t, the principal has room to offer a richer contract menu. In particular, if t is a mass point of the distribution F, the function $\xi(\cdot)$ is discontinuous at t, and the principal can assign different contracts to different $x \in (\xi(t-), \xi(t)]$. Thus, in Figure 1, he is assumed to offer $(\hat{w}(x), \hat{y}(x))$ for $x \in (\xi(t-), \xi(t)]$, with wages that are uniformly distributed between w_1 and w_2. However, such an arrangement cannot be optimal for him. Because all $x \in (\xi(t-), \xi(t)]$ correspond to the same "real" type $\tau(x) = t$, incentive compatibility requires that all the contracts $(\hat{w}(x), \hat{y}(x))$ for $x \in (\xi(t-), \xi(t)]$ provide the agent with the same utility when his type is t. Thus, in Figure 1, the contract offers $(\hat{w}(x), \hat{y}(x))$ for $x \in (\xi(t-), \xi(t)]$ all lie on the same indifference curve $I(t)$ for this type. Strict quasi-concavity of u implies that the indifference curve $I(t)$ is strictly convex. If the principal replaces the wage offers $\hat{w}(x)$ for $x \in (\xi(t-), \xi(t)]$ by their (conditional) expectation $w^* = (w_1 + w_2)/2$, he can ask for an output y^* that is strictly

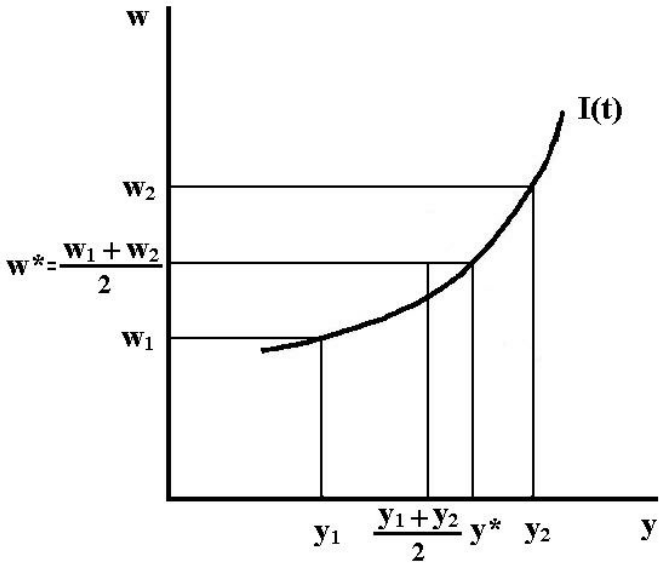

Figure 1: Multiple contracts for different people of the same type are sub-optimal.

greater than the (conditional) expectation of $\hat{y}(x)$, $x \in (\xi(t-), \xi(t)]$. By introducing heterogeneity into the contract offers to people with the same "real" type t, the principal would just harm himself.

For lack of a better term, I refer to the problem of maximizing (3.15) under the constraints that $\hat{y}(\cdot)$ be nondecreasing and that $\hat{v}(\cdot)$ satisfy (3.16) - (3.18) as *the principal's modified problem.* From Lemma 3.2, one immediately obtains.

Proposition 3.3 *A contract menu $(w(\cdot), y(\cdot))$, with associated indirect utility function $v(\cdot)$, solves the principal's problem, if and only if the functions $\hat{w}(\cdot), \hat{y}(\cdot)$, and $\hat{v}(\cdot)$ that are given by (3.1), (3.2), and (3.14) solve the principal's modified problem.*

In the remainder of the paper, I study the principal's modified problem. The purpose is to show that any solution must exhibit the analogues of Properties A - C. These analogues are written as:

Property $\hat{\mathbf{A}}$ There is no distortion at the top: For any sequence $\{x^k\}$ in $[x_0, x_1]$ that converges to x_1 from below, one has

$$\lim_{k\to\infty} (\hat{w}(x^k), \hat{y}(x^k)) = (w^*(\tau(x_1), \hat{v}(x_1)), y^*(\tau(x_1), \hat{v}(x_1))). \tag{3.26}$$

Property $\hat{\mathbf{B}}$ There are downward distortions below the top: For $x \in [x_0, x_1)$, $\tau(x) < t_1$ implies

$$(\hat{w}(x), \hat{y}(x)) << (w^*(\tau(x), \hat{v}(x)), y^*(\tau(x), \hat{v}(x))). \tag{3.27}$$

Property $\hat{\mathbf{C}}$ If $x \in [x_0, x_1)$ and $\hat{x} \in (x, x_1)$ are such that $\tau(x) = \tau(\hat{x}) < t_1$, then there exists $\bar{x} \in (\hat{x}, x_1)$ such that $\tau(\hat{x}) < \tau(\bar{x})$ and, moreover, $(\hat{w}(x'), \hat{y}(x')) = (\hat{w}(x), \hat{y}(x))$ for all $x' \in (x, \bar{x})$..

Proposition 3.4 *A contract menu $(w(\cdot), y(\cdot))$, with associated indirect utility function $v(\cdot)$, exhibits Properties A - C if and only if the functions $\hat{w}(\cdot), \hat{y}(\cdot)$, and $\hat{v}(\cdot)$ that are given by (3.1), (3.2), and (3.14) exhibit Properties $\hat{A}$ - $\hat{C}$.*

The proof of Proposition 3.4 is left to the reader. The only part that is not immediately obvious from the definitions concerns the equivalence of Properties A and $\hat{\text{A}}$ when t_1 is a mass point of the distribution F. If

$F(\{t_1\}) > 0$, one has $\tau(x') = t_1$ for all x' in some nondegenerate interval $[x, x_1]$. By Lemma 3.2 and the monotonicity of $\hat{y}(\cdot)$, it follows that $(\hat{w}(x'), \hat{y}(x'))$ and $\hat{v}(x')$ are the same for all $x' \in (x, x_1)$. Property Â therefore implies that

$$(\hat{w}(x'), \hat{y}(x')) = (w^*(t_1, \hat{v}(x_1)), y^*(t_1, \hat{v}(x_1)))$$

for all $x' \in (x, x_1)$ and, hence, by (3.1) and (3.2), that

$$(w(t_1), y(t_1)) = (w^*(t_1, v(x_1)), y^*(t_1, v(x_1))),$$

which is Property A for this case.

4 Analysis of the Principal's Modified Problem

4.1 Preliminaries

The principal's modified problem has the same formal structure as the principal's problem itself when F has a density. However, the density $g(\cdot)$ in (3.15) is not, in general, continuous. Moreover, $\hat{u}$, $\hat{u}_w$, and $\hat{u}_y$ are not, in general, continuously differentiable with respect to x. From (3.16), one easily verifies that $\hat{u}$, like u, is twice continuously differentiable and strictly quasi-concave in w and y jointly, as well as increasing in w and decreasing in y. In particular, one has

$$\hat{u}_w(w, y, x) = u_w(w, y, \tau(x)) \tag{4.1}$$

and

$$\hat{u}_y(w, y, x) = u_y(w, y, \tau(x)) \tag{4.2}$$

for any w, y, and x. However, from (4.1) and (4.2), one obtains

$$\frac{\partial}{\partial x} \frac{|\hat{u}_y(w, y, x)|}{\hat{u}_w(w, y, x)} = \frac{\partial}{\partial t} \frac{|u_y(w, y, \tau(x))|}{u_w(w, y, \tau(x))} \tau'(x), \tag{4.3}$$

which has a discontinuity with respect to x whenever τ' and the density $g = 1 - \tau'$ have a discontinuity. By (2.2) and Lemma 3.1, (4.3) yields

$$\frac{\partial}{\partial x} \frac{|\hat{u}_y(w, y, x)|}{\hat{u}_w(w, y, x)} \leq 0 \tag{4.4}$$

for all w, y, and x. Thus, $\hat{u}$ satisfies a *weak* single-crossing condition with respect to x. However, for any x for which $\tau(x)$ is a mass point of the

distribution F, this weak single-crossing condition cannot be strict. For such x, one has $\tau'(x) = 0$, and hence,

$$\frac{\partial}{\partial x}\frac{|\hat{u}_y(w,y,x)|}{\hat{u}_w(w,y,x)} = 0 \tag{4.5}$$

for all w and y, and all $x \in (\xi(t-), \xi(t)]$.

4.2 Optimality Conditions

Despite the discontinuities associated with the derivative of the function τ, the principal's modified problem can be handled by control-theoretic methods. If $\hat{y}(\cdot)$ was known to be absolutely continuous, it would actually be a standard control problem with $\hat{v}(\cdot)$ and $\hat{y}(\cdot)$ as state variables and with $\hat{w}(\cdot)$ and $q(\cdot) := \hat{y}'(\cdot)$, as control variables, with the constraint that $q(x) \geq 0$ for all x. The Hamiltonian for this control problem would be

$$\begin{aligned} H(x) \quad = \quad & (\hat{y}(x) - \hat{w}(x))g(x) + \hat{\mu}(x)(\hat{u}(\hat{w}(x), \hat{y}(x), x) - \hat{v}(x)) \\ & + \hat{\varphi}(x)\hat{u}_x(\hat{w}(x), \hat{y}(x), x) + [\hat{\psi}(x) - \hat{\chi}(x)]q(x), \end{aligned} \tag{4.6}$$

where $\hat{\mu}(\cdot)$ is the Lagrange multiplier function for equation (3.16), $\hat{\varphi}(\cdot)$, $\hat{\psi}(\cdot)$ are the costate variables associated with the state variables $\hat{v}(\cdot)$ and $\hat{y}(\cdot)$, and $\hat{\chi}(\cdot)$ is the distribution function of a nonnegative measure that is concentrated on the set $\{x \in [x_0, x_1] | \hat{y}(x) = 0\}$.

Maximization of $H(x)$ with respect to the controls requires that

$$-g(x) + \hat{\mu}(x)\hat{u}_w(\hat{w}(x), \hat{y}(x), x) + \hat{\varphi}(x)\hat{u}_{xw}(\hat{w}(x), \hat{y}(x), x) \leq 0, \tag{4.7}$$
$$\text{with equality if } \hat{w}(x) > 0;$$

$$\hat{\psi}(x) - \hat{\chi}(x) \leq 0, \tag{4.8}$$
$$\text{with equality if } q(x) > 0.$$

Further, the costate variables $\hat{\varphi}(\cdot)$ and $\hat{\psi}(\cdot)$ must be absolutely continuous, with derivatives satisfying

$$\hat{\varphi}'(x) = \hat{\mu}(x) \tag{4.9}$$

and

$$\hat{\psi}'(x) = -g(x) - \hat{\mu}(x)\hat{u}_y(\hat{w}(x), \hat{y}(x), x) - \hat{\varphi}(x)\hat{u}_{xy}(\hat{w}(x), \hat{y}(x), x) \tag{4.10}$$

for almost all $x \in [x_0, x_1]$. The costate variables $\hat{\varphi}(\cdot)$ and $\hat{\psi}(\cdot)$ must also satisfy the transversality conditions

$$\hat{\varphi}(x_0)\hat{v}(x_0) = \hat{\varphi}(x_1) = 0 \tag{4.11}$$

and

$$[\hat{\psi}(x_0) - \hat{\chi}(x_0)]\hat{y}(x_0) = [\hat{\psi}(x_1) - \hat{\chi}(x_1)]\hat{y}(x_1) = 0. \tag{4.12}$$

These conditions correspond to those formulated by Guesnerie and Laffont (1984) under the assumption that the variable on which the monotonicity constraint is imposed is piecewise continuously differentiable, as well as continuous, in the exogenous parameter.[3]

One cannot generally presume that $\hat{y}(\cdot)$ is continuous, let alone differentiable. However, as shown in Hellwig (2006), these properties are not actually needed. Monotonicity itself provides enough structure to permit the formulation of a maximum principle for the given control problem.

If no additional restriction is imposed on $\hat{y}(\cdot)$, the necessary conditions for optimality are *identical* to the ones just given, *with one exception*: (4.8) must hold with equality whenever $\hat{y}(\cdot)$ is strictly increasing at x; this requirements extends the maximum principle for the "slope" of $\hat{y}(\cdot)$ from the case where this slope is well defined and equal to some number $q(x)$ to the case where this "slope" can be infinite, e.g. because the function $\hat{y}(\cdot)$ can be discontinuous. Even in this case, the "slope" of $\hat{y}(\cdot)$ at x must maximize the product of this "slope" with the costate variable of $\hat{y}(\cdot)$ at x. Thus, (4.8) is replaced by

$$\hat{\psi}(x) - \hat{\chi}(x) \leq 0, \tag{4.13}$$
$$\text{with equality if } \hat{y}(\cdot) \text{ is strictly increasing at } x.$$

In the following, I will suppose that $\hat{w}(\cdot), \hat{y}(\cdot)$, and $\hat{v}(\cdot)$ provide a solution to the principal's modified problem, and that $\hat{\mu}(\cdot), \hat{\varphi}(\cdot)$, and $\hat{\psi}(\cdot)$ are the associated Lagrange multiplier and costate variables.

Given the Inada condition $u_y(w, 0, t) = 0$ for all w and t, one does not have to worry about boundary solutions.

Lemma 4.1 *Any solution to the principal's modified problem satisfies* $\hat{w}(x) > 0$ *and* $\hat{y}(x) > 0$ *for all* $x \in (x_0, x_1]$.

Proof. I first show that $\hat{y}(x) > 0$ for all $x \in (x_0, x_1]$. For suppose that this claim is false and that $\hat{y}(x) = 0$ for some $x \in (x_0, x_1]$. Let $\bar{x} := \sup\{x \in [x_0, x_1] | \hat{y}(x) = 0\}$. Because $\hat{y}(\cdot)$ is nondecreasing, one has $\hat{y}(x) = 0$ for all

[3] This assumption allows the application of the maximum principle in the original version of Pontryagin et al. (1965). In fact, the results of Clarke (1976, 1983) imply that only absolute continuity is needed.

$x \in [x_0, \bar{x})$. One also has $\hat{y}(\bar{x} + \varepsilon) > 0$ for all $\varepsilon > 0$. Thus, $\hat{y}(\cdot)$ is strictly increasing at x, and (4.13) yields

$$\hat{\psi}(\bar{x}) - \hat{\chi}(\bar{x}) = 0. \tag{4.14}$$

Because $u_y(w, 0, t) = 0$ for all w and t, $\hat{y}(x) = 0$ for $x \in [x_0, \bar{x})$ also implies

$$\hat{u}_y(\hat{w}(x), \hat{y}(x), x) = u_y(\hat{w}(x), 0, \tau(x)) = 0$$

and

$$\hat{u}_{xy}(\hat{w}(x), \hat{y}(x), x) = u_{yt}(\hat{w}(x), 0, \tau(x))\tau'(x) = 0$$

for all $x \in [x_0, \bar{x})$. By (4.10) and (4.13), it follows that

$$\hat{\psi}(x) = \hat{\psi}(\bar{x}) - \int_x^{\bar{x}} \hat{\psi}'(x')dx' = \int_x^{\bar{x}} g(x')dx'$$

for all $x \in [x_0, \bar{x})$. Because $\hat{\chi}(\cdot)$ is the distribution function of a nonnegative measure, one also has $\hat{\chi}(x) \leq \hat{\chi}(\bar{x})$ for all $x \in [x_0, \bar{x})$. Thus,

$$\hat{\psi}(x) - \hat{\chi}(x) \geq \int_x^{\bar{x}} g(x')dx' + \hat{\psi}(\bar{x}) - \hat{\chi}(\bar{x}),$$

and, by (4.14),

$$\hat{\psi}(x) - \hat{\chi}(x) \geq \int_x^{\bar{x}} g(x')dx' = G(\bar{x}) - G(x) \tag{4.15}$$

for all $x \in [x_0, \bar{x})$. By the definition of $G(\cdot)$, one has

$$G(\bar{x}) - G(x) = F(\tau(\bar{x})) - F(\tau(x)) > 0,$$

hence $\hat{\psi}(x) - \hat{\chi}(x) > 0$ whenever x is sufficiently close to x_0. This is incompatible with the first-order condition (4.8) for the choice of $q(x)$. The assumption that $\hat{y}(x) = 0$ for some $x \in (x_0, x_1]$ has thus led to a contradiction and must be false.

Given that $\hat{y}(x) > 0$ for $x \in (x_0, x_1]$, the individual-rationality condition $\hat{u}(\hat{w}(x), \hat{y}(x), x) \geq 0$ also yields $\hat{w}(x) > 0$ for $x \in (x_0, x_1]$. ■

Lemma 4.1 implies that condition (4.7) must hold as an equation. Moreover, the function $\hat{\chi}(\cdot)$ must be constant, and there is no loss of generality in setting $\hat{\chi}(x) = 0$ for all x. Using (4.9), one can rewrite (4.7) and (4.10) as

$$\hat{\varphi}'(x)\hat{u}_w + \hat{\varphi}(x)\hat{u}_{xw} = g(x) \tag{4.16}$$

and

$$\hat{\psi}'(x) = -g(x) - \hat{\varphi}'(x)\hat{u}_y - \hat{\varphi}(x)\hat{u}_{xy}, \tag{4.17}$$

where $\hat{u}_w, \hat{u}_y, \hat{u}_{wx}$, and $\hat{u}_{yx}$ are all evaluated at $(\hat{w}(x), \hat{y}(x), x)$. If one uses (4.16) to substitute for $\hat{\varphi}'(x)$ in (4.17), one obtains:

$$\hat{\psi}'(x) = -\left(1 + \frac{\hat{u}_y}{\hat{u}_w}\right) g(x) - \hat{\varphi}(x)\left(\hat{u}_{yx} - \frac{\hat{u}_y}{\hat{u}_w}\hat{u}_{wx}\right),$$

or, equivalently,

$$\hat{\psi}'(x) = -\frac{\hat{u}_w + \hat{u}_y}{\hat{u}_w} g(x) + \hat{\varphi}(x)\hat{u}_w \frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w}. \tag{4.18}$$

4.3 Analysis of the Optimality Conditions

Equation (4.18) is the key condition for assessing the efficiency properties of the contracts $(\hat{w}(x), \hat{y}(x))$, $x \in (x_0, x_1]$. A contract $(\hat{w}(x), \hat{y}(x)) \in \Re^2_{++}$ is efficient for x if

$$\hat{u}_w(\hat{w}(x), \hat{y}(x), x) + \hat{u}_y(\hat{w}(x), \hat{y}(x), x) = 0, \tag{4.19}$$

distorted downwards from efficiency for x if

$$\hat{u}_w(\hat{w}(x), \hat{y}(x), x) + \hat{u}_y(\hat{w}(x), \hat{y}(x), x) > 0, \tag{4.20}$$

and distorted upwards from efficiency for x if

$$\hat{u}_w(\hat{w}(x), \hat{y}(x), x) + \hat{u}_y(\hat{w}(x), \hat{y}(x), x) < 0. \tag{4.21}$$

By (4.18), therefore, $(\hat{w}(x), \hat{y}(x))$ is efficient for x if

$$\hat{\psi}'(x) - \hat{\varphi}(x)\hat{u}_w \frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w} = 0, \tag{4.22}$$

distorted downwards from efficiency for x if

$$\hat{\psi}'(x) - \hat{\varphi}(x)\hat{u}_w \frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w} < 0, \tag{4.23}$$

and distorted upwards from efficiency for x if

$$\hat{\psi}'(x) - \hat{\varphi}(x)\hat{u}_w \frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w} > 0. \tag{4.24}$$

The function $\hat{\varphi}(\cdot)$ is given as the solution to the differential equation (4.16) that satisfies the transversality condition $\hat{\varphi}(x_1) = 0$. This solution is computed as

$$\hat{\varphi}(x) = -\int_x^{x_1} \frac{g(x')}{\hat{u}_w(\hat{w}(x'), \hat{y}(x'), x')} \exp\left(\int_x^{x'} \frac{\hat{u}_{xw}(\hat{w}(x''), \hat{y}(x''), x'')}{\hat{u}_w(\hat{w}(x''), \hat{y}(x''), x'')} dx'' \right) dx', \tag{4.25}$$

so that one obtains:

Lemma 4.2 *The costate variable $\hat{\varphi}(\cdot)$ satisfies $\hat{\varphi}(x) < 0$ for all $x \in [x_0, x_1)$.*

At this point, a standard argument, along the lines of Mirrlees (1971, 1976) or Seade (1977, 1982), might go as follows: If there is no pooling of types, so that $\hat{\psi}'(x) = 0$, and if the single-crossing condition (4.4) is strict, i.e., if $\frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w} < 0$, then, for $x \in [x_0, x_1)$, Lemma 4.2 implies (4.23). Therefore, the contract $(\hat{w}(x), \hat{y}(x))$ is distorted downwards from efficiency if $x \in [x_0, x_1)$.

However, the presumptions that $\hat{\psi}'(x) = 0$ and that the single-crossing condition (4.4) is strict are both not justified. As mentioned above, the inequality (4.4) cannot be strict if $\tau(x)$ is a mass point of the distribution $F(\cdot)$. I will also show that, if $\tau(x)$ is a mass point of the distribution $F(\cdot)$, one must have $\hat{\psi}'(x) < 0$. A mass point $\tau(x) < t_1$ for which $\hat{\psi}'(x) = 0$ would satisfy (4.22) so that the contract $(\hat{w}(x), \hat{y}(x))$ would be efficient, contrary to what one wants to prove.

Whereas the traditional argument for downward distortions has focussed on the sign of $\hat{\varphi}(x)$, the argument here focusses on the behaviour of the costate variable $\hat{\psi}(\cdot)$ that corresponds to the monotonicity constraint. To reconcile the monotonicity constraint with the prevalence of downward distortions for types between the mass point $\tau(x)$ and t_1, the optimal conrtact menu must pool $\tau(x)$ with adjacent higher types.

By (4.18), one finds that

$$\hat{\psi}(\hat{x}) - \hat{\psi}(x) = \int_x^{\hat{x}} \left[\hat{\varphi}(x)\hat{u}_w \frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w} - \frac{\hat{u}_w + \hat{u}_y}{\hat{u}_w} g(x') \right] dx' \tag{4.26}$$

for any x and $\hat{x}$. The following lemma relates the behaviour of $\hat{\psi}(\cdot)$ to the efficiency properties of the contracts $(\hat{w}(x), \hat{y}(x))$.

Lemma 4.3 *For any $x \in [x_0, x_1]$, the costate variable $\hat{\psi}(\cdot)$ satisfies $\hat{\psi}(x) = 0$ unless the contract $(\hat{w}(x), \hat{y}(x))$ is distorted downward from efficiency.*

Proof. I first show that $\hat{\psi}(x_0) = 0$. If $\hat{y}(x_0) > 0$, this follows from the transversality condition (4.12). If $\hat{y}(x_0) = 0$, Lemma 4.1 implies that the function $\hat{y}(\cdot)$ is strictly increasing at x_0; in this case, $\hat{\psi}(x_0) = 0$ follows from the optimality condition (4.13).

Suppose that one has $\hat{\psi}(\hat{x}) \neq 0$ for some $\hat{x}$ where $(\hat{w}(\hat{x}), \hat{y}(\hat{x}))$ is *not* distorted downwards from efficiency. Since $\hat{\psi}(x_0) = 0$ and $\hat{\psi}(\cdot)$ is continuous, there exists $\bar{x} \in [x_0, \hat{x})$ such that $\hat{\psi}(\bar{x}) = 0$ and $\hat{\psi}(x) \neq 0$ for all $x \in (\bar{x}, \hat{x}]$. By (4.13), one actually has $\hat{\psi}(x) < 0$ and $\hat{y}(x) = \hat{y}(\hat{x})$ for all $x \in (\bar{x}, \hat{x}]$. By (3.17), one also $\hat{w}(x) = \hat{w}(\hat{x})$ for all $x \in (\bar{x}, \hat{x}]$, i.e., all types t with $\xi(t) \in (\bar{x}, \hat{x}]$ must get the same contract.

By the weak single-crossing condition (4.4), it follows that, for any $x' \in (\bar{x}, \hat{x}]$, the contract $(\hat{w}(x'), \hat{y}(x')) = (\hat{w}(\hat{x}), \hat{y}(\hat{x}))$ is not distorted downwards from efficiency for x. By (4.19) - (4.21), it follows that

$$\hat{u}_w(\hat{w}(x'), \hat{y}(x'), x') + \hat{u}_y(\hat{w}(x'), \hat{y}(x'), x') \leq 0$$

for all $x' \in (\bar{x}, \hat{x}]$. From (4.26), one therefore obtains

$$\hat{\psi}(\hat{x}) - \hat{\psi}(\bar{x}) \geq -\int_x^{\hat{x}} -\hat{\varphi}(x)\hat{u}_w \, \frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w} dx'. \tag{4.27}$$

By (4.4) and Lemma 4.2, (4.27) in turn yields $\hat{\psi}(\hat{x}) - \hat{\psi}(\bar{x}) \geq 0$, which is incompatible with the assumption that $\hat{\psi}(\hat{x}) < 0$ and $\hat{\psi}(\bar{x}) = 0$. The assumption that $\hat{\psi}(\hat{x}) \neq 0$ for some $\hat{x}$ where $(\hat{w}(\hat{x}), \hat{y}(\hat{x}))$ is *not* distorted downwards from efficiency has thus led to a contradiction and must be false. ■

Lemma 4.3 implies that, for any x for which $(\hat{w}(x), \hat{y}(x))$ is *not* downward distorted, the term $\hat{\psi}(x)$ in (4.26) vanishes. Because, by (4.13), the term $\hat{\psi}(\hat{x})$ is nonpositive, it follows that, for any such x and any $\hat{x} > x$, one must have

$$\int_x^{\hat{x}} \hat{\varphi}(x')\hat{u}_w \, \frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w} dx' \leq \int_x^{\hat{x}} \frac{\hat{u}_w + \hat{u}_y}{\hat{u}_w} g(x') dx'. \tag{4.28}$$

Since $\hat{u}_w > 0$ and $\hat{u}_y \leq 0$, this is equivalent to the requirement that

$$\int_x^{\hat{x}} \hat{\varphi}(x')\hat{u}_w \, \frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w} dx' \leq \int_x^{\hat{x}} \left(1 - \frac{|\hat{u}_y|}{\hat{u}_w}\right) g(x') dx'. \tag{4.29}$$

By Lemma 4.2 and the single-crossing condition (4.4), the integrand on the left-hand side is everywhere nonnegative. What about the integrand

on the right-hand side? If $(\hat{w}(x), \hat{y}(x))$ is *not* downward distorted, then, at x, the integrand on the right-hand side is zero or negative. For $x' > x$, the integrand on the right-hand side of (4.29) may change, first, because $\frac{|\hat{u}_y|}{\hat{u}_w}$ depends on x' directly, and, second, because the contract $(\hat{w}(x'), \hat{y}(x'))$ depends on x'. The following lemma exploits the monotonicity of $\hat{y}(\cdot)$ and the quasi-concavity of u and $\hat{u}$ to provide a bound on the change that depends only on the direct effect of x' on the marginal rate of substitution.

Lemma 4.4 *For any $x \in [x_0, x_1)$ and any $x' \in (x, x_1]$,*

$$\frac{|\hat{u}_y(\hat{w}(x'), \hat{y}(x'), x')|}{\hat{u}_w(\hat{w}(x'), \hat{y}(x'), x')} \geq \frac{|\hat{u}_y(\hat{w}(x), \hat{y}(x), x)|}{\hat{u}_w(\hat{w}(x), \hat{y}(x), x)} + \int_x^{x'} \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx''. \tag{4.30}$$

The inequality is strict if $\hat{y}(x') > \hat{y}(x)$.

Proof. Incentive compatibility implies that the functions $\hat{y}(\cdot)$ and $\hat{w}(\cdot)$ are co-monotonic. Indeed, by (3.16) and the incentive compatibility condition (3.17), one must have $d\hat{w}(x'') + \frac{\hat{u}_y}{\hat{u}_w} d\hat{y}(x'') = 0$ for almost all $x'' \in [x_0, x_1]$. By standard calculus, one therefore has

$$\begin{aligned} &\frac{|\hat{u}_y(\hat{w}(x'), \hat{y}(x'), x')|}{\hat{u}_w(\hat{w}(x'), \hat{y}(x'), x')} - \frac{|\hat{u}_y(\hat{w}(x), \hat{y}(x), x)|}{\hat{u}_w(\hat{w}(x), \hat{y}(x), x)} - \int_x^{x'} \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx'' \\ &\qquad = \int_x^{x'} \left[\frac{\partial}{\partial w} \frac{|\hat{u}_y|}{\hat{u}_w} \left(-\frac{\hat{u}_y}{\hat{u}_w} \right) + \frac{\partial}{\partial y} \frac{|\hat{u}_y|}{\hat{u}_w} \right] d\hat{y}(x'') \end{aligned} \tag{4.31}$$

for all $x \in [x_0, x_1)$ and all $x' \in (x, x_1]$. The right-hand side of (4.31) is computed as

$$\int_x^{x'} \left(-\frac{1}{\hat{u}_w^3} \right) \left[\hat{u}_y^2 \hat{u}_{ww} - \hat{u}_y \hat{u}_w (\hat{u}_{wy} + \hat{u}_{yw}) + \hat{u}_w^2 \hat{u}_{yy} \right] d\hat{y}(x''). \tag{4.32}$$

Because u and $\hat{u}$ are quasi-concave in w and y and because $\hat{y}(\cdot)$ is nondecreasing, expression (4.32) is nonnegative. Moreover, if $\hat{y}(x') > \hat{y}(x)$, then, because the quasi-concavity of u and $\hat{u}$ in w and y is strict, expression (4.32) is strictly positive. The left-hand side of (4.31) is therefore nonnegative. It is positive if $\hat{y}(x') > \hat{y}(x)$. ■

Upon combining (4.29) and (4.30), one finds that, for any x for which $(\hat{w}(x), \hat{y}(x))$ is *not* downward distorted, one must have

$$\begin{aligned} \int_x^{\hat{x}} \hat{\varphi}(x') \hat{u}_w \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx' \quad \leq \quad & \left(1 - \frac{|\hat{u}_y(\hat{w}(x), \hat{y}(x), x)|}{\hat{u}_w(\hat{w}(x), \hat{y}(x), x)} \right) \int_x^{\hat{x}} g(x') dx' \\ & - \int_x^{\hat{x}} \int_x^{x'} \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx'' \, g(x') dx' \end{aligned} \tag{4.33}$$

for all $\hat{x} > x$. Upon combining (4.33) with (4.4) and Lemma 4.2, one obtains

Lemma 4.5 *None of the contracts* $(\hat{w}(x), \hat{y}(x))$ *for* $x \in [x_0, x_1)$ *is distorted upwards from efficiency, i.e.,*

$$\hat{u}_w(\hat{w}(x), \hat{y}(x), x) + \hat{u}_y(\hat{w}(x), \hat{y}(x), x) \geq 0$$

for all $x \in [x_0, x_1)$.

Proof. If the lemma is false, then, by (4.21), one has

$$\frac{|\hat{u}_y(\hat{w}(x), \hat{y}(x), x)|}{\hat{u}_w(\hat{w}(x), \hat{y}(x), x)} > 1 \tag{4.34}$$

for some $x \in [x_0, x_1)$. Moreover, Lemmas 4.3 and 4.4 imply that (4.33) must hold for all $\hat{x} \in (x, x_1]$. However, (4.34) implies that, for $\hat{x}$ sufficiently close to x, the right-hand side of (4.33) is negative. By (4.4) and Lemma 4.2, the left-hand side of (4.33) is nonnegative. The assumption that the lemma is false thus leads to a contradiction. ■

By contrast, the possibility that $(\hat{w}(x), \hat{y}(x))$ might be efficient for x cannot be entirely ruled out. The following lemma and its corollary show that, if this is the case, then for any $x' \in [x, x_1]$, the contract $(\hat{w}(x'), \hat{y}(x'))$ must be efficient for x'. Moreover, one must have pooling of all types between $\tau(x)$ and t_1, the top of the type set.

Lemma 4.6 *If, for some* $x \in [x_0, x_1)$, *the contract* $(\hat{w}(x), \hat{y}(x))$ *is efficient for* x, *then, for every* $\hat{x} \in [x, x_1]$, *the contract* $(\hat{w}(\hat{x}), \hat{y}(\hat{x}))$ *is efficient for* $\hat{x}$, *i.e.,*

$$\hat{u}_w(\hat{w}(x), \hat{y}(x), x) + \hat{u}_y(\hat{w}(x), \hat{y}(x), x) = 0 \tag{4.35}$$

implies

$$\hat{u}_w(\hat{w}(\hat{x}), \hat{y}(\hat{x}), \hat{x}) + \hat{u}_y(\hat{w}(\hat{x}), \hat{y}(\hat{x}), \hat{x}) = 0. \tag{4.36}$$

Proof. If the lemma is false, there exist $x \in [x_0, x_1)$ and $\hat{x} \in [x, x_1]$ such that $(\hat{w}(x), \hat{y}(x))$ is efficient and $(\hat{w}(\hat{x}), \hat{y}(\hat{x}))$ is distorted downwards from efficiency. Let $\bar{x} \geq x$ be the infimum of the set of $\hat{x} \in [x, x_1)$ for which (4.36) fails to hold. I first show that

$$\hat{u}_w(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x}) + \hat{u}_y(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x}) = 0. \tag{4.37}$$

If $\bar{x} = x$, the claim is trivial. If $\bar{x} > x$, the definition of $\bar{x}$ implies that (4.36) holds for all $x' \in [x, \bar{x})$. By Lemma 4.4, it follows that

$$\frac{|\hat{u}_y(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x})|}{\hat{u}_w(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x})} \geq 1 + \int_{x'}^{\bar{x}} \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx''$$

for all $x' \in [x, \bar{x})$. Upon taking limits as x' converges to $\bar{x}$ from below, one obtains $|\hat{u}_y(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x}| \geq \hat{u}_w(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x})$. By Lemma 4.5, one also has $|\hat{u}_y(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x}| \leq \hat{u}_w(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x})$. (4.37) follows immediately.

Because the contract $(\hat{w}(\bar{x}), \hat{y}(\bar{x}))$ satisfies (4.37), Lemmas 4.3 and 4.4 imply that (4.33) must hold for $x = \bar{x}$ and any $\hat{x} \in (\bar{x}, x_1]$; moreover, by (4.37), the first term on the right-hand side of (4.33) is zero. Thus, one must have

$$\int_{\bar{x}}^{\hat{x}} \hat{\varphi}(x') \hat{u}_w \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx' \leq - \int_{\bar{x}}^{\hat{x}} \int_{\bar{x}}^{x'} \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx''\, g(x') dx' \tag{4.38}$$

for all $\hat{x} \in (x, x_1]$. By (4.4) and (3.7), the right-hand side of (4.38) is no greater than $-\int_{\bar{x}}^{\hat{x}} \int_{\bar{x}}^{\hat{x}} \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx''\, dx'$. Therefore, (4.38) implies that

$$\int_{\bar{x}}^{\hat{x}} \hat{\varphi}(x') \hat{u}_w \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx' \leq -(\hat{x} - \bar{x}) \int_{\bar{x}}^{\hat{x}} \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx'',$$

or, equivalently, that

$$\int_{\bar{x}}^{\hat{x}} [\hat{\varphi}(x') \hat{u}_w + \hat{x} - x] \frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} dx' \leq 0 \tag{4.39}$$

for all $\hat{x} \in (\bar{x}, x_1]$. By Lemma 4.2, there exists $A > 0$ such that $\hat{\varphi}(x') \hat{u}_w \leq -A$ if x' is sufficiently close to $\bar{x}$. If $\hat{x} - \bar{x} < A$, the integrand in (4.39) is everywhere nonnegative; moreover, it is strictly positive if $\frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} < 0$. For (4.39) to hold, one must therefore have $\frac{\partial}{\partial x} \frac{|\hat{u}_y|}{\hat{u}_w} = 0$ for all $x' \in [\bar{x}, \hat{x}]$. By Lemma 4.4, it follows that

$$\frac{|\hat{u}_y(\hat{w}(x'), \hat{y}(x'), x')|}{\hat{u}_w(\hat{w}(x'), \hat{y}(x'), x')} \geq \frac{|\hat{u}_y(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x})|}{\hat{u}_w(\hat{w}(\bar{x}), \hat{y}(\bar{x}), \bar{x})} \tag{4.40}$$

for all $x' \in [\bar{x}, \hat{x}]$. By (4.37), this implies $|\hat{u}_y(\hat{w}(x'), \hat{y}(x'), x')| \geq \hat{u}_w(\hat{w}(x'), \hat{y}(x'), x')$. By Lemma 4.5, one also has $|\hat{u}_y(\hat{w}(x'), \hat{y}(x'), x')| \leq \hat{u}_w(\hat{w}(x'), \hat{y}(x'), x')$. Therefore, (4.36) holds for all $x' \in [\bar{x}, \hat{x}]$, contrary to the assumption that $(\hat{w}(\hat{x}), \hat{y}(\hat{x}))$ is distorted downwards from efficiency. The assumption that the lemma is false has thus led to a contradiction. ■

Thus, if $(\hat{w}(x), \hat{y}(x))$ is efficient for $x < x_1$, one must have

$$\frac{|\hat{u}_y(\hat{w}(\hat{x}), \hat{y}(\hat{x}), \hat{x})|}{\hat{u}_w(\hat{w}(\hat{x}), \hat{y}(\hat{x}), \hat{x})} = \frac{|\hat{u}_y(\hat{w}(x), \hat{y}(x), x)|}{\hat{u}_w(\hat{w}(x), \hat{y}(x), x)} = 1 \tag{4.41}$$

for all $\hat{x} \in [x, x_1]$. By Lemma 4.4 and (4.4), one must then also have $\hat{y}(\hat{x}) = \hat{y}(x)$ and

$$\frac{\partial}{\partial x'} \frac{|\hat{u}_y(\hat{w}(x), \hat{y}(x), x')|}{\hat{u}_w(\hat{w}(x), \hat{y}(x), x')} = 0 \tag{4.42}$$

for all $x' \in (x, \hat{x})$ and all $\hat{x} \in [x, x_1]$. By (4.3) and (2.2), it follows that

$$\tau'(x') = 0 \tag{4.43}$$

for all $x' \in [x, x_1]$. By integrating (4.43) between x and x_1, one obtains:

Lemma 4.7 *The functions $\hat{w}(\cdot), \hat{y}(\cdot)$, and $\hat{v}(\cdot)$ exhibit Property $\hat{B}$: For any $x \in [x_0, x_1]$, the contract $(\hat{w}(x), \hat{y}(x))$ is distorted downwards from efficiency, i.e.,*

$$\hat{u}_w(\hat{w}(x), \hat{y}(x), x) + \hat{u}_y(\hat{w}(x), \hat{y}(x), x) > 0,$$

if $\tau(x) < t_1$.

For mass points of the distribution F below t_1, Lemma 4.7 has the further implication that there must be some pooling of types:

Lemma 4.8 *The functions $\hat{w}(\cdot)$ and $\hat{y}(\cdot)$ exhibit Property $\hat{C}$: If $x \in [x_0, x_1)$ and $\hat{x} \in (x, x_1)$ are such that $\tau(x) = \tau(\hat{x}) < t_1$, then there exists $\bar{x} \in (\hat{x}, x_1)$ such that $\tau(\hat{x}) < \tau(\bar{x})$ and, moreover, $(\hat{w}(x'), \hat{y}(x')) = (\hat{w}(x), \hat{y}(x))$ for all $x' \in (x, \bar{x})$.*

Proof. Let $\check{x} := \sup\{x' | \tau(x') = \tau(x)\}$. For any $x' \in (x, \check{x})$, one has $\tau'(x') = 0$. By (4.3), this implies

$$\frac{\partial}{\partial x} \frac{|\hat{u}_y(\hat{w}(x'), \hat{y}(x'), x')|}{\hat{u}_w(\hat{w}(x'), \hat{y}(x'), x')} = 0.$$

By (3.7), $\tau'(x') = 0$ also implies $g(x') = 1$. By (4.18) and Lemma 4.7, it follows that $\hat{\psi}'(x') < 0$ for all $x' \in (x, \check{x})$, and, therefore, that $\hat{\psi}(\check{x}) < 0$. Therefore, there exists $\bar{x} > \check{x}$ such that $(\hat{w}(x'), \hat{y}(x')) = (\hat{w}(x), \hat{y}(x))$ for all $x' \in (x, \bar{x})$. Because $\bar{x} > \check{x}$, one must have $\tau(\bar{x}) > \tau(x)$ and, hence, $\tau(\bar{x}) > \tau(\hat{x})$. ■

In the last step of the analysis, I show that there is no distortion at the top.

Lemma 4.9 *The functions $\hat{w}(\cdot), \hat{y}(\cdot)$, and $\hat{v}(\cdot)$ exhibit Property $\hat{A}$: For any sequence $\{x^k\}$ that converges to x_1 from below, one has*

$$\lim_{k\to\infty}\left[\hat{u}_w(\hat{w}(x^k),\hat{y}(x^k),x^k)+\hat{u}_y(\hat{w}(x^k),\hat{y}(x^k),x^k)\right]=0. \tag{4.44}$$

Proof. Let $\{x^k\}$ be any sequence that converges to x_1 from below. Without loss of generality, one may assume that the sequence is nondecreasing. The contract sequence $\{(\hat{w}(x^k),\hat{y}(x^k))\}$ is then also nondecreasing. Given that this sequence is bounded by $(\hat{w}(x_1),\hat{y}(x_1))$, it must have a limit $(\bar{w},\bar{y})$.

Recall that, from the transversality condition (4.12) and the optimality condition (4.13), one has $\hat{\psi}(x_1)=0$ and $\hat{\psi}(x)\leq 0$. For any k, therefore, condition (4.26) yields

$$\frac{1}{x_1-x^k}\int_{x^k}^{x_1}\hat{\varphi}(x)\hat{u}_w\,\frac{\partial}{\partial x}\frac{|\hat{u}_y|}{\hat{u}_w}dx'\geq\frac{1}{x_1-x^k}\int_{x^k}^{x_1}\frac{\hat{u}_w+\hat{u}_y}{\hat{u}_w}g(x')dx'. \tag{4.45}$$

Upon taking limits as $k\to\infty$ and noting that, by the continuity of $\hat{\varphi}(\cdot)$ and the transversality condition (4.11), the integrand on the left-hand side converges to zero if x' converges to x_1, one obtains

$$0\leq\hat{u}_w(\bar{w},\bar{y},x_1)+\hat{u}_y(\bar{w},\bar{y},x_1). \tag{4.46}$$

By Lemma 4.5, one also has

$$0\geq\hat{u}_w(\bar{w},\bar{y},x_1)+\hat{u}_y(\bar{w},\bar{y},x_1). \tag{4.47}$$

The lemma follows immediately. ■

Theorem 2.1 now follows from Propositions 3.3 and 3.4 and the fact that a solution to the principal's modified problem exhibits properties Â - Ĉ.

A Appendix

Lemma A.1 *A contract menu is incentive-compatible on T if and only if there exists an extension $(w(\cdot),y(\cdot))$ of the contract menu to the interval $[t_0,t_1]$ that is incentive-compatible on $[t_0,t_1]$.*

Proof. The "if"-part of this statement is trivial. To prove the "only if"-part, consider any contract menu $(w(\cdot),y(\cdot))$ that is incentive-compatible on T. Let $\underline{t}\in T$ and $\bar{t}\in T$ be such that $(\underline{t},\bar{t})\cap T=\emptyset$. Incentive compatibility of

$(w(\cdot), y(\cdot))$ on T implies that the functions $w(\cdot)$ and $y(\cdot)$ are nondecreasing and co-monotonic on T. Moreover,

$$u(w(\underline{t}), y(\underline{t}), \underline{t}) \geq u(w(\bar{t}), y(\bar{t}), \underline{t}) \tag{A.1}$$

and

$$u(w(\bar{t}), y(\bar{t}), \bar{t}) \geq u(w(\underline{t}), y(\underline{t}), \bar{t}). \tag{A.2}$$

Because u is continuous, there exists $\hat{t} \in [\underline{t}, \bar{t}]$ such that

$$u(w(\underline{t}), y(\underline{t}), \hat{t}) = u(w(\bar{t}), y(\bar{t}), \hat{t}). \tag{A.3}$$

Extend $w(\cdot)$ and $y(\cdot)$ to the interval $(\underline{t}, \bar{t})$ by setting

$$(w(t), y(t)) = (w(\underline{t}), y(\underline{t})) \text{ for } t \in (\underline{t}, \hat{t}), \tag{A.4}$$

$$(w(t), y(t)) = (w(\bar{t}), y(\bar{t})) \text{ for } t \in (\hat{t}, \bar{t}), \tag{A.5}$$

and

$$(w(\hat{t}), y(\hat{t})) = (w(\bar{t}), y(\bar{t})) \text{ if } \hat{t} > \underline{t}. \tag{A.6}$$

By standard arguments, the extended contract menu is incentive-compatible. Thus, by the single-crossing condition, (A.3) implies

$$u(w(\underline{t}), y(\underline{t}), t) \geq u(w(\bar{t}), y(\bar{t}), t) \tag{A.7}$$

for all $t \in (\underline{t}, \hat{t})$ and

$$u(w(\bar{t}), y(\bar{t}), t) \geq u(w(\underline{t}), y(\underline{t}), t) \tag{A.8}$$

for all $t \in (\hat{t}, \bar{t}]$. Further, incentive compatibility of $(w(\cdot), y(\cdot))$ on T implies that

$$u(w(\underline{t}), y(\underline{t}), \underline{t}) \geq u(w(t'), y(t'), \underline{t}) \tag{A.9}$$

for all $t' \in T \cap [t_0, \underline{t})$. By the monotonicity of $(w(\cdot), y(\cdot))$ on T and the single-crossing condition, it follows that

$$u(w(t), y(t), t) \geq u(w(t'), y(t'), t). \tag{A.10}$$

for all $t \in [\underline{t}, \bar{t}]$ for all $t' \in T \cap [t_0, \underline{t})$. A precisely symmetric argument shows that (A.10) must also hold for $t \in [\underline{t}, \bar{t}]$ and $t' \in T \cap (\bar{t}, t_1]$.

For $t \in T$ and $t' \in T \cup (\underline{t}, \bar{t})$, the validity of (2.5) follows trivially from the incentive compatibility of $(w(\cdot), y(\cdot))$ on T and the observation that the extension to $T \cup (\underline{t}, \bar{t})$ has not affected the range of the contract menu.

If one applies the preceding procedure to any interval whose intersection with T is empty, one obtains an extension of $w(\cdot)$ and $y(\cdot)$ that is incentive compatible on a dense subset of $[t_0, t_1]$. This extension is obviously nondecreasing. For any $t \in [t_0, t_1]$, one can therefore define

$$(w(t), y(t)) = \lim_{t^k \downarrow t} (w(t^k), y(t^k)), \tag{A.11}$$

where the limit is taken along a decreasing sequence of elements of the dense set to which $w(\cdot)$ and $y(\cdot)$ have been extended. The functions $w(\cdot)$ and $y(\cdot)$ are thus extended to the entire interval $[t_0, t_1]$. Incentive compatibility follows from (A.11) by continuity considerations. ■

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