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Neoclassical Growth with Long-Term One-Sided Commitment Contracts*

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Abstract

This paper characterizes the stationary equilibrium of a continuous-time neoclassical production economy with capital accumulation in which households can insure against idiosyncratic income risk through long-term insurance contracts. Insurance companies operating in perfectly competitive markets can commit to future contractual obligations, whereas households cannot. For the case in which household labor productivity takes two values, one of which is zero, and where households have log-utility we provide a complete analytical characterization of the optimal consumption insurance contract, the stationary consumption distribution and the equilibrium aggregate capital stock and interest rate. Under parameter restrictions, there is a unique stationary equilibrium with partial consumption insurance and a stationary consumption distribution that takes a truncated Pareto form. The unique equilibrium interest rate (capital stock) is strictly decreasing (increasing) in income risk. The paper provides an analytically tractable alternative to the standard incomplete markets general equilibrium model developed in Aiyagari (1994) by retaining its physical structure, but substituting the assumed incomplete asset markets structure with one in which limits to consumption insurance emerge endogenously, as in Krueger and Uhlig (2006).

JEL Codes: E21, D11, D91, G22
Keywords: Idiosyncratic Risk, Limited Commitment, Stationary Equilibrium

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1 Introduction

In this paper we provide a fully micro-founded, analytically tractable general equilibrium macroeconomic model of neoclassical investment, production and the cross-sectional consumption distribution in which the limits to insurance to idiosynchratic income risk are explicitly derived from contractual frictions.

With this model we seek to integrate two foundational strands of the literature on macroeconomics with household heterogeneity. The first strand has developed and applied the standard incomplete markets model with uninsurable idiosyncratic income shocks and neoclassical production, see Bewley (1986), Imrohoroglu (1989), Uhlig (1990), Huggett (1993) and Aiyagari (1994). In that model, households can trade assets to self-insure against income fluctuations, but these assets are not permitted to pay out contingent on a household's individual income realization, thereby ruling out explicit insurance against income risk. The second branch is the broad literature on recursive contracts and endogenously incomplete markets which permits explicit insurance but in which the extent of such insurance is restricted by informational or contract enforcement frictions. Specifically, in this paper we incorporate dynamic insurance contracts offered by competitive financial intermediaries, as analyzed in Krueger and Uhlig (2006), into a neoclassical production economy. Financial intermediaries can commit to long-term financial contracts, but households cannot.

As a result we make three contributions: one substantive, one methodological and one technical in nature. On the substantive side, we provide a macroeconomic model with household heterogeneity that links the accumulation of the aggregate capital stock in the economy to the insurance provided by financial intermediaries to households. In practice, capital held for financing insurance commitments is a substantial part of the capital stock. In our model we make the arguably extreme assumption that this accounts for all of it.[1] On the methodological side we fully analytically (as well as numerically) characterize a dynamic optimal insurance model with one-sided limited commitment and production as well as capital accumulation. On the technical side, we extend the discrete-time analysis of recursive dynamic contracting problems in Marcet and Marimon (2019) to a continuous time setting as well as establish the appropriate mathematical framework and key results, see Appendix A and Online Appendices B, C and D.

[1] One can argue that models of the Aiyagari-Huggett-Uhlig variety also assume that insurance against idiosyncratic income fluctuations accounts for the entire holdings of capital: agents with constant income and the same discount factor would not accumulate capital and financial institutions are absent in these models.

In a seminal paper, Aiyagari (1994) analyzed an economy in which households self-insure against idiosyncratic income fluctuations by purchasing shares of the aggregate capital stock. Variants of the model differ in the set of assets households can trade, but by assumption agents do not have access to financial instruments that provide direct insurance against the idiosyncratic income risk they face, despite the fact that such insurance would be mutually beneficial, given the underlying physical environment. A large literature is now building on that model to link microeconomic inequality to macroeconomic performance, including applied policy (reform) analysis.[2] Any analysis of welfare in such models then necessarily comes with the caveat that households may already be able to do better for themselves if only the model builder allowed them to do so. As parameters or policies change, one may be concerned that these missing gains from trade shift, too.[3] Alternative general equilibrium workhorse models are therefore needed, in which households are allowed to pursue all contractual possibilities, limited only by informational or commitment constraints. The purpose of this paper is to provide one such alternative model.

The contractual friction in our model arises from the inability of households to commit to future obligations implied by full-insurance risk sharing contracts. We postulate financial markets in which perfectly competitive intermediaries offer long-term insurance contracts to households. These financial intermediaries receive all incomes from a customer that has signed a contract, and can commit perfectly to future state-contingent consumption payments. Competition among intermediaries implies that the present discounted value of profits from these contracts is zero at the time of contract signing. The crucial friction that prevents perfect consumption insurance in the model is that *households,* at any moment, can costlessly switch to another intermediary, signing a new contract there. That is, we model relationships between financial intermediaries and private households as long-term contracts with one-sided limited commitment: the intermediary is fully committed, the household is not. This structure of financial markets is identical to the one assumed in the discrete-time, partial equilibrium model of Krueger and Uhlig (2006), which in turn builds on the seminal work of Harris and Holmstrom (1982), Thomas and Worrall (1988), Kehoe and Levine (1993, 2001), Phelan (1995), Kocherlakota (1996) and Alvarez and Jermann (2000), and in economies with storage, by Abraham and Laczo (2018).

In our previous paper, and in accordance with the contract theory literature, we showed

[2] See surveys by Heathcote, Storesletten and Violante (2011) and Krueger, Mitman and Perri (2016, 2017).

[3] A recent and general approach to assessing welfare consequences in models with heterogeneous agents is contained in Dàvila and Schaab (2022b)

that the one-sided limited commitment friction induces contracts with payments from the household to the intermediary that are front-loaded: when income is high, the household effectively builds up a stock of savings with the intermediary, which then finances the insurance offered by the intermediary against low income realizations down the road. In this paper we embed these contracts and the implied asset demand by the intermediaries in a neoclassical production economy, as in Aiyagari (1994). The contractual savings implied by back-loaded insurance contracts fund the aggregate capital stock of the economy. Financial intermediaries buy shares of the capital stock to finance their future liabilities from the insurance contracts they have signed with households. Aggregate capital itself is accumulated and used together with inelastically supplied labor in an aggregate neoclassical production function by a competitive sector of production firms.

Households supply labor inelastically to these firms, but as in Bewley (1986), Imrohoroglu (1989), Uhlig (1990), Huggett (1993) and Aiyagari (1994), their labor productivity and thus earnings are subject to idiosyncratic risk. This risk induces household insurance needs and thus generates a savings motive, which in turn finances the capital stock. Our model therefore provides a third (and intermediate) alternative neoclassical production economy with capital, relative to the self-insurance framework of Aiyagari (1994) and the full-insurance (representative agent) framework.

As a methodological innovation to the limited commitment general equilibrium literature we describe our model in continuous time. This is useful since an optimal insurance contract is akin to an optimal stopping problem, and the use of continuous time avoids integer problems (the optimal stopping time falling in between two periods) that arise in a discrete time setting. In order to obtain our sharp analytical characterization of the equilibrium for a full understanding of the forces at work, we focus on the case where households have logarithmic utility and labor productivity can take only two values, one of which is zero. For this case, we provide a complete analytical characterization of the optimal consumption insurance contract as well as the stationary consumption distribution. Under restrictions on the parameters, we show that there is a unique equilibrium that features partial consumption insurance. We provide explicit closed-form expressions for this equilibrium, including the steady-state capital stock and its rate of return. The stationary consumption distribution is also available in closed form, and we show that this distribution has a Pareto form, truncated by an upper mass point. Comparative statics with respect to the deep parameters of the model (and, specifically, the parameters determining income risk, preferences and production technologies) deliver unambiguous results. We submit that this full analytical

characterization of a stationary equilibrium is an additional, attractive benefit of our model, and a welcome methodological advance, noting that Aiyagari-type models (as standard limited commitment economies with a continuum of households, as in Krueger and Perri, 2006) and Broer, 2013) typically require numerical solutions.

Our results complement work on and provide a foundation for understanding properties of richer structures, which one may confront directly with the data, but are likely to be less tractable. The paper shares that ambition with the characterization of the two-state continuous-time Aiyagari model in Achdou et al. (2022). Like us, they aim for a deeper understanding of these models rather than an empirically appealing quantification. As here, they characterize the equilibrium by two differential equations: one governing the optimal solution of the consumption insurance problem, and one characterizing the associated stationary distribution. They derive an analytical characterization of the wealth distribution, given the savings function. While the latter cannot be characterized analytically there, we achieve complete characterization in this paper, and thus can proceed all the way to provide closed-form solutions for all equilibrium objects. Methodologically, the papers complement each other by characterizing equilibria in the same physical environment, but under two fundamentally different market structures.

In principle, contracts can depend on the entire history up to the present. As a technical innovation, we thus provide a mathematical framework and appropriate language to describe histories and measurability in continuous time (see Online Appendix C) and then proceed to expand the discrete-time analysis of recursive dynamic contracting problems in Marcet and Marimon (2019) to a continuous time setting (see Appendix A.1 and Online Appendix D.2). This is in contrast to the recursive representation with current capital and productivity as the state space in Achdou et al. (2022).[4] We show how to study expected payoffs by splitting the future into parts without a state change as well as the first state change, and use it to establish a number of non-trivial properties as well as the Hamilton-Jacobi-Bellman equations, see Appendix A.4 and Online Appendices D.4 and D.5. While tailored to the specific environment at hand, these techniques should prove considerably useful beyond the model studied here. A road map to these technical contributions is in Online Appendix B.

[4]We derive rather than assume a state space representation, and establish key properties without it.

1.1 Relation to the Literature

As discussed above, our broad aim in this paper is to connect the dynamic contracting literature with income risk and limited commitment to the quantitative general equilibrium literature in macroeconomics. Our dynamic limited commitment risk sharing contract model builds on the theoretical work characterizing optimal contracts in such environments. Especially relevant is the subset of the literature that has done so in continuous time.

As highlighted above, the paper by Achdou et al. (2022) is most closely related to our work. Zhang (2013) studies a consumption insurance model with limited commitment similar to that in Krueger and Uhlig (2006), but permits income to be a serially correlated finite state Markov chain, rather than a sequence of iid random variables. He also allows the household's outside option to be a general function of the current income state, rather than simply autarky. The author derives the optimal consumption insurance contract. Grochulski and Zhang (2012) characterize the optimal contract in continuous time, under the assumption that the market return equals the discount rate, the outside option is autarky, and the income process follows a general geometric Brownian motion. The work by Miao and Zhang (2015) shares related results with Grochulski and Zhang (2012).

Like us, Dàvila and Schaab (2022a) generalize Marcet and Marimon (2019) to continuous-time heterogeneous-agent settings, but in a rather different context. They introduce "timeless penalties" in order to analyze Ramsey optimal policies. Our explicit derivation of the cumulative Lagrange multipliers complements their formulation and can aid in providing a foundation. Overall, our approach is related in spirit to recent approaches such as Achdou et al (2022), Alvarez and Lippi (2022) and Alvarez, Lippi and Souganidis (2022) who explicitly characterize equilibrium quantities by pushing far the analytics of aggregating the continuous-time dynamics of heterogeneous actors and exploring mean field games.

Turning to general equilibrium treatments, in the context of the sovereign debt and default literature, Hellwig and Lorenzoni (2009) and the generalization in Martins-da-Rocha and Santos (2019) consider an endowment economy, in which two agents optimally share their risky income stream over time, subject to contractual constraints. The market return in their economy is shown to be zero under appropriate assumptions. Gottardi and Kubler (2015) study an endowment economy with finitely many (types of) agents and complete markets, but under the assumption that the short sales of the Arrow securities have have to be collateralized. Default on debt results in the loss of the collateral, but as in our work there is no additional punishment. The focus of their work is to study the existence and the efficiency properties of equilibria in their model without capital. Although our focus is dif-

ferent, the long-term risk-sharing consumption allocations we characterize and then embed in a neoclassical production economy with capital accumulation can also be decentralized as competitive equilibria in a model where households trade a full set of Arrow securities and physical capital, and where the short sales of the Arrow securities have to be collateralized by capital, as in the market structure of Gottardi and Kubler (2015). We explore this formulation in Ando, Krueger and Uhlig (2022) and Krueger, Li and Uhlig (2022).

On the applied side, there is now considerable evidence that individual consumption smoothing is larger than what standard approaches of self-insurance via asset savings would generate. In a benchmark contribution, Blundell, Pistaferri and Preston (2008) have shown that there is a fairly low pass-through of income shocks to consumption. Using improved methods and data as well as alternative approaches, their results have been largely confirmed by the more recent literature such as Arellano, Blundell and Bonhomme (2017), Eika et al. (2020), Chatterjee, Morley and Sigh (2020), Braxton et al. (2021), Commault (2021), and Balke and Lamadon (2022). Thus, alternatives to the conventional self-insurance approach are needed. Our paper connects to this literature by allowing for endogenously incomplete insurance against income risk.

One interpretation of the contractual arrangements of our paper is that of firms that provide workers with long-term employment-wage contracts. A recent literature, building on the work of Harris and Holmstrom (1982), emphasizes that firms provide insurance to its workers against idiosyncratic productivity fluctuations. Lamadon (2016) and Balke and Lamadon (2022) have calculated the optimal within-firm insurance mechanism, in the presence of a variety of sources of risk, including firm-specific risk, worker productivity risk and unobservable effort. Guiso, Pistaferri and Schivardi (2005) also argue, empirically, that the insurance of worker productivity by firms is an important mechanism to insulate workers from idiosyncratic shocks. Finally, Saporta-Eksten (2014) shows that wages are lower after a spell of unemployment, which he interprets as a loss in productivity. In the context of our model this observation can also be rationalized as part of the optimal consumption insurance contract, in the event that the productivity of the worker has dropped temporarily.

In Section 2 we set out the model, and Section 3 characterizes the optimal risk-sharing contract. Section 4 derives the associated stationary consumption distribution and Section 5 characterizes the stationary general equilibrium. We contrast this stationary equilibrium with one emerging in the standard incomplete markets model in Section 6. Section 7 discusses the possibility of multiple stationary equilibria when we depart from log-utility. Section 8 concludes. The Appendix and Online Appendices provide the formal analysis.

2 The Model

2.1 Preferences and Endowments

Time is continuous. There is a population of a continuum of infinitely lived agents of mass 1. Agents have a strictly increasing, strictly concave and twice continuously differentiable period utility function $u : \mathbf{R}_{++} \to \mathbf{R}$ and discount the future at rate $\rho > 0$. Expected lifetime utility of a newborn household is given by

$$E\left[\int_0^\infty e^{-\rho t} u(c(t))dt\right].$$

For our analytical results, we impose that

$$u(c) = \log(c)$$

Labor productivity z_{it} of an individual agent i at time t is assumed to follow a two-state Markov process that is independent across agents. Productivity can be either high, $z_{it} = \zeta > 0$ or zero $z_{it} = 0$. Let $Z = \{0, \zeta\}$. The transition from high to low productivity occurs at rate $\xi > 0$, whereas the transition from low to high productivity occurs at rate $\nu > 0$. Since labor income will equal labor productivity times a common wage w for each household, we will use the terms (labor) productivity and income interchangeably.[5]

Given the stochastic structure of the endowment process, the share of households with low and high income is equal to

$$(\psi_l, \psi_h) = \left(\frac{\xi}{\xi + \nu}, \frac{\nu}{\xi + \nu}\right)$$

We assume that newborn households draw their productivity from the stationary income distribution and that the average labor productivity in the economy is equal to 1. Thus we assume that

$$\frac{\nu}{\xi + \nu}\zeta = 1. \tag{1}$$

[5]We assume that households with low labor productivity also have some nontradable endowment $\chi > 0$ that they can consume if they do not sign up for a consumption risk-sharing contract. This assumption avoids the complication that individuals who initially have not yet received the high income realization at least once and thus will not be provided with consumption insurance (as we will show) are forced to consume 0. Denote the utility from consuming the nontradable endowment by $\underline{u} = u(\chi) > -\infty$. In the steady-state equilibrium the mass of these individuals will be zero, of course, and thus this assumption is irrelevant for the remainder of this paper focusing on long-run stationary equilibria.

2.2 Technology

There is a competitive sector of production firms that uses labor and potentially capital to produce the final output good according to the Cobb-Douglas production function

$$AF(K, L) = AK^{\theta}L^{1-\theta}.$$

where $\theta \in (0, 1)$ denotes the capital share. Production firms seek to maximize profits, taking as given the market spot wage w per efficiency unit of labor and the market rental rate per unit of capital. Capital accumulation is linear and depreciates at rate δ. There is a resulting equilibrium rate of return or interest rate r for investing in capital. We dropped the subscript t to economize on notation, since we shall concern ourselves only with stationary equilibria in which aggregate variables are constant.

There is a competitive sector of risk-neutral intermediaries who seek to maximize profits. Agents seek to insure themselves against these income fluctuations with financial intermediaries. However, the commitment is one-sided only: while the intermediary can commit to the contract for the entire future, agents are free to leave the contract at any time they please without punishment and sign up with the next intermediary. Intermediaries compete for agents, and do not have resources on their own. Similar to Krueger and Uhlig (2006), these assumptions will imply that newborn agents will have to wait until the first time they receive the high income before signing an insurance contract. They then provide their chosen intermediary with a stream of "insurance premium payments"while in the high income state, to finance subsequent payments for a potential "dry spell" of low productivity, until they transit to high income again. We assume that the law of large numbers applies to each individual intermediary or, alternatively, that there is full mutual insurance among intermediaries, so that intermediaries are not exposed to any risk The intermediaries invest the premium payments in capital and therefore discount future streams of payments and incomes at the rate of return r on capital.

2.3 Timing of Events

At time zero, a newborn household first draws labor productivity z from the stationary income distribution and then signs a long-term consumption insurance contract with one of the many competing financial intermediaries, delivering lifetime utility $U^{out}(z)$. At any subsequent instant $t > 0$, first the current labor productivity z is realized from the

household-level Markov process. The household then has the option of sticking with its current intermediary or signing up with another intermediary, in the latter case receiving a contract delivering lifetime utility $U^{out}(z)$. Consumption is then allocated to the household according to the consumption insurance contract this household has signed.

2.4 Stationary Equilibrium

Intermediary contracts promise some lifetime utility U for the household from delivering a stochastic stream of future consumption. Given U and the current labor productivity z of the household, the profit maximization objective of intermediaries is equivalent to minimizing the net present value $V(z,U)$ of the contract costs, i.e., to minimize the net present value of the difference between the household's stream of consumption and its income. The income is given by the labor productivity $z(\tau)$ at future dates τ multiplied by the wage w. It will likewise be convenient to scale consumption by the wage level. In slight abuse of notation, let $c(\tau)w$ denote the consumption of the household at date τ. It is an adapted process, that is, it may depend on events known at date τ. In particular, it will depend on the history of the productivity process $z(s), s \leq \tau$ for that agent, up to and including τ. In Online Appendix C, we provide precise notation to express this history dependence, but skip it in the main text for ease of notation. In designing the contract, the intermediary needs to take into account that the household will leave whenever residual lifetime utility drops below the outside option $U^{out}(z)$ that is available to the agents when signing a new contract with a competing intermediary.

Definition 1. *For fixed outside options $U^{out}(z)$, with $z \in Z$, a starting date t and a fixed wage w and rate of return on capital or interest rate r, an* **optimal consumption insurance contract** *$c(\tau; z, U)$ and the* **cost function** *$V(z,U)$ solve*

$$V(z,U) = \min_{\langle c(\tau)\rangle \geq 0} \mathbf{E}\left[\int_t^\infty e^{-r(\tau-t)}\left[wc(\tau) - wz(\tau)\right]d\tau \middle| z(t) = z\right]$$

subject to the **promise-keeping constraint**

$$\mathbf{E}\left[\int_t^\infty e^{-\rho(\tau-t)}u(wc(\tau))d\tau \middle| z(t) = z\right] \quad \geq \quad U \tag{2}$$

and the **limited commitment constraints**

$$\mathbf{E}\left[\int_s^\infty e^{-\rho(\tau-s)}u(wc(\tau))d\tau\middle| z(s)\right] \;\geq\; U^{out}(z(s)) \textit{ for all } s>t \tag{3}$$

for all $\tau \geq t$, *for all* $z \in Z$ *and all* $U \in \left[U^{out}(z), \frac{\bar{u}}{\rho}\right)$.

Note that the stationary structure of the model ensures that the optimal consumption insurance contract does not depend on calendar time, but rather only on the income z with which the household starts the contract. Moreover, there will never be a reason to leave the current contract and take the outside option, restarting a contract at some particular date. For the equilibrium definition, we therefore implicitly confine ourselves to contracts starting at date $t = 0$.

Definition 2. *A stationary equilibrium consists of outside options* $\{U^{out}(z)\}_{z\in Z}$, *consumption insurance contracts* $c(\tau, z, U) : \mathbf{R}_+ \times Z \times \left[U^{out}(z), \frac{\bar{u}}{\rho}\right) \to \mathbf{R}_+$ *and* $V : Z \times \left[U^{out}(z), \frac{\bar{u}}{\rho}\right) \to \mathbf{R}$, *an equilibrium wage* w *and interest rate* r *and a stationary consumption probability density function* $\phi(c)$ *such that*

1. *Given* $\{U^{out}(z)\}_{z\in Z}$ *and* r, *the consumption insurance contract* $c(\tau, z, U), V(z, U)$ *is optimal in the sense of definition 1.*

2. *The outside options lead to zero profits of the financial intermediaries: for all* $z \in Z$

$$V(z, U^{out}(z)) = 0.$$

3. *The interest rate and wage* (r, w) *satisfy*

$$r = AF_K(K, 1) - \delta \tag{4}$$
$$w = AF_L(K, 1) \tag{5}$$

4. *The goods market clears*

$$\int wc\phi(c)dc + \delta K = AF(K, 1). \tag{6}$$

5. *The capital market clears*

$$\frac{w\left[\int c\phi(c)dc - 1\right]}{r} = K \tag{7}$$

6. *The stationary consumption probability density function is consistent with the dynamics of the optimal consumption contract as well as the stochastic structure of birth and death in the model.*

Several elements of this definition are noteworthy. The first two items formalize the notion that financial intermediaries compete for households by offering optimal consumption insurance contracts (item 1), and that their profits are driven to zero by perfect competition (item 2). These equilibrium requirements are identical to those in the endowment economy of Krueger and Uhlig (2006), but accounting for the fact that the current model is cast in continuous time. Whereas item 3 contains the standard optimality conditions of the representative production firm, the statement of the capital market clearing condition (7), as well as the inclusion of both the capital market clearing condition and the goods market clearing condition (6) require further discussion.

In the capital market clearing condition (7), the right-hand side $K = K^d$ is the demand for capital by the representative firm. The numerator on the left-hand side is the excess consumption, relative to labor income, of all households, that is, the capital income required to finance the consumption that exceeds labor income. Dividing by the return to capital r gives the capital stock that households, or financial intermediaries on behalf of households, need to own to deliver the required capital income. Thus we can think of

$$K^s = \frac{w\left[\int c\phi(c)dc - 1\right]}{r} \tag{8}$$

as the supply of capital by the household sector, intermediated through financial markets by the intermediaries. By restating the capital market clearing condition as

$$K^s(r) = K^d(r)$$

where $K^s(r)$ is defined in (8) and $K^d(r)$ is defined through (4) we can provide a graphical analysis of the existence and uniqueness of the stationary equilibrium in the (K, r) space, analogously to the well-known figure from Aiyagari (1994) for the standard incomplete markets model.

Finally, we note that as long as $r \neq 0$, the usual logic of Walras' law applies and one of the two market clearing conditions is redundant. To see this, note that the right-hand side of equation (6) can be written as

$$AF(K,1) = AF_L(K,1) + AF_K(K,1)K$$

and from equations (4) and (5) it follows that

$$AF(K,1) = w + (r+\delta)K.$$

Using this in equation (6) and rearranging implies, for $r \neq 0$, the capital market clearing condition (7). Thus for all $r \neq 0$ we can use either of the market clearing conditions in our analysis. The case $r = 0$, however, will require special attention, and we will argue in Section 5 that even though the goods market clears for $r = 0$ under fairly general conditions, the capital market generically does not, indicating that a) $r = 0$ is generically not a stationary equilibrium interest rate and b) at $r = 0$ we need to study both the goods and the capital market clearing conditions when analyzing a stationary equilibrium.

In order to do so, in the next sections we now aim to characterize the entire steady-state equilibrium, including the stationary consumption distribution whose cumulative distribution function we denote by Φ (with associated probability density function ϕ). First, we characterize the optimal consumption contract under various assumptions on the relationship between the constant interest rate r and the constant time discount rate ρ of the household. Then we discuss aggregation and the equilibrium determination of interest rates.

3 The Optimal Risk-Sharing Contract

The nature of the optimal consumption insurance contract depends crucially on the relationship between the subjective time discount factor ρ and the endogenous stationary equilibrium interest rate r. We discuss the relevant cases in turn. First, we discuss the case $r = \rho$, which will deliver a sharp and very simple characterization of the optimal consumption contract that features full consumption insurance of the household after the first instance of having received high income. We then analyze the case $r < \rho$, which will result in partial consumption insurance, the relevant case for the general equilibrium of the model

in a wide range of model parameterizations.[6]. We shall examine the range $r \in (-\delta, \rho]$ for the equilibrium interest rate. In order to ensure that capital supply is well defined over that range, we need the following assumption (see Online Appendix E.1.1).[7]

Assumption 1.

$$\xi > \delta \tag{9}$$

The key property of the optimal contract is that the limited commitment constraint is binding for individuals with high productivity $z = \zeta$, whereas the constraint is slack for low-productivity agents and a standard complete markets Euler equation holds. For this characterization, the following useful and intuitively appealing result is crucial. It says that intermediaries have to offer, in equilibrium, a contract to high-productivity individuals that yields higher lifetime utility than the one for low-productivity agents.

Lemma 1.

$$U^{out}(\zeta) > U^{out}(0) \tag{10}$$

The proof is in Appendix A.3 and Online Appendix E.1. The key idea of the proof is that an agent with currently high productivity (and thus with higher expected lifetime income transferred to the intermediary) can be provided with the contract of the low-productivity agent, delivering the same utility and a profit to the principal, a contradiction to perfect competition between (and, thus, zero profits of) the intermediaries.[8]

3.1 Full Insurance in the Long Run: $\rho = r$

We first characterize the optimal consumption insurance contract under the assumption $r = \rho$, and for productivity $z = \zeta$ and outside option $U^{out}(\zeta))$, and then discuss the relevance of other (z, U) combinations. A visual representation is provided in the left panel of Figure 1.

[6]In our model we cannot a priori exclude the possibility of equilibria in which the real interest rate exceeds the household time discount factor, and we analyze the optimal consumption contract under the assumption that $r > \rho$, a case we call superinsurance, in Online Appendix G. There we also argue that this case cannot result in a stationary general equilibrium.

[7]Ultimately, we only need that the equilibrium interest rate r^* calculated in (36) satisfies $r^* > -\xi$. The exposition of the theory is more transparent with the stronger assumption in equation (9), though.

[8]The details of the construction of the contract are subtle and require the construction of a three-state stochastic process in order to make starting a contract in the high and the low productivity state formally comparable. There we also describe how the three-state process generates the two-state process $z(s)$ assumed throughout the main body of the paper.

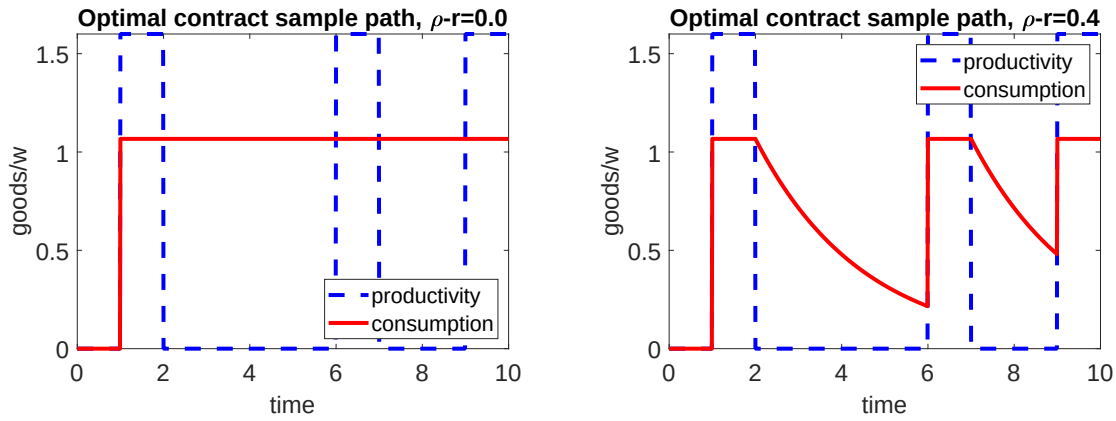


Figure 1: *These two figures show the implied path for the optimal contract consumption, given a sample path for productivity. If the agent always had (near) zero income in the past, the agent will also consume (near) zero. Upon the first instance of high productivity, the agent signs a long-term contract, surrendering part of his current income for consumption insurance in the future. When $r = \rho$ as in the left panel, consumption is then constant forever. While productivity is high, consumption is also constant for $r < \rho$ as shown in the right panel, since an otherwise optimally declining consumption path would lead agents to abandon the current contract and sign up with a new intermediary at a higher starting consumption amount. When productivity switches to zero, consumption follows a standard continuous-time Euler equation. These properties are established in Lemma 2.*

Proposition 1. *Suppose that $\rho = r$. In that case, the household consumes the nontradable endowment $c_l = \chi$ as long as $z_{it} = 0$, and signs a consumption contract that has constant consumption $c_h = \left(\frac{\rho+\nu}{\rho+\nu+\xi}\right)\zeta$ and remains there forever the instant labor productivity rises to ζ. Households born with income ζ immediately sign a contract and consume c_h forever.*

The formal proof is in Online Appendix E.2.1; here we give a heuristic derivation of the main components of the contract. To do so, in what follows, let the wage-deflated cost of the consumption insurance contracts be denoted by $v = V/w$. As further shorthand, denote as

$$\begin{aligned} v_l &= V(0, U^{out}(0))/w \\ v_h &= V(z, U^{out}(\zeta))/w \end{aligned}$$

and let v_{hl} denote the wage-deflated cost of a contract for the financial intermediary in which the household had high income in some previous periods (and thus currently consumes c_h) but currently has productivity $z = 0$ and, thus, no labor income.

Generally per Proposition 12 in Appendix A.4 or specifically per Corollary 3 in Online Appendix E.2.1, the flow costs of the financial intermediary satisfy the Hamilton-Jacobi-Bellman equations

$$\begin{aligned} rv_l &= c_l + \nu(v_h - v_l) \\ rv_h &= c_h - \zeta + \xi(v_{hl} - v_h) \\ rv_{hl} &= c_h + \nu(v_h - v_{hl}) \end{aligned}$$

Due to perfect competition of financial intermediaries (item 2 of the equilibrium definition) $v_h = v_l = 0$. Using this in the above equations to solve for (c_l, c_h, v_{hl}) and imposing $r = \rho$ delivers:

$$c_l = 0 \tag{11}$$

$$c_h = \left(\frac{\rho + \nu}{\rho + \nu + \xi}\right)\zeta = c_h(\rho) \tag{12}$$

$$v_{hl} = \frac{c_h}{\rho + \nu} = \left(\frac{1}{\rho + \nu + \xi}\right)\zeta \tag{13}$$

Thus the optimal risk-sharing contract collects a net insurance premium

$$\zeta - c_h = \left(\frac{\xi}{\rho + \nu + \xi}\right)\zeta$$

from households with high income realizations and uses it to pay consumption insurance c_h to those households that have obtained insurance (those with previously high income realizations) and have currently low income. The expected net present discount value of this insurance, recognizing that with Poisson intensity ν the household receives high income and leaves the current insurance spell, is given by v_{hl} in equation (13).

3.2 Partial Insurance: $r < \rho$

We now characterize the optimal consumption contract when $r < \rho$. A visual representation is provided in the right panel of Figure 1.

Proposition 2. *Suppose $r < \rho$ and*

$$u(c) = \log(c) \tag{14}$$

1. *Whenever a household has high productivity, it consumes a constant wage-deflated amount* $c_h = \left(\frac{\rho+\nu}{\rho+\nu+\xi}\right)\zeta$.

2. *When productivity switches to* 0*, consumption is continuous and subsequently drifts down according to the full-insurance Euler equation*

$$\frac{\dot{c}(t)}{c(t)} = r - \rho < 0 \tag{15}$$

3. *Let* τ *denote the time elapsed since productivity last switched from* $z = \zeta$ *to* 0*. Then,*

$$c(\tau) = c_h e^{(r-\rho)\tau} \tag{16}$$

Equation (15) follows from (55) for the log-case $\sigma = 1$. Note that the proposition implies that consumption jumps back up to c_h upon a switch to high productivity, a property established in the key lemma 2. The complete proof for a more general utility function including the CRRA case, is in Online Appendix E.2.2.

We now again heuristically derive this result for logarithmic period utility. By perfect competition, contract costs are zero when entering the contract with high income, $v_h = 0$, and similarly for entering the consumption contract with low income, $v_l = 0$. Denote by τ the time elapsed since having had the high productivity and by $v_{hl}(\tau)$ the remaining wage-deflated costs of the contract, at that point. Asymptotically, consumption $c(\tau)$ converges to $c_l = 0$, as $\tau \to \infty$ and as long as no switch back to high productivity occurs. Generally per Proposition 12 in Appendix A.4 or specifically per Corollary 4 in Online Appendix E.2.2, the Hamilton-Jacobi-Bellman equations characterizing the wage-deflated costs in the high-productivity state, the low-productivity state prior to having had a high-productivity realization, and after time τ since having had high productivity read as

$$\begin{aligned} rv_h &= c_h - \zeta + \xi(v(0) - v_h) && (17)\\ rv_l &= c_l + \nu(v_h - v_l) && (18)\\ rv_{hl}(\tau) &= c(\tau) + \nu(v_h - v_{hl}(\tau)) + \dot{v}_{hl}(\tau) && (19) \end{aligned}$$

with terminal condition

$$v_{hl}(\infty) = v_l = 0.$$

Simplifying equations (17) to (19) again delivers $c_l = 0$. As before $c_l = 0$, and individ-

uals with initially low income do not obtain any consumption insurance in the risk-sharing contract. Insurance would require prepayment by the insurance company, and perfect competition plus limited commitment on the household side implies that this prepayment cannot be recouped later. The other two equations simplify to

$$\begin{aligned} \xi v_{hl}(0) &= \zeta - c_h && (20) \\ (r+\nu)v_{hl}(\tau) &= c(\tau) + \dot{v}_{hl}(\tau) && (21) \end{aligned}$$

The first equation states that in the case of high income, the household, as before in the case where $r = \rho$, pays an insurance premium $\zeta - c_h$ that has to compensate the financial intermediary for the cost incurred during the low-income spell in which the losses for the intermediary amount to $v_{hl}(0)$. This equation relates the two endogenous variables c_h and $v_{hl}(0)$ to each other.

Equation (21) is a linear ordinary differential equation and can be integrated using the consumption path in (16) to obtain

$$v_{hl}(\tau) = \frac{c(\tau)}{\rho+\nu} = \frac{c_h e^{(r-\rho)\tau}}{\rho+\nu} \tag{22}$$

This result can be verified by differentiating (22) and verifying that it solves equation (21). Additional details and results for the general CRRA case are provided in Online Appendix E.2.2.

We can evaluate (22) at $t = 0$ to obtain[9]

$$v_{hl}(0) = \frac{c_h}{\rho+\nu} \tag{23}$$

The optimal consumption path drifts downward at rate $r-\rho$ from c_h toward $c_l = 0$. Thus the entry consumption level c_h fully characterizes the consumption contract. Using equation (20) to substitute out $v_{hl}(0)$ in equation (23) delivers this consumption level as

$$c_h = \left(\frac{\rho+\nu}{\rho+\nu+\xi}\right)\zeta \tag{24}$$

exactly as (12) in the full-insurance case. Notably, for logarithmic utility c_h only depends

[9]Note that this cost $v(0)$ is the counterpart to the insurance cost in equation (13) for the full-insurance case; if $r = \rho$ then $v(0) = v_{hl}$ where v_{hl} was defined in (13).

on exogenous parameters, but not the equilibrium interest rate $r < \rho$. This makes the log-case especially tractable and does not hold true for general CRRA period utility.

We summarize the optimal consumption contract for the full-insurance and the partial-insurance cases and for log utility in the following proposition.

Proposition 3. *If $r \leq \rho$ and $u(c) = \log(c)$, then there exists a unique consumption level*

$$c_h = \left(\frac{\rho + \nu}{\rho + \nu + \xi}\right)\zeta = \left(\frac{1}{1 + \frac{\xi}{\rho+\nu}}\right)\zeta = c_h(\rho)$$

with the following properties:

1. *Agents with currently high productivity receive the wage-deflated consumption c_h.*
2. *Agents with currently low productivity, who switched from high productivity τ periods ago, receive the wage-deflated consumption*

$$c(t) = c_h e^{(r-\rho)\tau}$$

3. *c_h is independent of the interest rate r, proportional to ζ, strictly decreasing in ξ and strictly increasing in $\rho + \nu$.*

Individuals who never had high income consume the nontradable endowment $c_l = \chi$ until the first occurrence of high income and then sign the consumption risk-sharing contract.

The proof is provided through the calculations above. Proposition 18 in Online Appendix E.2.2 provides the generalization to the CRRA case.[10]

4 The Invariant Consumption Distribution

In the previous section we have shown that the optimal consumption insurance contract depends on the relationship between the endogenous market interest rate r and the subjec-

[10]Note that the expected present discounted value of the cost of the consumption contract is always finite as equation (22) reveals. That the expected present discounted value of the revenue from the first phase of the contract (when the agent has high productivity but consumes $c_h < \zeta$) remains finite requires that $r > -\xi$, but this will be ensured with Assumption 1 in equilibrium.

Finally, note that optimal insurance contracts can also be characterized for the case in which the interest rate exceeds the time preference rate. Online Appendix G argues that the optimal consumption contract has the same features as the one for full insurance, but that consumption grows at rate $r - \rho >$ after the first time productivity turns high. We show in the Online Appendix that in this case no stationary consumption distribution with finite aggregate consumption exists.

tive discount factor ρ, which determines whether the contract is characterized by full or partial consumption insurance. The risk-sharing contract in turn determines the long-run, stationary consumption distribution, which we now derive.

4.1 Full Insurance in the Long Run: $\rho = r$

In this case, the optimal consumption contract has only two consumption levels, $c_l = 0$ and c_h, as characterized in Section 3.1. Since individuals flow out of c_l at positive rate ν and there is no inflow to this consumption level, the stationary consumption distribution places all mass $\phi_h = 1$ on c_h; in the long run, consumption of all individuals is constant at c_h.

4.2 Partial Insurance: $r < \rho$

In Section 3.2 we characterized the optimal consumption contract under the parametric restriction that $r < \rho$. We showed that all households with high income consume $c_h = \left(\frac{\rho}{r} \cdot \frac{r+\nu}{r+\nu+\xi}\right)\zeta$, independent of the interest rate. Thus the stationary consumption distribution has a mass point at c_h with mass $\phi(c_h) = \frac{\nu}{\nu+\xi}$.

Households with currently low income have a consumption process that satisfies

$$\dot{c}(t) = (r - \rho)c(t). \tag{25}$$

Finally, since there is positive outflow out of consumption level $c_l = 0$ at rate ν and no inflow, the invariant consumption distribution has no second mass point at c_l.

Proposition 4. *On $(0, c_h)$ the stationary consumption distribution satisfies the Kolmogorov forward equation or Fokker-Planck equation*

$$0 = -\frac{d\left[(r - \rho)c\phi(c)\right]}{dc} - \nu\phi(c) \tag{26}$$

Proof. The easiest way to see this is to note that the equation is the Kolmogorov forward equation for a drift-diffusion process with negative drift $(r - \rho)c$ and zero diffusion (see Theorem 2.8 in Pavliotis (2014) or equation (3.40) in Stokey, 2009), and thus for the process (25), if there were no jumps. The second term then arises from the fact that the household switches to high income with Poisson intensity ν. Alternatively, this is a version of equation (8) in Achdou et al. (2022). □

Since

$$\frac{d\left[(r-\rho)c\phi(c)\right]}{dc} = (r-\rho)\left[\phi(c) + c\phi'(c)\right]$$

we find that on $c \in (0, c_h)$ the stationary distribution satisfies

$$(\rho - r)\left[\phi(c) + c\phi'(c)\right] = \nu\phi(c)$$

and thus

$$\frac{c\phi'(c)}{\phi(c)} = \frac{\nu}{\rho - r} - 1.$$

Thus on this interval the stationary consumption distribution is Pareto with tail parameter $\frac{\nu}{\rho-r} - 1$, that is

$$\phi(c) = \phi_1 c^{\left(\frac{\nu}{\rho-r}-1\right)}$$

where ϕ_1 is a constant that is determined by the requirement that the stationary consumption distribution integrates to the share $\xi/(\nu+\xi)$ of zero productivity households over the interval $c \in (0, c_h)$. A straightforward calculation delivers:

Proposition 5. *For any given $r < \rho$, the stationary consumption distribution is given by a mass point at c_h of mass $\nu/(\nu+\xi)$ and a Pareto density below this mass point,*

$$\phi_r(c) = \begin{cases} \frac{\xi\nu(c_h)^{-\frac{\nu}{\rho-r}}}{(\rho-r)(\nu+\xi)} c^{\frac{\nu}{\rho-r}-1} & \text{if} \quad c \in (0, c_h) \\ \frac{\nu}{\nu+\xi}\boldsymbol{\delta}_{c_h} & \text{if} \quad c = c_h \end{cases}$$

where $\boldsymbol{\delta}_{c_h}$ indicates a Dirac mass point at c_h.

Aside from ρ and c_h, the shape of the consumption probability function for zero productivity depends on ν, which governs the hazard rate of moving to high productivity, the ratio ν/ξ of the two exit rates as well as the interest rate r. Figure 2 shows three examples when r is varied and all other parameters are held constant. The growth rate of the pdf is given by

$$\frac{d\log\phi_r(c)}{d\log c} = \frac{\nu}{\rho - r} - 1$$

We therefore have the following corollary.

Corollary 1. *If $\nu < \rho - r$, then the pdf is strictly decreasing in c. If $\nu > \rho - r$ then the pdf is strictly increasing in c. If $\nu \in (\rho - r, 2(\rho - r))$, then the pdf is strictly increasing and strictly concave in c. Finally, if $\nu > 2(\rho - r)$ then the pdf is strictly increasing and strictly convex in c.*

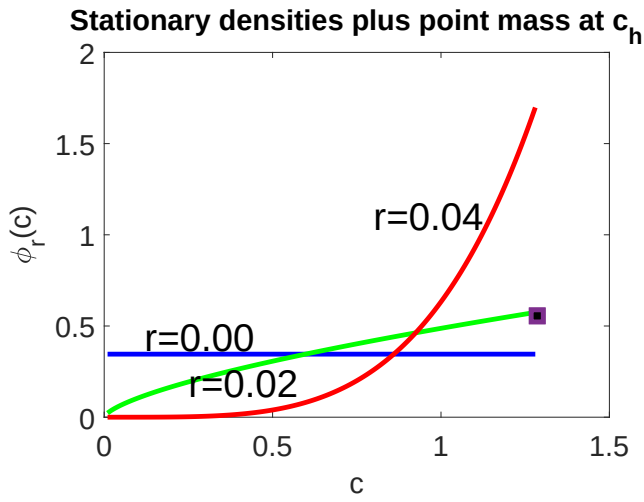


Figure 2: *Consumption distributions as stated in Proposition 5 for three different values of* r*, when* $\rho = 0.05, \nu = 0.05$ *and* $\xi = 0.04$*. There is a mass point at the same* $c_h \approx 1.3$ *of mass* ≈ 0.56 *independent of* r *(see equation* (24)*) and indicated by a black square. For zero productivity, consumption has a density that depends on* r*. Corollary 1 informs us about the shape. For* $r = 0$*, we have* $\nu = \rho - r$ *and the density is flat. For* $r = 0.02$*,* $\rho - r < \nu < 2(\rho - r)$*, and the density is strictly concave. For* $r = 0.04$*,* $\nu > 2(\rho - r)$ *and the density is strictly convex.*

5 General Equilibrium: Market Clearing Interest Rate r

In equilibrium, the goods market clearing condition (6) and the capital market clearing condition (7) have to hold and these are the remaining equations to satisfy. By Walras' law, the latter implies the former. We proceed by parameterizing both sides of these equations and hence demand and supply for capital and consumption goods with the equilibrium interest rate r. It will be convenient to always divide by the equilibrium wage $w = w(r)$.

5.1 Supply of Consumption Goods and Demand for Capital

The supply of consumption goods and the demand for capital can be derived in a straightforward fashion from the production side of the economy. Exploiting the production first-order conditions (4) and (5) and as in Aiyagari (1994) we can express aggregate demand for capital and the wage as a function of the interest rate, $K = K^d(r)$ and

$w = w(r) = AF_L(K^d(r), 1)$. Define the capital demand normalized by the wage, as

$$\kappa^d(r) = \frac{K^d(r)}{w(r)}$$

and the consumption goods supply, normalized by the wage, as

$$G(r) = \frac{AF(K^d(r), 1) - \delta K^d(r)}{w(r)}$$

The following result then immediately follows from straightforward calculations (see Online Appendix E.3):

Proposition 6. *Let the production function be of the form*

$$Y = AK^\theta L^{1-\theta}.$$

Then

$$G(r) = 1 + \frac{\theta r}{(1-\theta)(r+\delta)} \tag{27}$$

$$\kappa^d(r) := \frac{K^d(r)}{w(r)} = \frac{\theta}{(1-\theta)(r+\delta)} \tag{28}$$

The functions $G(r), \kappa^d(r)$ are continuously differentiable on $r \in (-\delta, \infty)$, and $G(r)$ is strictly increasing, with $\lim_{r \searrow -\delta} G(r) = -\infty, G(r = 0) = 1$ and $\lim_{r \nearrow \infty} G(r) = 1 + \frac{\theta}{1-\theta}$ and $\kappa^d(r)$ is strictly decreasing, with $\lim_{r \searrow -\delta} \kappa^d(r) = \infty, \kappa^d(r = 0) = \frac{\theta}{(1-\theta)\delta}$ and $\lim_{r \nearrow \infty} \kappa^d(r) = 0$.

5.2 Demand for Consumption Goods and Supply of Capital

Aggregate consumption, normalized by the aggregate wage w, is

$$C(r) = \int c\phi_r(c)dc \tag{29}$$

where ϕ_r was calculated explicitly in Section 4. Aggregate consumption in excess of total wage earnings is financed by returns on capital. Define capital supply, normalized by wage, as $\kappa^s(r) = K^s(r)/w(r)$. Per the left-hand side of the capital market clearing condition (7),

it is given by

$$\kappa^s(r) = \frac{C(r) - 1}{r} \tag{30}$$

For $r = 0$, we need to determine $\kappa^s(0)$ through an application of L'Hopital's rule.

In the case of $r = \rho$ and thus full insurance, the invariant consumption distribution puts unit mass on consumption c_h given by (12). Substituting out ζ in equation (12) per the normalization in equation (1), the wage-normalized aggregate consumption demand and the wage-normalized aggregate capital supply in the case of $r = \rho$ are

$$C(\rho) \quad = c_h \quad = 1 + \frac{\rho\xi}{\nu(\nu + \rho + \xi)} \tag{31}$$

$$\kappa^s(\rho) \quad := \kappa^{FI} \quad = \frac{\xi}{\nu(\nu + \rho + \xi)} \tag{32}$$

In the case of $r < \rho$, there is partial insurance. After tedious algebra (see Online Appendix E.4), we obtain

$$C(r) \quad = \quad 1 + \frac{r\xi}{(\nu + \rho - r)(\nu + \rho + \xi)} \tag{33}$$

$$\kappa^s(r) \quad = \quad \frac{\xi}{(\nu + \rho - r)\,(\nu + \rho + \xi)} \tag{34}$$

The next proposition is proved in Online Appendix E.4 and summarizes useful properties of the capital supply function.

Proposition 7. *The capital supply function $\kappa^s(r)$ is continuously differentiable and strictly increasing on $r \in [-\delta, \rho)$, with*

$$\kappa^s(-\delta) = \frac{\xi}{(\nu + \rho + \delta)\,(\nu + \rho + \xi)} < \infty \quad \textit{and} \quad \lim_{r \nearrow \rho} \kappa^s(r) = \kappa^{FI}.$$

5.3 Characterization of the Equilibrium and Comparative Statics

There is a unique time discount factor ρ^{FI} such that capital demand (28) equals full-insurance capital supply (32). It satisfies

$$\frac{\theta}{(1-\theta)(\rho^{FI} + \delta)} = \frac{\xi}{\nu(\nu + \rho^{FI} + \xi)}$$

In that case, consumption supply (27) is equal to full-insurance consumption demand (31). The following assumption will ensure the existence of a unique equilibrium:

Assumption 2. *Let the exogenous parameters of the model satisfy $\theta, \nu, \xi, \rho > 0$ and*

$$\frac{\theta}{(1-\theta)(\rho+\delta)} \leq \frac{\xi}{\nu(\nu+\rho+\xi)} \tag{35}$$

Theorem 1. *Let Assumption 2 be satisfied. Then there exists a unique stationary equilibrium. If $\rho = \rho^{FI}$ then the equilibrium features full insurance. If $\rho \neq \rho^{FI}$, then the equilibrium features partial insurance. In contrast, if Assumption 2 is violated, then no stationary equilibrium exists.*

Proof. The proof builds on Propositions 6 and 7 as well as the calculations above. If Assumption 2 holds with equality, the previous discussion showed that in this knife-edge case, the unique stationary equilibrium satisfies full insurance with $r^* = \rho$. Suppose then that Assumption 2 holds with strict inequality and that therefore

$$\kappa^d(\rho) < \lim_{r \nearrow \rho} \kappa^s(r) = \kappa^{FI}$$

Since $\kappa^s(r)$ and $\kappa^d(r)$ are continuous on $(-\delta, \rho)$ and since $\kappa^s(r)$ is strictly increasing, while $\kappa^d(r)$ is strictly decreasing, and since $\kappa^s(r = -\delta) < \infty = \lim_{r \searrow -\delta} \kappa^d(r)$, the intermediate value theorem implies that there exists a unique $r^* \in (-\delta, \rho)$ such that

$$\kappa^s(r^*) = \kappa^d(r^*)$$

Finally, if instead Assumption 2 is violated, then

$$\lim_{r \nearrow \rho} \kappa^s(r) = \kappa^{FI} < \kappa^d(\rho)$$

and thus any stationary equilibrium must satisfy $r^* > \rho$. However, for any $r > \rho$, as argued in Online Appendix G, there is no stationary equilibrium. □

The unique equilibrium interest rate satisfies $\kappa^s(r^*) = \kappa^d(r^*)$. Exploiting equations (28) and (34), we find

$$r^* = \frac{\theta(\nu+\rho+\xi)(\nu+\rho) - \xi\delta(1-\theta)}{\xi + \theta(\nu+\rho)} \tag{36}$$

This is our central result: the equilibrium interest rate and, with it, all other equilibrium quantities can be calculated explicitly. Associated with this interest rate is a stationary consumption distribution with mass point at c_h and a truncated Pareto distribution below $c_h$with Pareto coefficient $\kappa = \frac{\nu}{\rho - r^*} - 1$, given in Proposition 5 at $r = r^*$. The comparative statics of this unique equilibrium are immediate and summarized in the following proposition:

Proposition 8. *Let Assumption 2 be satisfied with strict inequality. Then the unique equilibrium interest rate $r^* \in (-\delta, \rho)$ is a strictly increasing function of $\rho + \nu$ and θ and a strictly decreasing function of ξ and δ. The associated equilibrium capital stock $K^* > K^{FI} =$ is a strictly increasing function of ξ and a strictly decreasing function of $\rho + \nu$ as well as δ.*

Proof. Write

$$r^* = \frac{(\nu + \rho + \xi)(\nu + \rho) + \xi\delta - \frac{\xi\delta}{\theta}}{\frac{\xi}{\theta} + \nu + \rho}$$

to see that r^* is increasing in θ, since the numerator is increasing in θ and the denominator is decreasing in θ. Proceed likewise for the other claims, except calculating the derivative for the dependence on ξ. □

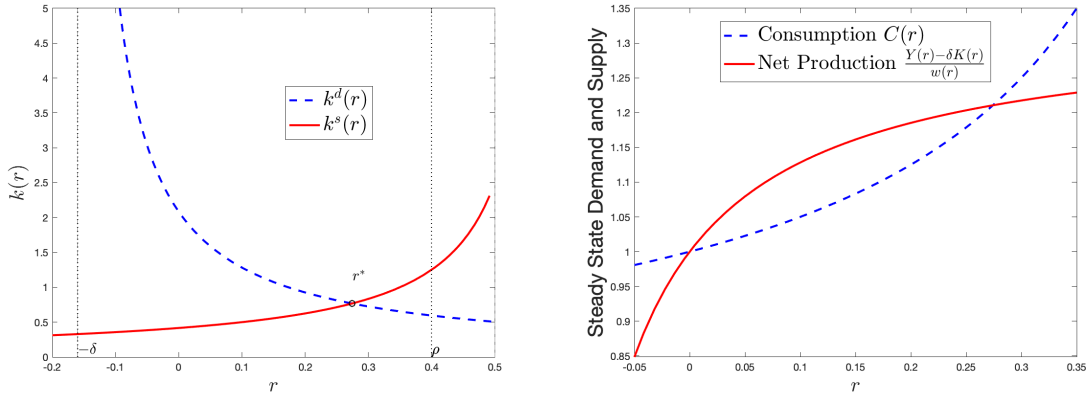
The unique equilibrium can be represented graphically, as in the standard incomplete markets models. Aiyagari (1994) plots asset demand and supply in (r, K) space. We do the same here, in Figure 3a for a specific parameterization chosen in the welfare analysis conducted in the next section. As shown above, there is a unique equilibrium with an interest rate $r < \rho$ that clears the capital market.

Figure 3b and direct calculation show that the goods market clears at $r = 0$. However[11] and as shown in Figure 3a, capital demand by firms differs from capital supplied by households through the financial intermediaries at $r = 0$ and there is no equilibrium at $r = 0$, except for the knife-edge case where the capital market also clears at $r = 0$, i.e.,

$$\frac{\xi}{(\nu + \rho + \xi)(\nu + \rho)} = \frac{\theta}{(1 - \theta)\delta}$$

In the case where $r^* > 0$ and therefore $\kappa^s(0) < \kappa^d(0)$ as in Figure 3a, one could slightly expand the model and implement $r = 0$ as an equilibrium by having a government own just

[11] We thank Marcus Hagedorn and Matt Rognlie for very helpful discussions on this issue. Auclert and Rognlie (2020) show that the same argument applies to the standard incomplete markets model (as originally described in Aiyagari, 1994). Our discussion here is an adaptation of their argument.



(a) Capital Demand $\kappa^d(r)$ and Supply $\kappa^s(r)$ as a Function of the Interest Rate r

(b) Goods Demand $w(r)C(r)$ and Net Supply $Y(r) - \delta K(r)$ as a Function of r

Figure 3: *The left panel shows wage-normalized capital demand by production firms $\kappa^d(r)$ and capital supply by financial intermediaries $\kappa^s(r)$, as a function of the interest rate. As proved in the previous subsection, $\kappa^d(r)$ is strictly decreasing on $[-\delta, \rho]$ and approaches ∞ as r approaches $-\delta$. Capital supply is strictly increasing on $[-\delta, \rho]$, and the figure is drawn with Assumption 1 in place, guaranteeing a unique intersection and thus a unique stationary equilibrium interest rate $r^* < \rho$. The right panel plots consumption demand $C(r)$ by the household sector versus consumption goods supply $G(r)$. There are two intersections: one at the stationary equilibrium interest rate r^* (in this case positive) and one at $r = 0$; we argue below that, generically, $r = 0$ is not an equilibrium interest rate.*

the right amount of capital $K^g > 0$ such that

$$K^s(0) + K^g = K^d(0) \tag{37}$$

Since $r = 0$, the government does not collect any revenue from this ownership that would need to be distributed, and thus a simple adjustment of the equilibrium definition that has the government own just the right amount of the capital stock would implement $r = 0$ as an equilibrium, with associated partial-insurance consumption allocation.[12] For the case $r^* < 0$ and therefore $\kappa^s(0) > \kappa^d(0)$, the government could issue bonds at zero interest to implement the $r = 0$ equilibrium; that is, $K^g < 0$ in equation (37) is now a liability rather

[12]This discussion appears to suggest that Walras' law breaks down at $r = 0$. We observe that capital market clearing always implies consumption goods market clearing. If $r \neq 0$ the reverse is also true, but not at $r = 0$. Walras' law $p * z(p) = 0$ only implies the statement that if N-1 markets clear (excess demand $d_i(p) = 0$ for all $i = 1, ...N - 1$), then the N-th market clears *if* the price vector p has *only non-zero elements*. It simply does not follow from Walras' law at $r = 0$ that one market clearing condition implies the other.

than an asset of the government.

6 Comparison to the Standard Incomplete Markets Model

Our model presents an alternative general equilibrium model with idiosyncratic income shocks to the canonical standard incomplete markets model. It is therefore instructive to compare stationary equilibria in both models. To clarify the sources of the differences it is instructive to formulate the Hamilton-Jacobi-Bellman equations for both versions of the model.

Consider first the household problem in the standard representative agent neoclassical growth model. For that model, the Hamilton-Jacobi-Bellman equation reads as

$$\begin{aligned} \rho U(k) &= \max_{c,x} \{u(c) + U'(k)x\} \\ \text{s.t.} &\quad c + x = rk + w \end{aligned}$$

or plugging in the budget constraint to substitute out the (marginal) change in the capital stock x, one obtains the perhaps more familiar form

$$\rho U(k) = \max_{c} \{u(c) + U'(k)(rk + w - c)\}$$

Introducing idiosyncratic productivity risk z with Poisson transitions as above, but under the assumption of incomplete insurance markets we obtain the HJB equation (see, e.g., Achdou et al., 2022):

$$\rho U(k,z) = \max_{c,x} \{u(c) + U'(k)x + p_z(U(k,\tilde{z}) - U(k,z))\} \tag{38}$$

$$\text{s.t.} \quad c + x = rk + wz \tag{39}$$

where the value function $U(k,z)$ now depends on the idiosyncratic productivity state z in addition to the capital stock owned by the household and p_z denotes the Poisson intensity with which the productivity state changes from the current state z to the other state $\tilde{z}$, i.e., $p_z = \xi$ if $z = \zeta$ and $p_z = \nu$ if $z = 0$. With intensity p_z lifetime utility changes from $U(k,z)$ to $U(k,\tilde{z})$, and the incomplete markets assumption is reflected in the fact that the capital stock upon a state change from z to $\tilde{z}$ remains the same.

Our model instead has complete markets but limited commitment. In Krueger and Uh-

lig (2006) we show that one way to interpret (or decentralize) the optimal consumption contract is through a financial market in the spirit of Alvarez and Jermann (2000) in which households trade Arrow securities that pay off contingent on the realization of the idiosyncratic productivity state, but with endogenously determined shortsale constraints. Without any punishment for default, Krueger and Uhlig (2006) show that these shortsale constraints prevent negative asset positions altogether. The prices of the Arrow securities reflect the transition rates across idiosyncratic states. As we show in Proposition 13 of Appendix A.4, the HJB equation in this decentralized version of our model then reads as

$$\begin{aligned} \rho U(k,z) &= \max_{c,x,\tilde{k}} \Big\{ u(c) + U'(k)x + p_z(U(\tilde{k},\tilde{z}) - U(k,z)) \Big\} && (40) \\ \text{s.t.} \quad & c + x + p_z(\tilde{k} - k) = rk + wz && (41) \\ & \tilde{k} \geq 0, x \geq 0 \text{ if } k = 0 && (42) \end{aligned}$$

In contrast to the HBJ equation in (38), the capital stock with which the household enters the next period is state-contingent and thus allowed to differ between the contingency of no state transition (lifetime utility $U(k,z)$) and a state transition (lifetime utility $U(\tilde{k},\tilde{z})$). These state-contingent capital stocks are reflected in equation (40) and contrast with (38) for the standard incomplete markets model where the capital stock is restricted to be the same across the two productivity states. In the budget constraint (41), the term $p_z(\tilde{k} - k)$ reflects the state-contingent addition (or subtraction) of capital, at the actuarially fair price p_z, and the constraints in equation (42) ensure that the capital stock in the case of a state transition cannot go negative, and that the capital stock, conditional on remaining in the same state, cannot go from zero to negative.

In Figure 4 we plot the (normalized by the wage) capital demand by firms and the capital supply, and display the market clearing real interest rate, both in our model and in the standard incomplete markets model, as pioneered by Aiyagari (1994), and characterized in continuous time with two income shocks by Achdou et al. (2022).[13]

We observe that for every interest rate, the supply of assets from the household sector is higher in the standard incomplete markets model than in the limited commitment model with endogenous consumption insurance contracts. In the presence of explicit income insurance (subject to the endogenous limit that state-contingent assets cannot become negative) the need to accumulate capital for precautionary reasons is reduced. As a conse-

[13] See also Sargent et al. (2021)

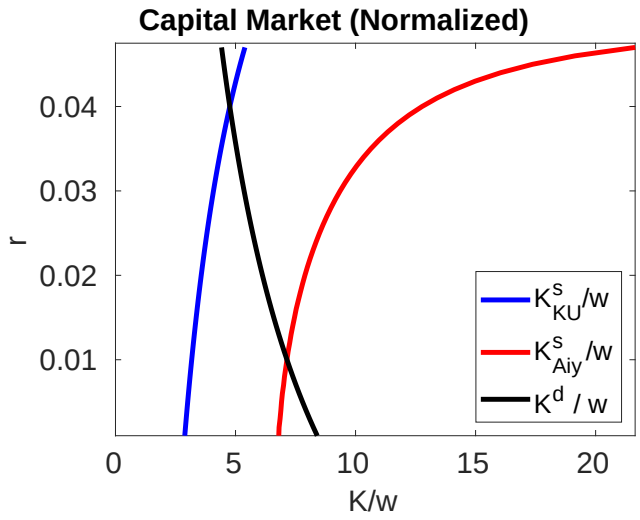


Figure 4: *This figure compares equilibria in our economy and the Aiyagari (1994) model. It displays the equilibrium determination in the capital market. Following Aiyagari (1994) it has the interest rate on the y-axis and (normalized by the wage) capital demand by firms and capital supply by the household sector on the x-axis. Capital demand (in black) is common between both models. Capital supply in our model was already plotted in Figure 3a, as was the equilibrium (but with x-axis and y-axis interchanged). Figure 4 also shows the familiar asset supply curve from the Aiyagari model that diverges to ∞ as r approaches the time discount rate ρ from below. Asset supply in the standard incomplete markets economy is larger for every interest rate, and the resulting equilibrium interest rate is lower, and equilibrium capital stock is higher in that model, relative to ours.*

quence of the larger supply of capital for a given r, the equilibrium interest rate is lower and the equilibrium capital stock is higher in the Aiyagari model.

7 Multiple Partial-Insurance Equilibria When the Elasticity of Substitution Is Not Unity

In the previous sections we have shown that with log-utility at most one stationary equilibrium exists. We now argue that deviating from a unit elasticity of substitution raises the possibility of multiple stationary equilibria. Assume now that the period utility function is given by

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

where $\sigma \neq 1$ is the coefficient of relative risk aversion (and the inverse of the intertemporal elasticity of substitution). All other model elements are completely unchanged.

Evidently, the normalized capital demand function $\kappa^d(r)$ is unaffected since it is determined purely from the production side of the economy. The argument that there cannot be a stationary equilibrium with $r > \rho$ and the condition for a full-insurance equilibrium remain unchanged as well, since the full-insurance allocation does not depend on σ. Thus, we continue to assume that Assumption 2 holds with strict inequality, so that we can focus on partial-insurance equilibria with $r < \rho$.

In Online Appendix F we show that the optimal consumption insurance contract has exactly the same properties as in the log-case: consumption jumps up upon receiving high productivity and drifts down at a constant rate when productivity turns low as long as it remains low. The key difference is that this decay rate is now given by rate $\frac{r-\rho}{\sigma} < 0$ instead of the rate $r - \rho$. The stationary consumption distribution is still characterized by a mass point at the top and a truncated Pareto distribution below the top. In the Online Appendix we also show that the normalized supply of capital is now given by

$$\kappa^s(r) = \frac{\xi}{\left(\nu + \frac{\rho-r}{\sigma} + r + \xi\right)\left(\nu + \frac{\rho-r}{\sigma}\right)}$$

which of course specializes to the log-case analyzed above for $\sigma = 1$. Capital demand $\kappa^d(r)$ remains as in equation (28). The capital market clearing condition $\kappa^s(r) = \kappa^d(r)$ now reads[14]

$$\frac{\xi}{\left(\nu + \frac{\rho-r}{\sigma} + r + \xi\right)\left(\nu + \frac{\rho-r}{\sigma}\right)} = \frac{\theta}{(1-\theta)(r+\delta)} \tag{43}$$

The characterization of equilibrium remains fully analytically tractable since any equilibrium interest rate is a solution r to this equation, which can be rewritten as a quadratic equation in r for all $\sigma \in (0, \infty)$ (see Online Appendix F.3.2). Only when $\sigma = 1$ and the income and substitution effects cancel in the optimal consumption contract, the term $\frac{\rho-r}{\sigma} + r$ in (43) vanishes, capital supply is unambiguously increasing in the interest rate and the market clearing condition becomes linear in r.

[14]Note that the total cost of the optimal consumption contract is only finite, and thus the capital supply function $\kappa^s(r)$ for the partial-insurance case is only well-defined for interest rates satisfying $\nu + \frac{\rho-r}{\sigma} + r > 0$. For $\sigma \leq 1$, this imposes no further restrictions and $\kappa^s(r)$ is well-defined on $(-\infty, \rho)$. However, for $\sigma > 1$ the domain of $\kappa^s(r)$ is restricted since the interest rate cannot be too negative. Concretely, it is given by $\left(-\frac{\sigma\nu+\rho}{\sigma-1}, \rho\right)$.

We will now demonstrate that as long as σ is sufficiently small (the IES is sufficiently large and the substitution effect is strong relative to the income effect), the capital supply function $\kappa^s(r)$ is upward-sloping in the interest rate on $r \in [-\delta, \rho]$ and the equilibrium remains unique. While there are two solutions to the quadratic equation in principle, only one of them is of economic relevance, corresponding to a positive amount of capital. On the other hand, for large enough σ (small enough IES and thus small enough substitution effect), capital supply might be downward sloping, and is downward sloping if $\sigma = \infty$ and the lifetime utility function is of Leontieff form. This opens up the possibility of multiple stationary equilibria. We now discuss these results more formally.

7.1 Intertemporal Elasticity of Substitution Equal to Zero: Leontieff Preferences

In the limit case $\sigma = \infty$, households are not willing to intertemporally substitute, the optimal consumption contract resembles that of the full-insurance case (consumption jumps upon the receipt of first high income and stays constant thereafter), and the stationary consumption distribution has unit mass at this consumption level. Equation (43) becomes

$$\frac{\xi}{\nu\left(\nu + r + \xi\right)} = \frac{\theta}{(1-\theta)(r+\delta)} \tag{44}$$

It can be rewritten as a linear equation in r. The following result is the limit case $\sigma \to \infty$ of Proposition 10 in the next subsection, and stated here to motivate that proposition.

Proposition 9. *Let $\sigma = \infty$ and thus the lifetime utility function is Leontieff. Suppose that*[15] *$\nu > \delta$. Then $\kappa^s(r)$ is well-defined and strictly decreasing in the interest rate on the interval $(-\nu, \infty)$. There is a unique equilibrium interest rate $r^* > -\delta$.*

Proof. Examine $\kappa^s(r)$ on the left-hand side of (44). Calculate the unique solution to this linear equation in r. □

7.2 General IES $\sigma \neq 1$

The possibility that normalized capital supply is downward sloping in the interest rate for sufficiently large values of σ (sufficiently weak substitution effect) admits the possibility

[15] This is needed for the NPV calculations in Section F.1 and is implied by (45) as $\sigma \to \infty$.

of multiple equilibria, as already suggested by the (very) special case in the previous subsection. The next proposition shows that the equilibrium remains unique for $\sigma \leq 2$, but the possibility of exactly two stationary equilibria emerges for larger σ. These equilibrium interest rates are solutions to a quadratic equation, and thus the characterization of equilibrium remains analytically tractable even for the case $\sigma \neq 1$, although we might lose the uniqueness of a partial-insurance stationary equilibrium.

Proposition 10. *Let Assumption 2 be satisfied with strict inequality.*

1. *If $\sigma < 1$, then $\kappa^s(r)$ is well-defined, continuous and strictly increasing on $r \in [-\delta, \rho]$. There exists a unique stationary equilibrium with interest rate $r \in (-\delta, \rho)$.*

2. *Let $\sigma > 1$ and*
$$\frac{\sigma\nu + \rho}{\sigma - 1} > \delta. \tag{45}$$
Then $\kappa^s(r)$ is well-defined[16] *and continuous on $r \in [-\delta, \rho]$. There exists at least one stationary equilibrium with $r \in (-\delta, \rho)$.*

   (a) *Suppose $\sigma \in (1, 2]$ and $\xi \geq \delta$. Then $\kappa^s(r)$ is increasing on $r \in [-\delta, \rho)$ and the stationary equilibrium with interest rate $r \in (-\delta, \rho)$ is unique.*

   (b) *There exist parameter combinations with $2 < \sigma < \infty$ such that $\kappa^s(r)$ has decreasing parts on $[-\delta, \rho)$ and that there are two stationary equilibria with $r \in (-\delta, \rho)$ solving (43).*

Proof. See Online Appendix F.3.1. For the last part, see the example in Figure 5. □

This proposition shows that for wide parameter combinations, uniqueness of equilibrium can be guaranteed (parts 1 and 2a), and identifies (in part 2b) the range of parameters where multiple equilibria can emerge. The condition in part 2a of the proposition ensures that $\kappa^s(r)$ is increasing at $r = -\delta$ (and thus is increasing for all $r \in [-\delta, \rho]$).

Prior to exploring the multiplicity of stationary equilibria numerically, we observe that equilibrium interest rate(s) scale in the parameters representing rates per unit of time, i.e., the time discount rate, the income transition rates and the depreciation rate. Cutting each of these rates in half will cut the equilibrium interest rate in half, and will also preserve the number of equilibria.

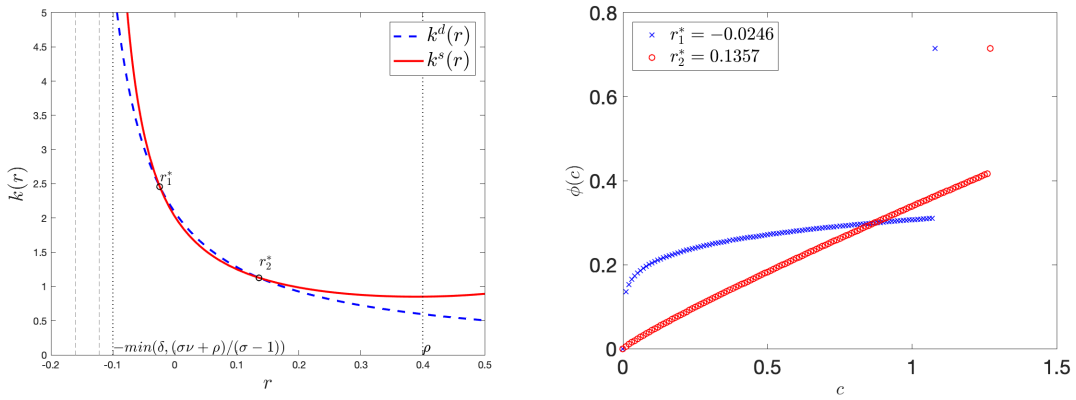
[16] This last assumption ensures that the effective discount rate $r + \nu + g(r)$ used to determine c_h is positive at $r = -\delta$, and thus c_h is finite at that interest rate and at all higher interest rates.

Corollary 2. *Suppose that $r^* \in (-\delta, \rho)$ is an equilibrium interest rate for parameters $\rho, \delta, \xi, \nu, \sigma, \theta, A$. Let $\alpha > 0$. Then $r = \alpha r^* \in (-\alpha\delta, \alpha\rho]$ is an equilibrium interest rate for parameters $\alpha\rho, \alpha\delta, \alpha\xi, \alpha\nu, \sigma, \theta, A$.*

Proof. See Online Appendix F.3.2 □

Since all equilibrium interest rates are solutions to the quadratic equation, we could in principle characterize regions of the six-dimensional parameter space $(\sigma, \theta, \delta, \nu, \xi, \rho)$ for which multiple equilibria emerge. Rather than doing so, we display an example parameter combination with $\sigma = 10$ that exhibits the two stationary equilibria in Figure 5. This example is not meant to be empirically realistic, but rather to demonstrate graphically that our model can indeed have multiple stationary equilibria.

Figure 5: Two equilibria with partial insurance when $\sigma > 2$.



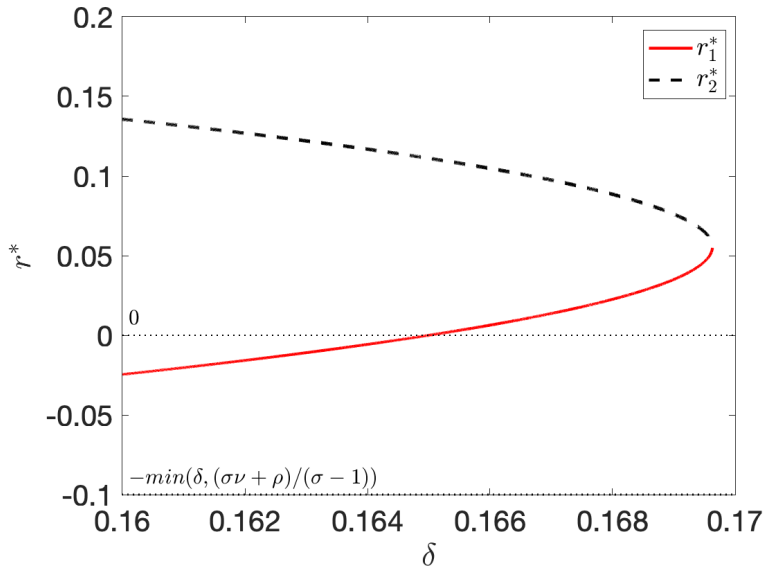
(a) Capital Market Clearing (b) Equilibrium Consumption Distributions

This figure plots an example of two equilibria, both with partial insurance, under parameter values $\sigma = 10, \theta = 0.25, \delta = 0.16, \nu = 0.05, \xi = 0.02, \rho = 0.4$. The two equilibrium interest rates are given by $r_1^* = -0.0246, r_2^* = 0.1357$. Left panel: solid line represents the capital supply curve $k^s(r)$, dashed line represents the capital demand curve $k^d(r)$. The right panel displays the two equilibrium consumption distributions.

Figure 5a plots normalized capital demand $\kappa^d(r)$ and supply $\kappa^s(r)$ against the interest rate r. As shown in the proposition above, since $\sigma > 2$, both capital demand and supply are downward-sloping in the interest rate, and thus can intersect more than once. Figure 5b displays the consumption (normalized by the wage) distributions $\phi_r(c)$ associated with the two equilibrium interest rates. The blue x-ed line corresponds to the low equilibrium interest rate and the red circled line to the high equilibrium interest rate. Both distributions have a mass point equal to $\psi_h = \frac{\nu}{\xi+\nu}$ and a truncated power distribution below this mass

point. The consumption mass point $c_h(r)$ is increasing in the (equilibrium) interest rate as long as the IES $1/\sigma$ is less than 1, and thus the remaining probability mass spreads out over a larger support of the consumption distribution in the high interest rate equilibrium, falling more rapidly as consumption approaches zero. Thus, the consumption distribution has fewer individuals with very low consumption in the equilibrium with the high interest rate, and therefore better consumption insurance. Note that by Proposition 6 aggregate normalized consumption $C(r) = \int c\phi_r dc$ is increasing in the interest rate r, a fact clearly visible when comparing the two consumption distributions.

Figure 6: Equilibrium Set as Function of Depreciation Rate.



The figure plots the equilibrium interest rates r_1^*, r_2^* as δ changes. Other parameters are $\sigma = 10, \theta = 0.25, \nu = 0.05, \xi = 0.02, \rho = 0.4$.

Finally, we display how the set of equilibrium interest rates changes as we change parameters. Specifically, we vary the depreciation rate and keep all other parameters constant in Figure 6. The figure shows that the example above with two stationary equilibria is not a knife-edge case, but rather emerges for a range of parameter values, as long as σ is sufficiently large, and therefore the intertemporal elasticity of substitution is sufficiently small and the income effect sufficiently potent relative to the substitution effect. The figure also shows that both equilibrium interest rates are positive for a positive-length interval of δ-values. Finally, note that condition (45) in Proposition 10 fails and a partial-insurance equilibrium ceases to exist, when the deprecation rate δ becomes too large.

7.3 Welfare Ranking of Stationary Equilibria

Given that we have identified the possibility of multiple stationary equilibria, the natural question arises as to whether they can be ranked in terms of their welfare properties. On one hand, a lower interest rate is associated with a higher capital stock and thus higher wages. On the other hand, a lower interest rate implies a faster decline in consumption after receiving adverse income shocks, and thus potentially less consumption insurance, also depending on the entry consumption level c_h. One of the benefits of our environment is that we can characterize both components of welfare in closed form.

With our focus on stationary equilibria, individual welfare in such an equilibrium can be defined as expected utility at birth in the stationary equilibrium.[17] To rank the normative properties of two equilibria, one can ask by what constant $\alpha > 0$ one has to scale consumption in a low interest rate equilibrium to be indifferent to being born into a high interest rate equilibrium. This consumption equivalent welfare measure is derived in Online Appendix H and characterized in the following proposition.

Proposition 11. *Assume $\sigma > 1$ and suppose there exist two stationary equilibria, with $r_1 < r_2$. Then the equivalent variation α is given by*

$$\alpha = \frac{w(r_2)}{w(r_1)} \cdot \left[\frac{c_h(r_2)}{c_h(r_1)}\right]^{\frac{\sigma}{\sigma-1}} = \alpha_{wage} \cdot \alpha_{contract} \tag{46}$$

The aggregate wage component satisfies $\alpha_{wage} = \frac{w(r_2)}{w(r_1)} < 1$ and the contract component satisfies $\alpha_{contract} = \left[\frac{c_h(r_2)}{c_h(r_1)}\right]^{\frac{\sigma}{\sigma-1}} > 1$.

In principle, the wage effect might dominate and thus the low interest rate equilibrium has higher steady-state welfare ($\alpha < 1$) for some set of parameter values exhibiting multiplicity of equilibria, while the better consumption insurance effect dominates and the high interest rate equilibrium has higher steady-state welfare ($\alpha > 1$) for the complementary set of parameter values. While one could characterize these sets precisely to establish which of them are nonempty and under what conditions, the economic insight obtained is of more value: in this model, when there are multiple stationary equilibria, there is a welfare trade-off between higher aggregate wages and consumption and more consumption inequality.

It is important to note that the focus on stationary equilibria in this paper ignores the potential benefits of consuming part of the capital stock along the transition, and thus a

[17] Alternatively, we could define it as expected period utility. The characterization below goes through almost unchanged under this alternative notion of equilibrium welfare, as Online Appendix H demonstrates.

finding that attaches higher welfare to the low interest (high capital and wage) steady state is subject to the usual caveat that it ignores transitional dynamics during which this higher capital stock needs to be accumulated.[18]

8 Conclusion

In this paper we have analytically characterized stationary equilibria in a neoclassical production economy with idiosyncratic income shocks and long-term one-sided limited commitment contracts. For an important special case (log-utility, two income states, zero income in the lower state) the equilibrium is unique and can be given in closed form, with complete comparative statics results.

Given these findings, we can identify three immediately relevant next questions for further investigation. First, on account of our use of a continuous time setting, the endogenous optimal contract length is analytically tractable even outside the special case we have focused on thus far. However, this length will in general depend on the interest rate in the economy, which complicates the analytical aggregation step of the analysis.

Second, thus far we have focused on stationary equilibria, thereby sidestepping the question of whether this stationary equilibrium is reached from a given initial aggregate stock, and what the qualitative properties of the associated transition path are. This question is especially relevant for a full welfare (and possibly associated policy) analysis.

Finally, thus far we have focused on an environment that has idiosyncratic but no aggregate shocks, rendering the macro economy deterministic. Given our sharp analytical characterization of the equilibrium in the absence of aggregate shocks, we conjecture that the economy with aggregate shocks might be at least partially analytically tractable as well. We view these questions as important topics for future research and take a first step toward them in Krueger, Li and Uhlig (2022) as well as Ando, Krueger and Uhlig (2022).

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Appendix

A A selection of results and proofs.

A precise analysis of the model requires an exact mathematical underpinning, and a tight proof of each claim. We provide a full version in the Online Appendix, see the road map Section B there. Here, we highlight a few key results and their proofs, focusing on parts that are of particular significance or are less straightforward to establish.

A.1 Histories, Contracts and Multipliers

The stochastics for an agent is governed by a finite-state continuous time Markov processes in a state $x \in X$. Productivity at date t is $z_t = \mathbf{z}(x_t) \in \{0, \zeta\}$. Contracts will generally depend on more than just current productivity. They should not be constrained a priori to depend on some additional low-dimensional state, but rather may depend on the entire history, up to that date. Building on, e.g., Puterman (2005) and as explained in greater detail and with additional notation in Online Appendix C, histories are denoted by their beginning and end date as well as all the Markov switching dates and the values of the Markov process at these dates,

$$h_{t,\tau} = (\tau, n, t_0 = t, t_1, \ldots, t_n, x_0 = x, x_1, \ldots, x_n) \tag{47}$$

All histories between t and τ starting from some Markov state x shall be denoted with $\mathcal{H}_{t,\tau}(x)$. $P_{t,\tau}$ denotes the probability distribution across histories. With this, contracts can be more formally defined as mappings $\mathbf{c}(h_{t,\tau}; x, U)$ from histories $h_{t,\tau}$, the initial state x and promised utility U, see definition 3. The cost minimization problem can be more formally defined. For example, the **limited commitment constraints** are now stated as

$$\int_s^\infty \int_{\mathcal{H}_{s,\tau}(x(h_{t,s}))} e^{-\rho(\tau-s)} u(w\mathbf{c}([h_{t,s}, h_{s,\tau}]; x, U)) dP_{s,\tau} d\tau \quad \geq \quad U^{out}(\mathbf{z}(\mathbf{x}(h_{t,s}))) \\ \text{for all } s > t \text{ and } h_{t,s} \in \mathcal{H}_{t,s}(x) \tag{48}$$

for all $x \in X$ and $U \in \mathcal{U}(x)$. We will make repeated use of this constraint below. Basic properties of the cost function such as concavity and differentiability are established (see Online Appendix D.1). The contract cost minimization problem can be rewritten as a

Lagrangian, (see Online Appendix D.2). Using partial integration, the Lagrangian can be rewritten with **cumulative Lagrange multipliers** $\boldsymbol{\lambda}(h_{t,\tau})$ and the history notation as

$$\begin{aligned} L &= \int_t^\infty \int_{\mathcal{H}_{t,\tau}} e^{-r(\tau-t)} \left[w\mathbf{c}(h_{t,\tau}) - w\mathbf{z}(\mathbf{x}(h_{t,\tau})) \right] dP_{t,\tau} d\tau \\ &\quad - \int_t^\infty \int_{\mathcal{H}_{t,\tau}} \boldsymbol{\lambda}(h_{t,\tau}) e^{-\rho(\tau-t)} u(w\mathbf{c}(h_{t,\tau})) dP_{t,\tau} d\tau \\ &\quad + \boldsymbol{\lambda}(h_{t,t}) U + \int_{s=t}^\infty \int_{\mathcal{H}_{t,s}} e^{-\rho(s-t)} U^{out}(\mathbf{z}(\mathbf{x}(h_{t,s}))) dP_{t,s} \times d\lambda, \end{aligned} \tag{49}$$

restated with further explanations as equation (49) in the Online Appendix D.2. The cumulative Lagrange multiplier reformulation in (49) provides a version of Marcet and Marimon (2019) in continuous time. This formulation, together with differentiability properties, yield the first-order condition

$$e^{(\rho-r)(\tau-t)} = \boldsymbol{\lambda}(h_{t,\tau}) u'(w\mathbf{c}(h_{t,\tau}; x, U)) \tag{50}$$

A.2 A Key Property of the Contracts

The following central lemma, also in Online Appendix D.3, expresses key properties of the contract. The proof is non-trivial. It draws on some material in the Online Appendices C and D, which should be consulted for additional detail and explanations.

Lemma 2 (key properties of the contract). *1. Suppose that the constraint* (48) *does not bind at history $h_{t,s}$. Then $\dot{\boldsymbol{\lambda}}(h_{t,s})_+ = 0$ and the derivative $\dot{\mathbf{c}}_+(h_{t,s})$ exists. If the last jump occurred strictly before date s, i.e., if $t_n < s$, then $\dot{\boldsymbol{\lambda}}(h_{t,s}) = 0$ and $\mathbf{c}$ is differentiable at $h_{t,s}$.*

2. *Suppose $\mathbf{c}$ is differentiable at history $h_{t,s}$. Then $\boldsymbol{\lambda}$ is differentiable at history $h_{t,s}$ and*

$$\rho - r = \left(\frac{u''(wc)wc}{u'(w\mathbf{c}(h_{t,s}))} \right) \frac{\dot{\mathbf{c}}(h_{t,s})}{\mathbf{c}(h_{t,s})} + \frac{\dot{\boldsymbol{\lambda}}(h_{t,s})}{\boldsymbol{\lambda}(h_{t,s})} \tag{51}$$

The statement and equation likewise hold for the left-derivatives, if $\mathbf{c}$ is left-differentiable at history $h_{t,s}$, and for the right-derivatives, if $\mathbf{c}$ is right-differentiable at history $h_{t,s}$.

3. *Suppose that the limited commitment constraint* (48) *binds at history $h_{t,s}$. Suppose that $\rho = r$. Alternatively, suppose that $\rho > r$ and that Assumption 3 holds. Then $\mathbf{c}(h_{t,s;\Delta})$ is constant in $\Delta \geq 0$ and $\dot{\mathbf{c}}_+(h_{t,s}) = 0$.*

4. $\boldsymbol{\lambda}_-(h_{t,s}) \neq \boldsymbol{\lambda}(h_{t,s})$, *i.e.* $\boldsymbol{\lambda}$ *is discontinuous at history* $h_{t,s}$*, if and only if* $\mathbf{c}_-(h_{t,s}) \neq \mathbf{c}(h_{t,s})$. *In that case,* $\mathbf{c}_-(h_{t,s}) < \mathbf{c}(h_{t,s})$, $\boldsymbol{\lambda}_-(h_{t,s}) < \boldsymbol{\lambda}(h_{t,s})$ *and* (48) *binds at history* $h_{t,s}$ *with* $t_n = s$*, i.e., just when the state change occurred.*

Proof. 1. If the constraint does not bind at $h_{t,s}$, then it will not bind either for the no-state-change history extensions $h_{t,s;\Delta}$, provided $\Delta > 0$ is sufficiently small. Thus, $\boldsymbol{\lambda}(h_{t,s}) = \boldsymbol{\lambda}(h_{t,s;\Delta})$ is locally constant[19] and thus $\dot{\boldsymbol{\lambda}}_+(h_{t,s}) = 0$. The existence of $\dot{\mathbf{c}}_+$ at $h_{t,s}$ now follows from (50). If $t_n < s$, the same argument applied to the truncated histories $h_{t,s}(t, s-\Delta)$ shows that $\dot{\boldsymbol{\lambda}}(h_{t,s}) = 0$ and the differentiability of $\mathbf{c}$ at $h_{t,s}$.

2. Differentiation of (50) with respect to τ shows that λ is also differentiable[20] at τ and delivers (51), when replacing τ with s.

3. By assumption, the limited commitment constraint (48) binds at $h_{t,s}$. For ease of notation, write x for $x(s)$ and z for $\mathbf{z}(x)$. Consider the no-state-change history extension $h_{t,s;\Delta} = [h_{t,s}, (s+\Delta, 0, s, x)]$ for $\Delta > 0$. The proof proceeds in two parts. For part A, suppose that the limited commitment constraint (48) binds again at $h_{t,s;\bar{\Delta}}$ for some $\bar{\Delta} > 0$. We use an averaging argument to establish that consumption must be the same at s and at $s+\bar{\Delta}$. With the help of the first two parts as well as some careful analysis, we then show that the limited commitment constraint (48) must bind for all $h_{t,s;Delta}$ and $0 < \Delta \leq \bar{\Delta}$ and thus establish the claim for part A. For part B, suppose that the constraint (48) never binds again for $h_{t,s;\Delta}$ at any $\Delta > 0$. We show that this leads to a contradiction.

   A. For the first part, suppose that the limited commitment constraint (48) binds at $h_{t,s}$ as well as at $h_{t,s;\bar{\Delta}}$ for some (possibly large) $\bar{\Delta} > 0$. Compare the contract going forward conditional on these two histories: we will argue that one can do better by averaging them, should they be different. To that end and to express this precisely, let $h_{s,\tau}$ be some continuation at the current state x of the history $h_{t,s}$ to $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$: for a graphical illustration, see Figure 7 in the Online Appendix. Construct the corresponding continuation $h_{t,\tau+\bar{\Delta}} = [h_{t,s;\bar{\Delta}}, h^{\bar{\Delta}}_{s,\tau}]$ of $h_{t,s;\bar{\Delta}}$ with the $\bar{\Delta}$-time-shifted history $h^{\bar{\Delta}}_{s,\tau}$ (see equation (58)). This correspondence is one-one and measure preserving. Suppose now that the contract $\mathbf{c}(h_{t,\tau})$ differs from the corresponding $\mathbf{c}(h_{t,\tau+\bar{\Delta}})$ for a set $\mathcal{S}$ of extensions $h_{s,\tau}$

[19] Returning to our original Lagrange multipliers, $\boldsymbol{\mu}(h_{t,s;\Delta}) = 0$ for $\Delta \geq 0$ sufficiently small.
[20] This is a standard calculus argument.

with positive measure, i.e., suppose that $\int_{\tau=s}^{\infty}\int_{\mathcal{H}_{s,\tau}} 1_{h_{s,\tau}\in\mathcal{S}} dP_{s,\tau}d\tau > 0$. Consider then a new contract, which is the average between the original contract and the contract following $h_{t,s;\bar{\Delta}}$, i.e., consider

$$\tilde{\mathbf{c}}(h_{t,\tau}) = \left(\mathbf{c}([h_{t,s}, h_{s,\tau}]) + \mathbf{c}([h_{t,s;\bar{\Delta}}, h^{\bar{\Delta}}_{s,\tau}])\right)/2 \tag{52}$$

defined for all continuations $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ of $h_{t,s}$. In words, $\tilde{\mathbf{c}}$ is the average between the current contract as well as the contract portion following $h_{t,s;\bar{\Delta}}$ shifted backward by $\bar{\Delta}$. Since utility is strictly concave, this contract now delivers strictly higher continuation utility at history $h_{t,s}$, while its costs stay unchanged, a contradiction to the hypothesis that the constraint (48) binds at $h_{t,s}$, i.e. a contradiction to the assertion that the original contract was cost-minimizing, see Lemma 3. It follows that consumption at $s+\bar{\Delta}$ will be the same as at s for any $\bar{\Delta} > 0$, where (48) binds: let us denote that consumption level as c.

If the limited commitment constraint (48) binds for all $0 < \tilde{\Delta} < \bar{\Delta}$, we would be done with this part, since consumption would then be constant at $\mathbf{c}(h_{t,s;\tilde{\Delta}}) \equiv c$. Indeed, we would be done, if this is true for some sufficiently small $\bar{\Delta} > 0$, since it must then be true for all $\bar{\Delta}$ per "shifting" the contract by $\bar{\Delta}/2$ into the future. Suppose thus that (48) binds at some[21] $\Delta^* \leq \bar{\Delta}$, but does not bind for all $0 < \tilde{\Delta} < \Delta^*$. According to the first part of the lemma, the derivative $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}})$ exists and $\dot{\boldsymbol{\lambda}}(h_{t,s;\tilde{\Delta}}) = 0$.

i. Consider the case $\rho = r$. According to the second part of the lemma, $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}}) = 0$. Thus, consumption is constant at $\mathbf{c}(h_{t,s;\tilde{\Delta}}) \equiv c$ for all $0 < \tilde{\Delta} < \bar{\Delta}$, regardless of whether the constraint (48) binds or does not bind at $\tilde{\Delta}$, establishing our claim.

ii. Consider the case $\rho > r$ and current state x. We will show that we arrive at a contradiction; see Figure 8 in the Online Appendix. According to the second part of the lemma, $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}}) < 0$ for $0 < \tilde{\Delta} < \Delta^*$. Fix such a $\tilde{\Delta}$. It follows that $\mathbf{c}(h_{t,s;\tilde{\Delta}}) < \mathbf{c}(h_{t,s;\Delta^*}) = c$; that is, consumption jumps up at Δ^*, even though $U_{h_{t,s;\tilde{\Delta}}} > U_{h_{t,s;\Delta^*}} = U^{out}(\mathbf{z}(x))$. We will show that the contract at history $h_{t,s;\tilde{\Delta}}$ can therefore not have been cost-minimizing. For $x' \neq x$ and $\Delta > 0$, let $h_{t,s;\Delta,x'} = [h_{t,s}, (s+\Delta, 1, s, x, x(s+\Delta) = x']$ be

[21] Δ^* exists, because $U_{h_{t,s;\tilde{\Delta}}}$ is continuous in $\tilde{\Delta}$.

the extensions of the original history $h_{t,s}$ with a first state change to a new state x' occurring at date $s + \Delta$.

Define $U_{h_{t,s;\tilde{\Delta}}}$ as the continuation utility starting at the history $h_{t,s;\tilde{\Delta}}$. Deploying the construction of Online Appendix D.4 leading up to equation (82), it is given by

$$\begin{aligned} U_{h_{t,s;\tilde{\Delta}}} &= \int_0^{\Delta^*-\tilde{\Delta}} e^{(\alpha_{x,x}-\rho)\tau} u(w\mathbf{c}(h_{t,s;\tilde{\Delta}+\tau}))d\tau \\ &\quad + e^{(\alpha_{x,x}-\rho)(\Delta^*-\tilde{\Delta})} U^{out}(\mathbf{z}(x)) \\ &\quad + \sum_{x'\neq x} \int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\tilde{\Delta}+\tau,x'}} d\tau \end{aligned} \tag{53}$$

The first term captures the present discounted utility over the time interval from $t + s + \tilde{\Delta}$ to $t + s + \Delta^*$, conditional on no state change, the second term captures the associated continuation utility from $t + s + \Delta^*$ onward in that case; and the last term captures the expected continuation utility conditional on some state change from x to x' during the time interval $\Delta^* - \tilde{\Delta}$ following history $h_{t,s;\tilde{\Delta}}$.

Compare this to the similar continuation at $s + \Delta^*$,

$$\begin{aligned} U_{h_{t,s;\Delta^*}} &= \int_0^{\Delta^*-\tilde{\Delta}} e^{(\alpha_{x,x}-\rho)\tau} u(w\mathbf{c}(h_{t,s;\Delta^*+\tau}))d\tau \\ &\quad + e^{(\alpha_{x,x}-\rho)(\Delta^*-\tilde{\Delta})} U_{h_{t,s;\Delta^*+(\Delta^*-\tilde{\Delta})}} \\ &\quad + \sum_{x'\neq x} \int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\Delta^*+\tau,x'}} d\tau \end{aligned} \tag{54}$$

Note that $\mathbf{c}(h_{t,s;\tilde{\Delta}+\tau}) < \mathbf{c}(h_{t,s;\Delta^*+\tau})$ and that $U^{out}(z) \leq U_{h_{t,s;\Delta^*+(\Delta^*-\tilde{\Delta})}}$. Since $U_{h_{t,s;\tilde{\Delta}}} > U_{h_{t,s;\Delta^*}} = U^{out}(z)$, it must be the case that the inequality is reversed for the last term,

$$\begin{aligned} &\sum_{x'\neq x} \int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\tilde{\Delta}+\tau,x'}} d\tau \\ &\qquad > \sum_{x'\neq x} \int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\Delta^*+\tau,x'}} d\tau \end{aligned}$$

Recall, though, that the contract is cost-minimizing at history $h_{t,s;\tilde{\Delta}}$. Per the principle of optimality established in Lemma 9 of Online Appendix D.4, we thus arrive at a contradiction.

B. For the second part, suppose instead that (48) never binds for any $\Delta > 0$ at the no-state-change history extensions $h_{t,s;\Delta} = [h_{t,s}, (s+\Delta, 0, s, x(s))]$. The first part of the lemma shows that $\dot{\lambda}(h_{t,s;\Delta}) = 0$ for all $\Delta > 0$ as well as $\dot{\lambda}_+(h_{t,s}) = 0$. If $\rho = r$, then the second part of the lemma shows that $\dot{\mathbf{c}}_+(h_{t,s}) = 0$ and hence the claim. If $\rho > r$, then (76) shows that $\dot{\mathbf{c}}(h_{t,s;\Delta})/\mathbf{c}(h_{t,s;\Delta}) < (r-\rho)/\bar{\sigma} < 0$ and hence $\mathbf{c}(h_{t,s;\Delta}) \to 0$ as $\Delta \to \infty$.

   i. Suppose then that there is some $\underline{\Delta} > 0$, so that the continuation utility promises upon a state change to $x' \neq x(s)$ binds at all extended histories

$$h_{t,s;\Delta,x'} = [h_{t,s}, (s+\Delta, 1, s, s+\Delta, x(s), x')]$$

   for $\Delta > \underline{\Delta}$ and any $x' \in X$. The continuation contract at the no-state-change history extension $h_{t,s;\underline{\Delta}}$ is feasible when shifted backward in time to s, i.e., consider the contract

$$\tilde{\mathbf{c}}(h_{t,\tau}) = \mathbf{c}([h_{t,s;\underline{\Delta}}, h^{\Delta}_{s,\tau}])$$

   defined for all continuations $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ of $h_{t,s}$. Contract $\tilde{\mathbf{c}}$ is cheaper for the principal than contract $\mathbf{c}$, since consumption along the $h_{t,s;\Delta}$-histories keeps declining and since one cannot do better upon a state change than to achieve a binding constraint there. This is a contradiction to the assertion that the contract was cost-minimizing $\mathbf{c}$.

   ii. Suppose instead that for any $\underline{\Delta} > 0$, there is a positive measure of dates $s+\Delta$ with $\Delta > \underline{\Delta}$, at which the utility promised upon a state change is not binding. But then and with sufficiently large Δ and thus sufficiently small $\mathbf{c}(h_{t,s;\Delta})$ along the no-state-change path, the principal can achieve a higher promised utility for the agent by promising less consumption upon the state change for some positive interval of time and more consumption along the no-state-change path, again a contradiction to the contract being cost-minimizing.

4. Since λ is weakly increasing, $\lambda(h_{t,s}) - \lambda_-(h_{t,s}) > 0$. The claim now follows from

(50) and exploiting the fact that $u'(\cdot)$ is strictly decreasing, as well as from noting that $\lambda(h_{t,s})$ is only increasing if (48) binds. Furthermore, it must be the case that $t_n = s$, i.e., that the state change just occurred on date s, since otherwise the derivative of consumption would have been zero per the third part of the lemma.

□

For the CRRA utility function $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, equation (51) implies

$$\frac{\dot{c}(s)}{c(s)} = -\frac{\rho - r}{\sigma} + \frac{1}{\sigma}\frac{\dot{\lambda}(s)}{\lambda(s)} \tag{55}$$

A.3 Three States and Ordering of Outside Options

While Lemma 1 is intuitive the proof requires the comparison of a contract starting at high productivity to a contract starting at low productivity. This in turn requires the expansion of the state space to some underlying state x that can take three values $x(t) \in X = \{0, 1, 2\}$, evolving independently from each other (See also Online Appendix E). The transition rates $\alpha_{i,j}$ to transit from state $x = i$ to $x = j$ are $\alpha_{0,1} = \alpha_{2,1} = \nu$, $\alpha_{1,0} = \alpha_{2,0} = \xi$ and $\alpha_{0,2} = \alpha_{1,2} = 0$. Let $\alpha_{i,i} = -\sum_{j \neq i} \alpha_{i,j}$, so that α is a intensity matrix or infinitesimal generator matrix. The following proof of Lemma 1 assumes $0 < r \leq \rho$ and is also in Online Appendix E.1, together with explanations on how to handle the case of $r < 0 < \rho$.

Proof of Lemma 1. The key idea is that an agent currently at high productivity can be provided with the contract of the low-productivity agent, delivering the same utility and a profit to the principal, a contradiction to perfect competition between the principals. Some care needs to be taken to implement this idea, however. Contracts depend on the history of states. Thus, if the history was expressed only in terms of productivities, it would be meaningless to give an agent starting with high productivity "the same" contract as an agent starting with low productivity. The underlying state and the corresponding productivity need to be decoupled. It is here where the three-state construction described at the beginning of this section and the careful distinction between the state and the productivity at that state as described at the beginning of Online Appendix C pay off.

Suppose by contradiction to the claim (10) that

$$U^{out}(0) \geq U^{out}(\zeta) \tag{56}$$

Fix the productivity mapping $\mathbf{z} : X \to Z$ to be $\mathbf{z}_A$. Recall that $\mathbf{z}_A(0) = \mathbf{z}_A(2) = 0$ and $\mathbf{z}_A(1) = \zeta$, and that the three-state process starting at $x(t) = 0$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z(t) = 0$. Consider an optimal consumption contract $c(\tau; 0, U^{out}(0))$ given to an agent at date $t = 0$, say, and starting off with productivity $z(0) = 0$, delivering date-0 promised utility $U = U^{out}(0)$ in (2) and generating costs $V(0, U^{out}(0)) = 0$. Wlog, we shall impose the condition that $x(0) = 2$: any contract as defined per history dependence in Online Appendix C and starting at $x(0) = 0$ can be written[22] as a contract starting at $x(0) = 2$ delivering the same outcomes, per ignoring transitions from $x = 2$ to $x = 0$. Thus, the optimal consumption contract $c(\tau; 0, U^{out}(0))$ is a mapping $\mathbf{c} : \mathcal{H}_0 \to \mathbf{R}_+$ from x-histories into consumption outcomes, where all $h_{s,0} \in \mathcal{H}_0$ satisfy $x(0) = 2$, and which satisfies the constraints (48).

Next, fix the productivity mapping $\mathbf{z} : X \to Z$ to be $\mathbf{z}_B$. Recall that $\mathbf{z}_B(0) = 0$ and $\mathbf{z}_B(1) = \mathbf{z}_B(2) = \zeta$, and that the three-state process starting at $x(t) = 1$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z(t) = \zeta$. The contract $\mathbf{c}$ delivers the same expected utility $U^{out}(0)$. The contract $\mathbf{c}$ satisfies the constraints (48) for states $x(s) = 0$ and states $x(s) = 1$, where $\mathbf{z}_A$ and $\mathbf{z}_B$ coincide. With equation (103), the constraints are also satisfied for the state $x(s) = 2$ and $\mathbf{z}_B(2) = \zeta$ rather than $\mathbf{z}_A(2) = 0$. The consumption portion generates the same costs for the principal, as nothing has changed regarding the consumption process, but the expected revenue from productivity income is now strictly higher per Lemma 10. It follows, that the contract $\mathbf{c}$ now delivers strictly negative costs $V(\zeta, U^{out}(0))$. Per Lemma 3 and equation (103), $0 > V(\zeta, U^{out}(0)) \geq V(\zeta, U^{out}(\zeta))$. However, $V(\zeta, U^{out}(\zeta)) = 0$ per the definition of equilibrium. With that, we have arrived at a contradiction. □

A.4 The Hamilton-Jacobi-Bellman Equations

Proposition 12 (the cost-minimizing HJB equation). *The cost function $V(x, U)$ solves the Hamilton-Jacobi-Bellman equation*

$$rV(x,U) \;=\; \min_{c,\dot{U},(U(x'))_{x'\in X/\{x\}}} wc - w\mathbf{z}(x) + V'_-(x,U)\dot{U} + \sum_{x'\neq x} \alpha_{x,x'}(V(x', U(x')) - V(x,U))$$

[22] This argument can be made precise with some tedious notation.

subject to

$$\begin{aligned} \rho U &= u(wc) + \dot{U} + \sum_{x' \neq x} \alpha_{x,x'}(U(x') - U) \\ \dot{U} &\geq 0, \text{ if } U = U^{\textbf{Out}}(\mathbf{z}(x)) \\ \frac{\bar{u}}{\rho} > U(x') &\geq U^{\textbf{Out}}(\mathbf{z}(x')) \end{aligned}$$

for all $x(t) = x \in X$ *and all* $U \in \left[U^{out}(\mathbf{z}(x)), \frac{\bar{u}}{\rho}\right)$*, provided that (2) binds.*

This is a restatement of Proposition 14 in the Online Appendix D.4 which also contains the proof. For the dual perspective of maximizing utility, subject to the costs expressed as capital, one obtains the following version, see Online Appendix D.5 for details.

Proposition 13 (the utility-minimizing HJB equation). *The utility function* $U(k;x)$ *solves the Hamilton-Jacobi-Bellman equation*

$$\rho U(k;x) = \max_{c,\dot{k},(k(x'))_{x' \in X/\{x\}}} u(c) + \frac{\partial U(k;x)}{\partial k}\dot{k} + \sum_{x' \neq x} \alpha_{x,x'}(U(k(x');x') - U(k;x))$$

subject to

$$\begin{aligned} c + \dot{k} + \sum_{x' \neq x} \alpha_{x,x'}(k(x') - k) &= rk + w\mathbf{z}(\mathbf{x}) \\ k(x') \geq 0 &\qquad \text{for all } x' \in X/\{x\} \\ \dot{k} \geq 0 &\qquad \text{if } k = 0 \end{aligned}$$

for all $x \in X$ *and all* $k \geq 0$*.*

Online Appendix

B Road Map

The following Sections C and D provide a precise language to analyze contracts in continuous time with finite state stochastic processes and establish key results of the optimal contract. They provide the basis for proving the specific lemmas and propositions in the paper, starting with Section E, but are more general than that. As such, Sections C and D are useful considerably beyond the particular model studied in the paper.

Section C describes the mathematical framework for finite state continuous time Markov processes and history dependence of allocations. Such a framework is necessary for the mathematically precise description of contracts in Section D . Histories are encoded by a beginning and end date, the number and dates of Markov switches in between and the value of the Markov process for all episodes (see equation (57)). Notation is introduced to describe smaller segments of a history or for how to concatenate two adjacent histories. With that, the Markov transition probability law can be stated as a probability measure on the set of all histories (see equation (59)). We proceed, using this as the appropriate probability space. Equation (61) states how to calculate conditional expectations of functions of future histories, using this probability law.

Section D describes and analyzes contracts for finite-state continuous time Markov processes in five subsections. The key properties of the contracts are established in Lemma 7 of subsection D.3. That lemma is foundational for the description of the contract properties in the main body of the text as well as for proving the results in Section E. Subsections D.1, D.2 and D.4 are the necessary preliminaries to establish this lemma, but useful in their own right. In particular, subsection D.2 extends Marimon-Marcet (2019) to continuous time. Subsection D.5 provides the dual perspective of utility maximization as an "add-on."

1. Subsection D.1 introduces contracts as mappings from histories, the current state and a promised utility level U in definition 3 and defines optimal contracts as minimizing the appropriate cost function of the principal in definition 4. Three lemmas establish that the cost function is increasing, convex and differentiable in U.

2. Subsection D.2 describes a Lagrangian approach to the cost minimization problem of subsection D.1. Starting from the somewhat heuristic formulation in equation (67), it provides for a precise definition in equation (71), using recursive Lagrange

multipliers. This extends the Marimon-Marcet (2019) approach to the continuous-time finite-state Markov case. It leads to the key first-order condition (74) and the contract property in Lemma 6 that consumption is either constant or declining when $\rho \geq r$.

3. The key Lemma 7 is established in subsection D.3. It establishes several differentiability properties, necessary for equation (80) following that lemma. That equation, together with the properties of the cumulative Lagrange multiplier and differentiability results provided by Lemma 7 in turn, is foundational for the derivatives-based analysis in the main body of the paper.

   Part 3 of the lemma establishes that consumption remains constant if the limited commitment constraint binds. That property and its proof is a central piece of the analysis and not trivial. It frequently resorts to a technique of splitting the future life of the contract into a short and immediate future of length Δ and the subsequent future history as in equation (78). That technique is more formally established and studied in subsection D.4. The proof of the lemma here builds on results established there, in particular the crucial principle of optimality in Lemma 9. From the perspective of mathematical logic, subsection D.4 precedes subsection D.3, but is only necessary for understanding the proof of Lemma 7 here. In the interest of readability, we therefore chose the current ordering.

4. Subsection D.4 proceeds to establish an equivalent recursive formulation (see definition 5). It proceeds by splitting the future into three parts as in equation (82). First, there is a short time interval $\Delta > 0$ without a state change. Second, there is the future beyond Δ and no state change until Δ. Finally, there are all of the first state changes before Δ and their continuation values. Lemma 9 establishes the principle of optimality, i.e., a key monotonicity result of the optimal contract: if the promised utility is higher, then consumption during the no-state-change epoch Δ as well as promised utility upon the first state change as well as the continuation utility beyond Δ will be higher. The proof is not entirely straightforward and requires a careful examination of inequalities and the Lagrange multipliers provided in subsection D.2. Almost as a by-product of the recursive formulation, we establish the Hamilton-Jacobi-Bellman, or HJB, equation in Proposition 14.

5. Subsection D.5 considers the dual perspective of maximizing utility, given costs.

Much of the properties here parallel the developments before, allowing us to be brief. Proposition 15 establishes equivalence. Proposition 16 provides the HJB equation.

C Mathematical Preliminaries

The purpose of this section is to provide a precise mathematical framework to describe the stochastic nature of consumption contracts in the next sections. It will turn out that we need to allow the contracts to depend on a bit more than just the history of productivities, see in particular the proof of Lemma 3 below. Furthermore, we provide these mathematical preliminaries for more than just two productivity states, in order to allow building on these in future work. The material here is not easily available elsewhere in concise form. The approach taken here and some of the material are in chapter 11 of Puterman (2005), though we need a bit more for the analysis in subsequent sections.

We assume throughout that there is a k-state Markov process for an underlying state $x(t) \in X = \{0, \ldots, k-1\}$ for each agent. The state for one agent evolves independently from that of any other agent and with constant Markov transition rates $\alpha_{i,j}$ from state $x = i$ to state $x = j$. We impose that $\alpha_{i,i} = -\sum_{j \neq i} \alpha_{i,j}$, so that α is an intensity matrix or infinitesimal generator matrix. We assume that there is a mapping $\mathbf{z} : X \to Z$ determining individual labor productivity $z = \mathbf{z}(x)$ if the individual state is s. Note that k may be larger than the cardinality of Z. The beginning of Section E provides the specific details for the case with two labor productivities used in the main text and an underlying state x that can take three values. This construction will be used in the proof of Lemma 1 (see subsection E.1).

Given dates $t < \tau < \infty$, let $x_0 = x(t)$ be the state at date $t_0 = t$. Suppose there are $n \geq 0$ switches between t and τ at switch dates $t_0 < t_1 \ldots < t_n \leq \tau$. Let $x(t_j) = x_j$ for $j > 0$ denote the new values of the state at these switching dates. The history of the state between time t and time $\tau > t$, denoted compactly as $h_{t,\tau}$, and explicitly given by

$$h_{t,\tau} = (\tau, n, t_0 = t, t_1, \ldots, t_n, x_0 = x, x_1, \ldots, x_n) \tag{57}$$

keeps track of all this information. The starting history at time t is $h_{t,t} = (t, 0, t, x)$, when the state is at $x(t) = x$ and by construction no state change has occurred yet. Generally, when $n = 0$, no switch occurs and the state remains at the initial state x_0 from t to τ. We

impose[23] the condition that $n < \infty$; i.e., we only examine histories between the two dates t and τ with finitely many switching dates. This is true with probability 1.

Given some history $h_{t,\tau}$ as in (57) and some $\Delta > 0$, define the time-shifted history

$$h^{\Delta}_{t,\tau} = (\tau + \Delta, n, t_0 = t + \Delta, t_1 + \Delta, \ldots, t_n + \Delta, x_0 = x, x_1, \ldots, x_n) \tag{58}$$

This construction will be of help in the proof of Lemma 7. Given some history $h_{t,\tau}$ between dates t and τ and any two intermittent dates s, s' with $t \leq s \leq s' \leq \tau$, it will be useful to construct the **history between** s **and** s' and denote it by $h_{t,\tau}(s, s')$. To do so, starting from (57), let $m = \text{argmax}\{j \mid t_j \leq s\}$ be the index of the last switching date before the date s. Likewise, let $m' = \text{argmax}\{j \mid t_j \leq s'\}$ be the index of the last switching date before the date s'. Therefore there are $m' - m$ state transitions between dates s and s'. Starting from the new initial date $\tilde{t}_0 = s$ rewrite the dates of these state transitions as $\tilde{t}_1 = t_{m+1}, \ldots, \tilde{t}_{m'-m} = t_{m'}$. Likewise, denote the states at these redefined transition dates as $\tilde{x}_0 = x_m, \ldots, \tilde{x}_{m'-m} = x_{m'}$. Using this construction we can now define the history $h_{t,\tau}(s, s')$ between s and s' implied by the history $h_{t,\tau}$ between t and τ as

$$h_{t,\tau}(s, s') = (s', m' - m, \tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_{m'-m}, \tilde{x}_0, \ldots, \tilde{x}_{m'-m})$$

The most relevant purpose of this construction is to split the history $h_{t,\tau}$ into two non-overlapping parts $h_{t,\tau}(t, s)$ and $h_{t,\tau}(s, \tau)$, where $t \leq s \leq \tau$, or conversely, define a **concatenated history** $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ by gluing two histories $h_{t,s}$ and $h_{s,\tau}$

$$\begin{aligned} h_{t,s} &= (s, m, t^a_0 = t, t^a_1, \ldots, t^a_m, x^a_0, x^a_1, \ldots, x^a_m) \\ h_{s,\tau} &= (\tau, n, t^b_0 = s, t^b_1, \ldots, t^b_n, x^b_0, x^b_1, \ldots, x^b_n) \end{aligned}$$

together. This construction requires that the state x^a_m at the end of history $h_{t,s}$ equals the state $x^b_0 = x^a_m$ at the beginning of history $h_{s,\tau}$.[24] Note that the last switch date before or including s is the date t^a_m, while the first subsequent switch date is t^b_1. The date s itself drops out from the switching date history, and is contained in the interval $t^a_m \leq s \leq t^b_1$. The

[23] Or better: we work with a subset of the probability space, so that this is true.

[24] We stipulate that this condition must be satisfied whenever the notation for concatenation is utilized. A more explicit notation would be cumbersome.

concatenated history is then given explicitly as

$$\begin{aligned} h_{t,\tau} &= [h_{t,s}, h_{s,\tau}] \\ &= (\tau, m+n, t_0 = t, t_1 = t_1^a, \ldots, t_m = t_m^a, t_{m+1} = t_1^b, \ldots, t_{m+n} = t_n^b, \\ &\qquad x_0^a, x_1^a, \ldots, x_m^a, x_1^b, \ldots, x_n^b) \end{aligned}$$

In particular, note that

$$h_{t,\tau} = [h_{t,\tau}(t,s), h_{t,\tau}(s,\tau)]$$

for $t \leq s \leq \tau$.

Let $\mathcal{H}_{t,\tau}(x)$ be the set of all possible histories $h_{t,\tau}$ between two given dates $\tau \geq t$, starting at $x_0 = x$. Let $\mathcal{H}_t(x)$ be their union across τ, i.e. the set of all histories $h_{t,\tau}$ for any date $\tau \geq t$, given t, and starting at $x_0 = x$. Let $\mathcal{H}_t$ be the unions of all $\mathcal{H}_t(x)$ across all $x \in X$. The transition rates $\alpha_{i,j}$ deliver a probability measure $P_{t,\tau}$ on $\mathcal{H}_{t,\tau}(x)$ for histories $h_{t,\tau}$ between two dates t and τ, given by

$$dP_{t,\tau}(h_{t,\tau}) = \exp\left((\tau - t_n)\alpha_{x_n,x_n}\right) \prod_{j=1}^{n} \exp\left((t_j - t_{j-1})\alpha_{x_{j-1},x_{j-1}}\right) \alpha_{x_{j-1},x_j} dt_j \tag{59}$$

where the dt_j are to be arranged in the sequence $dt_n\, dt_{n-1} \ldots dt_1$, when writing this out explicitly. This is important for appropriately writing the integral in equation (61). Note that a history $h_{t,\tau} = (\tau, 0, t, x)$ with $n = 0$ and thus without transitions has the point mass

$$dP_{t,\tau}(\tau, 0, t, x) = \exp((\tau - t)\alpha_{x,x})$$

More generally, and as an arbitrary example, consider a history $h_{0,3}$ between the two dates $t = 0$ and $\tau = 3$ and two switching dates, given by

$$h_{0,3} = (\tau = 3, n = 2, t_0 = 0, t_1 = 1.3, t_2 = 2.3, x_0 = 0, x_1 = 1, x_2 = 0)$$

The probability for this history is

$$\begin{aligned} dP_{t,\tau}(h_{0,3}) &= \exp((3 - 2.3)\alpha_{0,0} + (2.3 - 1.3)\alpha_{1,1} + (1.3 - 0)\alpha_{0,0})\alpha_{1,0} dt_2 \alpha_{0,1} dt_1 \\ &= \alpha_{0,1}\alpha_{1,0} \exp(2\alpha_{0,0} + \alpha_{1,1}) dt_2 dt_1 \end{aligned}$$

Note that[25]

$$P_{t,\tau}(h_{t,\tau}) = P_{t,s}(h_{t,\tau}(t,s))P_{s,\tau}(h_{t,\tau}(s,\tau)) \tag{60}$$

Write P_t for the overall probability measure $P_t(h_{t,\tau}) = P_{t,\tau}(h_{t,\tau})$. While not essential, this also allows a precise construction of a suitable probability space. Formally, let $\mathcal{H}_t(x)$ be the set of underlying events. Note that $\mathcal{H}_t(x)$ can be written as the countable union of the sets

$$\begin{aligned} H_{t,n}(x) &= \{(\tau, n, t, t_1, \ldots, t_n, x, x_1, \ldots, x_n) \mid \\ &\qquad (\tau, t_1 \ldots, t_n, x_1, \ldots, x_n) \in \mathbf{R}^{n+1} \times X^n, t < t_1 \ldots < t_n \leq \tau\} \end{aligned}$$

of $\mathbf{R}^{n+1} \times X^n$, $n \geq 0$. The sets $H_{t,n}(x)$ have the usual Borel-σ-algebra[26] of subsets, which we shall denote with $\mathcal{B}_{t,n}(x)$. Their union $\mathcal{B}_t(x)$ is the Borel-σ-algebra of the measurable subsets of $\mathcal{H}_t(x)$. With the probability measure P_t defined above, $(\mathcal{H}_t(x), \mathcal{B}_t(x), P_t)$, becomes a probability space.

The set $\mathcal{S}(x)$ of stochastic processes on $\mathcal{H}_t(x)$ is the set of measurable functions from $\mathcal{H}_t(x)$ to the real line,

$$\mathcal{S}(x) = \{\mathbf{f} : \mathcal{H}_t(x) \to \mathbf{R} \mid \mathbf{f}^{-1}([a,b]) \in \mathcal{B}_t(x) \text{ for } a, b \in \mathbf{R}, a \leq b\}$$

The value $f(\tau) = \mathbf{f}(h_{t,\tau})$ is the value of the stochastic process $\mathbf{f}$ at date τ, given the history up to and including τ. (Re-)define the stochastic process $\mathbf{x} \in \mathcal{S}(x)$ as the mapping defined by $\mathbf{x}(h_{t,\tau}) = x_n$ for $h_{t,\tau}$ as described in (57). Proceeding this way, the number of switches between two dates is finite by construction. The stochastic process $\mathbf{x}$ generates the σ-algebra $\mathcal{B}_t(x)$.

Equipped with these probabilities, one can define expectations and conditional expectations. For example, the expectation of the stochastic process $f(\tau)$ at some date $\tau > t$ and

[25] One could start from a description of the probability law, imposing this consistency condition and some other mild assumptions, allowing us to move beyond the Markov structure imposed in (59). We do not pursue this here.

[26] It is generated by the Cartesian products of open subsets of $\mathbf{R}^{n+1}$ with any subset of X.

$f \in \mathcal{S}(x)$ amounts to integration with respect to $P_{t,\tau}$ over the set $\mathcal{H}_{t,\tau}(x)$. It is given by

$$\begin{aligned} E[f(\tau) \mid x(t) = x] &= \int_{\mathcal{H}_{t,\tau}(x)} \mathbf{f}(h_{t,\tau}) dP_{t,\tau} \\ &= \sum_{n=0}^{\infty} \sum_{(x_1,\ldots,x_n)\in X^n} \int_{t_1=t}^{t_1=\tau} \int_{t_2=t_1}^{t_2=\tau} \cdots \int_{t_n=t_{n-1}}^{t_n=\tau} \mathbf{f}(\tau, n, t, t_1, t_2, \ldots, t_n, x, x_1, \ldots, x_n) \\ &\qquad dP_{t,\tau}(\tau, n, t, t_1, t_2, \ldots, t_n, x, x_1, \ldots, x_n) \end{aligned} \tag{61}$$

As an arbitrary example with $t = 0, \tau = 3$ and $x(0) = 0$, the term for $n = 2$ and $x_1 = 1, x_2 = 0$ becomes

$$\int_0^3 \int_{t_1}^3 \mathbf{f}(3, 2, 0, t_1, t_2, 0, 1, 0)\alpha_{0,1}\alpha_{1,0} \exp((2 - t_2 + t_1)\alpha_{0,0} + (t_2 - t_1)\alpha_{1,1}) dt_2 dt_1,$$

integrating the value of the stochastic process $f(3)$ over all histories from $t = 0$ to $\tau = 3$ with exactly $n = 2$ switches from $x_0 = 0$ to $x_1 = 1$ and back to $x_2 = 0$ across all the switching dates.

D Analysis of the Optimal Risk-Sharing Contract

We assume $0 < r \leq \rho$ throughout this section.

D.1 Basic Properties

Given the apparatus of Appendix C, we can restate definition 1 more formally. We impose the condition that a consumption contract starting at date t is a mapping from histories in $\mathcal{H}_t$, starting state x and promised utility levels U to the real line, with $\mathbf{c}(h_{t,\tau}; x, U)$ the amount of consumption at date τ, after observing the history $h_{t,\tau}$. The resulting stochastic process[27] $c(\tau; x, U) = \mathbf{c}(h_{t,\tau}; x, U)$ is adapted to the stochastic process $(x_\tau)_{\tau \geq t}$. Note that this allows for processes that are functions of the last switch date t_n, the current date τ as well as the current state $x_n = x(\tau)$. These suffice to describe the contracts in the main body of the paper. Here we repeat this definition, but now we use the notation of Appendix C.

Definition 3 (contracts). *For fixed outside options $U^{out}(z)$, with $z \in Z$ and a starting date*

[27] Formally, given τ, x and U, $c(\tau; x, U)$ is a random variable, mapping $\mathcal{H}_{t,\tau}(x)$ into $\mathbf{R}$.

t, let $\mathcal{U}(x) = \left[U^{out}(\mathbf{z}(x)), \frac{\bar{u}}{\rho}\right)$ *for* $x \in X$ *and let* $\mathcal{C}$ *be the set of all consumption contracts,*

$$\mathcal{C} = \{\mathbf{c} \mid \mathbf{c} : \{(h_{t,\tau}; x, U) \mid x \in X, h_{t,\tau} \in \mathcal{H}_t(x), U \in \mathcal{U}(x)\} \to \mathbf{R}_+\} \tag{62}$$

Definition 4 (**cost-minimizing contracts).** *For a fixed wage* w *and rate of return on capital or interest rate* r*, an* **optimal consumption insurance contract** $\mathbf{c} \in \mathcal{C}$ *and the* **cost function** $V : \{(x, U) \mid x \in X, U \in \mathcal{U}(x)\} \to \mathbf{R}$ *solve*

$$V(x, U) = \min_{\mathbf{c}\in\mathcal{C}} \int_t^\infty \int_{\mathcal{H}_{t,\tau}(x)} e^{-r(\tau-t)} \left[w\mathbf{c}(h_{t,\tau}; x, U) - w\mathbf{z}(\mathbf{x}(h_{t,\tau}))\right] dP_{t,\tau} d\tau \tag{63}$$

subject to the **promise keeping constraint**

$$\int_t^\infty \int_{\mathcal{H}_{t,\tau}(x)} e^{-\rho(\tau-t)} u(w\mathbf{c}(h_{t,\tau}; x, U)) dP_{t,\tau} d\tau \quad \geq \quad U \tag{64}$$

and the **limited commitment constraints**

$$\int_s^\infty \int_{\mathcal{H}_{s,\tau}(x(h_{t,s}))} e^{-\rho(\tau-s)} u(w\mathbf{c}([h_{t,s}, h_{s,\tau}]; x, U)) dP_{s,\tau} d\tau \quad \geq \quad U^{out}(\mathbf{z}(\mathbf{x}(h_{t,s})))$$
$$\text{for all } s > t \text{ and } h_{t,s} \in \mathcal{H}_{t,s}(x) \tag{65}$$

for all $x \in X$ *and* $U \in \mathcal{U}(x)$*.*

Lemma 3 (**monotonicity of the cost function).** $V(x, U)$ *is increasing in* U*. It is strictly increasing in* U*, if (64) binds.*

Proof of Lemma 3. Consider two levels $U = U^A$ and $U = U^B$, with $U^A > U^B$. Any consumption contract that satisfies (64) for $U = U^A$ as well as (65) also satisfies (64) for $U = U^B$ as well as (65). This is true in particular for the cost-minimizing contract at $U = U^A$. Thus, the costs for U^B cannot be larger, that is, $V(x, U^B) \leq V(x, U^A)$.

Now suppose that $V(x, U^B) = V(x, U^A)$. That means that utility level $U = U^A$ could have been delivered for the same costs when a contract delivering $U = U^B$ is sought; that is, (64) could not have been binding at $U = U^B$, and thus $V(x, U^B) < V(x, U^A)$; that is, V is strictly increasing in U as along as (64) is binding at $U = U^B$. □

Lemma 4 (**convexity of the cost function).** $V(z, U)$ *is convex in* U*. It is strictly convex in* U*, if (64) binds.*

Proof of Lemma 4. Let $\lambda \in [0,1]$. Consider two levels $U = U^A$ and $U = U^B$. Suppose w.l.o.g. that $U^B < U^A$. Now consider the linear combination $\mathbf{c}^\lambda = \lambda \mathbf{c}^A + (1-\lambda)\mathbf{c}^B$. Since $u(\cdot)$ is concave, $\mathbf{c}^\lambda$ will satisfy constraint (65), since $\mathbf{c}^A$ and $\mathbf{c}^B$ both do. Let $V(\mathbf{c}^\lambda)$ be the costs of the contract $\mathbf{c}^\lambda$ per the right hand side of equation (92). Likewise, let $U(\mathbf{c}^\lambda)$ be the present value of the utility of the contract $\mathbf{c}^\lambda$ per the right hand side of equation (65). Note that $V(\mathbf{c}^\lambda) = \lambda V(x, U^A) + (1-\lambda)V(x, U^B)$. Since $u(\cdot)$ is strictly concave, $U(\mathbf{c}^\lambda) \geq U^\lambda$. The inequality is strict, if $\mathbf{c}^B \neq \mathbf{c}^A$ on a set of positive measure. Thus, $V(\mathbf{c}^\lambda) \geq V(x, U^\lambda)$ and strictly so, if (64) binds for U^A, per Lemma 3.

If (64) binds for U^B, it also binds for U^A and $U^\lambda = \lambda U^A + (1-\lambda)U^B$. Let $\mathbf{c}^A = \mathbf{c}(\cdot; x, U^A)$ and $\mathbf{c}^B = \mathbf{c}(\cdot; x, U^B)$ be the parts of the optimal consumption contract solving the cost minimization problems for (x, U^A) and (x, U^B). If (64) binds for U^B, then $\mathbf{c}^A$ cannot be the solution for U^B, i.e. $\mathbf{c}^B \neq \mathbf{c}^A$ on a set of positive measure. □

Lemma 5 (differentiability of the cost function). *The cost function $V(x,U)$ is continuous. It is differentiable in U on the right and left.*

Proof of Lemma 5. Continuity and right- as well as left-differentiability follow from the concavity of $V(x,\cdot)$ for interior points. Continuity and differentiability to the right at the lower bound $U^{out}(\mathbf{z}(x))$ follow, because $V(x,\cdot)$ is increasing and convex. □

We shall denote the right-hand side derivatives and the left-hand side derivatives of V with V'_+ and V'_-, Let

$$V'_+(x,U) = \lim_{h>0, h\to 0} \frac{V(x,U+h) - V(x,U)}{h},\ V'_-(x,U) = \lim_{h<0, h\to 0} \frac{V(x,U+h) - V(x,U)}{h}. \tag{66}$$

For $U = U^{out}(\mathbf{z}(\mathbf{x}))$, we define the left-hand side derivative as $V'_-(U^{out}(\mathbf{z}(\mathbf{x}))) = \mathbf{0}$, since $V(x,\cdot)$ is an increasing function.

D.2 A Lagrangian Approach

The analysis of the optimal contract and the derivation of the relevant first-order conditions follows in spirit the approach of Marimon and Marcet (2019) (see also Golosov et al. (2016) as well as the generalization of Marcet-Marimon (2019) to continous-time heterogeneous-agent settings and the introduction of "timeless penalties" in Dàvila and Schaab (2022a)). We first restate the optimization problem as a Lagrangian in this subsection, provide a

recursive perspective in subsection D.4 and then establish some key properties in subsection D.3.

As a first pass at the problem and for notational clarity, we shall drop the explicit history dependence and conditioning information in the constraint (3). Heuristically, let ζ be the Lagrange multiplier on (2) and let $\mu(s)$ be the Lagrange multiplier on (3). Integrating the constraints (3) discounted at $e^{-\rho(s-t)}$, the Lagrangian would then be

$$\begin{aligned} L &= \mathbf{E}\left[\int_t^\infty e^{-r(\tau-t)}\left(wc(\tau)-wz(\tau)\right)d\tau \right. \\ &\quad -\zeta\left(\int_t^\infty e^{-\rho(\tau-t)}u(wc(\tau))d\tau - U\right) \\ &\quad \left. -\int_t^\infty e^{-\rho(s-t)}\mu(s)\left(\int_s^\infty e^{-\rho(\tau-s)}u(wc(\tau))d\tau - U^{out}(z(s))\right)ds\right] \end{aligned} \tag{67}$$

This can be rewritten as

$$\begin{aligned} L &= \mathbf{E}\left[\int_t^\infty e^{-r(\tau-t)}\left[wc(\tau)-wz(\tau)\right]d\tau \right. \\ &\quad -\int_t^\infty \lambda(\tau)e^{-\rho(\tau-t)}u(wc(\tau))d\tau \\ &\quad \left. +\lambda(t)U+\int_t^\infty e^{-\rho(s-t)}U^{out}(z(s))d\lambda(s)\right] \end{aligned} \tag{68}$$

provided that

$$\lambda(\tau) = \zeta + \int_t^\tau \mu(s)ds \tag{69}$$

We shall call $\lambda(\cdot)$ the cumulative Lagrange multiplier. We proceed with (68) as the Lagrangian function without imposing that $\lambda(\tau)$ is differentiable or even continuous. We drop the multipliers $\mu(s)$ from the problem, though we keep the restriction from (69) that $\lambda(\tau)$ is a weakly increasing and nonnegative function of τ and that it is only increasing, if (3) binds. Given a path for λ up to date s, the integral with respect to $d\lambda(s)$ on the last line of (68) is a Riemann-Stieltjes integral, given the state history up to s.

$\lambda(\tau)$ is an adapted stochastic process and depends on the history of the state. To be more precise, we build on the formulation in definition 4. Let ζ be the Lagrange multiplier on (64) and let $\boldsymbol{\mu}(h_{t,s})$ be the Lagrange multiplier on (65). Integrating the constraints (65) discounted at $e^{-\rho(s-t)}$, the Lagrangian is

$$
\begin{aligned}
L &= \int_t^\infty \int_{\mathcal{H}_{t,\tau}} e^{-r(\tau-t)} \left(w\mathbf{c}(h_{t,\tau}) - w\mathbf{z}(\mathbf{x}(h_{t,\tau}))\right) dP_{t,\tau} d\tau \qquad (70)\\
&\quad -\zeta \left(\int_t^\infty \int_{\mathcal{H}_{t,\tau}} e^{-\rho(\tau-t)} u(w\mathbf{c}(h_{t,\tau})) dP_{t,\tau} d\tau - U \right)\\
&- \int_t^\infty \int_{\mathcal{H}_{t,s}} e^{-\rho(s-t)} \boldsymbol{\mu}(h_{t,s}) \left(\int_s^\infty \int_{\mathcal{H}_{s,\tau}} e^{-\rho(\tau-s)} u(w\mathbf{c}([h_{t,s}, h_{s,\tau}])) dP_{s,\tau} d\tau - U^{out}(\mathbf{z}(\mathbf{x}(h_{t,s}))) \right) dP_{t,s} ds
\end{aligned}
$$

This can be rewritten as

$$
\begin{aligned}
L &= \int_t^\infty \int_{\mathcal{H}_{t,\tau}} e^{-r(\tau-t)} \left[w\mathbf{c}(h_{t,\tau}) - w\mathbf{z}(\mathbf{x}(h_{t,\tau}))\right] dP_{t,\tau} d\tau \qquad (71)\\
&\quad - \int_t^\infty \int_{\mathcal{H}_{t,\tau}} \boldsymbol{\lambda}(h_{t,\tau}) e^{-\rho(\tau-t)} u(w\mathbf{c}(h_{t,\tau})) dP_{t,\tau} d\tau\\
&\quad + \boldsymbol{\lambda}(h_{t,t}) U + \int_{s=t}^\infty \int_{\mathcal{H}_{t,s}} e^{-\rho(s-t)} U^{out}(\mathbf{z}(\mathbf{x}(h_{t,s}))) dP_{t,s} \times d\lambda
\end{aligned}
$$

provided[28] that **the cumulative Lagrange multiplier** is given by

$$
\boldsymbol{\lambda}(h_{t,\tau}) = \zeta + \int_t^\tau \boldsymbol{\mu}(h_{t,\tau}(t,s)) ds \qquad (72)
$$

The cumulative Lagrange multiplier reformulation in (71) provides a version of Marcet and Marimon (2019) in continuous time. We proceed with (71) as the Lagrangian function without imposing that the mapping $\tau \mapsto \lambda(h_{t,s}[t,\tau])$ for some $h_{t,s}$ and $t \leq \tau \leq s$ is differentiable or even continuous, but keep the restriction from (72) that these mappings are weakly increasing and nonnegative and that they are only increasing, if (3) binds.

Differentiating the Lagrangian (68) with respect to $c(\tau)$ resp (71) with respect to $\mathbf{c}(h_{t,\tau}; x, U)$ yields the first-order condition

$$
e^{(\rho-r)(\tau-t)} = \lambda(\tau) u'(wc(\tau)) \qquad (73)
$$

[28]Regarding the measure $dP_{t,s} \times d\lambda$ for the double integral $s \in [t,\infty)$, $h_{t,s} \in \mathcal{H}_{t,s}$, use (61), replace τ with s, and integrate over $s \in [t,\infty)$. As one of the terms, consider $n = 2$ and $x_0 = 2, x_1 = 0, x_2 = 1$. Exchange the order of integration and calculate $\int_{t_1=t}^\infty \int_{t_2=t_1}^\infty \int_{s=t_2}^\infty f(t_1,t_2,s) d\lambda(s,2,t,t_1,t_2,2,0,1) dt_2 dt_1$, where $f(t_1,t_2,s)$ collects the remaining terms and where the integral with respect to $d\lambda(s,2,t,t_1,t_2,2,0,1)$ is a Riemann-Stieltjes integral, using the weakly increasing and possibly discontinuous function $s \mapsto \lambda(s,2,t,t_1,t_2,2,0,1)$ of $s \geq t_2$. Proceed likewise with all other terms. We drop a further discussion, as the integral with respect to $dP_{t,s} \times d\lambda$ is a constant and drops out in the first-order conditions.

or

$$e^{(\rho-r)(\tau-t)} = \boldsymbol{\lambda}(h_{t,\tau})u'(w\mathbf{c}(h_{t,\tau}; x, U)) \tag{74}$$

Lemma 6 (consumption is not increasing). $\mathbf{c}(h_{t,\tau}; x, U)$ *is decreasing at* τ *for* $r < \rho$ *and constant for* $r = \rho$*, when the limited commitment constraint (65) does not bind at* τ*.*

Proof. In that case, the Lagrange multiplier $\boldsymbol{\lambda}(h_{t,\tau})$ remains constant. The claim now follows from (74) and $u'' < 0$. □

D.3 Key Properties of the Optimal Contract

From here on, we drop the dependency on x and U in the consumption contract, in order to save on notation.

For the next result, the following assumption is helpful.

Assumption 3 (bounded risk aversion). *The utility function* $u(\cdot)$ *satisfies*

$$0 < -\frac{u''(x)x}{u'(x)} < \bar{\sigma} < \infty \tag{75}$$

for all $x > 0$ *and some* $\bar{\sigma}$*.*

Assumption 3 is obviously satisfied for all nonlinear CRRA utility functions. For $\mathbf{c}$ and a given history $h_{t,s}$, $s > t$, define the left limit

$$\mathbf{c}_-(h_{t,s}) = \lim_{\Delta\to 0, \Delta>0} \mathbf{c}(h_{t,s}(t, s-\Delta))$$

and likewise the left derivative

$$\dot{\mathbf{c}}_-(h_{t,s}) = \lim_{\Delta\to 0, \Delta>0} \frac{\mathbf{c}(h_{t,s}) - \mathbf{c}(h_{t,s}(t, s-\Delta))}{\Delta},$$

provided they exist. For the right limit, observe that $\mathbf{c}(h_{t,s+\Delta})$ is stochastic, and a probabilistic limit needs to be taken. Since the probability of a state change within the time interval from s to $s + \Delta$ vanishes, the probabilistic limit is equal to the limit, when only the future histories without a state change are taken into account. Formally, let $x(s)$ be the current state, $x(s) = \mathbf{x}(h_{t,s}) = x_n$. The extension of the current history without a state change until Δ or **no-state-change history extension** can be written as the concatenated history

$$h_{t,s;\Delta} = [h_{t,s}, (s + \Delta, 0, s, x(s))].$$

Define[29]

$$\mathbf{c}_+(h_{t,s}) = \lim_{\Delta \to 0, \Delta \geq 0} \mathbf{c}(h_{t,s;\Delta})$$

and likewise the right derivative

$$\dot{\mathbf{c}}_+(h_{t,s}) = \lim_{\Delta \to 0, \Delta > 0} \frac{\mathbf{c}(h_{t,s;\Delta}) - \mathbf{c}(h_{t,s})}{\Delta},$$

provided they exist. Continuity as well as differentiability at a given history $h_{t,s}$, $s > t$ are defined, when the left and right limits exist and coincide. The derivative in that case will be denoted with $\dot{\mathbf{c}}(h_{t,s})$ or simply $\dot{c}(s)$. We proceed likewise for $\boldsymbol{\lambda}$.

Lemma 7 (key properties of the contract). *1. Suppose that the constraint* (65) *does not bind at history $h_{t,s}$. Then $\dot{\boldsymbol{\lambda}}(h_{t,s})_+ = 0$ and the derivative $\dot{\mathbf{c}}_+(h_{t,s})$ exists. If the last jump occurred strictly before date s, i.e., if $t_n < s$, then $\dot{\boldsymbol{\lambda}}(h_{t,s}) = 0$ and $\mathbf{c}$ is differentiable at $h_{t,s}$.*

2. *Suppose $\mathbf{c}$ is differentiable at history $h_{t,s}$. Then $\boldsymbol{\lambda}$ is differentiable at history $h_{t,s}$ and*

$$\rho - r = \left(\frac{u''(wc)wc}{u'(w\mathbf{c}(h_{t,s}))} \right) \frac{\dot{\mathbf{c}}(h_{t,s})}{\mathbf{c}(h_{t,s})} + \frac{\dot{\boldsymbol{\lambda}}(h_{t,s})}{\boldsymbol{\lambda}(h_{t,s})} \tag{76}$$

   The statement and equation likewise hold for the left-derivatives, if $\mathbf{c}$ is left-differentiable at history $h_{t,s}$, and for the right-derivatives, if $\mathbf{c}$ is right-differentiable at history $h_{t,s}$.

3. *Suppose that the limited commitment constraint (65) binds at history $h_{t,s}$. Suppose that $\rho = r$. Alternatively, suppose that $\rho > r$ and that Assumption 3 holds. Then $\mathbf{c}(h_{t,s;\Delta})$ is constant in $\Delta \geq 0$ and $\dot{\mathbf{c}}_+(h_{t,s}) = 0$.*

4. *$\boldsymbol{\lambda}_-(h_{t,s}) \neq \boldsymbol{\lambda}(h_{t,s})$, i.e. $\boldsymbol{\lambda}$ is discontinuous at history $h_{t,s}$, if and only if $\mathbf{c}_-(h_{t,s}) \neq \mathbf{c}(h_{t,s})$. In that case, $\mathbf{c}_-(h_{t,s}) < \mathbf{c}(h_{t,s})$, $\boldsymbol{\lambda}_-(h_{t,s}) < \boldsymbol{\lambda}(h_{t,s})$ and* (65) *binds at history $h_{t,s}$ with $t_n = s$, i.e. just when the state change occurred.*

The proof of the lemma builds on techniques and results for the subsequent subsection D.4 and the recursive formulations there. In terms of mathematical logic, that subsection precedes rather than builds on the material here. Since it is only necessary for understanding the proof here, we chose the current ordering in the interest of readability.

[29] Note that we use $\Delta \geq 0$ for defining $\mathbf{c}_+(h_{t,s})$, as the processes $c(s) = \mathbf{c}(h_{t,s})$ may generally often be cadlag, i.e. right-continuous, but with a left limit.

Proof. 1. If the constraint does not bind at $h_{t,s}$, then it will not bind either for the no-state-change history extensions $h_{t,s;\Delta}$, provided $\Delta > 0$ is sufficiently small. Thus, $\boldsymbol{\lambda}(h_{t,s}) = \boldsymbol{\lambda}(h_{t,s;\Delta})$ is locally constant[30] and thus $\dot{\boldsymbol{\lambda}}_+(h_{t,s}) = 0$. The existence of $\dot{\mathbf{c}}_+$ at $h_{t,s}$ now follows from (73). If $t_n < s$, the same argument applied to the truncated histories $h_{t,s}(t, s-\Delta)$ shows that $\dot{\boldsymbol{\lambda}}(h_{t,s}) = 0$ and the differentiability of $\mathbf{c}$ at $h_{t,s}$.

2. Differentiation of (74) with respect to τ shows that λ is also differentiable[31] at τ and delivers (76), when replacing τ with s.

3. By assumption, the limited commitment constraint (65) binds at $h_{t,s}$. For ease of notation, write x for $x(s)$ and z for $\mathbf{z}(x)$. Consider the no-state-change history extension $h_{t,s;\Delta} = [h_{t,s}, (s+\Delta, 0, s, x)]$ for $\Delta > 0$. The proof proceeds in two parts. For part A, suppose that the limited commitment constraint (65) binds again at $h_{t,s;\bar{\Delta}}$ for some $\bar{\Delta} > 0$. We use an averaging argument to establish that consumption must be the same at s and at $s+\bar{\Delta}$. With the help of the first two parts as well as some careful analysis, we then show that the limited commitment constraint (65) must bind for all $h_{t,s;Delta}$ and $0 < \Delta \leq \bar{\Delta}$ and thus establish the claim for part A. For part B, suppose that the constraint (65) never binds again for $h_{t,s;\Delta}$ at any $\Delta > 0$. We show that this leads to a contradiction.

   A. For the first part, suppose that the limited commitment constraint (65) binds at $h_{t,s}$ as well as at $h_{t,s;\bar{\Delta}}$ for some (possibly large) $\bar{\Delta} > 0$. Compare the contract going forward conditional on these two histories: we will argue that one can do better by averaging them, should they be different. To that end and to express this precisely, let $h_{s,\tau}$ be some continuation at the current state x of the history $h_{t,s}$ to $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$: for a graphical illustration, see Figure 7. Construct the corresponding continuation $h_{t,\tau+\bar{\Delta}} = [h_{t,s;\bar{\Delta}}, h^{\bar{\Delta}}_{s,\tau}]$ of $h_{t,s;\bar{\Delta}}$ with the $\bar{\Delta}$-time-shifted history $h^{\bar{\Delta}}_{s,\tau}$ (see equation (58)). This correspondence is one-one and measure preserving. Suppose now that the contract $\mathbf{c}(h_{t,\tau})$ differs from the corresponding $\mathbf{c}(h_{t,\tau+\bar{\Delta}})$ for a set $\mathcal{S}$ of extensions $h_{s,\tau}$ with positive measure, i.e., suppose that $\int_{\tau=s}^{\infty} \int_{\mathcal{H}_{s,\tau}} 1_{h_{s,\tau}\in\mathcal{S}} dP_{s,\tau} d\tau > 0$. Consider then a new contract, which is the average between the original contract and the contract following

[30] Returning to our original Lagrange multipliers, $\boldsymbol{\mu}(h_{t,s;\Delta}) = 0$ for $\Delta \geq 0$ sufficiently small.

[31] This is a standard calculus argument.

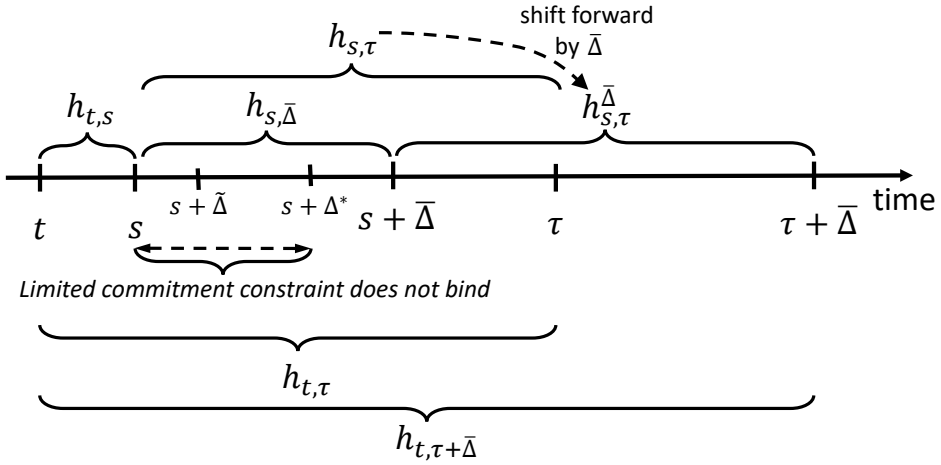

Figure 7: Timeline for part 3.A of the proof for Lemma 7. Starting point is the history $h_{t,s}$. Consider a time interval $\bar{\Delta}$ and the history extended to $h_{t,s;\Delta}$ without a state change between s and $s+\bar{\Delta}$. Suppose that the limited commitment constraint (65) binds at $h_{t,s}$ as well as at $h_{t,s;\bar{\Delta}}$. Consider some $\tau > s$ and history $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ coinciding with $h_{t,s}$ until s: there may be state changes at several points between s and τ. Shift the continuation piece $h_{s,\tau}$ forward by $\bar{\Delta}$ and append it to the history $h_{t,s;\bar{\Delta}}$. Consider the original contract for $h_{t,\tau}$ and the contract for this shifted-and-appended history. Averaging the original contract and the shifted contract as in (77) shows that consumption must be the same at s and $s+\bar{\Delta}$. Hence, if (65) binds for all $\bar{\Delta} > 0$, we'd be done: consumption would need to be constant, while the state does not change. Thus, suppose that for some Δ^* that the limited commitment constraint (65) does not bind at the no-change histories $h_{t,s;\Delta}$ for all $0 < \Delta < \Delta^*$. In part 3.A.i and 3.A.ii of the proof and illustrated in Figure 8, we show that this leads to a contradiction.

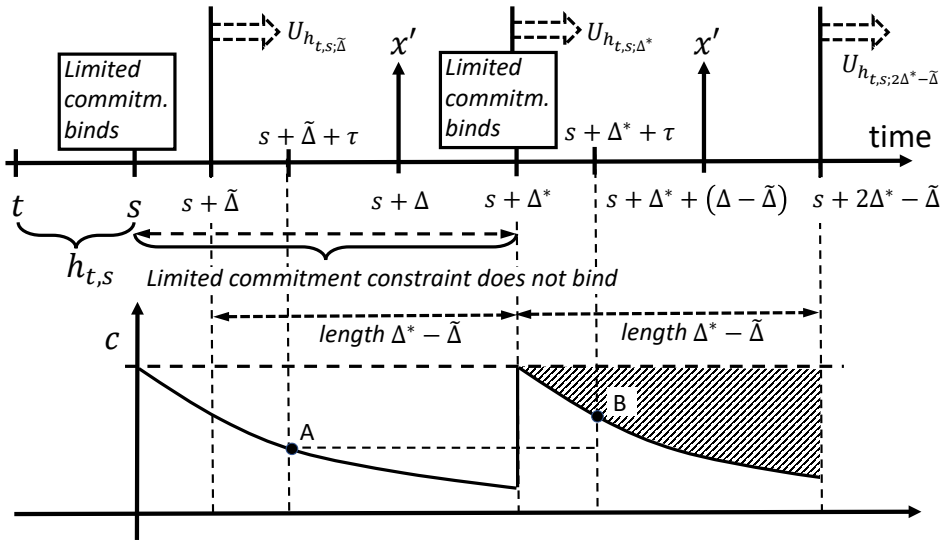

Figure 8: Timeline for part 3.A.ii of the proof for Lemma 7. This zooms in on a portion of Figure 7 and extends it with some additional detail. Suppose the limited commitment constraint (65) binds at $h_{t,s}$ as well as at $h_{t,s;\Delta^*}$, where Δ^* is chosen as small as possible. That means, that consumption must be declining between s and $s+\Delta^*$, when $r<\rho$, and jumps back up at $s+\Delta^*$ to the consumption level at s. Compare now the continuation utility $U_{h_{t,s;\tilde{\Delta}}}$ at some $s+\tilde{\Delta}$, $0<\tilde{\Delta}<\Delta^*$ to the continuation utility $U_{h_{t,s;\Delta^*}}$ at $s+\Delta^*$. Since the limited commitment constraint does not bind at $s+\tilde{\Delta}$, that continuation utility must be higher there than at $s+\Delta^*$. However, along the no-change-in-state, consumption at every $s+\tilde{\Delta}+\tau$ is smaller than at $s+\Delta^*+\tau$ as long as $\tau<\Delta^*-\tilde{\Delta}$. One can see this by comparing points A and B, where the shaded area indicates the range of consumption values beyond $s+\Delta^*$ and the lower bound results, if the outside option does not bind between $s+\Delta^*$ and $s+2\Delta^*-\tilde{\Delta}$. Furthermore, $U_{h_{t,s;\Delta^*}}=U^{out}(\mathbf{z}(x))<U_{h_{t,s;2\Delta^*-\tilde{\Delta}}}$. The principle of optimality of Lemma 9 delivers the result that this cannot be "compensated" for by the state change points, comparing $s+\Delta$ to $s+\Delta^*+(\Delta-\tilde{\Delta})$ or comparing the continuation utility at $s+\Delta^*$ (as a portion for the $s+\tilde{\Delta}$ calculation) to the continuation utility at $s+2\Delta^*-\tilde{\Delta}$ (as a portion for the $s+\Delta^*$ calculation). This is a contradiction.

$h_{t,s;\bar{\Delta}}$, i.e., consider

$$\tilde{\mathbf{c}}(h_{t,\tau}) = \left(\mathbf{c}([h_{t,s}, h_{s,\tau}]) + \mathbf{c}([h_{t,s;\bar{\Delta}}, h^{\bar{\Delta}}_{s,\tau}])\right)/2 \tag{77}$$

defined for all continuations $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ of $h_{t,s}$. In words, $\tilde{\mathbf{c}}$ is the average between the current contract as well as the contract portion following $h_{t,s;\bar{\Delta}}$ shifted backward by $\bar{\Delta}$. Since utility is strictly concave, this contract now delivers strictly higher continuation utility at history $h_{t,s}$, while its costs stay unchanged, a contradiction to the hypothesis, that the constraint (65) binds at $h_{t,s}$, i.e. a contradiction to the assertion that the original contract was cost-minimizing, see lemma 3. It follows that consumption at $s+\bar{\Delta}$ will be the same as at s for any $\bar{\Delta} > 0$, where (65) binds: let us denote that consumption level as c.

If the limited commitment constraint (65) binds for all $0 < \tilde{\Delta} < \bar{\Delta}$, we would be done with this part, since consumption would then be constant at $\mathbf{c}(h_{t,s;\tilde{\Delta}}) \equiv c$. Indeed, we would be done if this is true for some sufficiently small $\bar{\Delta} > 0$, since it must then be true for all $\bar{\Delta}$ per "shifting" the contract by $\bar{\Delta}/2$ into the future. Suppose thus that (65) binds at some[32] $\Delta^* \leq \bar{\Delta}$, but does not bind for all $0 < \tilde{\Delta} < \Delta^*$. According to the first part of the lemma, the derivative $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}})$ exists and $\dot{\boldsymbol{\lambda}}(h_{t,s;\tilde{\Delta}}) = 0$.

i. Consider the case $\rho = r$. According to the second part of the lemma, $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}}) = 0$. Thus, consumption is constant at $\mathbf{c}(h_{t,s;\tilde{\Delta}}) \equiv c$ for all $0 < \tilde{\Delta} < \bar{\Delta}$, regardless of whether the constraint (65) binds or does not bind at $\tilde{\Delta}$, establishing our claim.

ii. Consider the case $\rho > r$ and current state x. We will show that we arrive at a contradiction; see Figure 8. According to the second part of the lemma, $\dot{\mathbf{c}}(h_{t,s;\tilde{\Delta}}) < 0$ for $0 < \tilde{\Delta} < \Delta^*$. Fix such a $\tilde{\Delta}$. It follows that $\mathbf{c}(h_{t,s;\tilde{\Delta}}) < \mathbf{c}(h_{t,s;\Delta^*}) = c$; that is, consumption jumps up at Δ^*, even though $U_{h_{t,s;\tilde{\Delta}}} > U_{h_{t,s;\Delta^*}} = U^{out}(\mathbf{z}(x))$. We will show that the contract at history $h_{t,s;\tilde{\Delta}}$ can therefore not have been cost-minimizing. For $x' \neq x$ and $\Delta > 0$, let $h_{t,s;\Delta,x'} = [h_{t,s}, (s+\Delta, 1, s, x, x(s+\Delta) = x']$ be the extensions of the original history $h_{t,s}$ with a first state change to a new state x' occurring at date $s + \Delta$.

[32] Δ^* exists, because $U_{h_{t,s;\tilde{\Delta}}}$ is continuous in $\tilde{\Delta}$.

Define $U_{h_{t,s;\tilde{\Delta}}}$ as the continuation utility starting at the history $h_{t,s;\tilde{\Delta}}$. Deploying the construction of Appendix D.4 leading up to equation (82), it is given by

$$\begin{aligned} U_{h_{t,s;\tilde{\Delta}}} &= \int_0^{\Delta^*-\tilde{\Delta}} e^{(\alpha_{x,x}-\rho)\tau} u(w\mathbf{c}(h_{t,s;\tilde{\Delta}+\tau}))d\tau \\ &\quad + e^{(\alpha_{x,x}-\rho)(\Delta^*-\tilde{\Delta})} U^{out}(\mathbf{z}(x)) \\ &\quad + \sum_{x'\neq x} \int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\tilde{\Delta}+\tau,x'}} d\tau \end{aligned} \tag{78}$$

The first term captures the present discounted utility over the time interval from $t+s+\tilde{\Delta}$ to $t+s+\Delta^*$, conditional on no state change; the second term captures the associated continuation utility from $t+s+\Delta^*$ onward in that case; and the last term captures the expected continuation utility conditional on some state change from x to x' during the time interval $\Delta^*-\tilde{\Delta}$ following history $h_{t,s;\tilde{\Delta}}$.
Compare this to the similar continuation at $s+\Delta^*$,

$$\begin{aligned} U_{h_{t,s;\Delta^*}} &= \int_0^{\Delta^*-\tilde{\Delta}} e^{(\alpha_{x,x}-\rho)\tau} u(w\mathbf{c}(h_{t,s;\Delta^*+\tau}))d\tau \\ &\quad + e^{(\alpha_{x,x}-\rho)(\Delta^*-\tilde{\Delta})} U_{h_{t,s;\Delta^*+(\Delta^*-\tilde{\Delta})}} \\ &\quad + \sum_{x'\neq x} \int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\Delta^*+\tau,x'}} d\tau \end{aligned} \tag{79}$$

Note that $\mathbf{c}(h_{t,s;\tilde{\Delta}+\tau}) < \mathbf{c}(h_{t,s;\Delta^*+\tau})$ and that $U^{out}(z) \leq U_{h_{t,s;\Delta^*+(\Delta^*-\tilde{\Delta})}}$. Since $U_{h_{t,s;\tilde{\Delta}}} > U_{h_{t,s;\Delta^*}} = U^{out}(z)$, it must be the case that the inequality is reversed for the last term,

$$\begin{aligned} &\sum_{x'\neq x} \int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\tilde{\Delta}+\tau,x'}} d\tau \\ &\qquad > \sum_{x'\neq x} \int_0^{\Delta^*-\tilde{\Delta}} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)\tau} U_{h_{t,s;\Delta^*+\tau,x'}} d\tau \end{aligned}$$

Recall, though, that the contract is cost-minimizing at history $h_{t,s;\tilde{\Delta}}$. Per the principle of optimality established in Lemma 9 of Appendix D.4 below,

we thus arrive at a contradiction.

B. For the second part, suppose instead that (65) never binds for any $\Delta > 0$ at the no-state-change history extensions $h_{t,s;\Delta} = [h_{t,s}, (s+\Delta, 0, s, x(s))]$. The first part of the lemma shows that $\dot{\lambda}(h_{t,s;\Delta}) = 0$ for all $\Delta > 0$ as well as $\dot{\lambda}_+(h_{t,s}) = 0$. If $\rho = r$, then the second part of the lemma shows that $\dot{\mathbf{c}}_+(h_{t,s}) = 0$ and hence the claim. If $\rho > r$, then (76) shows that $\dot{\mathbf{c}}(h_{t,s;\Delta})/\mathbf{c}(h_{t,s;\Delta}) < (r-\rho)/\bar{\sigma} < 0$ and hence $\mathbf{c}(h_{t,s;\Delta}) \to 0$ as $\Delta \to \infty$.

   i. Suppose then that there is some $\underline{\Delta} > 0$, so that the continuation utility constraints upon a state change to $x' \neq x(s)$ bind at all extended histories

$$h_{t,s;\Delta,x'} = [h_{t,s}, (s+\Delta, 1, s, s+\Delta, x(s), x')]$$

   for $\Delta > \underline{\Delta}$ and any $x' \in X$. The continuation contract at the no-state-change history extension $h_{t,s;\underline{\Delta}}$ is feasible when shifted backward in time to s, i.e., consider the contract

$$\tilde{\mathbf{c}}(h_{t,\tau}) = \mathbf{c}([h_{t,s;\underline{\Delta}}, h^{\Delta}_{s,\tau}])$$

   defined for all continuations $h_{t,\tau} = [h_{t,s}, h_{s,\tau}]$ of $h_{t,s}$. Contract $\tilde{\mathbf{c}}$ is cheaper for the principal than contract $\mathbf{c}$, since consumption along the $h_{t,s;\Delta}$-histories keeps declining and since one cannot do better upon a state change than to achieve a binding constraint there. This is a contradiction to the assertion that the contract was cost-minimizing $\mathbf{c}$.

   ii. Suppose instead that for any $\underline{\Delta} > 0$, there is a positive measure of dates $s + \Delta$ with $\Delta > \underline{\Delta}$, at which the utility promised upon a state change is not binding. But then and with sufficiently large Δ and thus sufficiently small $\mathbf{c}(h_{t,s;\Delta})$ along the no-state-change path, the principal can achieve a higher promised utility for the agent by promising less consumption upon the state change for some positive interval of time and more consumption along the no-state-change path, again a contradiction to the contract being cost-minimizing.

4. Since λ is weakly increasing, $\lambda(h_{t,s}) - \lambda_-(h_{t,s}) > 0$. The claim now follows from (74) and, exploiting the fact that $u'(\cdot)$ is strictly decreasing, as well as from noting that $\lambda(h_{t,s})$ is only increasing if (65) binds. Furthermore, it must be the case that $t_n = s$,

i.e., that the state change just occurred on date s, since otherwise the derivative of consumption would have been zero per the third part of the lemma.

□

Using the simplified notation $c(s) = \mathbf{c}(h_{t,s})$ and $\lambda(s) = \boldsymbol{\lambda}(h_{t,s})$, note that (76) implies that

$$\frac{\dot{c}(s)}{c(s)} = -\frac{\rho - r}{\sigma} + \frac{1}{\sigma}\frac{\dot{\lambda}(s)}{\lambda(s)} \tag{80}$$

for the CRRA utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \tag{81}$$

The utility function $u(c) = \log(c)$ is the special case, where $\sigma = 1$.

D.4 A Recursive Approach

Consider some $\Delta > 0$. Using the principle of optimality, one can rewrite the cost minimization problem for the optimal insurance contract by examining $\tau \in [t, t + \Delta]$ and then use the minimized costs for $\tau \geq t + \Delta$. More precisely, let $x(t) = x \in X$ be the state at the beginning date t of the contract. The histories beyond t are of two kinds. There is the no-change-in-state path $h^0_{t,s} = (s, 0, t_0 = t, x_0 = x)$ all the way up to $s = t + \Delta$, where the superindex 0 indicates zero state changes.[33] This includes in particular the starting point $h^0_{t,t} = h_{t,t}$. Then there are paths with a jump to state $x' \neq x$ at some date $s \in [t, t + \Delta]$, starting with the histories $h^1_{t,s;x'} = (s, 1, t_0 = t, t_1 = s, x_0 = x, x_1 = x')$, where the superindex 1 indicates one state change. Consider the continuation costs and continuation utility promises following these histories discounted to the new starting dates. Writing $\mathbf{c}(h_{t,\tau})$ rather than $\mathbf{c}(h_{t,\tau}; x, U)$ to save on notation,

$$\begin{aligned}
V_{h^0_{t,t+\Delta}} &= \int_{t+\Delta}^{\infty} \int_{\mathcal{H}_{t+\Delta,\tau}(x)} e^{-r(\tau-t-\Delta)} \left(w\mathbf{c}([h^0_{t,t+\Delta}, h_{t+\Delta,\tau}]) - w\mathbf{z}([h^0_{t,t+\Delta}, h_{t+\Delta,\tau}]) \right) dP_{t+\Delta,\tau} d\tau \\
V_{h^1_{t,s;x'}} &= \int_{s}^{\infty} \int_{\mathcal{H}_{s,\tau}(x')} e^{-r(\tau-s)} \left(w\mathbf{c}([h^1_{t,s;x'}, h_{s,\tau}]) - w\mathbf{z}([h^1_{t,s;x'}, h_{s,\tau}]) \right) dP_{s,\tau} d\tau \\
U_{h^0_{t,t+\Delta}} &= \int_{t+\Delta}^{\infty} \int_{\mathcal{H}_{t+\Delta,\tau}(x)} e^{-\rho(\tau-t-\Delta)} u(w\mathbf{c}([h^0_{t,t+\Delta}, h_{t+\Delta,\tau}])) dP_{t+\Delta,\tau} d\tau \\
U_{h^1_{t,s;x'}} &= \int_{s}^{\infty} \int_{\mathcal{H}_{s,\tau}(x')} e^{-\rho(\tau-s)} u(w\mathbf{c}([h^1_{t,s;x'}, h_{s,\tau}])) dP_{s,\tau} d\tau
\end{aligned}$$

[33] The superindex notation was avoided in the proof of Lemma 7 in order to declutter the notation there.

With that and using the appropriate probabilities, the continuation utility after date $\tau \in [t, t+\Delta]$ and no-change-in-state between t and τ is

$$\begin{aligned} U_{h^0_{t,\tau}} &= \int_\tau^{t+\Delta} e^{(\alpha_{x,x}-\rho)(s-\tau)} u(w\mathbf{c}(h^0_{t,s}))ds \\ &\quad + e^{(\alpha_{x,x}-\rho)(t+\Delta-\tau)} U_{h^0_{t,t+\Delta}} \\ &\quad + \sum_{x' \neq x} \int_\tau^{t+\Delta} \alpha_{x,x'} e^{(\alpha_{x,x}-\rho)(s-\tau)} U_{h^1_{t,s;x'}} ds \end{aligned} \tag{82}$$

The cost function in definition 4 can likewise be rewritten. Formally,

Definition 5 (**recursive cost-minimization).** *For a fixed $\Delta > 0$, fixed outside options $U^{out}(z)$, with $z \in Z$, a starting date t, and a fixed wage w and rate of return on capital or interest rate r, a* **recursive cost function** *$V(x, U)$ optimally chooses $\mathbf{c}(h^0_{t,\tau})_{\tau \in [t,t+\Delta]} \geq 0$, $U_{h^0_{t,t+\Delta}}$ and $(U_{h^1_{t,s;x'}})_{s \in [t,t+\Delta], x' \in X/\{x\}}$ to solve*

$$\begin{aligned} V(x,U) &= \min \int_t^{t+\Delta} e^{(\alpha_{x,x}-r)(\tau-t)} \left[w\mathbf{c}(h^0_{t,\tau}) - w\mathbf{z}(x) \right] d\tau \\ &\quad + e^{(\alpha_{x,x}-r)\Delta} V(x, U_{h^0_{t,t+\Delta}}) \\ &\quad + \sum_{x' \neq x} \int_t^{t+\Delta} \alpha_{x,x'} e^{(\alpha_{x,x}-r)(s-t)} V(x', U_{h^1_{t,s;x'}}) ds \end{aligned} \tag{83}$$

subject to the **promise-keeping constraint**

$$U_{h_{t,t}} \geq U \tag{84}$$

and the **limited commitment constraints**

$$\frac{\bar{u}}{\rho} > U_{h^0_{t,\tau}} \geq U^{\mathit{Out}}(\mathbf{z}(x)) \text{ for all } \tau \in [t, t+\Delta] \tag{85}$$

$$\frac{\bar{u}}{\rho} > U_{h^1_{t,s;x'}} \geq U^{\mathit{Out}}(\mathbf{z}(x')) \text{ for all } s \in [t, t+\Delta] \tag{86}$$

for all $x(t) = x \in X$ and all $U \in \left[U^{out}(\mathbf{z}(x)), \frac{\bar{u}}{\rho} \right)$.

Lemma 8 (**equivalence).** *The two definitions 4 and 5 coincide.*

Proof. Clear from the calculations above. □

Lemma 9 (principle of optimality). *Fix the state x. Consider two utility levels $U^A > U^B$ and suppose that (84) binds at both. Suppose there is some $\Delta > 0$, so that (85) does not bind for all $\tau \in [t, t+\Delta]$ and the no-change histories $h^0_{t,s}$, starting from the promise*[34] *$U = U^B$. Consider the optimal recursive cost function choices $\mathbf{c}^\Psi(h^0_{t,\tau})_{\tau\in[t,t+\Delta]} \geq 0$, $\mathbf{c}^\Psi(h^0_{t,\tau})$ and $(U^\Psi_{h^1_{t,s;x'}})_{s\in[t,t+\Delta],x'\in X/\{x\}}$ for $\Psi \in \{A, B\}$. Then*

$$\begin{aligned} \mathbf{c}^A(h^0_{t,\tau}) &\geq \mathbf{c}^B(h^0_{t,\tau}) \text{ for almost all } \tau \in [t, t+\Delta] \\ U^A_{h^0_{t,t+\Delta}} &\geq U^B_{h^0_{t,t+\Delta}} \\ U^A_{h^1_{t,s;x'}} &\geq U^B_{h^1_{t,s;x'}} \text{ for almost all } s \in [t, t+\Delta] \text{ and all } x' \in X/\{x\} \end{aligned}$$

Proof of Lemma 9. This is due to the similar structure of the utility formula (82) and the cost function (83). With (83), only the constraints (84,85,86) have to be taken into account, using their Lagrange multipliers: the constraints beyond that are part of the continuation cost functions. Start at the promised utility U^B. Since (85) does not bind for all $\tau \in [t, t+\Delta]$ and the no-change histories $h^0_{t,s}$, $\lambda(h^0_{t,\tau}) \equiv \zeta$ for $\tau \in [t, t+\Delta]$ (where ζ is the Lagrange multiplier on (84)), utilizing that the Lagrange multipliers on the limited commitment constraint (85) for the no-state-change histories $h^0_{t,\tau}$ with $t \leq \tau \leq t+\Delta$ are zero, write the "original" Lagrangian (70) as

$$\begin{aligned} L \;=\; & \int_t^{t+\Delta} e^{(\alpha_{x,x}-r)(\tau-t)} \left(w\mathbf{c}(h^0_{t,\tau}) - w\mathbf{z}(x)\right) - \zeta e^{(\alpha_{x,x}-\rho)(\tau-t)} u(w\mathbf{c}(h^0_{t,\tau}))\, d\tau \\ & + e^{(\alpha_{x,x}-r)\Delta} V(x, U_{h^0_{t,t+\Delta}}) - \zeta e^{(\alpha_{x,x}-\rho)\Delta} U_{h^0_{t,t+\Delta}} + \text{const.} \\ & + \sum_{x'\neq x} \int_t^{t+\Delta} \alpha_{x,x'} e^{(\alpha_{x,x}-r)(s-t)} V(x', U_{h^1_{t,s;x'}}) - (\zeta + \mu(h^1_{t,s;x'}))\alpha_{x,x'} e^{(\alpha_{x,x}-\rho)(s-t)} U_{h^1_{t,s;x'}}\, ds \end{aligned} \tag{87}$$

This is legitimate at $U = U^B$: we will show that this is legitimate for all $U \in [U^B, U^A]$. Note that $\mu(h^1_{t,s;x'}) \neq 0$ only if (86) binds at $h^1_{t,s;x'}$: in that case, it must be the case that $U_{h^1_{t,s;x'}} = U^{out}(\mathbf{z}(\mathbf{x}'))$. Differentiate with respect to $\mathbf{c}(h^0_{t,\tau})_{\tau\in[t,t+\Delta]} \geq 0$, $\mathbf{c}(h^0_{t,\tau})$ and $(U_{h^1_{t,s;x'}})_{s\in[t,t+\Delta],x'\in X/\{x\}}$. Noting the dependency of the Lagrange multipliers on the

[34] Suppose that $U^B > U^{out}(\mathbf{z}(\mathbf{x}))$. Per continuity in τ of integrating the future consumption path starting at the lower bound τ, one can show that such a $\Delta > 0$ exists.

promised utility U by including it as an argument, the first-order conditions at $U = U^B$ are

$$\begin{aligned} e^{(\rho-r)(\tau-t)} &= \zeta(U)u'(w\mathbf{c}(h^0_{t,\tau})) && (88)\\ e^{(\rho-r)(\Delta-t)}V'_-(x, U_{h^0_{t,t+\Delta}}) &\leq \zeta(U) \leq e^{(\rho-r)(\Delta-t)}V'_+(x, U_{h^0_{t,t+\Delta}}) && (89)\\ e^{(\rho-r)(s-t)}V'_-(x', U_{h^1_{t,s;x'}}) &\leq \zeta(U) + \mu(h^1_{t,s;x'}; U) \leq e^{(\rho-r)(s-t)}V'_+(x', U_{h^1_{t,s;x'}}) && (90)\end{aligned}$$

Given the Lagrange multipliers $\zeta(U)$ and $\mu(h^1_{t,s;x'}; U)$, let $\mathbf{c}(h^0_{t,\tau}; U)$ for $\tau \in [t, t+\Delta]$, $U_{h^0_{t,t+\Delta};U}$ and $U_{h^1_{t,s;x'};U}$ for $s \in [t, t+\Delta]$, and $x' \in X/\{x\}$ be the solution to these equations. Given the strict concavity of $u(\cdot)$, $\mathbf{c}(h^0_{t,\tau}; U)$ for $\tau \in [t, t+\Delta]$ is strictly increasing in $\zeta(U)$. Given the convexity of $V(x, \cdot)$ according to Lemma 4, $U_{h^0_{t,t+\Delta};U}$ is weakly increasing in $\zeta(U)$ per equation (89). Likewise, equation (90) shows that $U_{h^1_{t,s;x'};U}$ is either weakly increasing in $\zeta(U)$ or constant and equal to the lower bound $U^{out}(\mathbf{z}(x'))$ of equation (86). It follows that $\zeta(U)$ is increasing in U. Note now that these statements are all correct at $U = U^B$ and that (85) does not bind for $\tau \in [t, t+\Delta]$ by assumption. Exploiting the local monotonicity of the solutions, equation (82) then shows that (85) does not bind for $\tau \in [t, t+\Delta]$ for all $U^B < U < U^B + \epsilon$, when $\epsilon > 0$ is sufficiently small, and that (87) is the appropriate Lagrangian for these U as well. Continuing that argument all the way to U^A shows that (85) does not bind for $\tau \in [t, t+\Delta]$ for any $U \in [U^B, U^A]$. Thus, $\mathbf{c}(h^0_{t,\tau}; U)$, $U_{h^0_{t,t+\Delta};U}$ and $U_{h^1_{t,s;x'};U}$ are weakly increasing functions of U. The statements comparing $\mathbf{c}^A(h^0_{t,\tau}) = \mathbf{c}(h^0_{t,\tau}; U^A)$ to $\mathbf{c}^B(h^0_{t,\tau}) = \mathbf{c}(h^0_{t,\tau}; U^B)$ now follow as do the others. □

As a consequence of (88), note that $\mathbf{c}(h^0_{t,\tau})$ is a weakly decreasing and continuous function of τ. As a consequence of (89), note that $U_{h^0_{t,t+\Delta}}$ is a weakly decreasing function of Δ. With some work, one can show that $U_{h^1_{t,s;x'}}$ for any $x' \in X$ is continuous in s at $s = t$. With that, we shall examine the limit, as $\Delta \to 0$.

Proposition 14 (the cost-minimizing HJB equation). *For fixed outside options $U^{out}(z)$, with $z \in Z$, a starting date t, and a fixed wage w and rate of return on capital or interest rate $r > 0$, a recursive cost function $V(x, U)$ solves the Hamilton-Jacobi-Bellman equation*

$$rV(x,U) = \min_{c,\dot{U},(U(x'))_{x'\in X/\{x\}}} wc - w\mathbf{z}(x) + V'_-(x,U)\dot{U} + \sum_{x'\neq x} \alpha_{x,x'}(V(x', U(x')) - V(x,U))$$

subject to

$$\begin{aligned} \rho U &= u(wc) + \dot{U} + \sum_{x' \neq x} \alpha_{x,x'}(U(x') - U) \\ \dot{U} &\geq 0, \text{ if } U = U^{\textbf{Out}}(\mathbf{z}(x)) \\ \frac{\bar{u}}{\rho} > U(x') &\geq U^{\textbf{Out}}(\mathbf{z}(x')) \end{aligned}$$

for all $x(t) = x \in X$ *and all* $U \in \left[U^{out}(\mathbf{z}(x)), \frac{\bar{u}}{\rho}\right)$*, provided that (84) binds.*

Proof of Proposition 14. The continuity of $\mathbf{c}(h^0_{t,s})$ and $U_{h^1_{t,s;x'}}$ in s at $s = t$, together with equation (82), shows that that $U_{h^0_{t,\tau}}$ is continuous in τ at $\tau = t$. Equation (82) furthermore implies that $U_{h^0_{t,\tau}}$ is differentiable with respect to τ at $\tau = t$. Denote that derivative by $\dot{U} = \dot{U}_{h_{t,t}}$. The arguments preceding the proposition imply $\dot{U} \leq 0$. Equation (82) implies that

$$\dot{U}_{h_{t,t}} = -u(w\mathbf{c}(h_{t,t})) + (\rho - \alpha_{x,x})U_{h_{t,t}} - \sum_{x' \neq x} \alpha_{x,x'} \lim_{s \to t} U_{h^1_{t,s;x'}}$$

Use the cost function definition (83) evaluated at the minimizing choices as well as $\dot{U} \leq 0$ and calculate

$$\begin{aligned} V(x, U_{h_{t,t}}) &= [w\mathbf{c}(h_{t,t}) - w\mathbf{z}(x)]\,\Delta \\ &\quad + (1 + (\alpha_{x,x} - r)\Delta)V(x, U_{h_{t,t}}) + V'_-(x, U_{h_{t,t}})\dot{U}_{h_{t,t}}\Delta \\ &\quad + \sum_{x' \neq x} \alpha_{x,x'} V(x', U_{h^1_{t,s;x'}})\Delta + o(\Delta), \end{aligned}$$

Subtract $V(x, U_{h_{t,t}})$, divide by Δ and let $\Delta \to 0$. Write x and x' in place of $h_{t,t}$ and $h_{t,t+\Delta;x'}$. The lemma follows, noting that $\alpha_{x,x} + \sum_{x' \neq x} \alpha_{x,x'} = 0$. □

The following will be useful. Let $\dot{V}$ denote the derivative of $V(x, U_{h^0_{t,\tau}})$ with respect to τ at $\tau = t$. Then,

$$\dot{V} = V'_-(x, U)\dot{U} \tag{91}$$

D.5 The Dual Problem: Utility Maximization

The dual problem to the contractual cost minimization problem above is a utility maximization problem, subject to a budget constraint. The budget is the resources provided by the intermediary. The intermediary uses capital in order to fund the consumption claims by contracted agents, effectively maintaining an account for each agent denoted in units of capital to do so. Thus, write k rather than v for the budget available. Rather than provide the contract formulation for arbitrary levels of outside options, we note that these outside options are available to agents starting from scratch, i.e. when signing up with a new intermediary. Starting from scratch is thus the same as starting from zero capital. With this, the dual problem becomes one of choosing state contingent capital and consumption subject to the constraint that capital must be non-negative: negative amounts would trigger the selection of the outside option and a default on future obligations.

Definition 6 (utility-maximizing contract). *For a starting date t, a starting state x and an initial amount of capital $k_t \geq 0$, a fixed wage w and rate of return on capital or interest rate r, an optimal consumption plan $\mathbf{c} : \mathcal{H}_t(x) \to \mathbf{R}_+$ and optimal savings plan $\mathbf{k} : \mathcal{H}_t(x) \to \mathbf{R}_+$ with $\mathbf{k}(h_{t,t}) = k_t$ solve*

$$\max_{\mathbf{c},\mathbf{k}} U(k_t; x) = \int_t^\infty \int_{\mathcal{H}_{t,\tau}(x)} e^{-\rho(\tau-t)} u(w\mathbf{c}(h_{t,\tau})) dP_{t,\tau} d\tau \tag{92}$$

subject to

$$\begin{aligned} \mathbf{k}(h_{t,\tau}) \;=\; & \int_{\mathcal{H}_{\tau,s}(x(h_{t,\tau}))} \Bigg(e^{-r(s-t)} \mathbf{k}([h_{t,\tau}, h_{\tau,s}]) \\ & + \int_\tau^s e^{-r(\tau-t)} \left(w\mathbf{c}([h_{t,\tau}, h_{\tau,s}(\tau, s')]) - w\mathbf{z}([h_{t,\tau}, h_{\tau,s}(\tau, s')]) \right) ds' \Bigg) dP_{t,s} \end{aligned} \tag{93}$$

for all $s \geq \tau \geq t$ and $h_{t,\tau} \in \mathcal{H}_{t,\tau}$.

Equation (93) is the budget constraint for the agent. For $\tau = t$ and $s \to \infty$, one obtains that k_t is the expected net present value of future consumption in excess of wage income,

$$k_t \;=\; \int_t^\infty \int_{\mathcal{H}_{t,\tau}(x)} e^{-r(\tau-t)} \left(w\mathbf{c}(h_{t,\tau}) - w\mathbf{z}(h_{t,\tau}) \right) dP_{t,\tau} d\tau, \tag{94}$$

changing the order of integration and index of integration and noting that $h_{t,s}[t, \tau] \in \mathcal{H}_{t,\tau}(x)$. However, (93) is a stricter constraint, since we have imposed the condition that

$\mathbf{k} : \mathcal{H}_t(x) \to \mathbf{R}_+$ is nonnegative. Conversely, it provides for more choices than the Aiyagari-style saving constraint

$$k_t = e^{-r(s-t)}\mathbf{k}(h_{t,s}) + \int_t^s e^{-r(\tau-t)} \left(w\mathbf{c}(h_{t,s}[t,\tau]) - w\mathbf{z}(h_{t,s}[t,\tau])\right) d\tau \tag{95}$$

and $\mathbf{k}(h_{t,s}) \geq 0$, for all s and $h_{t,s} \in \mathcal{H}_{t,s}(x)$, as (93) allows for state-contingent reallocation of capital and thus insurance against future state changes, subject to the constraint that capital cannot be negative. The next proposition establishes the equivalence between the cost-minimization problem in definition 4 and the utility-maximization problem in definition 6.

Proposition 15 (equivalence of cost-minimizing and utility-maximizing). *1. Given a cost-minimizing contract as defined in definition 4, suppose that $V(x, U^{out}(\mathbf{z}(x))) = 0$ for all $x \in X$. For $x \in X$ and some $U \in \mathcal{U}(x)$, define the continuation utilities,*

$$U(h_{t,\tau}) = \int_\tau^\infty \int_{\mathcal{H}_{\tau,s}(x(h_{t,\tau}))} e^{-\rho(s-\tau)} u(w\mathbf{c}([h_{t,\tau}, h_{\tau,s}]))) dP_{\tau,s} ds \tag{96}$$

Define

$$\mathbf{k}(h_{t,\tau}) = V(x(h_{t,\tau}), U(h_{t,\tau})) \text{ for all } \tau \text{ and } h_{t,\tau} \in \mathcal{H}_{t,\tau}(x) \tag{97}$$

Then $\mathbf{c}(\cdot; x, U)$ and $\mathbf{k}$ are utility-maximizing consumption and savings plans for the initial capital $k_t = V(x, U)$, as defined in definition 6, resulting in $U(k_t; x) = U(h_{t,t}) \geq U$.

2. *Suppose that the optimal cost function of definition 4 is continuous in U*[35]*. For all $x \in X$ and $k_t \geq 0$, calculate $U(k_t; x)$ and the optimal consumption and savings plans per definition 6. Denote them by $\mathbf{c}_{x,k_t}$ and $\mathbf{k}_{x,k_t}$. Let $U^{out}(\mathbf{z}(x)) = U(0; x)$ for all $x \in X$. Define*

$$\mathbf{c}(h_{t,\tau}; x, U(k_t; x)) = \mathbf{c}_{x,k_t}(h_{t,\tau}) \text{ for all } \tau \text{ and } h_{t,\tau} \in \mathcal{H}_{t,\tau}(x) \tag{98}$$

Then $\mathbf{c}$ is an optimal consumption contract as defined in definition 4, resulting in the cost

$$V(x, U(k_t; x)) = k_t \tag{99}$$

[35] With some work, this can probably be shown to be true.

Proof. 1. Similar to the recursive construction for (83), note that $\mathbf{k}(h_{t,\tau})$, defined as $V(x(h_{t,\tau}), U(h_{t,\tau}))$ satisfies (93). Since $V(x, U^{out}(\mathbf{z}(\mathbf{x}))) = \mathbf{0}$ and since $V(x, U)$ is increasing in U, it follows that $\mathbf{k}(h_{t,\tau}) \geq 0$. Since $V(x, \cdot)$ is strictly increasing at the promise $U(h_{t,t})$, there is no other consumption and savings plan, resulting in a higher utility.

2. Note that $V(x, U(k_t; x)) = k_t$ satisfies (94) and thus the equation for the cost function in definition 4. Suppose that the optimal contract achieves $U(k_t; x)$ at a lower cost. Since the optimal cost function is increasing and continuous in U, there is some utility level $U > U(k_t; x)$ resulting still in a cost below k_t. Exploiting the preceding reverse construction of proceeding from the cost-minimizing contract to a utility maximizing plan in the previous step then shows that the plan for k_t cannot have been optimal.

□

Similar to Proposition 14 and with the same assumptions, we obtain

Proposition 16 (the utility-minimizing HJB equation). *For a fixed wage w and rate of return on capital or interest rate $r > 0$, a recursive utility function $U(k; x)$ solves the Hamilton-Jacobi-Bellman equation*

$$\rho U(k;x) = \max_{c,\dot{k},(k(x'))_{x' \in X/\{x\}}} u(c) + \frac{\partial U(k;x)}{\partial k}\dot{k} + \sum_{x' \neq x} \alpha_{x,x'}(U(k(x');x') - U(k;x))$$

subject to

$$\begin{aligned} c + \dot{k} + \sum_{x' \neq x} \alpha_{x,x'}(k(x') - k) &= rk + w\mathbf{z}(\mathbf{x}) \\ k(x') &\geq 0 \qquad \textit{for all } x' \in X/\{x\} \\ \dot{k} &\geq 0 \qquad \textit{if } k = 0 \end{aligned}$$

for all $x \in X$ and all $k \geq 0$.

The proof is analogous to the proof of Proposition 14. We skip the details.

E Proofs of Lemmas and Propositions in the Main Text

In this section we provide the proofs for propositions in the main text as well as provide further details of the mathematical derivations. These are straightforward but tedious manipulations that were therefore excluded from the main text.

We wish to formally compare contracts starting from high productivity to those starting from low productivity. Therefore and from here on, we assume that there is a three-state Markov process for an underlying state $x(t) \in X = \{0, 1, 2\}$ for each agent, evolving independently from each other. The transition rates $\alpha_{i,j}$ to transit from state $x = i$ to $x = j$ are $\alpha_{0,1} = \alpha_{2,1} = \nu$, $\alpha_{1,0} = \alpha_{2,0} = \xi$ and $\alpha_{0,2} = \alpha_{1,2} = 0$. Let $\alpha_{i,i} = -\sum_{j \neq i} \alpha_{i,j}$, so that α is an intensity matrix or infinitesimal generator matrix.

Additionally, we assume that there is a mapping $\mathbf{z} : X \to Z$ so that the implied Markov process $z(t) = \mathbf{z}(X(t))$ has the transition rates ξ for transiting from $z = \zeta$ to $z = 0$ and ν for transiting from $z = 0$ to $z = \zeta$, as stated in subsection 2.1 and given some initial productivity. There are two options in particular. For the first option, let the mapping $\mathbf{z} = \mathbf{z}_A$ be given by $\mathbf{z}_A(0) = \mathbf{z}_A(2) = 0$ and $\mathbf{z}_A(1) = \zeta$. The three-state process starting at $x(t) = 0$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z(t) = 0$: the exit rate out of zero productivity is ν, regardless of whether the underlying state is $x = 0$ or $x = 2$, and the transitions between these two states play no role. The transition out of high productivity only happens from state $x = 1$ at rate ξ, exactly as in the two-state formulation. For the second option, let the mapping $\mathbf{z} = \mathbf{z}_B$ be given by $\mathbf{z}_B = 0$ and $\mathbf{z}_B(1) = \mathbf{z}_B(2) = \zeta$. The three-state process starting at $x(t) = 1$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z_t = \zeta$: the exit rate out of $z = \zeta$ productivity is ξ, regardless of whether the underlying state is $x = 1$ or $x = 2$, and the transitions between these two states play no role. The transition out of low productivity only happens from state $x = 0$ at rate ν, exactly as in the two-state formulation.

E.1 Ordering of the Outside Utilities

Assume $r > 0$. Let the net present value of future income be defined as

$$NPV(z) = \mathbf{E}\left[\int_t^\infty e^{-r(\tau - t)} w z(\tau) d\tau \middle| z(t) = z\right]$$

conditional on the starting income z at date t.

Lemma 10. *$NPV(z)$ is increasing in z. Specifically,*

$$NPV(z) = \begin{cases} \frac{\nu}{r+\nu+\xi}\frac{\zeta}{r} & \text{if } z = 0 \\ \frac{r+\nu}{r+\nu+\xi}\frac{\zeta}{r} & \text{if } z = \zeta \end{cases} \tag{100}$$

Proof of Lemma 10. Using Bellman logic, the two NPVs satisfy

$$(r+\nu)NPV(0) = \nu NPV(\zeta) \tag{101}$$
$$(r+\xi)NPV(\zeta) = \zeta + \xi NPV(0) \tag{102}$$

Solve. □

Proof of Lemma 1. The key idea is that an agent currently at high productivity can be provided with the contract of the low-productivity agent, delivering the same utility and a profit to the principal, a contradiction to perfect competition between the principals. Some care needs to be taken to implement this idea, however. Contracts depend on the history of states. Thus, if the history was expressed only in terms of productivities, it would be meaningless to give an agent starting with high productivity "the same" contract as an agent starting with low productivity. The underlying state and the corresponding productivity need to be decoupled. It is here where the three-state construction described at the beginning of this section and the careful distinction between the state and the productivity at that state as described at the beginning of Appendix C pay off.

Suppose by contradiction to the claim (10) that

$$U^{out}(0) \geq U^{out}(\zeta) \tag{103}$$

Fix the productivity mapping $\mathbf{z} : X \to Z$ to be $\mathbf{z}_A$. Recall that $\mathbf{z}_A(0) = \mathbf{z}_A(2) = 0$ and $\mathbf{z}_A(1) = \zeta$, and that the three-state process starting at $x(t) = 0$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z(t) = 0$. Consider an optimal consumption contract $c(\tau; 0, U^{out}(0))$ given to an agent at date $t = 0$, say, and starting off with productivity $z(0) = 0$, delivering date-0 promised utility $U = U^{out}(0)$ in (2) and generating costs $V(0, U^{out}(0)) = 0$. Wlog, we shall impose the condition that $x(0) = 2$: any contract as defined per history dependence in Appendix C and starting at $x(0) = 0$ can be written[36] as a contract starting at $x(0) = 2$ delivering the same outcomes, per ignoring transitions from $x = 2$ to $x = 0$. Thus, the optimal

[36]This argument can be made precise with some tedious notation.

consumption contract $c(\tau; 0, U^{out}(0))$ is a mapping $\mathbf{c} : \mathcal{H}_0 \to \mathbf{R}_+$ from x-histories into consumption outcomes, where all $h_{s,0} \in \mathcal{H}_0$ satisfy $x(0) = 2$, and which satisfies the constraints (65).

Next, fix the productivity mapping $\mathbf{z} : X \to Z$ to be $\mathbf{z}_B$. Recall that $\mathbf{z}_B(0) = 0$ and $\mathbf{z}_B(1) = \mathbf{z}_B(2) = \zeta$, and that the three-state process starting at $x(t) = 1$ or $x(t) = 2$ now generates the same stochastic process as the original two-state stochastic process for an agent starting at $z(t) = \zeta$. The contract $\mathbf{c}$ delivers the same expected utility $U^{out}(0)$. The contract $\mathbf{c}$ satisfies the constraints (65) for states $x(s) = 0$ and states $x(s) = 1$, where $\mathbf{z}_A$ and $\mathbf{z}_B$ coincide. With equation (103), the constraints are also satisfied for the state $x(s) = 2$ and $\mathbf{z}_B(2) = \zeta$ rather than $\mathbf{z}_A(2) = 0$. The consumption portion generates the same costs for the principal, as nothing has changed regarding the consumption process, but the expected revenue from productivity income is now strictly higher per Lemma 10. It follows, that the contract $\mathbf{c}$ now delivers strictly negative costs[37] $V(\zeta, U^{out}(0))$. Per Lemma 3 and equation (103), $0 > V(\zeta, U^{out}(0)) \geq V(\zeta, U^{out}(\zeta))$. However, $V(\zeta, U^{out}(\zeta)) = 0$ per the definition of equilibrium. With that, we have arrived at a contradiction. □

E.1.1 The Case of $-\xi < r < 0$

The proof of Lemma 10, and thus the proof of Lemma 1, required that $r > 0$ since consumption insurance contracts last forever and thus discounting has to be positive to render present discounted values of future incomes and costs of the contract finite. However, since contracts will effectively end and reset every time a high income shock is realized, the same arguments as in the proofs of Lemma 10, and thus of Lemma 1 can be used as long as $r > -\xi$.

The expected net present value of income during such a contract that starts with high income $z = \zeta$, extends through a spell of low income $z = 0$, and ends the instant a new high-income spell starts solves

$$rNPV(\zeta) = \zeta + \xi(NPV(0) - NPV(\zeta)) \tag{104}$$

as in equation (102), but now $NPV(0) = 0$ since the contract ends the next time high productivity is reached. Thus $NPV(\zeta) = \frac{\zeta}{r+\xi}$, which is finite as long as $r > -\xi$. Thus, for the class of contracts that turn out to be optimal (for which the incentive constraint is

[37] In slight abuse of notation, we calculate the costs, given a contract, rather than insisting that $V(\cdot, \cdot)$ are the minimized costs.

binding every time high productivity is realized, effectively resetting the contract), only the restriction $r > -\xi$ rather than the restriction $r > 0$ has to be imposed.

Note that all calculations that lead to Proposition 2 go through under this weaker restriction. During the high-productivity spell (which has a length with exponential distribution with parameter ξ) the intermediary makes expected discounted profits

$$\int_0^\infty (\zeta - c_h) e^{-rt} e^{-\xi t} dt = \frac{\zeta - c_h}{r + \xi} \tag{105}$$

which are finite as long as $r > \xi$. Similarly, the expected discounted cost of the low-productivity spell (in which consumption drifts down at rate $-(\rho - r) < 0$) that starts at random start date t and lasts a random, exponentially distributed (with parameter ν) time τ is given by

$$e^{-rt} \int_0^\infty e^{-r\tau} c_h e^{-(\rho - r)t} e^{-\nu t} dt = \frac{e^{-rt} c_h}{\rho + \nu} \tag{106}$$

and taking expectation with respect to the random time t at which productivity switches from high to low gives the expected cost of the low-productivity spell as

$$\int_0^\infty \frac{e^{-rt} c_h}{\rho + \nu} \xi e^{-\xi t} dt = \frac{\xi c_h}{(\rho + \nu)(r + \xi)} \tag{107}$$

which again is finite as long as $r > -\xi$. Equating expected profits and cost on the contract spell yields

$$\frac{\zeta - c_h}{r + \xi} = \frac{\xi c_h}{(\rho + \nu)(r + \xi)} \tag{108}$$

which yields c_h in Proposition 2 from the main text and shows that the relevant net present value calculations are all finite as long as $r > -\xi$.

The fact that the Poisson rate ξ of a productivity drop is helpful in relaxing the constraint required to keep present discounted values finite is intuitive since it determines the expected length of the initial high-income spell. What is perhaps surprising is that ν does not play a role in keeping the present discounted value of the cost of the low-income spell finite. This is because a low-income spell consumption is discounted at rate r and consumption itself falls at rate $-(\rho - r)$ and the spell ends at rate ν, and thus the effective discount rate is $r + \rho - r + \nu = \rho + \nu$ and thus the present discounted value is finite independent of the interest rate r. This in turn is a reflection of the income effect and substitution effect canceling out with log-utility (and would not be the case for $\sigma \neq 1$).

E.2 Optimal Consumption Insurance Contract

E.2.1 Full Insurance: $r = \rho$

Proof of Proposition 1: If (3) does not bind, then the first two parts of Lemma 7 show that consumption is constant. If (3) does bind, then consumption is locally constant to the right of $\dot{\mathbf{c}}_+(h_{t,s})$, as the third part of Lemma 7 shows. Note that the argument there does not require Assumption 3 in the case that $r = \rho$. The fourth part of the lemma shows that consumption may jump upward upon a state transition. Lemma 1 implies that the jump may occur for a transition from $z = 0$ to $z = \zeta$, but not vice versa. □

The optimal consumption contract is fully characterized by the constant consumption level and associated insurance premium charged to high-income households:

$$\begin{aligned} c_h(\rho) &= \frac{\rho + \nu}{\rho + \nu + \xi}\zeta \\ v_{hl} &= \frac{c_h}{\rho + \nu} > 0 \end{aligned}$$

Corollary 3. *Impose the conditions of Proposition 14. Define the wage-deflated contract costs by $v_h = V(x, U^{out}(\zeta))/w$, if $\mathbf{z}(x) = \zeta$ and $v_l = V(x, U^{out}(0))/w$, if $\mathbf{z}(x) = 0$ and there never was a high income in the past and finally $v_{hl} = V(x, U^{out}(\zeta))/w$, if $\mathbf{z}(x) = 0$, if there was. Then*

$$\begin{aligned} rv_l &= c_l + \nu(v_h - v_l) \\ rv_h &= c_h - \zeta + \xi(v_{hl} - v_h) \\ rv_{hl} &= c_h + \nu(v_h - v_{hl}) \end{aligned}$$

Proof. The transition rates $\alpha_{x,x'}$ correspond to the transition rates from productivity $\mathbf{z}(x)$ to $\mathbf{z}(x')$ per the construction in Appendix C. Note that $\dot{U} = 0$ in Proposition 14 for all U. Rewriting the Hamilton-Jacobi-Bellman equation in Proposition 14 at the optimal consumption choices c_l and c_h yields the equations here. □

E.2.2 Partial Insurance: $r < \rho$

In order to prove Proposition 2, we prove the more general version

Proposition 17. *Suppose that the utility function satisfies Assumption 3.*

1. *Whenever a household has high productivity, it consumes a constant wage-deflated amount* c_h.

2. *When productivity switches to* 0*, consumption is continuous and subsequently drifts down according to the full-insurance Euler equation*

$$\frac{\dot{c}(t)}{c(t)} = -g < 0 \tag{109}$$

where the negative of the consumption growth rate g *satisfies*

$$g = \left(\frac{-u''(wc(t))wc(t)}{u'(wc(t))} \right)^{-1} (\rho - r)$$

If the utility function is of the CRRA variety (81), then $g = (\rho - r)/\sigma$ *as in* (80)*. In that case, let* τ *be the time elapsed, since productivity last switched to* 0*. Then,*

$$c(t) = c_h e^{-g\tau} \tag{110}$$

Proof of Proposition 17. Let $c_h(0; \zeta, U^{out}(\zeta))$ be the consumption level at starting date $t = 0$ in a contract that just delivers the outside option $U^{out}(\zeta)$ at high productivity $z(0) = \zeta$. The limited commitment constraint (3) binds; see the proof of Lemma 3. Alternatively, note that it must bind, since otherwise consumption will drift down according to (76) with $\dot{\lambda} = 0$ and the outside option would be better shortly after $t = 0$, if no further state switch occurred. The third part of Lemma 7 thus implies that consumption is constant while productivity is high. Upon a switch to low productivity, the limited commitment constraint (3) never binds. Thus, the fourth part of Lemma 7 implies that consumption is continuous, that $\dot{\lambda}_+(h_{t,s}) = 0$ and that consumption drifts down according to (76) or (109), applied only to the right-derivatives at the date of the switch. The rest follows with some algebra. □

Corollary 4. *Impose the conditions of Proposition 14. Denote by* τ *the time elapsed since having had high productivity and by* $v_{hl}(\tau)$ *the remaining wage-deflated costs of the contract, at that point. The Hamilton-Jacobi-Bellman equations characterizing the wage-deflated costs in the high-productivity state, the low-productivity state prior to having had*

a high-productivity realization, and after time τ since having had high productivity read as

$$\begin{aligned} rv_h &= c_h - \zeta + \xi(v(0) - v_h) && (111)\\ rv_l &= c_l + \nu(v_h - v_l) && (112)\\ rv_{hl}(\tau) &= c(\tau) + \nu(v_h - v_{hl}(\tau)) + \dot{v}_{hl}(\tau) && (113) \end{aligned}$$

with terminal condition

$$v_{hl}(\infty) = v_l = 0.$$

Proof. The transition rates $\alpha_{x,x'}$ correspond to the transition rates from productivity $\mathbf{z}(x)$ to $\mathbf{z}(x')$ per the construction in Appendix C. Note that $\dot{U} = 0$ in Proposition 14, if $U = U^{out}(\zeta)$ and $U = U^{out}(0)$. Rewriting the Hamilton-Jacobi-Bellman equation in Proposition 14 yields equations (111) and 112. For equation (113), suppose that $U = U(\tau) > U^{out}(0)$, but that $\mathbf{z}(x) = 0$. Rewriting the Hamilton-Jacobi-Bellman equation in Proposition 14 at the optimal consumption choice $c(\tau)$ and exploiting equation (91) yields equation (113) here. □

We proceed to provide the details for the cost calculations, allowing the utility function $u(c)$ to be of the CRRA form (81). Equation (21) is a standard linear ODE. It can be integrated, using the fact that $c(\tau) = c_h e^{-g\tau}$ with $g = (\rho - r)/\sigma$ to obtain

$$v_{hl}(\tau) = \int_\tau^\infty e^{-(r+\nu)(s-\tau)} c_h e^{-gs} ds = c_h e^{-g\tau} \int_\tau^\infty e^{-(r+\nu+g)(s-\tau)} ds = \frac{e^{-g\tau}}{r+\nu+g} c_h \quad (114)$$

either using standard formulas for ODEs or checking the result per differentiating the solution to back out the original differential equation.

We can evaluate (114) at $t = 0$ to obtain[38]

$$v_{hl}(0) = \frac{c_h}{r+\nu+g} \quad (115)$$

The optimal consumption contract has consumption declining at rate $-g = r - \rho$ from c_h toward $c_l = 0$, and asymptotically it reaches $c_l = 0$. Thus the consumption level c_h fully characterizes the consumption contract. Using equation (20) to substitute out $v_{hl}(0)$

[38] Note that this cost $v(0)$ is the counterpart to the insurance cost in equation (13) for the full-insurance case; if $r = \rho$ and thus $g = 0$, $v(0) = v_{hl}$ in (13).

in equation (115) yields

$$\frac{c_h}{r+\nu+g} = \frac{\zeta - c_h}{\xi}$$

or

$$c_h = \frac{r+\nu+g}{r+\nu+g+\xi}\zeta \tag{116}$$

With this, we obtain a generalization of Proposition 3 to the CRRA case.

Proposition 18. *If $\rho > r$, there exists a unique consumption level*

$$c_h = \frac{r+\nu+g}{r+\nu+g+\xi}\zeta$$

which is strictly increasing in ζ and with the following properties:

1. *Agents with currently high productivity receive the wage-deflated consumption c_h.*
2. *Agents with currently low productivity who switched from high productivity τ periods ago receive the wage-deflated consumption*

$$c(t) = c_h e^{-g\tau}$$

Households that never have had high income consume the nontradable endowment $c_l = \chi$ until the first time they receive high income and sign the consumption risk-sharing contract.

E.3 Goods Supply and Capital Demand

Proof of Proposition 6: Calculating the capital stock and wages for Cobb-Douglas production from the production first-order conditions (4) and (5) yields

$$\begin{aligned} K(r) &= \left(\frac{\theta A}{r+\delta}\right)^{\frac{1}{1-\theta}} \\ w(r) &= (1-\theta)AK^{\theta} \end{aligned}$$

and thus

$$\frac{\left[AF_K(K(r),1)-\delta\right]K(r)}{AF_L(K(r),1)} = \frac{r}{(1-\theta)AK(r)^{\theta-1}} = \frac{r\theta}{(1-\theta)(r+\delta)}.$$

With Euler's theorem,

$$\begin{aligned} G(r) &= 1 + \frac{\left[AF_K(K(r),1) - \delta\right] K(r)}{AF_L(K(r),1)} \\ &= 1 + \frac{r\theta}{(1-\theta)\,(r+\delta)} \\ \kappa^d(r) &= \frac{K^d(r)}{w(r)} = \frac{\left[K^d(r)\right]^{1-\theta}}{(1-\theta)A} = \frac{\theta}{(1-\theta)(r+\delta)} \end{aligned}$$

The properties of these functions stated in the main text follow directly from inspection. □

E.4 Capital Supply and Consumption Demand for $r < \rho$

In this section we collect the details of the derivations about the properties of the capital supply function $\kappa^s(r)$ in the partial insurance case. Substitute ζ from equation (1) into equation (24) to obtain

$$c_h = \frac{\nu+\rho}{\xi+\nu+\rho}\,\frac{\xi+\nu}{\nu}$$

Direct calculations and exploiting the explicit functional form of ϕ_r in Proposition 5 reveal that wage-normalized aggregate consumption demand and capital supply are given by

$$\begin{aligned} C(r) &= \frac{\nu}{\nu+\xi}c_h + \int_0^{c_h} c\frac{\xi\nu\,(c_h)^{-\frac{\nu}{\rho-r}}}{(\rho-r)(\nu+\xi)}c^{\frac{\nu}{\rho-r}-1}dc = \frac{\nu}{\nu+\xi}\frac{\nu+\rho-r+\xi}{\nu+\rho-r}c_h \\ &= \frac{\nu+\rho-r+\xi}{\nu+\rho-r}\frac{\nu+\rho}{\nu+\rho+\xi} \\ &= \left(1+\frac{\xi}{\nu+\rho-r}\right)\left(1-\frac{\xi}{\nu+\rho+\xi}\right) \\ &= 1+\frac{\xi}{\nu+\rho-r}-\frac{\xi}{\nu+\rho+\xi}-\frac{\xi^2}{(\nu+\rho+\xi)(\nu+\rho-r)} \\ &= 1+\frac{r\xi}{(\nu+\rho+\xi)(\nu+\rho-r)} \\ \kappa^s(r) &= \frac{\xi}{(\nu+\rho+\xi)\,(\nu+\rho-r)} \end{aligned}$$

Proof of Proposition 7: It follows immediately from the equations above that the function $\kappa^s(r)$ is continuously differentiable and strictly increasing on $[-\delta,\rho)$. Aggregate consump-

tion demand and capital supply are continuous in the interest rate at $r = \rho$ since

$$\lim_{r \nearrow \rho} C(r) = 1 + \frac{\rho\xi}{\nu(\rho + \nu + \xi)} = C(r = \rho) \tag{117}$$

$$\lim_{r \nearrow \rho} \kappa^s(r) = \frac{\xi}{\nu(\xi + \nu + \rho)} = \kappa^s(r = \rho) \tag{118}$$

coincide with the values in equations (31) and (32) for the full-insurance case $r = \rho$. □

F General CRRA Utility

The analysis for the full-insurance case goes through completely unchanged, since at $r = \rho$ the growth rate of consumption and thus the aggregate consumption demand and capital supply are unaffected by the intertemporal elasticity of substitution $1/\sigma$. Here we focus on the case $\rho > r$.

F.1 Optimal Consumption Contract

As in the log-case, whenever a household has high income, it consumes c_h, and when income switches to 0, consumption drifts down according to the full-insurance Euler equation[39]

$$\frac{\dot{c}(t)}{c(t)} = -\frac{\rho - r}{\sigma} = -g < 0$$

where the growth rate of consumption is now defined as

$$g = \frac{\rho - r}{\sigma} > 0.$$

The log-utility case is of course just a special case where $\sigma = 1$ and thus $g = \rho - r$.

The steps of deriving the optimal consumption contract and associated cost then proceeds completely in parallel to the log-case. Consumption is given as

$$c(t) = c_h e^{-gt} \tag{119}$$

[39] The proof of Proposition 2 in Appendix E.2.2 is conducted for a general CRRA function and thus applies here.

and the cost of the contract is given by

$$v(t) = \frac{c_h e^{-gt}}{r + \nu + g} \tag{120}$$

Evaluating (22) at $t = 0$ gives

$$v(0) = \frac{c_h}{r + \nu + g} \tag{121}$$

Using equation (20), which continues to hold unchanged, to substitute out $v(0)$ into equation (23) yields

$$\frac{c_h}{r + \nu + g} = \frac{\zeta - c_h}{\xi}$$

or

$$c_h(r) = \frac{r + \nu + g}{r + \nu + g + \xi}\zeta = \frac{1}{1 + \frac{\xi}{r\left(1 - \frac{1}{\sigma}\right) + \nu + \frac{\rho}{\sigma}}}\zeta \tag{122}$$

Note that $c_h(r)$ together with $v(t) > 0$ requires $r + \nu + g > 0$. For $\sigma \leq 1$ this is satisfied for all $r \leq \rho$. For $\sigma > 1$, this is satisfied at $r = -\delta$ and thus for all $r \in [-\delta, \rho]$, if condition (45) of Proposition 10 holds. For $\sigma \to \infty$, condition (45) becomes $\delta < \nu$.

F.2 Invariant Consumption Distribution

As in the log-case, on $c \in (0, c_h)$ the consumption process follows a diffusion process with drift $-g$ (and no variance) and thus on $(0, c_h)$ the stationary consumption distribution satisfies the Kolmogorov forward equation (for the case of Poisson jump processes):

$$0 = -\frac{d\left[-gc\phi(c)\right]}{dc} - \nu\phi(c)$$

where the second term comes from the fact that with Poisson intensity ν the household has a switch to high income. Since

$$-\frac{d\left[-gc\phi(c)\right]}{dc} = -\left[-g\phi(c) - gc\phi'(c)\right] = g\left[\phi(c) + c\phi'(c)\right]$$

we find that on $c \in (0, c_h)$ the stationary distribution satisfies

$$\frac{c\phi'(c)}{\phi(c)} = \frac{\nu}{g} - 1$$

and thus on this interval the stationary consumption distribution is Pareto with tail parameter $\kappa = \frac{\nu}{g} - 1$, that is

$$\phi(c) = \phi_1 c^{\left(\frac{\nu}{g}-1\right)}$$

where ϕ_1 is a constant that needs to be determined. Now we need to determine the constant ϕ_1. Because of the mass point at c_h it is easier to think of the cdf for consumption on $(0, c_h)$ given by $\Phi(c) = \frac{\phi_1 (c)^{\kappa+1}}{\kappa+1}$. The inflow mass into this range is given by the mass of individuals at c_h given by $\phi(c_h) = \frac{\nu}{\nu+\xi}$ times the probability ξ of switching to the low-income state, whereas the outflow is due to receiving the high-income shock, and thus the stationary cdf has to satisfy

$$\nu \Phi(c_h) = \frac{\xi \nu}{\nu + \xi}$$

and therefore

$$\nu \frac{\phi_1 \left(c_h\right)^{\kappa+1}}{\kappa + 1} = \frac{\xi \nu}{\nu + \xi}$$

Exploiting the fact that $\kappa + 1 = \frac{\nu}{g}$ we find

$$\phi_1 g \left(c_h\right)^{\frac{\nu}{g}} = \frac{\xi \nu}{\nu + \xi}$$

and thus

$$\phi_1 = \frac{\xi \nu \left(c_h\right)^{-\frac{\nu}{g}}}{g(\nu + \xi)}$$

and therefore the density on $(0, c_h)$ is given by

$$\phi(c) = \frac{\xi \nu \left(c_h\right)^{-\frac{\nu}{g}}}{g(\nu + \xi)} c^{\frac{\nu}{g}-1}.$$

Therefore the stationary consumption distribution is now given by:

$$\phi_r(c) = \begin{cases} \frac{\xi \nu (c_h(r))^{-\frac{\nu}{g}}}{g(\nu+\xi)} c^{\frac{\nu}{g}-1} & \text{if} \quad c \in (0, c_h) \\ \frac{\nu}{\nu+\xi} \boldsymbol{\delta}_{c_h} & \text{if} \quad c = c_h \end{cases}$$

where $\boldsymbol{\delta}_{c_h}$ indicates a Dirac mass point at c_h. Thus, for a given interest rate r the invariant consumption distribution is completely characterized by the upper bound $c_h(r) = \frac{r+\nu+g}{r+\nu+g+\xi}\zeta$.

F.3 Equilibrium

We now determine the aggregate consumption demand $C(r)$ and the normalized capital supply function $\kappa^s(r)$. Direct calculations reveal that aggregate consumption demand and capital supply are given by:

$$\begin{aligned}
C(r) &= \frac{\nu}{\nu+\xi}c_h(r) + \int_0^{c_h(r)} c\frac{\xi\nu\left(c_h(r)\right)^{-\frac{\nu}{g}}}{g(\nu+\xi)}c^{\frac{\nu}{g}-1}dc = \frac{\nu}{\nu+\xi}\frac{\xi+\nu+g(r)}{\nu+g(r)}c_h(r) \\
&= \frac{\xi+\nu+g(r)}{\nu+g(r)}\frac{r+\nu+g(r)}{\xi+r+\nu+g(r)} \\
&= \left(1+\frac{\xi}{\nu+g(r)}\right)\left(1-\frac{\xi}{\xi+\nu+g(r)+r}\right) \\
&= 1+\frac{\xi}{\nu+g(r)}-\frac{\xi}{\xi+\nu+g(r)+r}-\frac{\xi^2}{(\xi+\nu+g(r)+r)(\nu+g(r))} \\
&= 1+\frac{r\xi}{(\xi+\nu+g(r)+r)(\nu+g(r))} \\
\kappa^s(r) &= \frac{\xi}{\left(\xi+\nu+\frac{\rho-r}{\sigma}+r\right)\left(\nu+\frac{\rho-r}{\sigma}\right)} \qquad (123)
\end{aligned}$$

where we have repeatedly used $g(r) = \frac{\rho-r}{\sigma}$.

F.3.1 Proof of Proposition 10

Proof. The first step of the proof is to establish that the normalized capital supply function is well-defined and continuous on $r \in [-\delta, \rho]$. The previous section gave $\kappa^s(r)$ in closed form, and it is evidently continuous and well-defined on $[-\delta, \rho]$ as long as both terms of the denominator are strictly positive. Since $r \leq \rho$, the second term in the denominator of equation (123) is always strictly positive. The first term is always positive for $\sigma \leq 1$ and $r \leq \rho$. For $\sigma > 1$, it is positive for $r \geq \delta$ due to condition (45). That condition is also needed for $c_h > 0$ and $v(t) > 0$; see the remarks at the end of Appendix F.1.

Since by Assumption 2 we have $\kappa^s(r = \rho) > \kappa^d(r = \rho)$ and since $\kappa^s(r = -\delta) < \infty = \kappa^d(r = -\delta)$, it follows that κ^s and κ^d intersect at least once in $(-\delta, \rho)$. This establishes the existence of a stationary equilibrium.

The uniqueness of equilibrium follows if $\kappa^s(r)$ is increasing (given that $\kappa^d(r)$ is strictly

decreasing). The derivative of $\kappa^s(r)$ is given by

$$\frac{d\kappa^s(r)}{dr} = \xi \frac{\left[\frac{2}{\sigma} - 1\right]\left[\frac{\rho - r}{\sigma} + \nu\right] + \frac{\xi + r}{\sigma}}{\left[\left(\xi + \nu + \frac{\rho - r}{\sigma} + r\right)\left(\nu + \frac{\rho - r}{\sigma}\right)\right]^2}$$

A sufficient condition for this expression to be positive is $\sigma < 1$ (part 1 of the proposition) or $\sigma \in (1, 2]$ and $\xi \geq \delta$ (part 2a of the proposition). Part 2b follows from the fact that equation (43) is a quadratic equation, and thus has at most two solutions (and we have already established that under the assumptions made it has at least one solution). The numerical example in the main text shows that the statement in 2b of the proposition is not vacuous. □

F.3.2 Proof of Corollary 2

Proof. In general equilibrium interest rates are real-valued solutions to the quadratic equation

$$0 = F(r) \equiv A_2 r^2 + A_1 r + A_0 \tag{124}$$

where

$$\begin{aligned}
A_0 &= (\sigma - 1)^2 \left[\xi\delta - \theta\nu^2 - \xi\theta(\delta + \nu)\right] + (\sigma - 1)\left[-2\theta\nu(\nu + \rho) - \xi(2\delta(\theta - 1) + \theta(2\nu + \rho))\right] \\
&\quad -\theta(\nu + \rho)^2 - \xi(\delta(\theta - 1) + \theta(\nu + \rho)) \\
A_1 &= -(\sigma - 1)^2(\theta(\nu + \xi) - \xi) - (\sigma - 1)(\theta(\rho + \xi) - 2\xi) + \theta(\rho + \nu) + \xi \\
A_2 &= \theta(\sigma - 1)
\end{aligned}$$

The coefficients A_0, A_1, A_2 defined above are functions of the parameters. Note that

$$\begin{aligned}
A_0(\alpha\rho, \alpha\delta, \alpha\xi, \alpha\nu; \sigma, \theta) &= \alpha^2 A_0(\rho, \delta, \xi, \nu; \sigma, \theta) \\
A_1(\alpha\rho, \alpha\xi, \alpha\nu; \sigma, \theta) &= \alpha A_1(\rho, \xi, \nu; \sigma, \theta)
\end{aligned}$$

and $A_2(\sigma, \theta)$ does not depend on ρ, δ, ξ, ν. Define

$$F(r; \alpha) = A_2(\sigma, \theta) r^2 + A_1(\alpha\rho, \alpha\xi, \alpha\nu; \sigma, \theta) r + A_0(\alpha\rho, \alpha\delta, \alpha\xi, \alpha\nu; \sigma, \theta)$$

Then

$$\alpha^2 F(r;1) \;=\; F(\alpha r;\alpha)$$

Hence, if $\bar{r}$ solves $F(\bar{r};1)=0$, then $r=\alpha\bar{r}$ solves $F(r;\alpha)=0$. □

G Superinsurance

In this appendix we characterize the optimal consumption insurance contract when the interest rate r exceeds the rate of time preference ρ, that is, $r>\rho$, and then discuss the possibility of a stationary distribution associated with that consumption contract.

G.1 The Optimal Contract for Superinsurance: $\rho < r$

If the limited commitment constraint is not binding, as in the partial-insurance case, consumption grows at a constant rate,

$$c_h(t) = c_h(0)e^{(r-\rho)t}$$

but now $\rho > r$; that is, consumption grows at a positive rate. As in the full and partial-insurance case, households born with low income cannot obtain insurance until their income switches to ζ, at which point it jumps to $c_h(0)$, as in the partial and full-insurance cases. From that point on, the household obtains income insurance (as in the full insurance case), but now consumption grows at rate $r-\rho>0$ (rather than remaining constant), until the household dies. The level $c_h(0)$ is determined by the zero profit condition of the intermediary, equating the expected revenue from the household's income stream with the expected cost of the consumption contract.

To determine this level, $c_h(0)$, first calculate the present discounted revenue (a_l, a_h) for the intermediary from a currently productive and currently unproductive individual (normalized by the wage) as follows. These PDV revenues satisfy

$$\begin{aligned} ra_h(t) &= \zeta + \xi(a_l(t)-a_h(t)) + \dot{a}_h(t) \\ ra_l(t) &= \nu(a_h(t)-a_l(t)) + \dot{a}_l(t) \end{aligned}$$

Evidently these two functions do not depend on time and solve

$$\begin{aligned} ra_h &= \zeta + \xi(a_l - a_h) \\ ra_l &= \nu(a_h - a_l) \end{aligned}$$

Solving yields

$$\begin{aligned} a_h &= \frac{r+\nu}{r(r+\nu+\xi)}\zeta \\ a_l &= \frac{\nu}{r(r+\nu+\xi)}\zeta \end{aligned}$$

both of which are finite since $r > \rho > 0$. Now we derive the present discounted value for the cost of the contract that starts at entry consumption $c_h(0)$ and grows at rate $r - \rho > 0$ over time. This gives

$$\kappa = \int_0^\infty e^{-r\tau} c_h e^{(r-\rho)\tau} d\tau = \frac{c_h(0)}{\rho}$$

Equating $\kappa = a_h$ delivers

$$c_h(0) = \left(\frac{\rho}{r} \cdot \frac{r+\nu}{r+\nu+\xi}\right)\zeta < c_h(\rho) \tag{125}$$

Note that the entry-level consumption in this case is smaller than in the full-insurance case $r = \rho$ in order to compensate for the higher cost of growing consumption. The household pays an insurance premium $\zeta - c_h(0)$ in exchange for future consumption insurance and consumption growth. Note that since $r > \rho$ and consumption grows along the contract, the insurance premium must be larger (and initial consumption $c_h(0)$ smaller) than in the full-insurance case $(r = \rho)$ to finance future consumption growth, and as the interest rate r converges to the time discount rate ρ from above, the entry-level consumption and the insurance premium converge to the full-insurance consumption level $c_h(\rho)$ from below.

G.2 A Stationary Consumption Distribution?

Although we can characterize the optimal consumption insurance contract in this case, since, conditional on having received the high income, once the consumption of all individuals continues to drift up at the constant (and identical) rate $r - \rho$, there is no stationary consumption distribution for the case $r > \rho$, and thus we can discard this case as a possi-

bility for a *stationary* equilibrium.

Formally, all households experiencing a jump to high income jump to $c_h(0)$ and immediately their consumption drifts up at rate $r - \rho > 0$, so there is no mass point at $c_h(0)$. Instead, there is a continuous consumption density on $[c_h(0), \infty)$ with power and scale parameters that need to determined in the same way as we did for the $r < \rho$ case.

In $c \in [c_h(0), \infty)$ the consumption process follows a diffusion process with drift $r - \rho > 0$ (and no variance) and thus on this interval the stationary consumption distribution satisfies the Kolmogorov forward equation

$$0 = -\frac{d\left[-gc\phi(c)\right]}{dc}$$

Since

$$-\frac{d\left[-gc\phi(c)\right]}{dc} = -\left[-g\phi(c) - gc\phi'(c)\right] = g\left[\phi(c) + c\phi'(c)\right]$$

we find that on $c \in (c_h(0), \infty)$ the stationary distribution satisfies

$$0 = g\left[\phi(c) + c\phi'(c)\right]$$

and thus the consumption distribution is Pareto on $[c_h(0), \infty)$ with power

$$-\frac{c\phi'(c)}{\phi(c)} = 1$$

But this implies that stationary aggregate consumption

$$\int_{c_h}^{\infty} c\phi(c)dc = \infty$$

(as a Pareto distribution with tail parameter 1 has infinite mean) and thus no stationary consumption distribution with finite aggregate consumption can exist in the case of $\rho < r$, ruling out the existence of a stationary equilibrium in this case.

H Welfare in Stationary Equilibrium for IES $\sigma \neq 1$

The wage-deflated consumption allocation in a partial-insurance stationary equilibrium is given by

$$\begin{aligned} c(t) &= c_h(r)e^{-g(r)t} \\ c_h(r) &= \frac{1}{1+\frac{\xi}{r+\nu+g(r)}}\zeta = \frac{1}{1+\frac{\xi}{r+\nu+\frac{\rho-r}{\sigma}}}\zeta = \frac{1}{1+\frac{\xi}{\rho+\nu+(\rho-r)\left(\frac{1}{\sigma}-1\right)}}\zeta \\ g(r) &= \frac{\rho-r}{\sigma} \end{aligned}$$

H.1 Lifetime Utility for Given Interest rate r

Expected lifetime utility is the weighted sum of lifetime utility from being born with low (no) income and being born with high income z. It is given, for interest rate r, by

$$EU(r) = \frac{\xi U_l(r) + \nu U_h(r)}{\xi + \nu}$$

where $U_i(r)$ is lifetime utility being born with income $i = l, h$. For the low-income state lifetime utility is given by

$$\rho U_l(r) = \underline{u} + \nu(U_h(r) - U_l(r))$$

and thus

$$U_l(r) = \frac{\underline{u} + \nu U_h(r)}{\rho + \nu}$$

Thus

$$EU(r) = \frac{\xi\frac{\underline{u}+\nu U_h(r)}{\rho+\nu} + \nu U_h(r)}{\xi+\nu} = \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)}U_h(r)$$

and lifetime utility is linear in lifetime utility conditional on being born with high income.

For being born with high income (for now, suppressing dependence on r), lifetime utility is given by

$$\rho U_h = u(w(r)c_h(r)) + \xi(U(0) - U_h)$$

where $U(t)$ is the lifetime continuation utility from the consumption contract after having

had low income for t units of time. It is given by the differential equation

$$\rho U(t) = u(w(r)c_h(r)e^{-g(r)t}) + \nu(U_h - U(t)) + \dot{U}(t)$$

Now define

$$\begin{aligned} u(t) &= \frac{U(t)}{w(r)^{1-\sigma}} \\ u_h(r) &= \frac{U_h(r)}{w(r)^{1-\sigma}} \end{aligned}$$

as wage-deflated lifetime utility. Lifetime utility can be decomposed in this way since the period utility function is CRRA (and thus lifetime utility is homothetic), and the aggregate wage is constant over time in a stationary equilibrium, and can be expressed as a function of the interest rate r (and exogenous parameters) only. The so-defined deflated lifetime utility function follows the Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \rho u_h(r) &= u(c_h(r)) + \xi(u(0) - u_h(r)) \\ \rho u(t) &= u(c_h(r))e^{-(1-\sigma)g(r)t} + \nu(u_h - u(t)) + \dot{u}(t) \end{aligned}$$

or rewriting the second equation

$$\dot{u}(t) = (\rho + \nu)u(t) - u(c_h(r))e^{-(1-\sigma)g(r)t} - \nu u_h$$

Solving the differential equation (one can differentiate with respect to time t using Leibnitz' rule to check that the solution is correct) yields, for now suppressing the dependence of $u_h(r)$ on r:

$$u(t) = \int_t^\infty e^{-(\rho+\nu)(s-t)} \left[\nu u_h + u(c_h(r))e^{-(1-\sigma)g(r)s}\right] ds.$$

Evaluating at $t = 0$ one obtains

$$\begin{aligned} u(0) &= \int_0^\infty e^{-(\rho+\nu)s}\left[\nu u_h + u(c_h(r))e^{-(1-\sigma)g(r)s}\right] ds \\ &= \nu u_h \int_0^\infty e^{-(\rho+\nu)s} ds + u(c_h(r)) \int_0^\infty e^{-[\rho+\nu+(1-\sigma)g(r)]s} ds \\ &= -\frac{\nu u_h}{\rho+\nu}\, e^{-(\rho+\nu)s}\Big|_0^\infty - \frac{u(c_h(r))}{\rho+\nu+(1-\sigma)g(r)}\, e^{-[\rho+\nu+(1-\sigma)g(r)]s}\Big|_0^\infty \\ &= \frac{\nu u_h}{\rho+\nu} + \frac{u(c_h(r))}{\rho+\nu+(1-\sigma)g(r)} \end{aligned}$$

and thus the two equations

$$\begin{aligned} u(0) &= \frac{\nu u_h}{\rho+\nu} + \frac{u(c_h(r))}{\rho+\nu+(1-\sigma)g(r)} \\ (\rho+\xi)u_h &= u(c_h(r)) + \xi u(0) \end{aligned}$$

can be solved for $u_h, u(0)$. This delivers

$$\begin{aligned} (\rho+\xi)u_h &= u(c_h(r)) + \frac{\xi\nu u_h}{\rho+\nu} + \frac{\xi u(c_h(r))}{\rho+\nu+(1-\sigma)g(r)} \\ \left[1+\frac{\xi}{\rho+\nu}\right]\rho u_h &= \left[1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r)}\right] u(c_h(r)) \end{aligned}$$

and thus

$$u_h(r) = \frac{1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r)}}{1+\frac{\xi}{\rho+\nu}} \frac{u(c_h(r))}{\rho} = \frac{1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r)}}{\left(1+\frac{\xi}{\rho+\nu}\right)\rho} \frac{c_h(r)^{1-\sigma}}{1-\sigma}$$

and

$$EU(r) = \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)} w(r)^{1-\sigma} u_h(r)$$

Now suppose we scale consumption in all periods by a factor $\alpha > 0$. Expected lifetime utility from this scaled consumption process, denoted by $EU(r;\alpha)$, is given by

$$\begin{aligned} EU(r;\alpha) &= \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)} w(r)^{1-\sigma} u_h(r;\alpha) \\ &= \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \alpha^{1-\sigma}\frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)} w(r)^{1-\sigma} u_h(r;1) \end{aligned}$$

since

$$u_h(r;\alpha) = \frac{1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r)}}{\left(1+\frac{\xi}{\rho+\nu}\right)\rho}\frac{[\alpha c_h(r)]^{1-\sigma}}{1-\sigma} = \alpha^{1-\sigma}\frac{1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r)}}{\left(1+\frac{\xi}{\rho+\nu}\right)\rho}\frac{[c_h(r)]^{1-\sigma}}{1-\sigma} = \alpha^{1-\sigma}u_h(r;1)$$

H.2 Comparing Welfare Across Equilibria with Interest Rates r_1, r_2

Now we want to compare welfare across two interest rates. For that, we ask by what factor α we have to scale equilibrium consumption under interest rate r_1 so that the household is indifferent to living under interest rate $r_2 > r_1$. That is, we are looking for α such that

$$EU(r_1;\alpha) = EU(r_2;1)$$

where $\alpha < 1$ indicates that the low interest rate equilibrium is preferred, and $\alpha > 1$ indicates that the high interest rate equilibrium is preferred. Using the results from the previous section, we solve for α such that

$$\begin{aligned} & \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \alpha^{1-\sigma}\frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)}w(r_1)^{1-\sigma}u_h(r_1;1) \\ = \quad & \frac{\xi}{(\xi+\nu)(\rho+\nu)}\underline{u} + \frac{(\xi+\rho+\nu)\nu}{(\xi+\nu)(\rho+\nu)}w(r_2)^{1-\sigma}u_h(r_2;1) \end{aligned}$$

and thus, using the expression for $c_h(r) = \frac{1}{1+\frac{\xi}{r\left(1-\frac{1}{\sigma}\right)+\nu+\frac{\rho}{\sigma}}}z$ from equation (24):

$$\alpha = \frac{w(r_2)}{w(r_2)}\cdot\left[\frac{u_h(r_2;1)}{u_h(r_1;1)}\right]^{\frac{1}{1-\sigma}} \tag{126}$$

$$= \frac{w(r_2)}{w(r_1)}\cdot\left[\frac{\left(1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r_2)}\right)}{\left(1+\frac{\xi}{\rho+\nu+(1-\sigma)g(r_1)}\right)}\right]^{\frac{1}{1-\sigma}}\frac{c_h(r_2)}{c_h(r_1)} \tag{127}$$

$$= \frac{w(r_2)}{w(r_1)}\cdot\left[\frac{c_h(r_2)}{c_h(r_1)}\right]^{\frac{\sigma}{\sigma-1}} = \alpha_{wage}\cdot\alpha_{contract} \tag{128}$$

where

$$\begin{aligned} \alpha_{wage} &= \frac{w(r_2)}{w(r_1)} \\ \alpha_{contract} &= \left[\frac{c_h(r_2)}{c_h(r_1)}\right]^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

A higher interest rate means a lower capital stock and thus lower wages. Therefore, unambiguously,

$$\alpha_{wage} = \frac{w(r_2)}{w(r_1)} < 1.$$

The second term captures lifetime utility from the *wage-deflated* consumption contract:

$$\alpha_{contract} = \left[\frac{c_h(r_2)}{c_h(r_1)}\right]^{\frac{\sigma}{\sigma-1}} > 1$$

A higher interest rate leads to a better consumption contract, since a higher interest rate is associated with better consumption insurance (consumption starts higher and falls less slowly).[40]

Given that $\alpha_{contract} > 1$ and $\alpha_{wage} < 1$, the overall welfare term α can be smaller or larger than 1. Since both $\alpha_{contract}, \alpha_{wage}$ are closed-form expressions of the two equilibrium interest rates, and these in turn are closed-form solutions of a quadratic equation, we could in principle give conditions on parameters under which the low interest rate yields higher welfare, and alternative conditions under which the reverse is true. However, that these parameter sub-spaces are both nonempty can also be verified numerically.

Note that we could also have defined welfare as expected period utility in the steady

[40]For $\sigma > 1$, we have $c_h(r_2) > c_h(r_1)$ and for $\sigma < 1$ the reverse is true (but then the ratio is taken to a negative exponent.)

state. Doing so, we obtain, when scaling consumption by a constant α

$$\begin{aligned} W(r,\alpha) &= \int_0^{c_h(r)} \frac{(\alpha w(r)\, c)^{1-\sigma}}{1-\sigma}\phi(c,r)\,dc + \frac{(\alpha w(r)\, c_h(r))^{1-\sigma}}{1-\sigma}\phi(c_h,r) \\ &= \int_0^{c_h(r)} \frac{(\alpha w(r)\, c)^{1-\sigma}}{1-\sigma}\frac{\xi\nu}{(\xi+\nu)\, g(r)} c_h(r)^{-\frac{\nu}{g(r)}}\, c^{\frac{\nu}{g(r)}-1} dc \\ &\quad + \frac{(\alpha w(r)\, c_h(r))^{1-\sigma}}{1-\sigma}\frac{\nu}{\xi+\nu} \\ &= \frac{(\alpha w(r))^{1-\sigma}}{1-\sigma}\frac{\nu}{\xi+\nu} c_h(r)^{1-\sigma}\left(1+\frac{\xi}{\nu+(1-\sigma)\, g(r)}\right) \end{aligned}$$

Again comparing welfare across two equilibria yields

$$\begin{aligned} & W(r_1,\alpha) = W(r_2,1) \\ \frac{(\alpha w(r_1))^{1-\sigma}}{1-\sigma} c_h(r_1)^{1-\sigma}\left(1+\frac{\xi}{\nu+(1-\sigma)\, g(r_1)}\right) &= \\ \frac{(w(r_2))^{1-\sigma}}{1-\sigma} c_h(r_2)^{1-\sigma}\left(1+\frac{\xi}{\nu+(1-\sigma)\, g(r_2)}\right) & \end{aligned}$$

$$\begin{aligned} \hat{\alpha} &= \frac{w(r_2)}{w(r_1)}\left(\frac{1+\frac{\xi}{\nu+(1-\sigma)g(r_2)}}{1+\frac{\xi}{\nu+(1-\sigma)g(r_1)}}\right)^{\frac{1}{1-\sigma}} \frac{c_h(r_2)}{c_h(r_1)} \\ &= \alpha_{wage}\cdot\hat{\alpha}_{contract} \end{aligned}$$

where

$$\begin{aligned} \alpha_{wage} &= \frac{w(r_2)}{w(r_1)} \\ \hat{\alpha}_{contract} &= \left[\frac{\left(1+\frac{\xi}{\nu+(1-\sigma)g(r_2)}\right)}{\left(1+\frac{\xi}{\nu+(1-\sigma)g(r_1)}\right)}\right]^{\frac{1}{1-\sigma}} \cdot \frac{c_h(r_2)}{c_h(r_1)} \end{aligned}$$

This alternative welfare measure therefore results in a similar decomposition. The aggregate wage factor α_{wage} is exactly the same, and the contract factor only differs by discounting. Thus both welfare measures give similar welfare comparisons; the second just contains an additional time discounting term, since in the first measure every contract starts with c_h, and in the other, the agent is randomly placed in the stationary consumption distribution.

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