Popoola, Osuolale Peter

Doctoral Thesis

Construction of Congruent Classes of Pairwise Balanced Design Using Lotto Design

CONSTRUCTION OF CONGRUENT CLASSES OF PAIRWISE BALANCED DESIGN USING LOTTO DESIGN

BY

POPOOLA, OSUOLALE PETER
HND, PGD (Statistics), PGD (Comp.Sc.), M. Sc. (Statistics)
MATRIC NO: 13/68DV003

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CERTIFICATION

This is to certify that this thesis is an original work carried out by Osuolale Peter POPOOLA with Matriculation Number 13/68DV003 in the Department of Statistics, Faculty of Physical Sciences, University of Ilorin. The thesis has been read and approved as meeting the requirements for the award of Doctor of Philosophy in Statistics of the University of Ilorin, Ilorin, Nigeria.

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Prof. B. A. Oyejola Date
(Supervisor)

---------- ----------

Dr. G. M. Oyeyemi Date
(Departmental Postgraduate Coordinator)

---------- ----------

Prof. A.O Adejumo Date
(Head of Department)

……. …….

Internal Examiner

…….. Date

External Examiner

DEDICATION

This thesis is dedicated to the Glory of God the Father, the Son and the Holy Spirit who has made it possible for me to accomplish this task and to the blessed memory of my Parents Late Mr. James Olabode and Late Mrs. Christiana Aduke Popoola.

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Abstract

Balanced Incomplete Block Designs (BIBDs) are commonly used when the number of treatments in an experiment is less than the block size. However, there are cases when the block sizes available for an experiment are not the same. Hence the use of Pairwise Balanced Design (PBD). A PBD (n, **K**, λ) is a block design where n is the number of treatments, **K**= $\{k_1, k_2 \ldots, k_b\}$ is the set of sizes of block and λ is number of time a pair of treatments appears together within blocks. Also, little is known about the construction of PBDs using Lotto Designs (LDs). Methods of constructing PBDs in literature are complex. The aim of this study therefore, was to provide a simple method for constructing PBDs when **K**= {3, 4} and {3, 4, 5}. The specific objectives were to: (i) investigate various methods of constructing PBDs (ii) establish conditions for the identification of LDs that qualify as PBDs; and (iii) provide simple method for constructing two classes of PBDs from LDs;

The study utilized the Li's inequality $\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \left(pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1) \right) < \binom{p}{2}\lambda$ to obtain LDs that are PBDs where p is the set of treatments that can intersect relevant k-blocks of an LD, t is the number of treatments in p that match k-blocks, r is the number of time a particular treatment appears in each block and λ is the number of time a pair of treatment appear within block. Some conditions were imposed on the generated LDs to obtain those that were qualified as PBDs. Hence, two classes of PBDs were constructed by partitioning n - treatments into set of blocks when K = 3, 4, 5 such that each pair of treatments is contained in precisely one block. A program to do this was written using C^{++}.

The following results were obtained:

(i) Various methods investigated shows that certain PBDs is difficult to construct especially when the number of treatment is large also, there is no standard techeque to generate block sizes of the PBDs.

(ii) Any LD(n, k, p, t) satisfying the Li inequality, k = 3, 4, 5 and n = p qualified as PBDs.

(iii) (a) 2-LDs(n, 3, p, 3), (n, 4, p, 4) can be used to construct PBD(n, {3, 4}) provided $n \equiv 0, 1 \pmod 3$

(b) 3-LDs(n, 3, p, 3), (n, 4, p, 4), (n, 5, p, 5) can be used to construct PBD(n, {3, 4, 5}) provided $n \equiv 2, 3 \pmod 4$ by partitioning n-treatments into block of sets K = 3, 4, 5 such that each pair of treatments is contained in precisely one block.

This study concluded that congruence classes of PBDs(n, {3, 4}, 1) and (n, {3, 4, 5}, 1) could be constructed from the appropriate LDs if the specified conditions hold. The study recommends the use of PBDs for constructing other important designs, and the use of LDs to construct PBDs.

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CHAPTER ONE
INTRODUCTION

1.0 Background of the Study

The modern foundations of Design of Experiments were laid by R. A. Fisher during the early part of the 20th century. Since then, design of Experiment has seen a phenomenal growth. Design of experiments has for long been an integral part of almost all scientific investigations and continues to be so. It has therefore played a fundamental role in statistical practice and research. Statistical training also has always emphasized the role of design of experiments in extracting correct information and making valid inference on the underlying problem and thus, design of experiments is an essential component of most statistics curricula (Cochran and Cox, 1957).

Design theory has its roots in recreational mathematics. Many types of designs that are studied today were first considered in the context of mathematical puzzles or brain-teasers in the eighteenth and nineteenth centuries. The study of design theory as a mathematical discipline really began in the twentieth century due to applications in the design and analysis of statistical experiments. Designs have many other applications as well, such as tournament scheduling, lotteries, mathematical biology, algorithm design and analysis, networking, group testing, and cryptography (Stinson, 2003).

In certain experimental situations, there may be systematic variations present among the experimental units. For example, in a field experiment, the experimental units are typically plots of land. In such an experiment, there may be a fertility gradient present such that plots on the same fertility level are more homogeneous than those which are at different fertility levels. In experiments with piglets as experimental units, it is very plausible that piglets belonging to the same litter are genetically closer to each other (being born to the same pair of parents) than those belonging to different litters. In the context of clinical trials with

patients forming the experimental units, the trial may be conducted at different centers (mainly to get enough number of observations) and patients from the same center may be more alike than those from different centers due to differences in treatment practices and or management procedures followed at different centers. The above examples, which are merely illustrative and by no means exhaustive, demonstrate that in many situations there is a systematic variation among the experimental units. In such situations, the use of a Completely Randomized Design (CRD) is not appropriate. Rather, one should take advantage of the a priori information about this systematic variation while designing the experiment in the sense that this information should be used while designing to eliminate the effect of such variability. The impact of this effort will be reflected in a reduced error, thereby increasing the sensitivity of the experiment. The above considerations led to the notion of local control or blocking. The groups of relatively homogeneous experimental units are called block*s*. When the blocking is done according to one attribute, we get a block design (Hinkelmann and Kempthorne, 1994). In a block design, the treatments are applied randomly to the experimental units within a block, the randomized allocation of treatments to experimental units within a block being done independently in each block (Dey, 1986). The simplest among the block designs is the Randomized Complete Block Design (RCBD). In such a design, each block is required to have as many experimental units as the number of treatments, i.e., the block size is equal to the number of treatments. However, it is not always possible to adopt a Randomized Complete Block Design in every experimental situation. Firstly, if one assumes that the intra-block variance is directly dependent on the block size, then adoption of a design with blocks of small sizes is preferable over one which has large block sizes. This restricts the use of Randomized Complete Block Designs in situations where the number of treatments is large. For example, in agronomic experiments, the experimenter generally chooses different block of sizes say 4, 5, 6 for his experiment, and if

this is accepted, then, one cannot adopt a randomized complete block design in situations where 20 treatments are to be allocated to different block sizes. Furthermore, in many experimental situations, the block size is determined by the nature of the experiment. For example, with some experiments in psychology, it is quite common to consider the two members of a twin pair as experimental units of a block. In that case, clearly a randomized complete block design cannot be prescribed if the number of treatments is larger than two. Similarly, it is reasonable to take litter-mates (of say mice) as units of a block and litter size may not be adequate to accommodate all the treatments under test. The few examples considered above clearly show that in many situations, one cannot adopt a randomized complete block design and thus, there is a need to look for designs where not all the treatments may appear in each block. Such designs are termed as Incomplete Block Designs (Wu and Hamada, 2000).

1.1 The Incomplete Block Designs(IBDs)

Incomplete Blocks Designs (IBDs) as their name implies, the block size is less than the number of treatments to be tested. These designs were introduced by Yates in order to eliminate heterogeneity to a greater extent than is possible with randomized blocks designs and Latin square designs when the number of treatments is large. If the number of treatments to be compared is large, then we need large number of blocks to accommodate all the treatments. This requires more experimental material and so the cost of experimentation becomes high which may be in terms of money, labor, time etc. The Completely Randomized Design (CRD) and Randomized Block Design (RCBD) may not be suitable in such situations because they will require large number of experimental units to accommodate all the treatments. In such situations when sufficient number of homogeneous experimental units are not available to accommodate all the treatments in a block, then incomplete block designs can be used (Bailey.2008).

In incomplete block designs, each block receives only some of the selected treatments and not all the treatments. The precision of the estimate of treatments effect depends on the number of replications of the treatment. If larger is the number of replications, the more the precision. Similarly, is the case for precision of estimate of difference between two treatments effects. If a pair of treatment occurs together more number of times in the design, the difference between these two treatments effect can be estimated with more precision. To ensure equal or nearly equal precision of comparisons of different pairs of treatment effect, the treatments are allocated to the experimental units in different blocks of equal sizes such that each treatment occurs at most once in a block and it has an equal number of replications and each pair of treatments has the same or nearly the same number of replications. Sometimes it is possible that the available blocks can accommodate only a limited number of treatments due to several reasons.

For example, the goodness of a car is judged by different features like fuel efficiency, engine performance, body structure etc. Each of this factor depends on many other factors, e.g., engine consists of many parts and the performance of every part combined together will result in the final performance of the engine. These factors can be treated as treatment effects. If all these factors are to be compared, then we need large number of cars to design a complete experiment. This may be an expensive affair.

The incomplete block designs overcome such problems. It is possible to use much less number of cars with the setup of an incomplete block design and all the treatments need not to be assigned to all the cars. Rather some treatments will be implemented in some cars and remaining treatments in other cars. The efficiency of such designs is, in general, not less than the efficiency of a complete block design. In another example, consider a situation of destructive experiments, e.g., testing the life of television sets, LCD panels, etc. If there are large number of treatments to be compared, then we need large number of television sets or

LCD panels. The incomplete block designs can use lesser number of television sets or LCD panels to conduct the test of significance of treatment effects without losing, in general, the efficiency of design of experiment. This also results in the reduction of experimental cost. This will result in the lower cost of experimentation. Incomplete Block Designs need less number of observations in a block than the observations in a complete block design to conduct the test of hypothesis without losing the efficiency of design of experiment (Wu and Hamada, 2000).

Also, in incomplete block designs, the precision of the estimate of a treatment effect depends on the number of replications of the treatment-the larger the number of replicates, the more is the precision. Similar is the case of the precision of estimate of the difference between two treatment effects. If a pair of two treatments effects can be estimated with more precision then, to ensure equal or nearly equal precision of comparisons of different pairs of treatment effects, the treatments are so allocated to the experimental units in different blocks of equal sizes such that each treatment occurs at most once in a block and it has equal number of replications and each pair of treatments has the same or nearly the same number of replications (Dean and Voss, 1999).

1.2 Complete vs Incomplete Block Designs (IBD):

The designs in which every block receives all the treatments are called Complete Block Designs (CRD) or if every treatment is used and replicated the same number of times in every block, the design is Complete Block Design. In CRD, all experimental units are considered the same and no division or grouping among them exist and the *n*- treatments are allocated randomly to the whole set of experimental units, without making any effort to group the experimental units in any way for more homogeneity. If each treatment is used once in every block, it is a Randomized Complete Block (RCBD) In RCBD, if large number of treatments are to be compared, then large number of experimental units are required. This will increase the

variation among the responses and RCRD may not be appropriate to use. In such a case when the experimental material is not homogeneous and there are *n*-treatments to be compared, then it may be possible to group the experimental material into blocks of sizes *n*- units. Blocks are constructed such that the experimental units within a block are relatively homogeneous and resemble to each other more closely than the units in the different blocks.

1.3 The Block Designs

Block Designs have applications in almost all areas of human investigation including agriculture, biology, engineering, medicine, physical and chemical sciences and industrial experimentation. The most primitive of the block designs is the randomized (complete) block design. However, in many practical situations, adoption of a complete block design is not appropriate and in some cases, not at all feasible. This fact prompted the development of various kinds of Incomplete Block Designs, which in turn have been used extensively for experiments in a variety of fields (Dey, 1986).

In Block Designs, blocks are built from the set of X (treatments), blocks are formed when pairs of treatments are chosen as subsets to satisfy some set of properties that are deemed useful for a particular application. Therefore, a design is a pair (X, Ɓ) such that the following properties are satisfied:

(i). X is a set of elements called treatments, and

(ii). Ɓ is a collection (i.e. multiset) of nonempty subsets of X called blocks.

The main parameters of a block design are:

n - Which represents number of treatments

b- Number of blocks

k- Block sizes.

1.4. The Balanced Incomplete Block Designs (BIBDs)

When the number of replication of all pairs of treatments in a design is the same, then we have an important class of IBD called BIBD. Thus, a BIBD is an arrangement of n-treatments in to blocks each of size k (< n) such that:

(i). Each treatment occurs almost once in a block

(ii). Each treatment occurs in exactly r blocks

(iii). Each pair of treatments occurs together in exactly λ blocks.

A Balanced Incomplete Block Designs (BIBD) is usually denoted by n, b, r, k, λ known as the parameters of the designs which satisfy the relations:

$$nr = bk \tag{1.1}$$

and,

$$\lambda(n-1) = r(k-1) \tag{1.2}$$

Each side of equation (1.1) represents the total number of experimental units or plots in the design.

Equation (1.2) can be established by noting that a given treatment occurs with k-1 other treatments in each r- blocks and also occurs with each of the other (n-1) treatments in λ blocks.

For example,

A design (6, 10, 5, 3, 2)-BIBD

Where, n = 6, b= 10, r =5, k = 3 and λ =2 and with the below allocation of treatments

Table 1.1: Allocation of Blocks of BIBD.

Blocks	Treatments
B1	T_1, T_2, T_3

B2	T_1, T_2, T_4
B3	T_1, T_3, T_4
B4	T_I, T_4, T_6
B5	T_1, T_5, T_6
B6	T_2, T_3, T_6
B7	T_2, T_4, T_5
B8	T_2, T_5, T_6
B9	T_3, T_4, T_5
B10	T_3, T_4, T_6

The parameters n, k, b, r, λ are not chosen arbitrarily. They satisfy the following relations:

a) $nr = bk$

e, g 6* 5= 10*3 = 30

b) $\lambda (n - 1) = r (k - 1)$

e.g 2* 5 = 5*2 = 10 and,

c) $b \geq n$

e.g 10 > 6

By relaxing the requirement that all blocks are the same but a set of blocks sizes, say **K** = {k_1, k_2, k_3..., k_b}, then we have a design known as Pairwise Balanced Design (PBD).

1.5 The Pairwise Balanced Design

Wilson, (1971) defined a pairwise balanced design (PBD) of order n as a pair (X, Ɓ) where Ɓ is a set of cardinality B, and B is a set of subsets of X (each of which is called a block) with the property that every 2-treatments subset of X is contained in a unique block.

For example, a PBD(5, {2, 3}, 1) where X= {1, 2, 3, 4, 5}, **K**={2, 3}, λ= 1 has the following set of blocks: Ɓ =.{ {1, 2, 3}, {1, 4, 5}, {2, 4}, {2, 5}, {3 ,4}, {3, 5} }

n is the total number of the treatment in the design therefore n = 5. And n = n(X)

K is the set of sizes of the block of the design.

Therefore, **K** = {2, 3}, clearly **K** = {k_1 ..., kb}

k_i is the block size as i= 1, 2, …, b, therefore k_1 = 2 and k_2 = 3

bi is the number of units in the block size of ki,

Therefore, b_1= 2 and b_2 = 4

Clearly b = $\sum_i bi$

λ is the number of time a pair of treatment appears together within blocks, therefore λ = 1.

In a Pairwise Balanced Design when λ = 1, it is usually written as PBD(n, **K**).

Ɓ is the collection of all the set of blocks (vector space) while B is the each of the block inside the set cardinality (vector). Therefore Ɓ = {{B_1, B_2…B_b}} and b is the number of blocks for each of the block sizes.

1.6 Pairwise Balanced Designs Concept

The concept of pairwise balanced designs was derived from linear space. A linear space of order n is a PBD on n points with λ =1 whose blocks have size at least 2. A linear space is nontrivial if it contains more than one block. The theory of block sizes of PBDs were also

derived from the concept of linear space. It is customary to refer to elements as points and blocks as lines when dealing with linear spaces and theorems are often stated in this geometric language. For example, in geometric terms, a linear space is a pair (P, Ɓ) where P is a set of points and Ɓ is a collection of subsets of P (the lines) for which every pair of points are contained on exactly one line and every line contains at least two points. Mathematically, a pairwise balanced design is a system of points and blocks (X, Ɓ), such that every pair of points in X is in exactly one block in B.

If $|X| = n$, we say (X, Ɓ) has order n where n is the number of points. Suppose **K** $\subseteq Z_{\geq 2}$ includes the set of all block sizes in B. A pairwise balanced design with order n and block sizes in **K** is called a PBD(n, **K**).

1.7 The Lotto Designs

Li (1999) describes a lotto design thus: Suppose n, k, p and t are integers and B is a collection of K-subsets of a set X of n elements (usually X is X (n)). Then B is an (n, k, p, t) Lotto Design (LD) if an arbitrary p-subset of X(n) intersects some K-set of B in at least t-elements. The K-sets in B are known as the blocks of the Lotto Designs, the elements X are known as the n -treatments of the design. Lotto design can also be denoted by (X, B) Where B denotes the sets of blocks of the design and X denotes the set from which the treatments of the blocks of B are chosen. More formally one could define an (n, k, p, t) Lotto Design to be a set of k-sets (blocks) of an n-set such that any p-set intersects at least one k-set in t or more elements. For example,

LD(13, 6, 5, 3) is a typical example of a Lotto Design.

Where n =13, k = 6, p = 5 and t = 3 has the following set of blocks

{{1, 2, 3, 4, 5, 6}, {1, 2, 3, 4, 5, 7},

{1, 2, 3, 4, 6, 7}, {1, 2, 3, 5, 6, 7}, {8, 9, 10, 11, 12, 13}}

1.8 Aim and Objectives of the Study

The aim of this study was to constructing two congruence classes of PBDs from LDs while the specific objectives were to:

(i) investigate various methods of constructing PBDs

(ii) establish conditions for the identification of LDs that qualify as PBDs;

(iii) provide a simple method for constructing two classes of PBDs when **K** = {3, 4} and {3, 4, 5} from the appropriate LDs

1.9 Statement of the Problem

Most construction of PBDs by earlier scholars could not: (i) specified the admissibility of n-treatments for a particular sets of block sizes for examples, In Deleting Element Methods by Roghavarao, (1998) who constructed a PBD(6, {2, 3}, 1) from a BIBD(7, 3, 1), in Method of Adjoining Elements by Andersen, (1990) who constructed a PBD(11, {2, 3, 4}, 1) from existing BIBD(9, 3, 1); e.t. c (ii) specified which of the resolvable BIBDs qualified as PBDs whereby making its construction somehow difficult e.g Bose, (1960) and Colbourn and Denitz, (1996); Also, Wilson fundamental constructed of PBDs could only be used for a small value of n-treatments and it constructions were based on the knowledge of GDDs which has different analogues, interpretations and meaning see Hanani,(1963); Beth, et.al (1998), Mullin, et. al, 1996 and Wilson, (1974) Therefore, this research work introduces a simple method for the construction of two congruent classes of PBD(n, {3, 4}), PBD(n, {3, 4, 5}) and specified the type of LDs that could be used for a particular class of PBDs, and specified the admissibility of n-treatments for a particular sets of block size of a particular PBDs using LDs .

1.10 Significance of the Study

Among all the Incomplete Block Designs (IBD), Balanced Incomplete Block Design (BIBD) is the easiest in terms of construction and analysis. BIBDs are commonly used when the number of treatments in an experiment is less than the block size. However, there are cases when the block sizes available for an experiment are not the same. Hence, the use of PBDs. PBDs are used to construct other designs such as Steiner Triple System, Pairwise Additive designs, Orthogonal Array, Partially Balanced Designs, Latin Square Designs etc. PBDs are also used to confirm the existence questions for other designs such as Balanced Incomplete Block Design (BIBD), Latin Square (LD), etc**.**

1.11 Relationship, similarity and Differences between BIBDs and PBDs

BIBDs usually emphasized the total number of the blocks 'b' of the deigns while PBDs are more concern about the set of block sizes '**K'** of the deigns. Also, block sizes in BIBDs are the same while block sizes in PBDs are not the same. Though both are Incomplete Block Designs (IBDs).

1.12 Definition of some Terms

Design of experiment. It means how to design an experiment in the sense that how the observations or measurements should be obtained to answer a query in a valid, efficient and economical way.

Experimental unit. For conducting an experiment, the experimental material is divided into smaller parts and each part is referred to as experimental unit. The experimental unit is randomly assigned to a treatment.

Treatment. Different objects or procedures which are to be compared in an experiment.

Experiment. A way of getting an answer to a question which the experimenter wants to know.

Replication. It is the repetition of the experimental situation by replicating the experimental unit.

In the replication principle, any treatment is repeated a number of times to obtain a valid and more reliable estimate than which is possible with one observation only. Replication provides an efficient way of increasing the precision of an experiment. The precision increases with the increase in the number of observations. Replication provides more observations when the same treatment is used, so it increases precision.

Block: The groups of relatively homogeneous experimental units are called blocks. When the blocking is done according to one attribute, we get a block design. In a block design, the treatments are applied randomly to the experimental units within a block, the randomized allocation of treatments to experimental units within a block being done independently in each block.

Linear Space. A linear space is an incidence structure (P, L, ι) such that:

(i) every pair of points in *P* is incident to exactly one line in P, and

(ii) every line in L is incident to at least two points.

We often write (P, L) to denote a linear space, or simply *P* when the set L is clear.

Resolvable Designs. Any block design is called resolvable if its blocks can be partitioned into parallel classes.

Mutually Orthogonal Latin Squares (MOLS). A Latin square of side n is an $n \times n$ array where each cell contains an entry from an n -set, say $[n]$. Every symbol occurs exactly once in each row and column of the array. Two Latin squares are orthogonal if the set of paired elements from corresponding cells exhausts the pairs in $[n]^2$. A set of mutually orthogonal Latin squares of side *n*, or MOLS(n), is a set of Latin squares that are pairwise orthogonal.

Transversal Design(TD). A transversal design, denoted TD(*k*, *n*), is a *k*-GDD of type n^k.

MOLS equivalence. A TD(k, n) is equivalent to a set of $k-2$ mutually orthogonal Latin squares of side n.

Group Divisible Design(GDD). A group divisible design, or GDD, is a triple (X, G, B) where X is a set of points and B is a set of blocks on X. The set G corresponds to a partition of the points into sets called 'groups'. The defining property of a GDD is that every pair of points is either on exactly one block, or is contained in a single group.

Deletion in Designs. A deletion in a linear space X is the removal of some set of points $S \subseteq X$ and all blocks incident to S. If a single point x is deleted from X then what remains is a GDD whose groups corresponding to a block incident to x.

Truncation in Designs. A truncation of (X, B) is the removal of some set of points from X. Unlike deletion, blocks that were incident to a truncated point remain in the new configuration with reduced size. We ignore blocks of size 1 in what is left.

Congruence. A mathematical relation between two numbers indicating they give the same reminder when divided by a modulus.

CHAPTER TWO
LITERATURE REVIEW

2.0 introduction

The aim of this section is to present a review of relevant research work done by other authors in the field of Incomplete Block Designs its relationship to classes PBDs, BIBDs and LDs.

2.1 The Origin of Block Designs

Historically, Incomplete Block Designs (IBDs) were introduced in the statistics literature by Yates (1936). However, combinatorial structures which are now recognize as Balanced Incomplete Block Designs (BIBDs) were known even in the 19th century. Kirkman (1850) solved the following problem, originally proposed by Woolhouse in 1844: A school mistress is in the habit of taking 15 girls of her school for a daily morning walk in 5 batches of 3 girls each, so that each girl has 2 companions. Is it possible to find an arrangement so that for 7 consecutive days, no girl walks with any of her companions in any batch more than once? The solution of the above problem (called the Kirkman's schoolgirl problem) has a one-one correspondence with the solution of a BIBD and such a BIBD is also called a Kirkman Triple System, KTS{15). A KTS{15) is shown below, where the schoolgirls are integers labeled 1, 2, ... , 15:

Table 2.1: Kirkman Triple System School Girls Solution

DAY 1	DAY 2	DAY 3	DAY 4	DAY 5	DAY 6	DAY 7
1,6,11	1,8,10	1,3,9	1,2,5	2,3,6	1,7,14	1,12,13
2,7,12	2,9,11	2,13,14	3,10,12	5,7,13	3,5,11	2,4,10
3,8,13	3,4,7	4,5,8	4,11,13	8,9,12	4,6,12	5,6,9
4,9,14	5,12,14	6,7,10	6,8,14	10,11,14	9,10,13	7,8,11
5,10,15	6,13,15	11,12,15	7,9,15	1,4,15	2,8,15	3,14,15

It is easily seen that the above plan is a BIBD with parameters n = 15, b = 35, r = 7, k = 3, and λ = 1 where triplets of girls are treated as blocks. Other solutions to KTS{15) were provided by several authors, including Cayley (1850), Peirce (1860) and Davis (1897). The solution of a Kirkman triple System KTS(n) for all n ≡ 3 (mod 6) was provided by Raychaudhuri and Wilson (1971). Steiner {1853} proposed the problem of arranging n objects in triplets (called Steiner's triple systems) such that every pair of objects appears in exactly one triplet. It is easy to see that Steiner's triples are in fact BIBDs with block size three. Early important contributions in respect of Block designs were made by Bose (1939, 1942, 1949), Fisher (1940) and Yates (1940). For an elegant description of the early history of combinatorial designs, including BIBDs, see Anderson, Colbourn, Dinitz and Griggs (2007). A Steiner triple system is an ordered pair (X, Ɓ), where X is a finite set of points (Treatments) and $\mathcal{B}$ is a set of all 3-element subsets of X called triples, such that each pair of distinct elements of X occurs together in exactly one triple of $\mathcal{B}$.

For examples,

i. An STS(7) where n = 7 therefore,

X = {0, 1, 2, 3, 4, 5, 6} and Ɓ = {{0, 1, 3}, {1, 2, 4}, {2, 3, 5}, {3, 4, 6}, {4, 5 0}, {5, 6, 1}, {6, 0, 2}}

ii. An STS(3) where n = 3 (a trivial example) therefore,

X = {0, 1, 2} and Ɓ = {{0, 1, 2}}

iii. An STS(1) where n = 1 (a more trivial example) therefore,

X = {1} and Ɓ = Ø (Popoola, et. al. 2019).

A resolvable *STS*(n) is called a Kirkman triple system, and it is denoted by KTS(n).

Later in 1853, Steiner discussed t -designs with k = t + 1 and λ = 1. When t = 2, these are triple systems with λ = 1. Thus, we call such designs Steiner triple systems. Put another way, they are 2-designs with parameters (n, 3, 1), and we denote such systems by S(2, 3, n).

Unaware of Kirkman's work, Steiner (1853) reintroduced triple systems, and as his work was more widely known, the systems were named in his honor (Steiner, 1853).

The existence question for which STS(n) does exist was first posed by W.S.B. woolhouse (Prize question 1733, Lady's and Gentlemen's Diary, 1844). The problem was solved in 1847 by Rev. T.P. Kirkman: Steiner (1847) further established the existence of a Steiner triple system of order n which exists if and only if $n \equiv 1$ or 3 (mod 6). Wilson (1973) and Hanani (1979) uses PBD's and GDD's to construct Steiner systems STS(2, k, n**).** A necessary and sufficient condition for the existence of an STS (l, **K**, n) is that n has an expression as a sum of members of **K**. A weaker condition, $n \equiv 0$ (mod g.c.d.(K)), is sufficient for large values of n. Necessary conditions for the existence of an S(t, **K**, n) with t > 1, are:

The existence of an S(t- 1, **K**- 1, n- 1) (where K- 1 denotes the set {k- 1: k ∈ **K**}), and $\binom{n}{t} \equiv 0(\ mod\ g.c.d\binom{k}{t}$ where k∈ ***K***. Wilson, (1975) was able to show that these two conditions are sufficient for the existence of an S(2, K, n) provided n is sufficiently large. One would like to prove a similar theorem for S(3, K, n), i.e. when n is sufficiently large, the necessary congruence conditions are sufficient for the existence of an S(3, **K**, n). Hanani, (1979) has shown that the necessary conditions given above for the existence of S(3, 4, n) and S(3, {4, 6}, n) are indeed sufficient. Hanani (1979) and Hanani (1971) also proved the result for some larger sets of block sizes. The paper of Hanani (1979) is also the state of the art as regards the existence of S(3, k, n) with $k \geq 5$, except for a few constructions of individual designs which are listed in (Beth, e. al, 1986).

2.1. The Block Designs

A block design is an incidence system (n, k, λ, b, r) in which a set *X* of n- treatments are partitioned into a family *B* of b subsets (blocks) in such a way that any two treatments determine λ blocks with k size in each block, and each treatment is contained in r different blocks. When k < n, and in this case the design is said to be incomplete e.g BIBD. When k

= n, is it a case of randomized completely block designs (RCBD). The actual study of block designs started with a study of algebraic curves by Plucker (1853), as cited in Colbourn and Dinitz (2007). He encountered a Steiner Triple System (STS) of order 9 and claimed that an STS could exist only when m ≡ 3 (mod 6). He correctly revised this condition to n ≡1, 3(mod 6) in 1839. Calyley (1863), as cited in Colbourn and Dinitz (2007), introduced the word tactic for the general area of designs. Also, Colbourn and Dinitz (2007), cited Reye (1876) as the scholar that gave a specific meaning to the word "configuration" in terms of geometrical structures. Reye (op.cit) defined a configuration as a system of v points and b lines with k points on each line and r lines through each point, with at most one line through any two points. If n = b (and therefore k = r), the configuration is symmetric and denoted by X_k; thus, for example, the Fano plane is 7_3, the Pappus configuration is 9_3 and the Desargues configuration is 103. The Fano plane 7_3 is a very good example of a symmetric design as it is the (7, 7, 3, 3,1) design. Levi (1942) unified the treatment of configurations with questions in algebra, geometry and design theory. Steiner (1853) investigated Steiner Triple Systems, (STS) systems, unaware of the ealier work of Kirkman (1847). This design occurs when k = 3, λ= 1, and n = 6t + 3 or n = 6t + 1 where t is any integer. Kirkman showed that a necessary and sufficient condition for a Steiner Tripple's is n ≡ 1 or 3 mod 6.

Balanced Incomplete Block Designs (BIBDs) were introduced by Yates (1936) where he drew attention to the importance of block designs for statistical design. Yates referred to them as symmetrical incomplete randomized blocks. John (1971) described several families of BIBDs these include the unreduced designs, the Symmetric designs, Steiner Triple System(STS), Hadamard 2-Designs, Mutually Orthogonal Latin Squares, Residual designs and so on.

Fisher and Yates (1938) published the "Statistical Tables for Biological, Agricultural and Medical Research". This book presented tables of BIBDs with replication number r up to 10.

Bose (1939) introduced the present terminology of balanced incomplete block designs although the present use of n, b, r, k, λ was due to Yates except that he originally used t for treatment. Street and Street (1987) identified BIBDs by their block structures as follows:

n - total number of treatments

b- total number of blocks

r_i- number of replicates per treatment i

k_j -size of block j

n_{ij} -number of plots carrying the treatment i in block j.

Fisher and Yates (1938) discussed the unreduced design. This type of BIBD is obtained by taking all possible combinations of k out of n treatments. The parameters of this design are n, b = $\binom{n}{k}$, r = $\binom{n-1}{k-1}$, λ= $\binom{n-2}{k-2}$. This design is easy to construct but it may become cumbersome and impracticable if there are so many treatments to consider.

Mathon and Rosa (2007) listed admissible sets of nontrivial BIBDs with r ≤ 41 and k≤ $\frac{n}{2}$.

Effanga et.al (2009) discussed alternative designs as design that are obtained by interchanging rows ior columns of the design matrix. They constructed Pairwise Balanced Incomplete Block Designs using a non – linear, non – preemptive binary integer goal programming model. Yokoya and Yamada (2009) used the tabu search approach to construct BIBDs using Mixed Interger Programming (MIP) solvers. The problem was formulated as a non-linear Mixed Integer Programming (MIP) problem and approached through a repeated solution of linear problems. Balanced incomplete block designs as a coherent theory was presented by Bose (1939). So many families of designs were constructed by him using Galois fields, difference methods and finite projective geometries. The notion of BIBD was generalized to Partially Balanced Incomplete Block Designs (PBIBD) by Bose and Nair (1939). This encompassed some lattice designs introduced earlier by Yates. Youden (1940) and Harshbarger (1947) generalized the concept of Latin square design by introducing

BIBDs for eliminating heterogeneity in two directions. Bose and Shimamoto (1952) introduced the concept of partially balanced association schemes and the definition of partially balanced design was rephrased in terms of this concept. Augmented designs were introduced by Federer (1955) to fill a need arising in screening new strains of sugar cane and soil fumigants used in growing pineapples. An augmented experimental design is any standard design augmented with additional treatments in the complete block, the incomplete blocks, the row, the column etc. Kempthorne (1953) and David (1963) investigated cyclic designs with only two plots per block. John (1966) discussed cyclic designs with larger blocks. These authors confined themselves to representing the varieties by the residue classes 0,1, … $n - 1$(mod n). Bruck and Ryser (1949), Chowla and Ryser (1950), Hall and Ryser (1951) gave important theorems on the existence of symmetric balanced incomplete block designs. Difference sets methods for the construction of Symmetric Balanced Incomplete Block designs were systematically studied by Bose (1939) but earliest results on difference method was developed by Netto (1893) and several research works have been carried out on the topic since then. Baumert (1971), in particular, studied cyclic different sets. Other series of designs based on difference set include the Bose's T_1 series, Bose (1939) and Sprotts A, B, C, D series, Sprott (1954). Other series such as E_1 and E_2 for $k = 3$, $\lambda = 1$ were given by Bose. Bhattacharya (1943) solved the auxiliary problem for the E_2 series.

A symmetric design occurs, when, in a BIBD, $b = n$ (or equivalently, $r = k$). An example of a symmetric design is the (7, 7, 3, 3, 1) BIBD. Symmetric BIBDs satisfies the arithmetic condition $\lambda(n - 1) = k(k - 1)$. Two methods, due to Bose (1939) are used for obtaining other BIBDs from a symmetric design. The first is the method of block section also called residual designs. The second is the method of block intersection which is also called derived designs.

Kleczkowski (1960) introduced a form of Nested Balanced Incomplete Block Design for laboratory studies on lesions produced by inoculating bean plants with tobacco necrosis

virus and it has parameters (r, n, b_1, k_1, λ_1, b_2, k_2, λ_2, m) where the n treatments are each replicated r times with two systems of blocks such that:

(a) The second system is nested within the first, with each block from the first system containing exactly m blocks from the second system (sub -blocks);

(b) Ignoring the second system leaves a BIBD with b_1 blocks each of k_1 units and with $\lambda_1$concurrences, and

(c) Ignoring the first system leaves a BIBD with b_2 blocks each of k_2 units and with $\lambda_2$concurrences

Thus, $nr = k_1b_1 = k_2mb_1 = k_2b_2$, $(n - 1)\lambda_1 = (k_1 - 1)r$, $(n - 1)\lambda_2 = (k_2 - 1)r$ so that $(n - 1)(\lambda_1 - m\lambda_2) = (m - 1)r$. Hanani (1961) proved that the admissible conditions for the existence of 2 –(n, 3, λ) results for block sizes 3,4 and 5; he also extended the results to block size 6 and any $\lambda \geq 2$, showing sufficiency of the basic necessary conditions for existence except for the nonexistent 2 – (21, 6, 2) design.

John and Mitchell (1977) constructed the duals of some incomplete block designs. A dual design is obtained by interchanging the roles of treatments and the blocks. So, if X is the incidence matrix of a design D, then X^T is the incidence matrix of the dual design. For a BIBD, the Fisher's inequality $n \leq b$ must hold, so in general, Fisher's inequality cannot hold for both a BIBD and its dual. The only exception occurs when the design is symmetric.

Arasu (1987) showed the finiteness of the number of symmetric designs for a given λ , when n is of a special form. Tierlinck (1987) showed that non – trivial t-designs without repeated blocks exist for all t.

Nurmela (1993) used probabilistic search heuristics for covering designs, packing designs and t- designs. Tiessen and van Rees (1997) listed many (22, 44, 14, 7, 4) and (15, 42, 14, 5,

4) BIBDs. Bofill (1997) in his Ph.D thesis described the evaluation of several search algorithms based on optimizing neural networks through its application to the generation of block designs. He established that generation of block designs was a good benchmark for neural optimization algorithms. A complete classification of (12, 4, 3) Resolvable BIBDs was done by Morales and Velarde (2001). Prestwich (2003) described a Constrained Local Search (CLS) implementation for PBD generation. The CLS method is well suited to large, structured combinatorial problems with few solutions. It combined the scalability of local search with the space pruning ability of constraint programming and PBD results he obtained supported this view.

2.2.0. The Pairwise Balanced Designs

In statistical term, Popoola and Oyejola (2016) defined PBD as an arrangement of n- treatments in b blocks of index λ if $(n; k_1, k_2, \ldots, k_b)$ and Provided:

(i) Each set contains $\{k_1, k_2 \ldots, k_b)$ symbols that are all distinct

(ii) $k_i \leq n;\ k_i \neq k_j$ and

(iii) every pair of distinct treatments occurs in exactly λ sets of the designs

Bose and Shrikhande, (1960) defined Pairwise Balanced Design (PBD) as a pair (X, Ɓ) where X is a set of n- treatments and Ɓ is a collection of subsets of X called blocks, such that each pair of treatments is contained in precisely one block.

Hanani (1963) defined a PBD of index unity is a pair (X, Ɓ) where X is a set (of *points)* and Ɓ a class of subsets B of X (called *blocks)* such that any pair of distinct points of X is contained in exactly one of the blocks of Ɓ (and we may also require /B/ ≥ 2 for each B ϵ Ɓ). Such systems are also known as linear spaces.

According to them, a characterization of pairwise balanced design in terms of NN^T matrix can be expressed in the following way. A block design *D* is called pairwise balanced design

if all the off-diagonal elements of NN^T matrix are same (constant) i.e. $NN^T = (r - \lambda) L_n + \lambda E_{nn}$. Where L_n is an identity matrix of order n, and E_{nn} is the unit matrix of order (n x n) row and column. Rao (1958) noted that, if the information matrix **C** of a block design satisfies $C = \theta \left[Ln - \frac{1}{n} Enn \right]$ where θ is the non-zero eigenvalue of the matrix **C** and PBDs satisfied this property and hence is a Variance Balanced (VB) design. Chakrabarti, (1963) stated that a block design is said to be variance-balanced if it permits the estimation of all estimable normalized treatment contrasts with the same variance, and gave useful concept of a **C**-matrix of design. Pearce (1964) obtained VB design with varying block sizes. Mukerjee and Kageyama (1985) obtained a resolvable VB design with unequal replications. Das and Ghosh (1985) obtained unequal replicated, unequal/equal block size VB design from BIBD and partially balanced incomplete block design (PBIBD) using augmented blocks and treatments. Rao (1958), Hedayat and Federer (1974), Raghavarao (1962), and Puri and Nigam (1977) defined that a design is said to be variance balanced if every normalized estimable linear function of treatment effect can be estimated with same precision. Kageyama (1988) discussed the construction of VB design using BIBD, PBIBD, and some incidence matrices. Khatri (1982) gave a formula to measure the overall A-efficiency of VB designs along with method of construction of VB designs. Gupta and Jones (1983) obtained VB designs using BIBD and PBIBD with two associate classes. Calvin and Sinha (1989) extended the technique of Calvin (1986) to construct VB designs with more than two distinct block sizes that permit fewer replications. Das and Ghosh (1985) defined generalized efficiency balanced (GEB) design which include both VB as well as efficiency balanced (EB) designs. Ghosh (1988), Ghosh and Karmoker (1988), Ghosh, Divecha, and Kageyama (1991), and Ghosh, Joshi, and Kageyama (1993) provided several methods for the construction of VB designs. Ghosh and Joshi (1995) constructed VB designs through a triangular design. Agarwal and Kumar (1985) constructed a VB design which is associated with group divisible

design. Ghosh and Joshi (1991) constructed a VB design through a group divisible (GD) design. The more general concept, where multiple block sizes are allowed, known as Pairwise Balanced Designs (PBD) was introduced by Bose and Shrikhande (1952) and Hanani (1963) had played important roles in their respective work on orthogonal Latin squares and BIBD's. The number k is called the replication number of the design. In a pair-wise balanced block design, any pair of treatments within the blocks occur equally often. The literature on combinatorial theory contains many contributions to the existence and construction of pair-wise balanced designs, including for settings with unequal block sizes. Bose and Shrikhande (1959) discussed about the pair-wise balanced design in addition to existence of orthogonal Latin square designs. Bose and Shrikhande (1960) obtained the various methods for the construction of pair-wise orthogonal sets of Latin square design. However, the detailed discussion and construction on pair-wise balanced design was studied by Bose and Shrikhande (1960). In other words, we can say that Bose and Shrikhande (1959, 1959, 1960) introduced a general class of incomplete block design which they called pair-wise balanced design of index λ. The pair-wise balanced design also shared some of the properties of BIB designs like every pair of treatments occurs together in λ blocks. The concept of pair-wise balanced design is merely the combinatorial interest in block designs. Because with the help of pair-wise balanced design many other incomplete block designs can be constructed. For example, Bose and Shrikhande (1960) used the pair-wise balanced design in the context of constructing Mutually Orthogonal Latin Square (MOLS). Hedayat and Stufken (1989) showed that the problems of constructing pair-wise balanced designs and variance balanced block designs are equivalent. Effanga, Ugboh, Enang and Eno (2009) developed a non-linear non-preemptive binary integer goal programming model for the construction of *D*-optimal pair-wise balanced incomplete block designs. The literature on

combinatorial theory contains many contributions to the existence and construction of pair-wise balanced designs with un-equal block sizes.

Smith, Blaikie and Taylor (1998) proved the existence of a PBD(n, K, λ) with bi blocks of size ki ϵ K and went further to present useful expression that connects PBD parameters with number of blocks of different sizes as follows:

$\lambda n(n-1) \geq \sum_i biki\ (ki-1)$ (2.1)

Clearly, b =$\sum_i bi$ (2.2)

2.2.1. PBDs Decomposition

Bose and Shrikhande (1959) showed the decomposition of PBD as follows: Let n be a positive integer and K ⊆ $Z_{\geq 2}$:= {2, 3, 4,...}. A pairwise balanced design PBD(n, K) is a pair (X, Ɓ), where

(i) X is a n-element set of treatments

(ii) Ɓ ⊆ $\bigcup_{k\in K}\binom{X}{k}$ is a family of subsets of X, called blocks; and

(iii) every two distinct points appear together in exactly one block

2.2.2. Existence Theory of a PBD.

One of the most important breakthroughs in design theory was made by Richard Wilson in the early 1970s. He showed that the trivial necessary conditions for the existence of various kinds of designs as asymptotically sufficient. Wilson (1971) proposed that there are necessary divisibility conditions for existence of PBD(n, K). In what follows, let $\alpha(K)$:= gcd$\{k-1 : k \in K\}$ and $\beta(K)$:= gcd$\{k(k-1) : k \in K\}$.We observe that the existence of a PBD(n, **K**) (with n > 0) implies

(i) $n \equiv 1 (\text{mod } \alpha(K))$, (2.3)

and

(ii) $n(n-1) \equiv 0 (\text{mod } \beta(K))$, (2.4)

where α(K) is the greatest common divisor of the integers {k-1; k ϵ **K**} and

β(K) is the greatest common divisor of the integers {k (k-l) : k ϵ **K**}. Here (2.3) follows from the fact that the blocks containing a given point of a PBD partition the remaining n-I points; and (2.4) follows since the pairs of points are partitioned by the blocks. The above conditions (2.3) and (2.4) are "asymptotically sufficient" for the existence of a PBD(n, **K**).

Wilson, (1972) proposed thus: Let K be a set of non-negative integers. Then all but finitely many integers n satisfying the conditions that α(K) divides n−1 and β(K) divides n (n− 1) belong to Ɓ (K). This powerful theorem has many consequences. For example, it shows that there are only countably many PBD-closed sets. It shows that the necessary conditions that k−1 divides n−1 and k(k−1) divides n (n−1) are sufficient for the existence of a Steiner system S(2, k, n) except for finitely many values of n, as claimed in the introduction.

2.2.3. Application of PBDs

The application of pairwise balanced designs in the construction of related combinatorial systems is of paramount important in the design theory. PBD's were used in the construction of most of resolvable and recursive designs. PBDs are used in the construction of other types of designs, and other types of combinatorial structures such as: Varieties of Short Conjugate, Orthogonal Quasi-Group Identities, Orthogonal arrays with interesting Conjugacy Properties, Egde-coloured Designs, and Mendelsohn Designs (Bennett, 1990). For example, Bose and Shrikhande (1960) used pairwise balanced design to construct Mutually Orthogonal Latin Square (MOLS). Hedayat and Stufken (1989) showed that the problems of constructing pairwise balanced designs and variance balanced block designs are equivalent. Effanga, Ugboh, Enang and Eno (2009) developed a non-linear non-preemptive binary integer goal programming model for the construction of *D*-optimal pairwise balanced incomplete block designs. PBDs can also be used to construct other designs such as Steiner Triple Systems (STS), Pairwise Additive Designs PBIB etc. (Cameron, 2003). It also has

greater significance in the application to the solution of existence questions for other types of designs such as t -Designs, Partially Balanced Designs etc. Ivan and Bane (2003) constructed Optical code-division Multiple-Access (CDMA). Their construction was based on combinatorial designs, more specifically using pairwise balanced designs (PBD). Three different constructions of novel Optical Orthogonal Codes (OOC) families were discussed, namely: (1) the method of adjoining elements; (2) the method of removing elements; and (3) difference-systems-based constructions. According to them, those PBD OOC families are able to support up to four different services. They further proposed an algorithm for constructing PBD OOCs from resolvable balanced incomplete block designs (BIBDs) supporting many arbitrary different services. The algorithm is illustrated using novel combinatorial construction—the integer lattice construction. This construction can be applied to both synchronous and asynchronous applications, and it is compatible with Spectral-Amplitude-Coding (SAC), fast-frequency-hopping, and time-spreading-encoding schemes. Matsubara and Kageyama, (2014) constructed Pairwise Additive Minimal BIB Design B(n, k, (k −1) 2) from the PBDs. In their paper the existence of pairwise additive B(n, 3, 1) was discussed. At first they showed that there are f pairwise additive B(n, 3, 1) for sufficiently large $n \equiv 1,3 \pmod 6$, even if $f > k$. Furthermore, the exact existence of 2 pairwise additive B(n, 3, 1) with $n \equiv 1,3 \pmod 6$ was discussed by providing direct and recursive constructions of pairwise additive B(n, 3, 1). Matsubara and Kageyama also showed that there are 2 pairwise additive B(n, 3, 1) for any $n \equiv 1 \pmod 6$ except possibly for 12 values. Popoola et.al. (2019) used a class of PBD(n, {3, 4}, 1) to construct STS(2n +1) and provides steps for the construction of all admissible values of n-treatments for the class.

2.3 Dimension of the Set of Blocks of PBDs

In the construction of PBDs one important question to be asked is how do we determine the minimum number of blocks in which the maximum size of a block is specified or in which the size of a particular block is specified. The dimension of a PBD is the maximum integer *d* such that any set of *d* treatment generates a proper flat. This definition, is taken from the context of linear spaces (Delandtsheer, 1995). For example, the flat generated by any two points is the line containing them. So every PBD(n, **K**) with more than one block has dimension at least two. Duke and Ling, (2009) purposed another theorem to collaborate Wilson theorem as follows: Given $K \subseteq Z_{\geq 2}$ and $d \in Z_{+}$, there exists a PBD(n, K) of dimension at least d for all sufficiently large admissible n. For a pairwise balanced design, (X, Ɓ), let *b* = |B| (i.e., *b* is the number of blocks in the PBD). Wilson (1971) proved the existence of a PBD(n, K), where n is the number of treatments and **K** is the set of blocks where $k \in \mathbf{K}$ and $k \geq 2$ ($\mathbf{K} \subseteq Z_{\geq 2}$) which means n -treatments contains the allowed block sizes. Since the set of blocks incident with any treatment must contain each other treatment once, and since the set of pairs of treatment must partition into the pairs covered in each block, thus, the 'divisibility' conditions (2.3) and (2.4). The integers n satisfying (2.3) and (2.4) are admissible. Wilson (1972) states that admissibility is sufficient for existence of a PBD(n, K), provided n is large. Wilson, (1972) further claimed that there exist PBD(n, **K**) for all sufficiently large admissible n and proved this as follows: Let (X, Ɓ) be a pairwise balanced design. A flat (or sub design) is a pair ($Y, Ɓ|_Y$), where $Y \subseteq X$ and $Ɓ|_Y := \{B \in Ɓ : B \subseteq Y\}$ have the property that any two distinct treatments in Y are together in a unique block of $Ɓ|_Y$. Flats in (X, Ɓ) form a lattice under intersection. As such, any set of points $S \subseteq X$ generates a flat (S) equal to the intersection of all flats containing S.

This theorem has it root from Group Divisible Designs (GDDs). GDD is a triple (X, G, Ɓ) where X is a set of treatments, G is a partition of X into groups (there need not be algebraic structure), and Ɓ is a set of blocks such that

- a group and a block intersect in at most one treatment; and
- every pair of treatments from distinct groups is together in exactly one block.

Writing T for the list of group sizes, we adopt the notation GDD(T, **K**) for similarity with the notation for PBDs. Typically, T is called the type of the GDD. When T contains, say, u copies of the integer g, this is abbreviated with 'exponential notation' as g^u. If the type is just g^u for some g, u, the resulting GDD is called uniform. A GDD(1^n, K) is just a PBD(n, K). Another abbreviation that could use is to write simply 'k' instead of '{k}' in the notation. Simple counting reveals the necessary divisibility conditions

$$k-1 \mid g(n-1) \text{ and } k(k-1) \mid g^2 n\,(n-1) \qquad (2.5)$$

on GDD(g^n, k). One could cite two useful asymptotic existence results, one for each parameter. Mohacsy**,** (2011) proposed that**:** Given integers $n \geq k \geq 2$, there exists a GDD(g^n, k) for all sufficiently large integers g satisfying (2.3). Chang, (1995) also, established that: Given k and g, there exists a GDD(g^n, k) for all sufficiently large integers u satisfying (2.3). Draganova, (2006, 2007) established that Given g and $\mathbf{K} \subseteq Z_{\geq 2}$, there exists a GDD($g^n$, K) for all sufficiently large n satisfying

$$\alpha(K) \mid g(n-1) \qquad (2.6)$$

and

$$\beta(K) \mid g^2 n\,(n-1) \qquad (2.7)$$

It is helpful to think of GDDs as 'holey' PBDs, in the sense that groups of a GDD can be 'filled' with appropriately-sized PBDs.

2.4.0. Review of Various Methods for the Construction of PBDs.

2.4.1. **Construction of PBDs using Deleting -Elements Method**

Raghavarao, (1988) constructed a PBD(6, {2, 3}, 1) using deleting -Elements Method. This should have been the simplest possible method of constructing PBDs from BIBD, and the main idea is to remove some elements from a BIBD(7, 3, 1). However, this method can only work for a small n- treatments, when K ={2, 3}and existing knowledge of BIBDs.

For example,

A BIBD(7, 3, 1)having the following blocks

{1, 2, 4}, {1, 3, 7}, {1, 6, 5}, {2, 5, 3}, {2, 6, 7}, {3, 6, 4}, and {4, 7, 5},

and deleting point 4 from all blocks containing it, we get a having {1, 2}, {1, 3, 7},

{1, 6, 5}, {2, 5, 3}, {2, 6, 7}, {3, 6}, and {7, 5} blocks.

which is a PBD(6, {2, 3}, 1).

Another disadvantage of this method is how do we determine a particular BIBDs that could be used to construct a particular PBD? And when the treatments to be allocated is large to generate the elements in each block may be difficult and complex.

2.4.2. **Construction of PBDs using the Method of Adjoining Elements**

PBD can be constructed using the method of adjoining elements based on the Group-Divisible Designs GDDs. According to Andersen, (1990) if a BIBD(n, k, λ) is a resolvable designs, then by adding a new symbol Θi to each block of the ith resolvability class (i = 1, 2, …,x ; 1 < x < r) and adding a new block $\{\Theta_1, \Theta_2, \ldots, \Theta_x\}$, we get a PBD(n + x, {k + 1, k, x}, 1) if x < r and a PBD(n, + r, {k + 1, r}, 1) if x = r

For Example,

A BIBD(9, 3, 1) can be grouped into the following resolution (parallel classes) with labels denoting the positions of '1' within the blocks:

{1, 2, 3} {1, 5, 8} {1, 4, 7} {1, 6, 9}

{4, 8, 9} {3, 4, 6} {2, 6, 8} {2, 4, 5}

{5, 6, 7} {2, 7, 9} {3, 5, 9} {3, 7, 8}

By adding Θ_1 to each block of the first resolution class, adding a block of $\{\Theta_1$, and $\Theta_2\}$, one can get a PBD(11, {2, 3, 4}, 1) as follows:

{1, 2, 3, Θ_1} {1, 5, 8, Θ_2} {1, 4, 7} {1, 6, 9} {Θ_1, Θ_2}

{4, 8, 9, Θ_1} {3, 4, 6, Θ_2} {2, 6, 8} {2, 4, 5}

{5, 6, 7, Θ_1} {2, 7, 9, Θ_2} {3, 5, 9} {3, 7, 8}

In which Θ_1, and Θ_2 are considered as 10^{th} and 11^{th} positions of '1's in corresponding blocks. This method also relied on the knowledge of GDD and the existing BIBDs with the following disadvantages: how do one know which of the GDD and BIBDs that will be used to construct a particular PBDs? also, there is no standard techeque to generate the elements of each of the block sizes of the PBDs which make it usage difficult.

2.4.3. **Construction of PBDs using Method of Difference-System**

The method of difference-system introduced by Bose, (1960) would has been an effective method for the construction of designs PBD(n, K, λ), especially for small values of K but this method relied on the Linear space. For example, if $D_1, D_2, \ldots, D_t$ are sets of size k in an additive abelian group G of order n such that differences arising from D_i give nonzero element of G exactly λ times, when $D_1, D_2, \ldots, D_t$ are said to form a (n, k, λ) difference system in G. It can be shown by Colbourn and Denitz, (1996) that sets $D_i + g_j$, $g_j \in G = \{g_0, g_1, \ldots g_{n-1}\}$, $1 \leq i \leq t$, $0 \leq j \leq n-1$ form blocks of BIBD(n, nt, kt, k, λ). Difference systems yield block designs because the sets of a difference system possess two properties: (1) they are all of the same size, and (2) they give every nonzero element as a difference between them exactly the same number of times (λ). When the sets are permitted to be of different sizes, but the balanced property is being held, the method yields PBDs.

2.4.5. **Construction of PBDs from Transversal Designs and Truncated Transversal Design**

According to Stinson (2003) Suppose that $k \geq 2$ and there is a TD $(k + 1, n)$. Then the following Pairwise Balanced Designs exist:

1. a $(kt + u, \{k, k+1, n, u\})$-PBD for all u such that $2 \leq u \leq t-1$
2. a $(kt+1, \{k, k+1, n\})$-PBD, and
3. a $((k +1) t, \{k+1, n\})$-PBD.

A truncation of (X, B) is the removal of some set of points from X. Unlike deletion, blocks that were incident to a truncated point remain in the new configuration with reduced size. By ignore blocks of size 1 in what is left. Truncating points is an important tool to find linear spaces with higher dimension. Note that truncation is the same as Wilson's Construction where each point is assigned a weight of 0 or 1. Dukes and Ling (2008) showed that truncating from a linear space preserves dimension, as long as the remaining points are not contained in a subspace of dimension less than *d*. For dimension at least 3, it is necessary to have points remaining from distinct sub-planes. They make use of truncation from the projective space $PG_d(q)$ whose largest subspace of dimension less than *d* has order $1 + q + q^2 + \cdots + q^{d-1}$.

Duke and Ling (2008) later presented the following Lemma known as 'Asymptotic Existence of PBDs with Prescribed Dimension'**:** For any positive integer d and $\mathbf{K} \subseteq Z_{\geq 2}$, there exists a PBD(n, K) of dimension at least d for all sufficiently large n satisfying 2.3 and 2.4 and later established the proved known as 'Truncation Preserves Dimension'.

Let W be the linear space resulting from truncating $PG_d(q)$. If the order of W is larger than $1+q+q^2+\cdots+q^{d-1}$, then W has dimension at least d.

Inflation kv+1**.** For any PBD(n,{3, 4, 5}) of dimension at least 3. Inflate each point in X by k, for $k \in \{2, 3, 4\}$. Replace blocks with {3, 4, 5} GDDs of required group

type which. This results in a {3, 4, 5}-GDD of type k^n and strong dimension at least 3. Add a new point x at infinity and fill in each group with a block of size k +1 containing x and each group. The resulting PBD($nk + 1$,{3, 4, 5}) has dimension at least 3. Duke and Ling (2009) latter proved the above Lemma thus: With x truncated, the configuration is a PBD(nk,{3, 4, 5}) of dimension at least 3, by a Theorem . Suppose this PBD of order kn has dimension $d \geq 3$ and call it X. As k + 1 is a valid block size, we only need to show that adding x to each block of inflated points in X preserves the dimension. Because X has dimension d, no set of d points in *X* generate the entire space. The GDD, where blocks on inflated points in *X* are taken as groups, has strong dimension *d*. Thus, by Truncation Preserves Dimension, therefore, the resulting PBD($nk + 1$,{3, 4, 5}) has dimension at least d.

2.4.5. **Constructions of PBDs using Projective Planes and Geometries**.

According to Colburn and Dinitz (2007) let A be a set of nonnegative integers, and let D be a PBD. Then a {w; A} arc, (or A-arc of order w) in a PBD (n, **K**), say D, is a set of w points S of D such that if B is a block, then $|B \cap S| \in A$.

Many PBDs and GDDs can be obtained by deleting judiciously chosen sets of points from other PBDs. Most frequently these are sets of points in projective or affine spaces. Where the results are obtained from deletion of points from desarguesian geometries, the results are stated in terms of q for q a prime or prime power. Suppose that a projective plane of order n contains a {w; A} arc. Then it also contains a complementary $\{n_2 + n + 1 - w; n + 1 - A\}$ arc, where $n + 1 - A = \{n + 1 - a: a \in A\}$. This method is based on the understanding of Linear Space and Angle with is to mathematical and the method only specified block sizes and failed to specified how to generate the elements of each of the block sizes.

2.4.6. **The Wilson's Fundamental Construction of PBDs**

Wilson, (1971) defined a pairwise balanced design (PBD) of order n as a pair (X, B) where X is a set of cardinality n, and B is a set of subsets of X (each of which is called a block) with the property that every 2-element subset of X is contained in a unique block. Wilson, (1971) further defined a group divisible design (GDD) of order n is a triple (X, G, B) with the property that (X, G ∪ B) is a PBD of order n, and each point of X is contained in a unique member of G, i.e. G is a partition of X. (Members of G are called groups) and a t-wise balanced design (t-BD) is defined to be a pair (X, B), where X is a finite set of points and B is a set of subsets of X, called blocks with the property that every t-element subset of X is contained in a unique block. Thus, a group divisible design is just a pair wise (=2-wise) balanced design with a distinguished set of blocks G ⊂ B such that (X, G) is a l-wise balanced design. The blocks of a l-wise balanced design form a partition of its point set. Note that we do insist that blocks be of size at least two.

For example,

Suppose there is a "master" GDD(X, G, B) (group divisible design) with a groups and a group size vector of g{Gj} : j = 1, …,g} g and a weighting that assigns a positive weight of w(x) to each point x. Let W(Bi) be the weight vector of the i-th block. If, for every block Bi, we have a K − GDD with a group size vector of W(Bi), then there exists a GDD(X, G, B) with a group size vector of $\sum_{x\in G_j} w(x)\,; j = 1, \dots, g$. The idea in the above example is that points of the original 'master' GDD get weighted, and blocks get replaced by small 'ingredient' GDDs. There is one noteworthy special case. A weighting with w (x) = 0 or 1 for all x ∈ X is called a truncation; in this case, blocks get replaced by smaller blocks. A careful truncation has a mild (or possibly no) effect on the set of allowed block sizes K. Also,

the difficulty in finding an appropriate analogue for GDD's with t = 3 is that there are several alternative definitions for a GDD which generalize in different ways when t = 3. The definition given above states that (X, G, **B)** is a GDD if (X, G) is a l-design, and (X, G ∪ B) is a PBD.

Group-divisible designs are extensively employed in the construction of pairwise balanced designs. A basic example is the singular direct product (Mullin, et.al., 1996). Most recursive constructions for pairwise balanced designs employ some application or variant of Wilson's fundamental construction. Wilson (1974) further showed that the trivial necessary conditions for the existence of various kinds of designs are asymptotically sufficient. For example, given a positive integer $k > 1$, the necessary conditions for the existence of a 2- (n, k, 1) design are that $k-1$ divides $n-1$ and $k(k-1)$ divides $n(n-1)$; it follows from Wilson's theorem that, given k, a design exists for all but finitely many n satisfying these conditions.

A group divisible design is a triple (X, G, B), where X is a set of points, G is a partition of X into groups, and B is a set of blocks such that

- a group and a block intersect in at most one point; and
- every pair of points from distinct groups is together in exactly one block.

We refer to this as a GDD or K-GDD, the latter emphasizing that the blocks have sizes in **K** $\subseteq Z_{\geq 2}$. The type of a GDD is the list of its group sizes. When this list contains, say, u copies of the integer g, this is abbreviated with 'exponential notation' as g^u. Another standard abbreviation is the use of k-GDD instead of {k}-GDD.

We can regard a GDD (X, G, B) as the linear space (or PBD) (X, B ∪ G), where groups and blocks are taken together to form lines. Alternatively, we could consider the linear space (X, B ∪ G_2), where G_2 denotes the set of all pairs of distinct points from common groups. Each of these interpretations allows one to talk about dimension for GDDs, the former being stronger (lower dimension) in general. For our purposes, though, we prefer to take an even

stronger notion for dimension. Given a GDD, say (X, G, B), let's call a subspace $X' \subset X$ strong if it intersects each group of G in either all points or no points. A strong subspace is then proper if it is disjoint from at least one group. (In practice, many groups will be missed.) Correspondingly, the strong dimension of a GDD is the maximum number of points which always generates a proper strong subspace. With a PBD(n, K) regarded as a K-GDD of type 1^n, strong dimension coincides with ordinary dimension in this case. On the other hand, the strong dimension of a transversal design is just 1, since two points from different groups generate a block of the TD, which in turn intersects all groups. Given a PBD, say (X, B), if we delete a point x and all incident blocks B_x, the result is a GDD $(X \setminus \{x\}, G_x, B, \setminus B_x)$. Here, the group partition G_x is given by the (now missing) punctured lines $B \setminus \{x\}$, where $B \in B_x$. Reversing this process, if we are given a GDD, say (X, G, B), we can add a point ∞ and replace groups with new blocks, all incident with ∞. One might abbreviate this PBD by (X^*, B^*), where $X^* = X \cup \{\infty\}$ and $B^* = B \cup \{X_i \cup \{\infty\}: X_i \in G\}$.

However, many researchers had tried to point out four possible analogues of GDD definitions which are feasible as given below which make it somehow difficult to relied on as a basis for the construction of PBDs. Hanani, (1963) defined A (X, G, T) as a G-design if (X, G) is a l-design and (X, G ∪ T) is a 3-wise balanced design. This definition includes definition of a G-design: G(m, r, ***k,***) (Mills, 1974). where Mills insists that there be ***m*** groups of size r, and that all blocks have size ***k).*** It is also related to Hanani's definition of the systems P_m *(K,* 1, n) Beth, et.al., (1986) defined a (X, B, T) as a distinguished PB design if (X, B) is a PBD and (X, B u T) is a 3-wise balanced design. This definition is correct if only if one wishes to generalize the idea that one obtains a GDD by deleting a point from a PBD. Also, Mullin, et.al., (1996) defined a (X, G, B, T) as a distinguished GD design if (X, G) is a l-design, (X, G u B) is a PBD and (X, G u B u T**)** is a 3-wise balanced design. This definition also is a realistic generalization of the idea of deleting a point from a 3-wise balanced design. It

requires, however, that the PBD derived from blocks containing the deleted point should actually be a GDD in that it contains a l-designs. Wilson, (1974) further defines a **(**X, G, B1, B2, T) as a 2-fan design if (X, G) is a l-design, (X, G u B,) is a PBD, (X, G u B,) is a PBD and (X, Gu *B1 u* B2 u *T)* is a 3-wise balanced design. This definition is appropriate when deleting two points from a 3-wise balanced design. Wilson, (1972) purposed that: There exists a 'master' GDD (X, G, B), where G = {X_1..., X_u}. Let ω : X → {0, 1, 2,...}, assigning nonnegative weights to each point in such a way that for every Ɓ ∈ B there exists an 'ingredient' K-GDD of type ω(B) := [ω(x) | x ∈ Ɓ]. Then there exists a K-GDD of type

$$\omega(\Pi) := \left[\sum_{x\in X_1} \omega(x), \ldots, \sum_{x\in X_u} \omega(x)\right].$$

Furthermore, if the master GDD has strong dimension d, then the resultant GDD has strong dimension ≥ d.

Wilson went further to Prove this as thus: Replacing each point x ∈ X by a new set of $x_{1,}$ ..., $x_{\omega(x)}$ of ω(x) points, maintaining the group partition. So, the type becomes ω(G). Every block of the master, say Ɓ ∈ B, is replaced by a copy of the GDD of type ω(Ɓ) as defined. In the resultant, if two points x_i, y_j from different groups are given, their 'projections' x, y belong to different groups, and therefore a unique block in the master. This block was replaced by a unique ingredient GDD. It follows that x_i,y_j appear together in a unique block in this ingredient, and therefore in the resultant. For the claim on dimension, suppose a set of d points is given in the resultant. They arose from at most d points, say Y ⊂ X in the master GDD. By assumption, Y is contained in a proper strong subspace X′ ⊂ X. It is clear that X′ lifts to a strong proper subspace {x_i : x ∈ X′,i = 1,...,ω(x)} in the resultant, since two points from different groups in X′ lie on a block of some ingredient GDD placed on X′.

2.5 The PBD-Closed Sets

In a sequence of three papers Wilson (1971, 1972, 1974) developed a theory of PBD-closed sets and the notation of PBD-closure. A set S of positive integers is said to be PBD-closed if the existence of a PBD(n, S) (n > 0) implies that n belongs to S.

Let K be a set of positive integers and let B(K) = {n > 0 ∈ PBD(n, K)}. Then B(K) is a PBD-closed set called the PBD-closure of **K**. According to Wilson's theory there exists a constant c_0(**K**) such that designs PBD(n, **K**) exist for all $n \geq c_0$(**K**) which satisfy the congruence $(n-1) \equiv 0 \bmod \alpha(\mathbf{K})$

And

$n(n-1) \equiv 0 \bmod \beta(\mathbf{K})$,

where

$\alpha(\mathbf{K}) = \gcd\{k-1 : k \in \mathbf{K}\}$ and

$\beta(\mathbf{K}) = \gcd\{k(k-1) : k \in \mathbf{K}\}$.

Concerning the structure of PBD-closed sets Wilson (1974) also showed that if S is a PBD-closed set, then S is eventually periodic with period β(S); that is, there exists a constant c_0(S) such that for every k ϵ S, $\{n : n \geq c_0(S), n \equiv k \bmod \beta(S)\} \subseteq S$.

Wilson (1974) further proves the necessary conditions and PBD- closure as follows: Suppose $\mathbf{K} \subseteq \{n \in \mathbf{X}: n \geq 2\}$, and define

B (**K**) = {n: there exists a (n, **K**)-PBD}.

Furthermore, define

$$\alpha(\mathbf{K}) = \gcd\{k-1 : k \in \mathbf{K} \quad (2.8)$$

and

$$\beta(\mathbf{K}) = \gcd\{k(k-1) : k \in \mathbf{K}. \quad (2.9)$$

Note that K ⊆ B (K) because a (trivial) (k, {k})-PBD exists for any integer $k \geq 2$.

Suppose K ⊆ {n ∈ X**:** n ≥ 2} and suppose that n ≥3 is an integer. Then n ∈ **B** $(K)_{only}$ if n − 1 ≡ 0 (mod α (K)) and n (n − 1) ≡ 0 (mod β(K)).

Suppose n ∈ **B**(K). Then there exists a (n, K)-PBD. Let X be the set of points in this design, suppose x ∈ X, and let r_x denote the number of blocks containing x. Let $B_1, \ldots, B_{rx}$ denote the blocks that contain x. Then,

$$\sum_{i=1}^{rx}(/Bi/-1) = n - 1 \qquad (2.10)$$

Clearly |*Bi*| −1 ≡ 0 (mod $\alpha(K)$) for all i, $1 \leq i \leq rx$, and hence $n - 1 \equiv 0$ This proves the first condition

In the second condition, let $B_1, \ldots, B_b$ be all the blocks in the Pairwise Balanced Design. Then,

$$\sum_{i=1}^{b}/B_i/\{/B_i/-1\} = X - 1 \qquad (2.11)$$

Clearly. /Bi/(/Bi/-1) ≡ 0 (mod $\beta(K)$) for all i, $1 \leq i \leq b$. Hence, n (n − 1) ≡ 0(mod $\beta(K)$)

This completes the proof.

If |K| = 1 (say **K** = {k}) and n > k, then a (n, K)-PBD is a (n, k, 1)-BIBD, and the conditions in the above become

$$n - 1 \equiv 0 \pmod{k-1} \qquad (2.12)$$

and

$$n(n-1) \equiv 0 \pmod{k(k-1)} \qquad (2.13)$$

These are precisely the conditions that the BIBD parameters r and b (respectively) be integers.

For example,

Suppose **K** = {3, 4, 6}. Then it is easy to compute

α(K) = gcd{2, 3, 5} = 1 (2.14)

and β(K) = gcd{6, 12, 30} = 6.

According to (2.3) and (2.4), necessary conditions for the existence of a PBD with block sizes 3, 4 or 6 are that:

$n-1 \equiv 0 \pmod 1$ and

$n(n-1) \equiv 0 \pmod 6$

The first conditions just say n is an integer. The second condition is satisfied if and only if

$n(n-1) \equiv 0 \pmod 3$, so we have $n \equiv 0$ or 1 mod 3. Then $n \geq 3$ follows since 3 is the smallest block size. The necessary conditions in the above, simplify to $n \equiv 0$ or 1 (mod 3). It follows that:

$\mathbf{B}(\{3, 4, 6\}) \subseteq \{n \in \mathbf{X}: n \equiv 0 \text{ or } 1 \pmod 3, k \geq 3\}$.

A PBD is a Closed set if B(Suppose that K) = K, i.e., if $n \in \subseteq$ {K whenever there exists a $n \in \mathbf{X}$: $k \geq 3$}. We say that K is a (n, K)-PBD.

The following definition is important. It is commonly called **"breaking up blocks**".

Wilson, 1973 ascertain that, Suppose $K \subseteq \{n \in \mathbf{X} : k \geq 3\}$. Then **B**(K) is PBD-closed.

He made further claims that Suppose $K \subseteq \{n \in \mathbf{X} : k \geq, 3\}$, and let (X, Ɓ) be any (n, B(K))-PBD.

Wilson, (1974) further gave the following examples of a closed PBD-sets:

(a) The set of n for which a 2-(n, k, λ) design exists, for any given k and. (Repeated blocks are permitted here.)

(b) The set of n for which s mutually orthogonal Latin squares of order n with a common transversal exist, for any given s.

(c) The set of n for which the edges of the complete graph K_n can be partitioned into copies of a given graph G.

(d) The set of r for which a resolvable 2-$(n, k, 1)$ design with r parallel classes exists (given k).

(e) The set of sides of Room squares.

The proof of part (a) is simple but illustrative. Suppose that $n \in B(K)$, and suppose that a 2-(l, k, λ) design exists for each $l \in K$. Take a PBD(K) on a set of v points. Each block of this PBD carries a 2-design with block size k; we just take all the k-sets occurring as blocks in these designs. Now any two distinct points lie in a unique block of the PBD, and hence in exactly λ blocks of the constructed design, as required. For example, the theorem asserts that B({4,7}) contains all but finitely many integers congruent to 1 mod 3; further analysis shows that the only exceptions are 10 and 19. Since there exist 2-(4, 4, 2) and 2-(7, 4, 2) designs, we conclude that there exist 2-(n, 4, 2) designs for all n congruent to 1 mod 3 except possibly $n = 10$ and $n = 19$. The general existence question is thus reduced to just two cases (in both of which the designs exist, as it happens). In the second example, the PBD-closed set K contains all prime powers $n > s+1$, and so $\alpha(K)=\beta(K)= 1$. We conclude that there is a number $n_0(s)$ such that s MOLS of order n exist for all $n \geq n_0(s)$. Another way of describing a PBD-closed set K is by giving its base X, the set of all elements $x \in K$ such that $x \in/ B(K \backslash \{x\})$. It can be shown that this set is finite, and is the unique minimal set satisfying B(X) = K. For example, the base for the set of orders of Steiner triple systems is {3}. It is not possible to give more than a brief outline of the proof of Wilson's theorem here. The proof, though elaborate, is in the spirit of many constructions of designs, and uses both direct and recursive construction methods. The direct methods show that PBDs with block sizes in K exist for all sufficiently large prime powers q satisfying the divisibility conditions. These constructions use cyclotomy to construct a family of subsets of GF(q), with sizes from K, such that any non-zero element of GF(q) is uniquely expressible as a difference of two elements from the same set. The recursive constructions then build new PBDs from old, showing that a PBD of any sufficiently large size satisfying the necessary conditions will exist. Many of the recursive constructions are conveniently expressed in terms of group

divisible designs (GDDs). We refer to Wilson's papers for details. There is an exposition of the proof in Wilson's survey article of Wilson, (1974). The theory is developed in a series of papers of Wilson, (1972).

2.6 Characteristics of Pairwise Balanced Designs

Street and Street (1987) identified PBDs by their block structures. Generally, there are:

n is the total number of treatments, **K** is the set of sizes of the block, therefore, **K** = {k1, k2, k3, …, kb} and λ is the number of time a pair of treatment appears within a block.

2.7. The Stanton-Kalbfleisch Bound

Stinson, (2003) derived and proved an equation to check the bound of minimum and maximum sizes of the block of a PBD as follows: Let k and n be integers such that $2 \leq k < n$. Suppose there is a $(n, \{2, \ldots, n-1\})$-PBD in which there exists a block containing exactly k treatments. Then,

$$b \geq SK(k,n) = 1 + \frac{k^2(n-k)}{n-1}$$

He when further to established this as follows: suppose that (X, Ɓ) is a $(n, \{2, \ldots, n-1\})$-PBD such that $B \in Ɓ$ is a block containing exactly k treatments. Denote the blocks of Ɓ by $B_1, \ldots, B_b$, where $B_b = Ɓ$.

Now construct a set system (Y, B) by deleting all the points in the block $Ɓ_b$ as follows:

$Y = X \backslash B_b$,

$B_i = B_i \backslash B_b$, $1 \leq i \leq b-1$, and

$Ɓ = \{B_i: 1 \leq i \leq b-1\}$

(Y, B) is a set system with n−k treatments and b−1 blocks in which every pair of treatments occurs in a unique block. (This set system may contained blocks of size one, so it need not be a PBD). For $1 \leq i \leq b-1$, denote $k_i = |B_i|$. Note that $k_i = |B_i|$ or $k_i = |B_i|-1$ for $1 \leq i \leq b-1$. Furthermore, $k_i = |B_i|-1$ if and only if B_i intersects B_b in a treatment.

Denote the treatments in Y by y_j, $1 \leq j \leq v-k$. For $1 \leq j \leq n-k$, define r_j = $|\{B_i \in \mathcal{B}: y_j \in B_i\}|$. Then a straightforward generalization of this claim shows that

$$\sum_{i=1}^{b-1} k_i = \sum_{j=1}^{b-1} r_j \qquad (2.14)$$

Now, in the pairwise balanced design $(X, \mathcal{B})$, every treatment y_j must occur in a unique block with each of the treatments in B_b. Hence $r_j \geq k$ for all j, $1 \leq j \leq n-k$. Substituting into (2.14), it follows that

$$\sum_{i=1}^{b-1} k_i \geq k(n-k) \qquad (1.15)$$

Every pair of treatments in Y occurs in exactly one of the B_i's, so it follows that

$$\sum_{i=1}^{b-1} k_i(k_i - 1) = (n-k)(n-k-1) \qquad (2.16)$$

Denote the mean of the integers $k_1, k_2, \ldots k_{b-1}$ to be

$$\overline{k} = \frac{\sum_{i=1}^{b-1} k_i}{b-1} \qquad (2.17)$$

Now, study quantity

$$S = \sum_{i=1}^{b-1} (k_i - \bar{k})^2 \qquad (2.\ 18)$$

We can use equations 2.16 and 2.16 to derive a formula for S:

$$S = \sum_{i=1}^{b-1} {k_i}^2 - 2\bar{k} \sum_{i=1}^{b-1} k_i + (b-1)(\bar{k})^2 \qquad (2.19)$$

$$= \sum_{i=1}^{b-1} k_i(k_i - 1) - (2\bar{k}-) \sum_{i=1}^{b-1} k_i + (b-1)(\bar{k})^2$$

$$(n-k)(n-k-1) - \left(2\bar{k} - 1\right)(b-1)\left(\bar{k}\right) + (b-1)(\bar{k})^2$$

$$(n-k)(n-k-1) - (b-1)\bar{k}(\bar{k} - 1)$$

Also, observe that S is a sum of non-negative terms, so clearly $s \geq 0$ therefore,

$0 \leq (n-k)(n-k-1) - (b-1)\bar{k}(\bar{k}-1)$ (2.20)

And that,

$\bar{k} \geq 1$ because $k_i \geq 1$ for all i, and hence, from 2.15, then,

$\bar{k}(\bar{k}-1) \geq \left(\frac{k(n-k)}{b-1}\right)\left(\frac{k(n-k)}{b-1} - 1\right)$ (2.21)

Substituting into equation 2.20 to obtain

$0 \leq (n-k)(n-k-) - (b-1)\left(\frac{k(n-k)}{b-1}\right)\left(\frac{k(n-k)}{b-1} - 1\right)$ (2.22)

Divide by a factor of n – k and simplifying, to obtain

$0 \leq n-k-1-(\frac{k}{b-1})(k(n-k)-(b-1)$ (2.23)

= n – 1- $k^2 \frac{(n-k)}{n-1}$

Hence, b $\geq 1 + k^2 \frac{(n-k)}{n-1}$

2.8 The formation of Lotto Designs.

Lotto Designs was derived from the game of lottery. For a small fee, a person chooses k numbers from n numbers or has the numbers chosen randomly for him. This constitutes the ticket. The sale of tickets is stopped at a certain point and the government or casino picks p numbers from the n numbers randomly. These p numbers are called the winning numbers. If any of the tickets sold match t or more of the winning numbers, a prize is given to the holder of the matching ticket. The larger the value of t, the larger the prize. Usually, t must be three or more to receive a prize.

2.9. The Lotto Designs (LD)

Li (1999) defines an LD(n, k, p, t) Lotto Design as a k-sets (blocks) of an n-set(treatments) such that any p-set intersects at least one k-set in t or more treatments.

Suppose n, k, p *and* t are integers and B is a collection of K-subsets of a set X (the block of n treatments) usually X is X (n). Then B is a (n, k, p, t) Lotto Design (LD) if an arbitrary p-subset of X (n) intersects some K-set of B in at least t-treatments. The K-sets in B are known

as the blocks of the Lotto design . The elements of *X* are known as the treatments of the design. The K-sets in B are known as the blocks of the lotto design B. The elements X are known as the treatments of the design. Lotto design can be denoted by (X, B) Where B denotes the blocks of the design and X denotes the set from which the elements of the blocks of B are chosen. For clarity, an (n, k, p t) Lotto Design (LD) is a set of k-blocks of an n-treatments such that any p-treatments intersect at least one k-block in t number of treatments.

For example,

An LD(7, 5, 4 , 3) where

n = 7, k= 5, p = 4, t =3 therefore,

X = {1, 2, 3, 4, 5, 6, 7}

The sets of K are all the blocks that could be formed from the set X chosen 5 at a time i.e $\binom{7}{5}$ we shall have the following blocks:

12345	12457	14567
12346	12467	23356
12347	12567	23457
12356	13456	23467
12357	13457	23567
12367	13467	24567
12456	13567	34567

The above 35 set of blocks (K) are the block B of the lotto design.

The next thing to do is to get all possible set of p blocks that can intersect relevant block of B therefore, a new set of blocks of p will be $\binom{7}{4}$ we shall have the following blocks

1234	1256	1557	2346	2467
1235	1257	1367	2347	2567
1236	1267	1456	2356	3456

1237	1345	1457	2357	3457
1245	1346	1467	2367	3467
1246	1347	1567	2456	3567
1247	1356	2345	2457	4567

$\binom{7}{5}$ is a collection of all k-sets(blocks), the number of blocks for intersecting is $\binom{7}{4}$ which is p and p = 4, the below blocks are the set of blocks when at least 3 treatments in p intersect k-blocks in B.

12345	12367	13457
12346	12456	13467
12347	12457	23456
12356	12467	23457
12357	13456	23467

Thus, in relating it to lottery game, the maximum number of tickets that must be bought by the player to guarantee a win is the 15 blocks above. Two special subclasses of lotto designs are covering designs and turan design A(n ,k ,t) covering designs is an (n, k, t, t) lotto design and a (n, p, t).Turan design is an (n ,t, p ,t)lotto design . The minimum number of blocks in any (n, k, t) is denoted by C(n, k, t) and the minimum number of blocks of any (n, p, t) Turan design is denoted by T (n, p, t) (Colbourn and Dinitz,1996). Upper bounds for lotto designs are often determined using computer search algorithms. Bate (1978). describe an exhaustive search algorithm to determine the value L (n, k, p, t). (Nurmela and Ostergard (1993) use simulated annealing to generate upper bounds for lotto design**.** Li (1999) discusses several other algorithms for constructing lotto designs the unfortunate feature of computer search algorithm is that, the search space tends to grow exponentially with respect to the value of n, k, p and t. Thus, more search algorithms are infeasible to run except small value of n, k, p and t. Thus, it is important to have techniques for constructing lotto designs

which can be stated as a formula which involve no searching. Li and Van Rees (2006) defined a (n, k, t)covering design or cover as an n-set X of elements and a set $\mathcal{B}$ of k-element subsets of X (blocks), such that every t-subset of X occurs in a block of $\mathcal{B}$. $C(n, k, t)$ Is the smallest number of blocks in a (n, k, t) covering design? An (n, k, t, t)-lotto design is a (n, k, t) covering design.

Li and Van Rees (2006) defined a (n, p, t) Turán system as an n-set of X of elements and a set $\mathcal{B}$ of t-element subsets of X (blocks), so that every p-subset of X contains a block of $\mathcal{B}$. $T(n, p, t)$ is the smallest number of blocks in an (n, p, t) Turán system an (n, t, p, t)-lotto design is an(n, p, t) Turán system. The earliest work in combinatorial literature on the lottery problem was due to Hananiet.al(1964).This work showed that $L(n, k, p, 2) \geq n(n-p-1)/k(k-1)(p-1)$.This was followed by Bate (1978) who studied the generalized (T, K, L, V) designs and the problem posed by Turán (1954) is equivalent to the determination of $B(T, T, L, V)$ design, a subcase of the generalized (T, K, L, V) designs. These (T, K, L, V) designs are actually (n, k, p, t) Lotto Designs even though Bate did not call them Lotto designs. Bate included in his thesis a computer program that can be used to construct, for small parameters, a minimal (n, k, p, t) lottery design. Lower and upper bounds for $B(T, K, L, V)$ where $V \leq 16$ and $K + L \leq V$ were also included in Bates thesis.

Brouwer (1978) derived this result independently using covering designs and it was stated in terms of covering numbers. Brouwer and Voorhoeve (1979) showed that$L(n, k, p, t) \geq$

$$\frac{T(n, p, t)}{C(k, t)}$$

De Caen (1983) determined a lower bound for Turan designs. It states: $T(n, p, t) \geq$

$$\frac{C(n, t)}{C(p-1, t-1)} \cdot \frac{n-p+1}{n-t+1}$$

Nurmela and Ostergard (1993) gave

$$L(n,k,p,t) \geq \frac{C(n,k)}{\sum_{i=t}^{\min(k,p)} C(k,i) \cdot C(n-k,\ p-i)}$$

As a lower bound for L(n, k, p, t). they used a program called "cover" to generate it. "Cover" is a probabilistic search technique based on simulated annealing. Excellent upper bounds have been got for many values of L (n, k, p, t) using simulated annealing as it is a useful tool for constructing Lotto Designs.

Because lower bounds are very difficult to compute for Lotto designs, Furedi et al (1996) solved for the case $t = 2$ using a multi-graph approach. The following lower bound for L (n, k, p, 2) was arrived at: $L(n,k,p,2) \geq \left(\sum_{t-1}^{p-1} ai\right) \frac{1}{k} \left(\sum_{i=1}^{p-1} ai \left\lceil \frac{ai-1}{k-1} \right\rceil\right)$ min

Colbourn and Dinitz (1996) gave an upper bound construction for L(n, k, p, t). It was shown that: If $n = n_1 + n_2$ and $p = p_1 + p_2 - 1$then $L(n,k,p_1,t) \leq L(n_1,k,p_1,t) + L(n_2,k,p_2,t)$. this result was applied to $L(49,6,3,3) + L(27,6,4,3) \leq 77 + 91 = 168$. This implies that in Canada's Lotto 6/49, a player is guaranteed to match 3 numbers if he/she buys a certain 169 tickets. Bate and van Rees (1998) determined the values for L(n, 6, 6, 2) for $n \leq 54$ by examining and analyzing the frequencies of elements in "nice" designs.

Li (1999**)** gave a complete proof of the result of Furediet.al in his thesis. He also gave a standard result for obtaining an upper bound for L(n, k, p, 2). Li determined which BIBDs are Lotto designs. He gave conditions that a BIBD must satisfy in order for it to be a Lotto design. Upper bounds for Lotto designs are given from constructions using BIBDs.

2.10 The Pairwise Balanced Designs and The Lotto Designs

Recalled that a PBD(n, **K**, λ) has the following parameters:

n is the total number of treatments

K is the set of the block sizes

λ is the number of time a pair of treatment appears within blocks

These parameters could be equated to that of lotto design parameters n, k, p, t as follows:

n is the total number of treatments in the PBDs and is equivalent to the n number of treatments of an LD.

K is the set of block sizes of the PBDs and is equivalent to a set of k-blocks **K** of an n-treatments of an LD.

p is the positive integer and it constitutes number of treatments that can intersect relevant k-blocks of an LD.

t is a positive integer and it is the number of treatments in p that match k-blocks.

2.11. **Optimality Criteria of PBDs.**

Let X = { $x_1, x_2, \ldots, x_n$} be a set of n- treatments; By a block design with parameters n, b, $k_1, k_2, \ldots, k_b$; N) and incidence matrix N denoted by BD(n, b, $r_1, r_2, \ldots, r_n$; $k_1, k_2, \ldots, k_b$; N) on X shall mean an allocation of elements of X-treatments one on each of the $m = \sum_{j=1}^{b} kj$ experimental units arranged in b blocks or groups of experimental units of size $k_1, k_2, \ldots, k_1$ such that x_1 is assigned into r_i experimental units. Thus, $\sum_{i=1}^{n} ri = \sum_{j=1}^{b} kj$. A block design is said to be proper if $k_j = k$, $j = 1, 2, \ldots, b$. A block design in which $r_i = r$, $i = 1, 2, \ldots, n$ holds will be said to be an equireplicate design. A block design is said to be Locally connected when any two treatments can be connected by a chain consisting alternatively of treatments and blocks, so that if treatment and block Bj are any two consecutive members of the chain, then treatment occurs in block B_j. Hedayat, (1972) has given a generalization of the concept of local connectedness and has introduced globally connected designs. A block design is said to be incomplete if there exists a block which does not contain all the elements of X i.e treatments. If we can assume that the position of experimental units in the blocks bears no information whatsoever, then the usual n x b incidence matrix N= (n_{ij}) where n_{ij} scores the number of experimental units in the j^{th} block receiving the i^{th} treatment, completely

characterizes the combinatorial arrangement of the design. A block design is said to be an n-array block design if the entries of N constitute n distinct integers D as and Rao (1986), Murty and Das (1967), Rao and Das (1969) and Tocher (1952).

A block design is said to be pairwise balanced if $NN' = T + \lambda J$, where N' is the transpose of N, T a diagonal matrix, λ a scalar and J a matrix with unit entries everywhere (Bose, and Shrikhande 1960).

CHAPTER THREE
RESEARCH METHODOLOGY

3.0 The Fundamentals

Numerous pairwise balanced designs constructed so far by various scholars were based on the understanding of Group Divisible Designs (GDDs) which have numerous analogues, interpretations and meanings: Linear Space which has many theorems and BIBDs which could not be used for large treatments. However, PBD plays important roles in the designs theory. More importantly, among the incomplete block designs PBDs have varieties of applications. The greatest significance of pairwise balanced designs is in its application to construct other designs and confirm the existence questions for other important of designs (Ronald, et.al., 1995). Therefore, this research work introduces a new method for the construction of Pairwise Balanced Designs (PBDs) from the Lotto Designs (LDs). In designs theory, if a new design is constructed from another design it is known as a recursive construction and if a specific fitted class of designs are generated with specific conditions imposed using mathematical relations of modulus system the design is known as congruent class, therefore, this research work presents a simple way of construction of PBDs from an appropriate LDs which is both recursive and congruent. The following stages shall be followed:

3.1. **The Construction Stages**

Stage One: Specification of the classes of the PBDs.

Stage Two: Identification of LDs that will qualify as PBDs. Computer program based on the Li inequality: $\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \left(pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1) \right) < \binom{p}{2}\lambda$ a concept in characterization of block designs was written so as to identify LDs that could produce two classes of the PBDs to be constructed.

Stage Three: Specify certain conditions and imposed those conditions on all the generated LDs from the Li inequality and Select all LDs that satisfy the specified conditions.

Stage Four: Derive steps for the construction of the classes of PBDs(n, {3, 4}) and (n, {3, 4, 5}) from LDs.

Stage five: The Actual Construction of PBDs from LDs

State Six: Confirmation of the results of the classes of PBDs constructed Using Smith at.al, (1998) and Stanton-Kalbfleisch Bound. Stinson (2003).

3.2.0. Stage one Explained.

An incidence structure is a triple (P, L, l), where P is a set of points (treatments), L is a set of lines, and $l \subset P \times L$ is a set of flags (sub designs). We say $p \in P$ is incident with or simply on $L \in l$ (and vice-versa) if and only if $(p, L) \in l$.

For example,

P be a set of points or (collection of points). Then, $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ and

Let L be a set of lines or (collection of lines. Then, $L = (l_1, l_2, l_{3\ldots}, l_n)$

Therefore,

$P \times L = \{((x_1, y_1), l_1), ((x_2, y_2), l_2), \ldots, ((x_n, y_n), l_n)\}$

A linear space is an incidence structure (P, L, l) with the property that every line is on at least two points and any two distinct points are both on exactly one line. In what follows P (and hence L) are assumed finite. The trivial case in which all points are on the same line is not excluded by our definition but it is effectively ruled out.)

Linear spaces appear in another context as Pairwise Balanced Designs (or PBDs)

For example,

given a set of points S = {(1, 2), (3, 6), (1, -4)} and plot these points on the graph and if all the points falls on the same line then we have what is known as a linear space which is related to PBDs.

Specifically, if n (number of treatment) is a positive integer and $K \subset Z_{\geq 2} := \{2,3,4,...\}$ a set of block sizes, a PBD(n, K) consists of a n-set X, together with a set Ɓ of blocks, where

(i) for each $B \in Ɓ$, we have $B \subset X$ with $|B| \in K$; and

(ii) any two distinct elements of X appear together in exactly one block.

Note that there are numerical constraints on n given K. First, the number of pairs of distinct points must be expressible as a (nonnegative) integral linear combination of the number of distinct pairs arising from blocks with sizes in K. This leads to what I called the general condition given by Wilson (1971) as

$n(n-1) \equiv 0 (\mathrm{mod}\ \beta(K))$, (general)

where,

$\beta(K) := \gcd\{k(k-1) : k \in K\}$.

Note that gcd is a mathematical term that means greatest common divisor

Also, deleting any point $x \in X$ from its incident blocks must partition the remaining points. That is, n−1 is an integral combination of $k - 1$, $k \in K$. This is what I called the local condition namely:

$n - 1 \equiv 0 (\mathrm{mod}\ \alpha(K))$, (local)

Where,

$\alpha(K) := \gcd\{k - 1 : k \in K\}$.

One can interchangeably discuss linear spaces and PBDs, identifying lines with the respective subsets of incident points as blocks. Then the notation such as (X, B) is used for PBDs and the associated linear spaces; the incidence relation l is seldom used from now on.

However, we occasionally retain some terminology from linear spaces (i.e. points, lines, spaces) when discussing PBDs.

Despite the similarity in the definitions, there is usually a difference in focus between the study of PBDs and linear spaces. The former is usually approached with a fixed K in mind, asking for which *n* we have existence. The latter often concerns additional structures such as configurations or localizations at points. Some features, such as parallelism, appear in both contexts.

3.2.1. Linear Space Dimension and PBDs

The dimension of a linear space is the maximum integer d such that any set of d points generates a proper subspace. For instance, the subspace generated by any two points is the line containing them. So, every nontrivial linear space has dimension at least two (Delandstsheer, 1995). Recall that a Steiner triple system is a PBD(n, {3}). It is well-known that Steiner triple systems on n points exist if and only if $n \equiv 1$ or 3 (mod 6). A Steiner space is defined to be a Steiner triple system of dimension at least 3. Teirlinck (1979) nearly completely settled the existence of Steiner spaces. The result is that for $n \equiv 1$ or 3 (mod 6) and $n \notin \{51, 67, 69,145\}$, there exists a Steiner space on n points if and only if n = 15,27,31,39, or $n \geq 45$. The four undecided cases are still open, to the best of my knowledge.

Another important family of linear spaces, especially in design theory, is that of the affine spaces. Let q be a prime power and F_q the finite field of order q. Consider the vector space X = F^d_q as points, together with all possible translates of subspaces x+ W $\subseteq$ X as 'flats'. This forms the affine space $AG_d(q)$.

Let $\mathcal{B}$ be the set of all lines in $AG_d(q)$. From basic linear algebra, we see that $(X, \mathcal{B})$ is a linear space (PBD) of dimension *d*, since *d*-point-generated subspaces correspond to proper

flats (and some d+1 points generate the whole space). There are $n = q^d$ point and every line has exactly $k = q$ points. In other words, this is a PBD($q^d, \{q\}$) of dimension d.

3.2.2. An existence theory that treats arbitrary block size(s) for all sufficiently large and admissible n = |*X*|.

Given $K \subseteq Z_{\geq 2}$ and $d \in Z_+$, there exists a PBD(n, K) of dimension at least d for all sufficiently large n satisfying (general) and (local).

When K = {k},

Then, we have $\alpha(K) = k - 1$ and

$\beta(K) = k(k - 1)$.

Since this is often the case of primary interest, and for clarity of presentation, we first prove this case separately.

For $k \in Z_{\geq 2}$ and $d \in Z_+$, there exists a PBD(n ,{k}) of dimension at least d for all sufficiently large n satisfying

$$n - 1 \equiv 0 \pmod{k-1}; \text{ and}$$

$$n(n-1) \equiv \pmod{k(k-1)}$$

Very broadly, the proofs proceed by applying some standard design-theoretic constructions to the affine space of dimension *d*, ensuring that the dimension stays preserved. First off, we state Wilson's famous 'asymptotic' existence result for PBDs.

Wilson**.** (1972) states that: Given $K \subseteq Z_{\geq 2}$, there exists n_0 such that a PBD(n, K) exists for all $n \geq n_0$ satisfying (general) and (local) conditions.

The replication number of a PBD(n {k}) is the common number r = n -1/ k -1 of blocks incident with each point. The local necessary condition for K = {k} amounts to $r \equiv 0 \pmod{}$

1). The general condition is easily seen as equivalent to $r(r-1) \equiv 0 \pmod{k}$. So one can restate Wilson's Theorem in terms of replication numbers.

Given $k \geq 2$, there exists $r_0(k)$ such that a PBD with block size k and replication number r exists for all $r \geq r_0$ satisfying $r(r-1) \equiv 0 \pmod{k}$.

It is clear that blocks of a PBD can be replaced by other PBDs. That is, the existence of a PBD(n, *K*) and, for each $k \in K$, a PBD(*k*, *L*) implies the existence of a PBD(n, *L*). This construction, which is usually known as 'breaking up blocks', respects dimension in a certain sense.

3.2.3. Proposition for PBDs dimension

Suppose there exists a PBD(n, K) of dimension d and, for each $k \in K$, any PBD(k, L). Then there exists a PBD(n, L) of dimension $\geq$ d.

Proof. In the PBD(n, K), say (X, Ɓ), replace each block B of size k with a PBD(k, L) on the points of B. The result is a PBD(n, L), say (X, B_1). It remains to check the dimension. Suppose a set *Y* of *d* points is given. They generate a proper subspace X' in (*X*, Ɓ) by hypothesis. But this remains a subspace in B_1 after replacement of blocks by PBDs.

3.2.4. The PBDs Dimension

The dimension of a linear space is the maximum positive integer *d* such that any *d* of its point generates a proper subspace. For a set *K* of integers at least two, recall that a pairwise balanced design PBD(*n*, *K*) is a linear space on *n* points whose lines (or blocks) have sizes belonging to *K*. We show that, for any prescribed set of sizes *K* and lower bound *d* on the dimension, there exists a PBD(n, *K*) of dimension at least *d* for all sufficiently large and numerically admissible n.

For example, consider the case d = 3, K = {3, 4, 5} one of the classes to be constructed. This K is interesting because α(K) = 1, β(K) = 2 and in fact any positive integer n ≡ 2, 6, 8 admits a PBD(n,{3 ,4 ,5}). This brings up an interesting side note.

Specifically, this research work attempted to construct two classes of PBDs where,

K = {3, 4};

And {3, 4, 5}

Using LDs and determined admissible value of n of dimension two or three. So every PBD(n, K) that has more than one block has dimension at least two. Recalled that a PBD(n, K), where n is the number of treatments and K is the set of blocks which must be ≥ 2 therefore, K ⊆ $Z_{\geq 2}$ which means n-treatments may contains the allowed block sizes.

3.3.0. The Li's inequality (stage Two)

The aim of using the Li-Inequality is to obtain LDs that qualify as PBD(n, K) where K = {3, 4} and {3, 4, 5} because it will be easier to use such LDs to construct PBDs. Li (1999) suggested that for any designs to be qualified as LDs, such a design must have satisfied the bellow inequality:

$$\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \left(pr - \left\lfloor \frac{pr}{t-1} \right\rfloor (t-1) \right) < \binom{p}{2} \lambda$$

The parameters p, r, t, λ of this inequality had been considered in written the program where PBDs where p is the set of treatments that can intersect relevant k-blocks of an LD, t is the number of treatments in p that match k-blocks, r is the number of time a particular treatment appears in each block and λ is the number of time a pair of treatment appear within block The ranges of p and t are selected to fit the range of most lottery formats available around the world. While r and λ are got from the classes of the specified PBDs.

The Li 's inequality could be broken into five components as follows:

Component 1: $\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2}$ = COMST

Component 2: $\binom{pr-\left\lfloor \frac{pr}{t-1} \right\rfloor}{2}(t-1)$ = COMBP

Component 3: $\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \binom{pr-\left\lfloor \frac{pr}{t-1} \right\rfloor}{2}(t-1) = COMBTT$

Component 4: $\binom{p}{2}\lambda$ = COMBL

Component 5: ANSW = Answer (with Yes or No)

3.3.1 The Li Algorithm

An Algorithm that will enable us to determine the PBDs that qualifies as Lotto Designs is as follows:

1. Define parameters p, t, r, λ

- P is fixed to run from 4 – 31
- t is fixed to run from 3-8
- r is calculated from a list of PBDs
- λ is selected from a list of PBDs

2, a Compute formula:

$$\left\lfloor \frac{pr}{t-1} \right\rfloor \binom{t-1}{2} + \binom{pr-\left\lfloor \frac{pr}{t-1} \right\rfloor}{2}(t-1) < \binom{p}{2}\lambda$$

2. b Identify items less than $\binom{p}{2}\lambda$

2. c Select elements using the criterion before

3 .Select sets qualify as lotto design.

3.3.2. Analysis of Algorithm

Step I

The choice of problem range has a significant effect on the result. Step 1 sets the initialization of working variables and boundary parameters. The time here is of the order 0(n).

Step 2

The iterative computation of Li's formula involves two loops. The outer loop runs for

$p(\cdots\ \cdots)$ while the inner loop run for t (… … . .). The computation time in this case is $O(n(-1))$. The external loop executes n times while the inner loop executes $n - 1$ times. The time for addition and multiplication and comparison within the formula is of the order $O(n)$ but is considered to be absorbed with the time set in $O(n(-1))$.

Step 3

Step 3 outputs the selected sets qualifying as lotto design. The execution time is $O(n)$ corresponding to the highest number of p. Further selections can be made by varying and refining the initial selection parameters and variables in step 1.

End of the algorithm.

3.4.0. Specification of conditions and imposition of the conditions (Stage Three)

Since not all the LDs that would be generated through the Li inequality may qualify as PBDs there is need to specify certain conditions that must satisfy for any LDs requirements that would meet the required PBDs classes. In doing this certain factor must be considered such as:

(i) General structures of the block designs

(ii) The general structures of LDs and

(iii) The general structures of the classes of the intended PBDs.

Based on the 3 conditions mentioned above, the following conditions shall be imposed on all the LDs generated from the Li inequalities program using r and λ of the two classes of the PBDs.

Any LD(n, k, p, t) satisfying the conditions:

(i) k = 3 or 5,

(ii) k < n, and

(iii) n = p .

3.4.1. The Conditions explained

Impose conditions k = 3 or 5, k < n and n = p on all the LDs produced. This imposition is based on the observation of the LDs taking into consideration the theory of lotto designs and the theory of combinatorial analysis. If $p \geq n$, the combination $\binom{n}{p}$ will not be defined. If $k \leq 3$ and $p = n$ then we shall select all LDs that certify these conditions from the qualified LDs taken into cognized the rage of k which must be less than n i.e major characteristics of the incomplete block design.

3.5. The Proposed Assertions (stage four)

Recall from the literature (Wilson, 1972) and Duke Ling (2008)**.** There exist PBD(n, K) for all sufficiently large admissible n. Let (X, Ɓ) be a pairwise balanced design. A flat (or sub design) is a pair (Y, $Ɓ|_Y$), where Y ⊆ X and $Ɓ|_Y$:= {B ∈ Ɓ : B ⊆ Y } have the property that any two distinct treatments in Y are together in a unique block of $Ɓ|_Y$. Flats in (X, Ɓ) form a lattice under intersection. As such, any set of points S ⊆ X generates a flat (S) equal to the intersection of all flats containing S.

Recall **(**Duke and Ling, 2009) proved that: Given K ⊆ $Z_{\geq 2}$ and d ∈ Z_+, there exists a PBD(n, K) of dimension at least d for all sufficiently large admissible n.

Putting together Wilson 1972 and Duke and Ling 2009 thus: For a given K and d, there exists, for all sufficiently large admissible n, a PBD(n, K) such that any n treatments generate a flat of size at most f(d, K), a constant independent of n. In this study two special classes of PBDs shall be considered which are PBD(n, {3, 4}) and PBD(n, {3, 4, 5})

1. where the set of sizes of K = {3, 4}; and
2. where the set sizes of K = {3, 4, 5}

Attempt was made to determine the value of n-treatments that could be admissible for each of the set of block sizes.

3.6. The PBD(n, K) as $K \subseteq Z_{\geq 2}$, Where K = {3, 4}

The Proposition One. There exists a PBD(n, {3, 4}) of dimension two if and only if $n \equiv 0, 1 \pmod 3$ provided $n \geq 6$ and $\lambda = 1$.

Therefore, for all n treatments equal or greater than 6 such as 6, 7, 9, 10, 12, 13 ..., satisfying these conditions could take set of K = {3, 4}.

Here r = 3 (Common Divisor) and

$\lambda = 1$.

3.6.1. The PBD(n, K), Where K = {3, 4, 5}

The Proposition Two. There exists a PBD(n, {3, 4, 5}) of dimension three if and only if $n \equiv 2, 3 \pmod 4$ provided $n \geq 11$ and $\lambda = 1$.

Therefore, for all n treatments equal or greater than 11 such as 11, 14, 15, 18, 19, 22, 23, 26, 27, 30, 31..., satisfying these conditions could take set of K = {3, 4, 5}.

Here r = 4(Common Divisor), and $\lambda = 1$

3.7.0. The Construction steps

3.7.1. Construction one (Pseudocode)

Pseudocode for a program that could be used to pair any n-treatment that satisfy n ≡ 0, 1 (mod3) to form the set of size 3 and 4.

```
Start
        'Type 1 pairing (set size 3, 4)
        get n;
' check ndiv3
        if ndiv3 equals 0 or 1 then
                do pairing1
        else
                cannot pair
        endif
end
```

3.7.2. **Construction two (Pseudocode)**

Pseudocode for a program that could be used to pair any n-treatment that satisfy n ≡ 2, 3 (mod 4) to form the set of size 3, 4, 5.

```
Start
        'Type 2 pairing (set size 3, 4, 5)
 get n;
 ' check n div3
 if n div3 equals 2 or 3 then
        do pairing2
 else
        cannot pair
 endif
end
```

3.8.0 **Confirmation of Results (stage five).**

The results of the constructed classes of the PBDs(n, {3, 4}) and (n, {3, 4, 5}) from LDs will be tested using

3.8.1 Smith, et. al. (1998) which states that there exists a PBD (n, {K}, λ) with b_i blocks of size k_i for each k_i, only and only if this connecting equation holds.

$$\lambda n(n-1) \geq \sum_i biki\ (ki-1)$$

3.8.2 Stanton-Kalbfleisch Bound(2003) which states that let k and n be integers such that 2 ≤ k < n, suppose there is (n, {2 …, n -1})PBD in which there exists a block containing exactly k treatments then

$$b \geq SK(k,n) = 1 + \frac{k2(n-k)}{n-1}$$

CHAPTER FOUR
RESULTS AND DISCURSION

In this chapter, all the stages outlined in the methodology shall be implemented.

4.0 Construction One

4.1.0. For A PBD with a set of **K** = {3, 4}

Using the ***Preliminary assertion 1.** There exists a PBD(n, {3, 4}) of dimension two if and only if $n \equiv 0, 1 \pmod 3$ provided $n \geq 6$

Here

r = 3 and

$\lambda = 1$.

n = 6, 7, 9, 10, 12, 13, ...

4.1.1. Using the Li Inequality program to generate LDs for the class of PBD(*n*, {3, 4} where λ =1 and r =3

Enter THE NUMBER R: 3

Enter THE NUMBER LAMDA: 1

P	T	R	L	COMBT	COMBP	COMTOT	COMBL	ANSW
3	3	3	1	4.00	0.00	4.00	3.00	NO
3	4	3	1	9.00	0.00	9.00	3.00	NO
3	5	3	1	12.00	0.00	12.00	3.00	NO
3	6	3	1	10.00	6.00	16.00	3.00	NO
3	7	3	1	15.00	3.00	18.00	3.00	NO
3	8	3	1	21.00	1.00	22.00	3.00	NO
4	3	3	1	6.00	0.00	6.00	6.00	NO
4	4	3	1	12.00	0.00	12.00	6.00	NO
4	5	3	1	18.00	0.00	18.00	6.00	NO
4	6	3	1	20.00	1.00	21.00	6.00	NO

4	7	3	1	30.00	0.00	30.00	6.00	NO
4	8	3	1	21.00	10.00	31.00	6.00	NO
5	3	3	1	7.00	0.00	7.00	10.00	YES
5	4	3	1	15.00	0.00	15.00	10.00	YES
5	5	3	1	18.00	3.00	21.00	10.00	NO
5	6	3	1	30.00	0.00	30.00	10.00	NO
5	7	3	1	30.00	3.00	33.00	10.00	NO
5	8	3	1	42.00	0.00	42.00	10.00	NO
6	3	3	1	9.00	0.00	9.00	15.00	YES
6	4	3	1	18.00	0.00	18.00	15.00	YES
6	5	3	1	24.00	1.00	25.00	15.00	NO
6	6	3	1	30.00	3.00	33.00	15.00	NO
6	7	3	1	45.00	0.00	45.00	15.00	NO
6	8	3	1	42.00	6.00	48.00	15.00	NO
7	3	3	1	10.00	0.00	10.00	21.00	YES
7	4	3	1	21.00	0.00	21.00	21.00	YES
7	5	3	1	30.00	0.00	30.00	21.00	NO
7	6	3	1	40.00	0.00	40.00	21.00	NO
7	7	3	1	45.00	3.00	48.00	21.00	NO
7	8	3	1	63.00	0.00	63.00	21.00	NO
8	3	3	1	12.00	0.00	12.00	28.00	NO
8	4	3	1	24.00	0.00	24.00	28.00	YES
8	5	3	1	36.00	0.00	36.00	28.00	NO
8	6	3	1	40.00	6.00	46.00	28.00	NO
8	7	3	1	60.00	0.00	60.00	28.00	NO

8	8	3	1	63.00	3.00	66.00	28.00	NO
9	3	3	1	13.00	0.00	13.00	36.00	YES
9	4	3	1	27.00	0.00	27.00	36.00	YES
9	5	3	1	36.00	3.00	39.00	36.00	NO
9	6	3	1	50.00	1.00	51.00	36.00	NO
9	7	3	1	60.00	3.00	63.00	36.00	NO
9	8	3	1	63.00	15.00	78.00	36.00	NO
10	3	3	1	15.00	0.00	15.00	45.00	YES
10	4	3	1	30.00	0.00	30.00	45.00	YES
10	5	3	1	42.00	1.00	43.00	45.00	YES
10	6	3	1	60.00	0.00	60.00	45.00	NO
10	7	3	1	75.00	0.00	75.00	45.00	NO
10	8	3	1	84.00	1.00	85.00	45.00	NO
11	3	3	1	16.00	0.00	16.00	55.00	YES
11	4	3	1	33.00	0.00	33.00	55.00	NO
11	5	3	1	48.00	0.00	48.00	55.00	NO
11	6	3	1	60.00	3.00	63.00	55.00	NO
11	7	3	1	75.00	3.00	78.00	55.00	NO
12	3	3	1	33.00	0.00	33.00	55.00	YES
12	4	3	1	33.00	0.00	33.00	55.00	YES
12	5	3	1	48.00	0.00	48.00	55.00	NO
12	6	3	1	60.00	3.00	63.00	55.00	NO
12	7	3	1	75.00	3.00	78.00	55.00	NO
12	8	3	1	84.00	10.00	94.00	55.00	NO
13	3	3	1	16.00	0.00	16.00	55.00	YES

13	4	3	1	33.00	0.00	33.00	55.00	YES
13	5	3	1	48.00	0.00	48.00	55.00	NO
13	6	3	1	60.00	3.00	63.00	55.00	NO
13	7	3	1	75.00	3.00	78.00	55.00	NO
13	8	3	1	84.00	10.00	94.00	55.00	NO
14	3	3	1	16.00	0.00	16.00	55.00	NO
14	4	3	1	33.00	0.00	33.00	55.00	NO
14	5	3	1	48.00	0.00	48.00	55.00	NO
14	6	3	1	60.00	3.00	63.00	55.00	NO
14	7	3	1	75.00	3.00	78.00	55.00	NO
14	8	3	1	84.00	10.00	94.00	55.00	NO
15	3	3	1	16.00	0.00	16.00	55.00	YES
15	4	3	1	33.00	0.00	33.00	55.00	YES
15	5	3	1	48.00	0.00	48.00	55.00	NO
15	6	3	1	60.00	3.00	63.00	55.00	NO
15	7	3	1	75.00	3.00	78.00	55.00	NO
15	8	3	1	84.00	10.00	94.00	55.00	NO
16	3	3	1	16.00	0.00	16.00	55.00	YES
16	4	3	1	33.00	0.00	33.00	55.00	YES
16	5	3	1	48.00	0.00	48.00	55.00	NO
16	6	3	1	60.00	3.00	63.00	55.00	NO
16	7	3	1	75.00	3.00	78.00	55.00	NO
16	8	3	1	84.00	10.00	94.00	55.00	NO
17	3	3	1	16.00	0.00	16.00	55.00	YES
17	4	3	1	33.00	0.00	33.00	55.00	NO

17	5	3	1	48.00	0.00	48.00	55.00	YES
17	6	3	1	60.00	3.00	63.00	55.00	NO
17	7	3	1	75.00	3.00	78.00	55.00	NO
17	8	3	1	84.00	10.00	94.00	55.00	NO
18	3	3	1	16.00	0.00	16.00	55.00	YES
18	4	3	1	33.00	0.00	33.00	55.00	YES
18	5	3	1	48.00	0.00	48.00	55.00	NO
18	6	3	1	60.00	3.00	63.00	55.00	NO
18	7	3	1	75.00	3.00	78.00	55.00	NO
18	8	3	1	84.00	10.00	94.00	55.00	NO
19	3	3	1	16.00	0.00	16.00	55.00	YES
19	4	3	1	33.00	0.00	33.00	55.00	YES
19	5	3	1	48.00	0.00	48.00	55.00	NO
19	6	3	1	60.00	3.00	63.00	55.00	NO
19	7	3	1	75.00	3.00	78.00	55.00	NO
19	8	3	1	84.00	10.00	94.00	55.00	NO
20	3	3	1	16.00	0.00	16.00	55.00	YES
20	4	3	1	33.00	0.00	33.00	55.00	NO
20	5	3	1	48.00	0.00	48.00	55.00	NO
20	6	3	1	60.00	3.00	63.00	55.00	NO
20	7	3	1	75.00	3.00	78.00	55.00	NO
20	8	3	1	84.00	10.00	94.00	55.00	NO
21	3	3	1	16.00	0.00	16.00	55.00	YES
21	4	3	1	33.00	0.00	33.00	55.00	YES
21	5	3	1	48.00	0.00	48.00	55.00	NO

21	6	3	1	60.00	3.00	63.00	55.00	NO
21	7	3	1	75.00	3.00	78.00	55.00	NO
21	8	3	1	84.00	10.00	94.00	55.00	NO
22	3	3	1	16.00	0.00	16.00	55.00	YES
22	4	3	1	33.00	0.00	33.00	55.00	YES
22	5	3	1	48.00	0.00	48.00	55.00	NO
22	6	3	1	60.00	3.00	63.00	55.00	NO
22	7	3	1	75.00	3.00	78.00	55.00	NO
22	8	3	1	84.00	10.00	94.00	55.00	NO
23	3	3	1	16.00	0.00	16.00	55.00	YES
23	4	3	1	33.00	0.00	33.00	55.00	NO
23	5	3	1	48.00	0.00	48.00	55.00	NO
23	6	3	1	60.00	3.00	63.00	55.00	NO
23	7	3	1	75.00	3.00	78.00	55.00	NO
23	8	3	1	84.00	10.00	94.00	55.00	NO
24	3	3	1	16.00	0.00	16.00	55.00	YES
24	4	3	1	33.00	0.00	33.00	55.00	YES
24	5	3	1	48.00	0.00	48.00	55.00	NO
24	6	3	1	60.00	3.00	63.00	55.00	NO
24	7	3	1	75.00	3.00	78.00	55.00	NO
24	8	3	1	84.00	10.00	94.00	55.00	NO
25	3	3	1	4.00	0.00	4.00	3.00	YES
25	4	3	1	9.00	0.00	9.00	3.00	YES
25	5	3	1	12.00	0.00	12.00	3.00	NO
25	6	3	1	10.00	6.00	16.00	3.00	NO

25	7	3	1	15.00	3.00	18.00	3.00	NO
25	8	3	1	21.00	1.00	22.00	3.00	NO
26	3	3	1	6.00	0.00	6.00	6.00	YES
26	4	3	1	12.00	0.00	12.00	6.00	NO
26	5	3	1	18.00	0.00	18.00	6.00	NO
26	6	3	1	20.00	1.00	21.00	6.00	NO
26	7	3	1	30.00	0.00	30.00	6.00	NO
26	8	3	1	21.00	10.00	31.00	6.00	NO
27	3	3	1	7.00	0.00	7.00	10.00	YES
27	4	3	1	15.00	0.00	15.00	10.00	YES
27	5	3	1	18.00	3.00	21.00	10.00	NO
27	6	3	1	30.00	0.00	30.00	10.00	NO
27	7	3	1	30.00	3.00	33.00	10.00	NO
27	8	3	1	42.00	0.00	42.00	10.00	NO
28	3	3	1	9.00	0.00	9.00	15.00	YES
28	4	3	1	18.00	0.00	18.00	15.00	YES
28	5	3	1	24.00	1.00	25.00	15.00	NO
28	6	3	1	30.00	3.00	33.00	15.00	NO
28	7	3	1	45.00	0.00	45.00	15.00	NO
28	8	3	1	42.00	6.00	48.00	15.00	NO
29	3	3	1	10.00	0.00	10.00	21.00	YES
29	4	3	1	21.00	0.00	21.00	21.00	YES
29	5	3	1	30.00	0.00	30.00	21.00	NO
29	6	3	1	40.00	0.00	40.00	21.00	NO
29	7	3	1	45.00	3.00	48.00	21.00	NO

29	8	3	1	63.00	0.00	63.00	21.00	NO
30	3	3	1	16.00	0.00	16.00	55.00	YES
30	4	3	1	33.00	0.00	33.00	55.00	YES
30	5	3	1	48.00	0.00	48.00	55.00	NO
30	6	3	1	60.00	3.00	63.00	55.00	NO
30	7	3	1	75.00	3.00	78.00	55.00	NO
30	8	3	1	84.00	10.00	94.00	55.00	NO
31	3	3	1	16.00	0.00	16.00	55.00	YES
31	4	3	1	33.00	0.00	33.00	55.00	YES
31	5	3	1	48.00	0.00	48.00	55.00	NO
31	6	3	1	60.00	3.00	63.00	55.00	NO
31	7	3	1	75.00	3.00	78.00	55.00	NO
31	8	3	1	84.00	10.00	94.00	55.00	NO

Want to run again (y/n) ? :

From the above results, select all p, t where we have YES. Since YES means ‘qualified as PBDs’ therefore, we have the followings:

Table 4.1: The Selected p and t from the Li Inequality.

p	t
5	3
5	4
5	5
6	3
6	4
7	3
7	4

7	5
8	4
9	3
9	4
10	3
10	4
10	5
11	3
12	3
12	4
12	5
13	3
13	4
13	5
14	3
14	4
15	3
15	4
15	5
16	3
16	4
17	3
18	3
18	4

19	3
19	5
20	3
21	3
21	4
22	3
22	4
23	3
24	3
24	4
25	3
25	4
26	3
27	3
27	4
28	3
28	4
29	3
29	4
30	3
30	4
31	3
31	4

Each of the p, t above is pair with k = 3 or 4, for all the values of *n from 5 to 31*. Thus,

the following LDs.

Table 4.2: The Generated Lotto Designs when n and k are combined with p and t

n	k	p	t
5	3	5	3
5	4	5	4
5	5	5	3
6	3	6	3
6	4	6	4
7	3	7	3
7	4	7	4
7	5	7	5
8	4	8	3
9	3	9	3
9	4	9	4
10	3	10	3
10	4	10	4
10	5	10	5
11	3	11	3
12	3	12	3
12	4	12	4
13	3	13	4
14	3	14	3
14	4	14	4
15	3	15	3

16	4	16	4
18	3	18	3
18	4	18	4
19	3	19	3
19	4	19	4
20	3	20	4
21	3	21	3
21	4	21	4
22	3	22	3
22	4	22	4
23	3	23	3
24	3	24	3
24	4	24	4
25	3	25	3
25	4	25	4
26	3	26	4
27	3	27	3
27	4	27	4
28	3	28	3
28	4	28	4
29	3	29	3
30	3	30	3
30	4	30	4
31	3	31	3

31	4	31	4

4.1.2. Conditions imposed

Several LDs were generated from using $\lambda = 1$ and $r = 3$ obtained from the classes of PBD(n, {3, 4}) on the Li inequality, this shows that some LDs were qualified as PBDs satisfying the proposition one.

Thus, the conditions:

i). k = 3 or 4 and

ii). n = p

Which give the following LDs:

The LDs(6, 3, 6, 3), (6, 4, 6, 4)

The LDs(7, 3, 7, 3), (7, 4, 7, 4),

The LDs(9, 3, 9, 3), (9, 4, 9, 4)

The LDs(10, 3, 10, 3), (10, 4, 10, 4)

The LDs(12, 3, 12, 3), (12, 4, 12, 4)

The LDs(13, 3, 13, 3), (13, 4, 13, 4)

The LDs(15, 3, 15, 3), (15, 4, 15, 4)

The LDs(16, 3, 16, 3), (16, 4, 16, 4),

The LDs(18, 3, 18, 3), (18, 4, 18, 4)

The LDs(19, 3, 19, 3), (19, 4, 19, 4)

The LDs(21, 3, 21, 3), (21, 4, 21, 4)

The LDs(22, 3, 22, 3), (22, 4, 22, 4)

The LDs(24, 3, 24, 3),(24, 4, 24, 4)

The LDs(25, 3, 25, 3), (25, 4, 25, 4),

The LDs(27, 3, 27, 3), (27, 4, 27, 4)

The LDs(28, 3, 28, 3), (28, 4, 28, 4)

The LDs(30, 3, 30, 3), (30, 4, 30, 4)

The LDs(31, 3, 31, 3), (31, 4, 31, 4)

All n -treatments satisfies: $n \equiv 0$ or 1(mod3), when $n \geq 6$ and $\lambda = 1$. And the imposed conditions for any LDs to be qualified as PBDs:

i). k = 3 or 4;

ii). n = p and

iii). $\lambda = 1$

Clearly, 2-LD(n, 3, p, 3)(n, 4 p, 4) = PBD(n, {3, 4}, 1).

4.2.0. The Assertion one: 2-LDs(n, 3, p, 3)(n, 4, p, 4) is a PBD(n, {3, 4}) if and only if the following conditions are satisfied**:**

i). for all $n \equiv 0, 1 \pmod 3$;

ii). k = 3 or 4;

iiii). n = p and

iv). $\lambda = 1$

4.2.1. Derived Steps for construction of congruent class of PBD(n, {3, 4}, 1) from any 2- LDs(n, 3, p, 3) (n, 4, p, 4).

(1). Select any n -treatments that satisfy the condition $n \equiv 0, 1 \pmod 3$.

(2). Select any 2LDs corresponds to n-treatments of the desired PBD that satisfy the followings conditions:

i). k = 3 or 4;

iii). n = p and

(3). Then, generate the blocks of the PBDs by Partitioning n - treatments into sets of blocks k = 3 or 4 such that each pair of treatments is contained in

precisely one block which follows that $n(n-1)$ is an integer linear combination of k(k − 1), k ∈ K.

4.2.2 The Construction One

4.2.3 **(i)** When $n = 7$ which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(7, 3, 7, 3),(7 4, 7, 4) satisfying (i). k = 3 or 4; and (ii). n = p

This can be used to construct PBD(7, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning 7- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of k(k − 1), k ∈ K.

Ɓ = {{1, 2, 3, 4},

{1, 5, 6},

{2, 5, 7}

{2, 6, 7}}

4.2.4 **(i)** When $n = 9$ which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(9, 3, 9, 3),(9, 4, 9, 4) satisfying (i). k = 3 or 4; and (ii). n = p

This can be used to construct PBD(9, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning 9- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of k(k − 1), k ∈ K.

Ɓ= {{1, 2, 3, 4},

{1, 5, 6, 7},

{1, 8, 9},

{2, 5, 8},

{2, 6, 9},

{3, 6, 8},

{4, 7, 8},

{4, 5, 9}}

4.2.5. **(i)** When n = 10 which satisfy the condition n ≡ 0, 1 (mod 3).

(ii) Select LDs(10, 3, 10, 3),(10 4, 10, 4) satisfying (i). k = 3 or 4; and (ii). n = p. This can be used to construct PBD(10, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning 10- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n - 1)$ is an integer linear combination of k(k − 1), k ∈ K.

Ɓ = {{1, 2, 3, 4},

{1, 5, 6, 7},

{1, 8, 9, 10};

{2, 5, 8},

{2, 6, 9},

{2, 7, 10},

{3, 5, 10},

{3, 6, 8},

{3, 7, 9},

{4, 5, 9},

{4, 6, 10},

{4, 7, 8}}.

4.2.6. **(i)** When n = 12 which satisfy the condition n ≡ 0, 1 (mod 3).

(ii) Select LDs(12, 3, 12, 3), (12, 4, 12, 4) satisfying (i). k = 3 or 4; and (ii). n = p. This can be used to construct PBD(12, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning 12- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of k(k − 1), k ∈ K.

B = {{1, 2, 3, 4},

{1, 5, 6, 7},

{1, 8, 9, 10},

{1, 11, 12},

{2, 5, 8},

{2, 6, 9},

{2, 7, 10},

{3, 5, 9},

{3, 6, 10},

{3, 7, 8},

{4, 5, 10},

{4, 6, 8},

{4, 7, 9}}

4.2.7 **(i)** When n = 13 which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(13, 3, 13, 3), (13, 4, 13, 4) satisfying (i). k = 3 or 4; and (ii). n = p. This can be used to construct PBD(13, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning 13- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in

precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$.

$\mathcal{B}$ = {{1, 2, 3, 4}

{1, 5, 6, 7}

{1, 8, 9, 10}

{1, 11, 12, 13}

{2, 6, 9}

{2, 10, 11}

{2, 5, 8}

{2, 6, 9}

{2, 7, 10}

{3, 5, 9}

{3, 6, 10}

{3, 7, 8}

{4, 5, 10}

{4, 6, 8}

{4, 7, 9}}

4.2.8. **(i)** When $n = 15$ which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(15, 3, 15, 3), (15, 4, 15, 4) satisfying (i). k = 3 or 4; and (ii). n = p. This can be used to construct PBD(15, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning 15- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$.

$\mathcal{B}$={{1, 2, 3, 4},

{1, 5, 6, 7},

{1, 8, 9, 10},

{1, 11, 12, 13},

{1, 14, 15},

{2, 11, 14},

{2, 12, 15},

{2, 5, 8},

{2, 6, 9},

{2, 7, 10},

{3, 5, 9},

{3, 4, 11},

{3, 6, 10},

{3, 7, 8},

{4, 5, 10},

{4, 6, 8},

{4, 7, 9}}

4.2.9. **(i)** When $n = 16$ which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(16, 3, 16, 3), (16, 4, 16, 4) satisfying (i). k = 3 or 4; and (ii). n = p. This can be used to construct PBD(16, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n - 1)$ is an integer linear combination of $k(k - 1)$, $k \in K$.

Ɓ={{1, 2, 3, 4}

{1, 5, 6, 7}

{1, 8, 9, 10}

{1, 11, 12, 13}

{1, 14, 15, 16}

{2, 5, 8}

{2, 11, 14}

{2, 12, 15}

{2, 6, 9}

{2, 7, 10}

{3, 5, 9}

{3, 6, 10}

{3, 9, 11}

{3, 10, 11}

{3, 12, 14}

{3, 11, 16}

{3, 7, 8}

{4, 5, 10}

{4, 6, 8}

{4, 7, 9}}

4.2.10. **(i)** When $n = 21$ which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(21, 3, 21, 3), (21, 4, 21, 4) satisfying (i). k = 3 or 4; and (ii). n = p. This can be used to construct PBD(21, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$.

$\mathcal{B}$ = {{1, 2, 3, 4}

{1, 5, 6, 7}

{1, 8, 9, 10}

{1, 11, 12, 13}

{1, 14, 15, 16}

{1, 17, 18, 19}

{1, 20, 21}

{2, 11, 14}

{2, 17, 20}

{2, 12, 21}

{2, 15, 18}

{2, 13, 16}

{2, 19, 20}

{2, 5, 8}

{2, 6, 9}

{2, 7, 10}

{3, 5, 9}

{3, 6, 10}

{3, 7, 8}

{4, 5, 10}

{4, 6, 8}

{4, 7, 9}}

4.2.11. **(i)** When $n = 22$ which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(22, 3, 22, 3), (22, 4, 22, 4) satisfying (i). k = 3 or 4; and (ii). n = p. This can be used to construct PBD(12, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$.

Ɓ={{1, 2, 3, 4}

{1, 5, 6, 7}

{1, 8, 9, 10}

{1, 11, 12, 13}

{1, 14, 15, 16}

{1, 17, 18, 19}

{1, 20, 21, 22}

{2, 5, 8}

{2, 6, 9}

{2, 7, 10}

{3, 5, 9}

{3, 6, 10}

{3, 7, 8}

{4, 5, 10}

{4, 6, 8}

{4, 7, 9}}

4.2.12. **(i)** When $n = 30$ which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(30, 3, 30, 3), (30, 4, 30, 4) satisfying (i). k = 3 or 4; and (ii). $n = p$. This can be used to construct PBD(30, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in

precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$.

$\mathcal{B}$ = {{1, 2, 3, 4}

{1, 5, 6, 7}

{1, 8, 9, 10}

{1, 11, 12, 13}

{1, 14, 15, 16}

{1, 17, 18, 19}

{1, 20, 21, 22}

{1, 23, 24, 25}

{1, 26, 27, 28}

{1, 29, 30}

{2, 11, 14}

{2, 17, 20}

{2, 23, 26}

{2, 29, 12}

{2, 15, 18}

{2, 21, 24}

{2, 27, 30}

{2, 13, 16}

{2, 19, 22}

{2, 25, 28}

{2, 5, 8}

{2, 6, 9}

{2, 7, 10}

{3, 5, 9}

{3, 6, 10}

{3, 7, 8}

{4, 5, 10}

{4, 6, 8}

{4, 7, 9}}

4.2.13. **(i)** When $n = 31$ which satisfy the condition $n \equiv 0, 1 \pmod 3$.

(ii) Select LDs(31, 3, 31, 3), (31, 4, 31, 4) satisfying (i). k = 3 or 4; and (ii). n = p. This can be used to construct PBD(31, {3, 4})

(iii) Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 such that each pair of treatments is contained in precisely one block which follows that $n(n - 1)$ is an integer linear combination of k(k − 1), k ∈ K.

Ɓ ={{1, 2, 3, 4}

{1, 5, 6, 7}

{1, 8, 9, 10}

{1, 11, 12, 13}

{1, 14, 15, 16}

{1, 17, 18, 19}

{1, 20, 21, 22}

{1, 23, 24, 25}

{1, 26, 27, 28}

{1, 29, 30, 31}

{2, 11, 14}

{2, 17, 20}

{2, 23, 26}

{2, 29, 12}

{2, 15, 18}

{2, 21, 24}

{2, 27, 30}

{2, 13, 16}

{2, 19, 22}

{2, 25, 28}

{2, 26, 31}

{2, 5, 8}

{2, 6, 9}

{2, 7, 10}

{3, 5, 9}

{3, 6, 10}

{3, 7, 8}

{4, 5, 10}

{4, 6, 8}

{4, 7, 9}}

Therefore,

$\mathcal{B} = \{B_I, B_2 \ldots, B_k\}$

$K = \{k_1, k_2 \ldots, k_b\}$ and

$b = \sum_i bi$

This shows that a Pairwise Balanced Design (PBD) is a pair (X, $\mathcal{B}$) where X is a set of n - treatments and $\mathcal{B}$ is a collection of subsets of X called blocks, such that each pair of treatments is contained in precisely one block.

4.3.0. Confirmation of the results

In order to confirm the results of construction one:

(a). Recall that Smith, et.al (1998) in 2.6 proposed an expression that connect the parameters of a PBD (n, {K}) which states that there exists a PBD (n, {K}, λ) if:

$$\lambda n(n-1) \geq \sum_i biki\ (ki-1)$$

(b). Recall also in 2.6.1, the Stanton-Kalbfleisch Bound by Wilson (1975) which states that let k and n be integers such that $2 \leq k < n$, suppose there is (n, {2 …, n-1})PBD in which there exists a block containing exactly k treatments then

$$b \geq SK(k,n) = 1 + \frac{k2(n-k)}{n-1}$$

4.3.1. Using Smith el.al. PBD(n, {3, 4}, 1) from the LDs(10, 3, 10, 3)(10, 4, 10, 4)

Where, $k_1 = 3$, $k_2 = 4$, $b_1 = 9$, $b_2 = 3$, $n = 10$, $\lambda = 1$

And $b = \sum_i bi = 12$,

Therefore,

1*10(10-1) = 9 *3 (3 – 1) + 3 * 4 (4 -1)

10(9) = 54 + 36

90 = 90

Thus,

$$\lambda n(n-1) = \sum_i biki\ (ki-1)$$

4.3.2. Using Stanton-Kalbfleisch Bound

$$b \geq SK(k,n) = 1 + \frac{k2(n-k)}{n-1}$$

$12 \geq SK(3, 10)$

= 1 + 9(10 -3) / 10 – 1

= 1 + 9(7) / 9

= 1+ 7

=8

Therefore,

$12 \geq 8$ also, when k = 4, and n = 10

$12 \geq SK(4, 10)$

= 1 + 16(6) / 9

= 1 + 10.7

= 1 + 10.7

=11. 7

Therefore,

$12 \geq 11.7$

This confirm construction 1.

4.3.3. Proof of the Proposition One

A PBD(n, {3, 4}) of dimension two is a 2-LDs(n, k, p, t) (n, k+1, p, t) if and only if

(i). for all $n \equiv 0, 1 (\text{mod } 3)$ provided $n \geq 6$,

ii). k = 3 or 4;

iiii). n = p and

iv). $\lambda = 1$

Proof :

Using Wilson 1972 (Necessary Condition for the Existence of a PBD(n, K) in (2.3) and (2.4).

When K = {3, 4}

Then, $\alpha(K) = \gcd\{2, 3\} = 1$

$\beta(K) = \gcd\{6, 12\} = 3$

According to Wilson, the first conditions are:

n -1≡ 0 (mod α) and

n (n -1) ≡ 0 (mod β)

Therefore,

n -1≡ 0 (mod 1) and

n (n -1) ≡ 0 (mod 3)

The first condition merely says that n is an integer. The second condition is satisfied if only if n (n -1) ≡ 0 or 1 mod 3. Therefore, all values of $n \geq 6$ such as 6, 7, 9, 10, 12, 13,… could have a set of block K = {3, 4}. Then, $n \geq 6$ follows since 3 is the smallest block size.

4.3.4. Characteristics of the constructed PBD(n, {3, 4}, 1) from the 2- LDs(n, 3, p, 3), (n, 4, p, 4)

(i) $K = \{k_1, k_2\ldots, k_b\}$

(ii) $k_i \leq n$; $k_i \neq k_j$

(iv) $b = \sum_i bi$ and

(v) $n \equiv 0, 1 (\text{mod } 3)$.

4.3.4 Summary of construction One

Table 4.3: Construction of PBD(n, {3, 4)}, 1) from 2-LD(n, 3, p, 3), (n, 4, p, 4)

Treatments n	K= {k1, k2}	Index (λ)	Replicate (r)	LD format 2-LD(n, 3, p, 3)	PBD format PBD(n,{3, 4}, λ)
6	{3, 4}	1	3	(6, 3, 6, 3), (6, 4, 6, 4)	(6, {3, 4}, 1)
7	{3, 4}	1	3	(7, 3, 7, 3), (7, 4, 7, 4)	(7, {3, 4}, 1)
9	{3, 4}	1	3	(9, 3, 9, 3), (9, 4, 9, 4)	(9, {3, 4}, 1)
10	{3, 4}	1	3	(10, 3, 10, 3), (10, 4, 10, 4)	(10, {3, 4}, 1)

12	{3, 4}	1	3	(12, 3, 12, 3), (12, 4, 12, 4)	(12, {3, 4}, 1)
13	{3, 4}	1	3	(13, 3, 13, 3), (13, 4, 13, 4)	(13, {3, 4}, 1)
15	{3, 4}	1	3	(15, 3, 15, 3), (15, 4, 15, 4)	(15, {3, 4}, 1)
16	{3, 4}	1	3	(16, 3, 16, 3), (16, 4, 16, 4)	(16, {3, 4}, 1)
18	{3, 4}	1	3	(18, 3, 18, 3), (18, 4, 18, 4)	(18, {3, 4}, 1)
19	{3, 4}	1	3	(19, 3, 19, 3), (19, 4, 19, 4)	(19, {3, 4}, 1)
21	{3, 4}	1	3	(21, 3, 21, 3), (21, 4, 21, 4)	(21, {3, 4}, 1)
22	{3, 4}	1	3	(22, 3, 22, 3), (22, 4, 13, 4)	(22, {3, 4}, 1)
24	{3, 4}	1	3	(24, 3, 24, 3), (13, 4, 13, 4)	(24, {3, 4}, 1)
25	{3, 4}	1	3	(25, 3, 25, 3), (25, 4, 13, 4)	(25, {3, 4}, 1)
27	{3, 4}	1	3	(27, 3, 27, 3), (27, 4, 27, 4)	(27, {3, 4}, 1)

30	{3, 4}	1	3	(30, 3, 10, 3), (30, 4, 30, 4)	(30, {3, 4}, 1)

4.4.0. Construction Two

4.4.1. For PBD(n, K) with a set block K = {3, 4, 5}

4.4.2. The Proposition Two. *There exists a PBD*(n, {3, 4, 5}) *of dimension three if and only if* $n \equiv 2, 3 \pmod 4$ *provided* $n \geq 11$ and $\lambda = 1$.

Here r = 4 and

$\lambda = 1$

4.2.3. The Generated LDs

Enter THE NUMBER R: 4

Enter THE NUMBER LAMDA: 1

Table 4.5: The Generated LDs When r = 4 and $\lambda = 1$

P	T	R	L	COMBT	COMBP	COMTOT	COMBL	ANSW
3	3	4	1	6.00	0.00	6.00	3.00	NO
3	4	4	1	12.00	0.00	12.00	3.00	NO
3	5	4	1	18.00	0.00	18.00	3.00	NO
3	6	4	1	20.00	1.00	21.00	3.00	NO
3	7	4	1	30.00	0.00	30.00	3.00	NO
3	8	4	1	21.00	10.00	31.00	3.00	NO
4	3	4	1	8.00	0.00	8.00	6.00	NO
4	4	4	1	15.00	0.00	15.00	6.00	NO
4	5	4	1	24.00	0.00	24.00	6.00	NO
4	6	4	1	30.00	0.00	30.00	6.00	NO

4	7	4	1	30.00	6.00	36.00	6.00	NO
4	8	4	1	42.00	1.00	43.00	6.00	NO
5	3	4	1	10.00	0.00	10.00	10.00	NO
5	4	4	1	18.00	1.00	19.00	10.00	NO
5	5	4	1	30.00	0.00	30.00	10.00	NO
5	6	4	1	40.00	0.00	40.00	10.00	NO
5	7	4	1	45.00	1.00	46.00	10.00	NO
5	8	4	1	42.00	15.00	57.00	10.00	NO
6	3	4	1	12.00	0.00	12.00	15.00	YES
6	4	4	1	24.00	0.00	24.00	15.00	NO
6	5	4	1	36.00	0.00	36.00	15.00	NO
6	6	4	1	40.00	6.00	46.00	15.00	NO
6	7	4	1	60.00	0.00	60.00	15.00	NO
6	8	4	1	63.00	3.00	66.00	15.00	NO
7	3	4	1	14.00	0.00	14.00	21.00	YES
7	4	4	1	27.00	0.00	27.00	21.00	NO
7	5	4	1	42.00	0.00	42.00	21.00	NO
7	6	4	1	50.00	3.00	53.00	21.00	NO
7	7	4	1	60.00	6.00	66.00	21.00	NO
7	8	4	1	84.00	0.00	84.00	21.00	NO
8	3	4	1	16.00	0.00	16.00	28.00	YES
8	4	4	1	30.00	1.00	31.00	28.00	NO
8	5	4	1	48.00	0.00	48.00	28.00	NO
8	6	4	1	60.00	1.00	61.00	28.00	NO
8	7	4	1	75.00	1.00	76.00	28.00	NO

8	8	4	1	84.00	6.00	90.00	28.00	NO
9	3	4	1	18.00	0.00	18.00	36.00	YES
9	4	4	1	36.00	0.00	36.00	36.00	NO
9	5	4	1	54.00	0.00	54.00	36.00	NO
9	6	4	1	70.00	0.00	70.00	36.00	NO
9	7	4	1	90.00	0.00	90.00	36.00	NO
9	8	4	1	105.00	0.00	105.00	36.00	NO
10	3	4	1	20.00	0.00	20.00	45.00	YES
10	4	4	1	39.00	0.00	39.00	45.00	YES
10	5	4	1	60.00	0.00	60.00	45.00	NO
10	6	4	1	80.00	0.00	80.00	45.00	NO
10	7	4	1	90.00	6.00	96.00	45.00	NO
10	8	4	1	105.00	10.00	115.00	45.00	NO
11	3	4	1	22.00	0.00	22.00	55.00	YES
11	4	4	1	42.00	1.00	43.00	55.00	YES
11	5	4	1	66.00	0.00	66.00	55.00	YES
11	6	4	1	.00	6.00	86.00	55.00	NO
11	7	4	1	105.00	1.00	106.00	55.00	NO
11	8	4	1	126.00	1.00	127.00	55.00	NO
12	3	4	1	22.00	0.00	22.00	55.00	NO
12	4	4	1	42.00	1.00	43.00	55.00	NO
12	5	4	1	66.00	0.00	66.00	55.00	YES
12	6	4	1	80.00	6.00	86.00	55.00	NO
12	7	4	1	105.00	1.00	106.00	55.00	NO
12	8	4	1	126.00	1.00	127.00	55.00	NO

13	3	4	1	22.00	0.00	22.00	55.00	NO
13	4	4	1	42.00	1.00	43.00	55.00	NO
13	5	4	1	66.00	0.00	66.00	55.00	YES
13	6	4	1	80.00	6.00	86.00	55.00	NO
13	7	4	1	105.00	1.00	106.00	55.00	NO
13	8	4	1	126.00	1.00	127.00	55.00	NO
14	3	4	1	22.00	0.00	22.00	55.00	YES
14	4	4	1	42.00	1.00	43.00	55.00	YES
14	5	4	1	66.00	0.00	66.00	55.00	YES
14	6	4	1	80.00	6.00	86.00	55.00	NO
14	7	4	1	105.00	1.00	106.00	55.00	NO
14	8	4	1	126.00	1.00	127.00	55.00	NO
15	3	4	1	6.00	0.00	6.00	3.00	YES
15	4	4	1	12.00	0.00	12.00	3.00	YES
15	5	4	1	18.00	0.00	18.00	3.00	YES
15	6	4	1	20.00	1.00	21.00	3.00	NO
15	7	4	1	30.00	0.00	30.00	3.00	NO
15	8	4	1	21.00	10.00	31.00	3.00	NO
17	3	4	1	8.00	0.00	8.00	6.00	YES
17	4	4	1	15.00	0.00	15.00	6.00	NO
17	5	4	1	24.00	0.00	24.00	6.00	NO
17	6	4	1	30.00	0.00	30.00	6.00	NO
17	7	4	1	30.00	6.00	36.00	6.00	NO
18	8	4	1	42.00	1.00	43.00	6.00	YES
18	3	4	1	10.00	0.00	10.00	10.00	YES

18	4	4	1	18.00	1.00	19.00	10.00	YES
18	5	4	1	30.00	0.00	30.00	10.00	NO
18	6	4	1	40.00	0.00	40.00	10.00	NO
18	7	4	1	45.00	1.00	46.00	10.00	NO
18	8	4	1	42.00	15.00	57.00	10.00	NO
19	3	4	1	12.00	0.00	12.00	15.00	YES
19	4	4	1	24.00	0.00	24.00	15.00	YES
19	5	4	1	36.00	0.00	36.00	15.00	YES
19	6	4	1	40.00	6.00	46.00	15.00	NO
19	7	4	1	60.00	0.00	60.00	15.00	NO
19	8	4	1	63.00	3.00	66.00	15.00	NO
20	3	4	1	14.00	0.00	14.00	21.00	NO
20	4	4	1	27.00	0.00	27.00	21.00	YES
20	5	4	1	42.00	0.00	42.00	21.00	NO
20	6	4	1	50.00	3.00	53.00	21.00	NO
20	7	4	1	60.00	6.00	66.00	21.00	NO
20	8	4	1	84.00	0.00	84.00	21.00	NO
21	3	4	1	16.00	0.00	16.00	28.00	YES
21	4	4	1	30.00	1.00	31.00	28.00	NO
21	5	4	1	48.00	0.00	48.00	28.00	NO
21	6	4	1	60.00	1.00	61.00	28.00	NO
21	7	4	1	75.00	1.00	76.00	28.00	NO
21	8	4	1	84.00	6.00	90.00	28.00	NO
22	3	4	1	18.00	0.00	18.00	36.00	YES
22	4	4	1	36.00	0.00	36.00	36.00	YES

22	5	4	1	54.00	0.00	54.00	36.00	YES
22	6	4	1	70.00	0.00	70.00	36.00	NO
22	7	4	1	90.00	0.00	90.00	36.00	NO
22	8	4	1	105.00	0.00	105.00	36.00	NO
23	3	4	1	20.00	0.00	20.00	45.00	YES
23	4	4	1	39.00	0.00	39.00	45.00	YES
23	5	4	1	60.00	0.00	60.00	45.00	YES
23	6	4	1	80.00	0.00	80.00	45.00	NO
23	7	4	1	90.00	6.00	96.00	45.00	NO
23	8	4	1	105.00	10.00	115.00	45.00	NO
24	3	4	1	22.00	0.00	22.00	55.00	NO
24	4	4	1	42.00	1.00	43.00	55.00	NO
24	5	4	1	66.00	0.00	66.00	55.00	YES
24	6	4	1	.00	6.00	86.00	55.00	NO
24	7	4	1	105.00	1.00	106.00	55.00	NO
24	8	4	1	126.00	1.00	127.00	55.00	NO
25	3	4	1	22.00	0.00	22.00	55.00	NO
25	4	4	1	42.00	1.00	43.00	55.00	NO
25	5	4	1	66.00	0.00	66.00	55.00	YES
25	6	4	1	80.00	6.00	86.00	55.00	NO
25	7	4	1	105.00	1.00	106.00	55.00	NO
25	8	4	1	126.00	1.00	127.00	55.00	NO
26	3	4	1	22.00	0.00	22.00	55.00	YES
26	4	4	1	42.00	1.00	43.00	55.00	YES
26	5	4	1	66.00	0.00	66.00	55.00	YES

26	6	4	1	80.00	6.00	86.00	55.00	NO
26	7	4	1	105.00	1.00	106.00	55.00	NO
26	8	4	1	126.00	1.00	127.00	55.00	NO

Want to run again (y/n) ? :

From the above results, select all p, t where we have YES. Since YES means 'qualified as PBDs' according to Li- inequality. Therefore, we have the followings:

Table 4.4: The Selected p and t from the Li Inequality When r = 5 and $\lambda = 1$

p	t
6	3
7	3
8	3
9	3
10	3
10	4
11	3
11	4
11	5
12	5
13	5
14	3
14	4
14	5
15	3
15	4

15	5
17	3
18	3
18	4
18	5
19	3
19	4
19	5
20	4
21	3
22	3
22	4
22	5
23	3
23	4
23	5
25	3
26	3
26	4
26	5

Each of the p, t above is pair with k = 3 or 4 or 5, when n is between 6 and 26 Thus, the following LDs.

Table 4.5: The Generated LDs when p and t are combined with the values of n and k = 3 or 4 or 5.

n	k	p	t
6	3	6	3
7	3	7	3
8	3	8	3
9	3	9	3
10	3	10	3
10	4	10	4
11	3	11	3
11	4	11	4
11	5	11	5
12	5	12	5
13	5	13	5
14	3	14	3
14	4	14	4
14	5	14	5
15	3	15	3
15	4	15	4
15	15	15	5
17	3	17	3
18	3	18	3
18	4	18	4
18	5	18	5
19	3	19	3
19	4	19	4

19	5	19	5
20	4	20	4
21	5	21	5
22	3	22	3
22	4	22	4
22	5	22	5
23	3	23	3
23	4	23	4
23	5	23	5
25	5	25	5
26	3	26	3
26	4	26	4
26	5	26	5

4.2.3. Conditions imposed

Several LDs were produced from using $\lambda = 1$ and $r = 4$ obtained from the classes of PBDs(n, {3, 4, 5}) for all n-treatments $n \equiv 2, 3 (\text{mod } 4)$ on the Li inequality, this shows that some LDs qualified as PBDs only and only if the following conditions holds:

i). $k = 3$ or 4 or 5 and

ii). $n = p$

Which give the following LDs:

The LDs(11, 3, 11, 3), (11, 4, 11, 4), (11, 5, 11, 5)

The LDs(14, 3, 14, 3), (14, 4, 14, 4), (14, 5, 14, 5)

The LDs(15, 3, 15, 3), (15, 4, 15, 4), (15, 5, 15, 5)

The LDs (18, 3, 18, 3), (18, 4, 18, 4), (18, 5, 18, 5)

The LDs(19, 3, 19, 3), (19, 4, 19, 4), (19, 5, 22, 5)

The LDs(22, 3, 22, 3), (22, 4, 22, 4), (22, 5, 14, 5)

The LDs(23, 3,23, 5, 3), (23, 4, 23, 4), (23, 5, 23, 5)

The LDs (26, 3, 26, 3), (26, 4, 26, 4), (26, 5, 26, 5)

Clearly, 3-LD(n, 3, p, 3), (n, 4, p, 4), (n, 5, p, 5)= PBD(n, {3, 4, 5}, 1).

4.4.3 The Proposition Two

A PBD(n, {3, 4, 5}) is a 3-LDs(n, 3, p, 3), (n, 4, p, 4), (n, 5, p, 5) if and only if the following conditions are satisfied:

i). $n \equiv 2, 3 \pmod 4$

ii). k = 3 or 4 or 5

iiii). n = p and

iv). $\lambda = 1$

4.5.0. Derived Steps for construction of PBD(n, {3, 4, 5}, 1) from any 3- LDs(n, 3, p, 3) (n, 4, p, 4) (n, 5, p, 5).

(1). Select any n-treatments that satisfy the condition $n \equiv 2, 3 \pmod 4$.

(2). Select any 3-LDs corresponds to n-treatments of the desired PBD that satisfy the followings conditions:

i). k = 3 or 4 or 5;

ii). n = p and

(3). Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$.

4.5.1. The Construction Two

(i) When n = 11 which satisfy the condition $n \equiv 2, 3 \pmod 4$.

(ii) Select LDs(11, 3, 11, 3), (11 4, 11, 4), (11, 5, 11, 5) satisfying (i). $3 \leq k \leq 5$; and (ii). $n = p$

This can be used to construct PBD(11, {3, 4, 5})

(iii) Then, generate the blocks of the PBDs by Partitioning n- treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$

Ɓ= {{1, 2, 3, 4, 5},

{1, 6, 7, 8},

{1, 9, 10, 11},

{2, 6, 9},

{2, 7, 10},

{3, 6, 10},

{3, 7, 11},

{3, 8, 9},

{4, 8, 10},

{4, 7, 9},

{5, 8, 11},

{4, 6, 11} }

4.5.2. **(i)** When $n = 14$ which satisfy the condition $n \equiv 2, 3 \pmod 4$.

(ii) Select LDs(14, 3, 14, 3), (14, 4, 14, 4), (14, 5, 14, 5) satisfying (i). k = 3 or 4 or 5; and (ii). $n = p$

This can be used to construct PBD(14, {3, 4, 5})

(iii) Then, generate the blocks of the PBDs by Partitioning *14*- treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$

Ɓ= {{1, 2, 3, 4, 5},

{1, 6, 7, 8, 9},

{1, 10, 11,12, 13},

{2, 6, 10, 14},

{3, 7, 11, 14},

{4, 8, 12, 14},

{5, 9, 13, 14}

{2, 7,12},

{2, 8, 13},

{2, 9, 10},

{3, 6, 11},

{3, 9, 12},

{4, 6, 12},

{4, 7, 13},

{5, 6,13},

{5, 7, 10},

{5, 8, 11}}

4.5.3. **(i)** When $n = 15$ which satisfy the condition $n \equiv 2, 3 \pmod 4$.

(ii) Select LDs(15, 3, 15, 3), (15, 4, 15, 4), (15, 5, 15, 5) satisfying (i). k = 3 or 4 or 5; and (ii). n = p

This can be used to construct PBD(15, {3, 4, 5})

(iii) Then, generate the blocks of the PBDs by Partitioning 15- treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$

Ɓ= {{1, 2, 3, 4, 5},

{1, 6, 7, 8, 9},

{1, 10, 11, 12, 13};

{2, 6, 10, 14},

{2, 7, 11, 15},

{3, 6, 12, 15},

{3, 7, 13, 14},

{4, 8, 10, 15},

{4, 9, 11, 14},

{5, 8, 12, 14},

{5, 9, 13, 15}

{1, 14, 15},

{2, 8, 13},

{2, 9, 12},

{3, 8, 11},

{3, 9, 10},

{4, 6, 13},

{4, 7, 12},

{5, 6, 11},

{5, 7, 10}}.

4.5.4. **(i)** When n = 19 which satisfy the condition $n \equiv 2, 3 \pmod 4$.

(ii) Select LDs(19, 3, 19, 3), (19, 4, 19, 4), (19, 5, 19, 5) satisfying (i). k = 3 or 4 or 5; and (ii). n = p

This can be used to construct PBD(19, {3, 4, 5})

(iii) Then, generate the blocks of the PBDs by Partitioning 19- treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$

Ɓ= {{1, 2, 3, 4, 5},

{1, 6, 7, 8, 9},

{1, 10, 11, 12, 13},

{1, 14, 15, 16, 17},

{2, 6,10,14, 18},

{2, 7, 11, 15, 19},

{3, 6, 11, 16, 19},

{3, 7, 12, 17, 18},

{4, 8, 12, 16},

{4, 9, 11, 18},

{5, 9, 13, 19},

{4, 7, 10},

{5, 6, 12},

{5, 8, 14},

{5, 7, 14}}

4.5.5. **(i)** When n = 23 which satisfy the condition $n \equiv 2, 3 \pmod 4$.

(ii) Select LDs(23, 3, 23, 3), (23, 4, 23, 4), (23, 5, 23, 5) satisfying (i). k = 3 or 4 or 5; and (ii). n = p

This can be used to construct PBD(23, {3, 4, 5})

(iii) Then, generate the blocks of the PBDs by Partitioning 23- treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$.

B= {{1, 2, 3, 4, 5},

{1, 6, 7, 8, 9},

{1, 10, 11, 12, 13},

{1, 14, 15, 16, 17},

{1, 18, 19, 20, 21},

{2, 6, 10,14, 18},

{2, 7, 11, 15, 19},

{2, 8, 12, 16, 20},

{2, 9, 13, 17, 21},

{3, 6, 11, 16, 21}

{3, 7, 12, 17, 18},

{3, 8, 10, 15, 22}

{4, 9, 10, 16, 19}

{4, 6, 12, 15, 23},

{4, 7, 13, 15, 20}

{4, 8, 11, 17},

{5, 6, 13, 22},

{5, 7, 10, 23},

{1, 22, 23},

{5, 9, 22}

{3, 9, 23}}

Therefore,

$\mathcal{B} = \{B_I, B_2 \ldots, B_k\}$

$K = \{k_1, k_2 \ldots, k_b\}$ and

$b = \sum_i bi$

This shows that a Pairwise Balanced Design (PBD) is a pair (X, $\mathcal{B}$) where X is a set of treatments and $\mathcal{B}$ is a collection of subsets of X called blocks, such that each pair of treatments is contained in precisely one block as shown above.

4.6.0. Confirmation of the results

In order to establish the derived Assertion Two:

(a). Recall that Smith, et.al (1998) in 2.6 proposed an expression that connect the parameters of a PBD (n, {K}) which states that there exists a PBD (n, {K}, λ) as thus:

$$\lambda n(n-1) = \sum_i biki\ (ki-1)$$

(b). Recall also in 2.6.1, the Stanton-Kalbfleisch Bound by Wilson (1975) which states that let k and n be integers such that $2 \leq k < n$, suppose there is (n, {2 …, n-1})PBD in which there exists a block containing exactly k treatments then

$$b \geq SK(k,n) = 1 + \frac{k2(n-k)}{n-1}$$

4.6.1. For example, using Smith el.al.

For the constructed PBD(n, {3, 4, 5}) from the 3-LDs(15, 3, 15, 3), (15, 4, 15, 4), (15, 5, 15, 5)

Where, $k_1 = 3$, $k_2 = 4$, $k_3 = 5$ $b_1 = 3$, $b_2 = 8$, $b_3 = 9$, $n = 15$, $\lambda = 1$

And $b = \sum_i bi = 20$,

Therefore,

1* 15(15 -1) = 9*3 (2) + 8 * 4 (3) + 3 * 5 (4)

210 = 54 + 96 +60

210 = 210

Thus, $\lambda n(n-1) = \sum_i biki\ (ki-1)$ Smith, et. al.(1998)

4.6.2. Using Stanton-Kalbfleisch Bound $b \geq SK(k,n) = 1 + \frac{k2(n-k)}{n-1}$

20 ≥ SK(3, 15)

= 1 + 9(17)/ 14

= 1 +10.9

= 11.9

Therefore

20 ≥ 11. 9

Also, when k = 4 and n = 15

20 ≥ SK(4, 15)

= 1 + 16(11)/ 14

= 1 + 12.6

= 13. 6

Therefore,

20 ≥ 13. 6

Also, when k = 5 and n = 15

20 ≥ SK(5, 15)

= 1 + 25 (10) / 14

= 1 + 17. 9

= 18. 9

Therefore,

$20 \geq 18.9$

This confirm construction 2.

4.6.3. Proof of the Proposition Two

A PBD(n, {3, 4, 5}) of dimension two is a 3-LDs(n, 3, p, 3), (n, 4, p, 4), (n, 5, p, 5) if and only if

(i). for all $n \equiv 0, 1 (\text{mod } 3)$ provided $n \geq 11$,

ii). k = 3 or 4 or 5;

iiii). n = p and

iv). $\lambda = 1$

Proof:

Using Wilson 1972 (Necessary Condition for the Existence of a PBD(n, K) in (2.3) and (2.4).

When K = {3, 4, 5}

Then, $\alpha (K) = \gcd\{2, 3, 4\} = 1$

$\beta (K) = \gcd\{6, 12, 20\} = 4$

According to Wilson, the first conditions are:

$n - 1 \equiv 0 (\text{mod } \alpha)$ and

$n(n-1) \equiv 0 (\text{mod } \beta)$

Therefore,

$n - 1 \equiv 0 (\text{mod } 1)$ and

$n(n-1) \equiv 0 (\text{mod } 4)$

The first condition merely says that n is an integer. The second condition is satisfied if only if $n(n-1) \equiv 2$ or 3 mod 4. Therefore, all values of $n \geq 11$ such as 11, 14, 15, … could have a set of block K = {3, 4, 5}. Then, $n \geq 11$ follows since 3 is the smallest block size.

4.6.4. Characteristics of the constructed PBD(n, {3, 4, 5}, 1) from the 3- LDs(n, 3, p, 3)(n, 4, p, 4)(n, 5, p, 5)

(i) $K = \{k_1, k_2, k_3\}$

(ii) $k_i \leq n$; $k_i \neq k_j$

(iv) $b = \sum_i bi$ and

(iv) $n \equiv 2, 3 (\text{mod } 4)$.

4.6.5. Summary Table for Constructions Two.

Table 4.6. Construction of PBD(n, {3, 4, 5}, 1) from 3-LD(n, 3, p, 3), (n, 4, p, 4), (n, 5, p, 5) .

Treatments n	K= {k1, k2, k3}	Index (λ)	Replicate (r)	LDs format 3-LD(n, 3, p, 3)(n,4, p, 4)(n, 5, p, 5) .	PBD format PBD(n,{3, 4, 5}, 1).
11	{3, 4, 5}	1	4	(11, 3, 11, 3),(11, 4, 11, 4), (11, 5, 115)	(11, {3, 4, 5}, 1)
14	{3, 4, 5}	1	4	(114, 3, 14, 3), (14, 4, 14, 4), (14, 5, 14, 5)	(14, {3, 4, 5}, 1)
15	{3, 4, 5}	1	4	(15, 3, 15, 3), (15, 4, 15, 4), (15, 5, 15, 5)	(15, {3, 4, 5}, 1)
18	{3, 4, 5}	1	4	(18, 3, 18, 3), (18, 4, 18, 4), (18, 5, 18, 5)	(18, {3, 4, 5}, 1)
19	{3, 4, 5}	1	4	(19, 3, 19, 3), (19, 4, 19, 4), (19, 5, 19, 5)	(19, {3, 4, 5}, 1)

22	{3, 4, 5}	1	4	(22, 3, 22, 3), (22, 4, 22, 4), (22, 5, 22, 5)	(22, {3, 4, 5}, 1)
23	{3, 4, 5}	1	4	(23, 3, 23, 3), (23, 4, 23, 4), (23, 5, 23, 5)	(23, {3, 4, 5}, 1)
26	{3, 4, 5}	1	4	(26, 3, 26, 3), (26, 4, 26, 4), (26, 5, 26, 5)	(26, {3, 4, 5}, 1)

CHAPTER FIVE
SUMMARY, CONCLUSION AND RECOMMENDATION

5.0 Summary

Pairwise Balanced Designs (PBD) is a pair (X, B) where X is a set of treatments and Ɓ is a collection of subsets of X called blocks, such that each pair of treatments is contained in precisely one block. PBDs plays important role in design theories, PBDs are used to construct other designs such as Steiner Triple System (STS), Pairwise Additive designs, Orthogonal Array Designs, Partially Balanced Designs e,t.c. PBDs are also used to confirm the existence questions for other designs such as Balanced Incomplete Block Design, Latin Square e.t.c.**.** Lotto designs (LDs) are combinatorial block designs with structures primary applied in lotteries. However, little is known about the construction of PBDs from the LDs. This study attempts to seek for the simple way to construct congruent classes of PBDs from an appropriate Lotto Designs (LDs). The study derived the following assertions for the constructions of the PBD(n, {3, 4}) and PBD(n, {3, 4, 5})

(a). A PBD(n, {3, 4}) is a 2-LDs(n, 3, p, 3), (n, 4, p, 4) if and only if $n \equiv 0, 1 \pmod 3$ and

(b) PBD(n, {3, 4, 5}) is a 3-LDs (n, 3, p, 3), (n, 4, p, 4), (n, 5, p, t) if and only if $n \equiv 2, 3 \pmod 4$. and proved the assertions respectively. The study also established conditions for that must be satisfied for any LDs to qualify as PBDs as follows:

(i) k = 3 or 4 or 5 and (ii) n = p. and derived the following simple steps for the construction of the congruent class of the PBD(n, {3, 4}), (n, {3, 4, 5}) as follow:

(a). Select any n -treatments that satisfy the condition $n \equiv 0, 1 \pmod 3$.

(b). Select any 2-LDs or 3-LDs corresponds to n -treatments of the desired PBD that satisfy the followings conditions in (i) and the Li inequality, and

(c). Then, generate the blocks of the PBDs by Partitioning n - treatments into sets of blocks of 3 or 4 or 5 such that each pair of treatments is contained in precisely one block which follows that $n(n-1)$ is an integer linear combination of $k(k-1)$, $k \in K$. Thus, congruence

classes of PBD(n, {3, 4}) and PBD(n, {3, 4, 5}) were constructed for all admissible n - treatments: 7, 9,…,31 and 14, 15,…, 26 respectively which satisfied specified conditions and the two propositions from the appropriate LDs. The two classes of the PBDs constructed were tested for validity using two claims: Smith, et. al. (1998) and Stanton-Kalbfleisch Bound (Stinson, 2003) which was used to confirm the validity of the constructions.

5.1 Conclusion

This study concluded that congruence classes of PBDs (n, {3, 4}, 1) and (n, {3, 4, 5}, 1) could be constructed from appropriate LDs if the specified conditions holds. The study also concluded that PBD as an example of incomplete block designs has varieties of applications than other incomplete block designs and is variance balanced and efficiently balanced compare to BIBD, Partially Balanced Designs et.c in terms of applications. PBDs has a wider application especially in constructions of other important types of Designs like Partially Balanced Designs, pairwise balanced Additive designs, Orthogonal Array designs e.t.c. and PBDs are also used to confirm the existence of other designs e, g, Balanced Incomplete Block Designs e. t. c. compare to BIBDs which have been widely studies by various scholars. Allocation of treatments to more than one block sizes in PBDs make PBDs to be better than BIBDs in terms of balanced properties of the block designs

5.2. Recommendation

The study recommends the use of appropriate LDs for the construction of congruence classes of PBD(n, {3, 4}) and PBD(n, {3, 4, 5}) when $\lambda = 1$, $k = 3$ or 4 or 5 and when n is large. And the use of PBD for the construction of other important designs since construction of PBDs has been simplified through this research.

5.3. Contribution to Knowledge

This research work has been able to contribute to knowledge in the following ways;

1. Specified conditions that must be satisfied for any LD(n, k, p, t) to be used to constructs two classes of PBDs.

2. Proposed two propositions for the constructions of two classes of PBD(n, {3, 4}, 1) and PBD(n, {3, 4, 5}, 1) from the 2-LDs(n, 3, p, 3), (n, 4, p, 4) and 3-LDs(n, 3, p, 3)(n, 4, p, 4), (n, 5, p, 5) respectively.

3. Derive simple steps for the construction of two congruent classes of PBDs from LDs.

5.4. Suggestion for Further Study

1. Other classes of PBDs where K = {5, 6, 7……} may be investigated using an appropriate LDs;

2. Wider application of PBD could be established especially using PBDs to construct other important designs and confirm the existence of other type of important designs;

3. Method of Data Analysis of PBD could be investigated: and

4. Optimality criterial of the constructed PBDs could also be investigated.

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APPENDIX 1

Program for Implementing Algorithm I.

This consists of one main program and two functions.

1……Main Program

```
        PROGRAM COMBI
C CALCULATES THE COMBINATORIAL OF C(N, P)
        IMPLICIT NONE
        INTEGER NTEST, PLOW, PHIGH, TLOW, THIGH, J, I, ICOMPL, R, LAMBDA, PR
        REAL*8 COMBT, COMBP, COMBL, COMTOT
        CHARACTER*1 affirm
            CHARACTER *3 ANSW
            CHARACTER *17 COMNT
        REAL *8 XFACT
        REAL *4 t
10      WRITE(*,20) 'Enter THE NUMBER R:'
        READ(*,*) R
12      WRITE (*,20) 'Enter THE NUMBER LAMDA:'
        READ(*,*) LAMBDA
```

```
20      FORMAT(1X,A,Ş)
C       PRINT 400, ‘PR’, ‘COMBT’, ‘ICOMP’, ‘COMBL’, ‘COMBP’, ‘COMTOT’,’ANSW’
        PRINT 400, ‘P’,’T’,’R’,’L’,’COMBT’,’COMBP’,’COMTOT’,’COMBL’,’ANSW’
C       LAMBDA=LAMBDA, P=I, T=J, R=R
400     FORMAT (A4, 2X, A4, 2X, A4, 2X,A4,2X, 4A8, A8)
        DO 200 I=3, 11
        DO 300 J= 3, 8
        THIGH=J-1
        TLOW = 2
        PHIGH=I
        PLOW=2
        PR=(PHIGH*R)/THIGH
        COMBT=(XFACT(THIHG)/(XFACT(THIGH-TLOW)*XFACT(TLOW)))*PR
        ICOMP=(PHIGH*R)-(PR*THIGH)
        NTEST=ICOMP-TLOW
        COMBL = (XFACT(PHIGH)/(XFACT(PHIGH-PLOW) * XFACT(PLOW))) *
        LAMBDA
        IF (NTEST.LT.0) GO TO 800
```

```
      COMBP = XFACT(ICOMP)/(XFACT(ICOMP-TLOW)*XFACT(TLOW))
      GO TO 900
800   COMBP =0.0
900   COMTOT=COMBT+COMBP
      IF (COMTOT.LT.COMBL) GO TO 700
      ANSW='NO'
      GO TO 701
700   ANSW= 'YES'
701   PRINT 401, I,J,R, LAMBDA, COMBT, COMBP, COMTOT, COMBL, ANSW
401   FORMAT(14, 2X, 14, 2X, 14, 2X, 14,2X, 4F8.2,4X, A)
      GO TO 300
C     PRINT*, COMBP
C 800 COMNT='DIVISON BY ZERO'
C     PRINT 801, I,J,R, LAMBDA, COMNT
C 801 FORMAT(14,2X, 14,2X, 14,2X, 14,2X, A)
300   CONTINUE
200   CONTINUE
      IF (AFFIRM('WANT TO RUN AGAIN').EQ.'Y') GOTO 10 ! REPEAT
```

```
      STOP
      END
```

2……..

```
      FUNCTION XFACT(N)
CFUNCTION FACTORIAL
C=======================
      DIMENSION S(N)
      XFACT=1.0
      K=0
      XN=N
10    IF (XN.EQ.0.00) GO TO 20
      K=K+1
      S(K)=XN
      XN=XN-1.0
      GO TO 10
20    IF (K.GT.0) GO TO 30
      RETURN
30    XFACT=XFACT*S(K)
      K=K-1
```

```
        GO TO 20
        END
3……..
        CHARACTER*1 FUNCTION AFFIRM(S)
C-------------------------------------------------
        IMPLICIT NONE
        CHARACTER  *(*)S
        CHARACTER*1   C

10      WRITE(*,20)S', (Y/N) ? :'
20      FORMAT(1X,A,A,Ṣ)
         READ(*,30)C
30      FORMAT(A1)
        IF (C.NE.'Y'.AND.C.NE.'Y'.AND. C.NE.'N'.AND. C.NE.'N') GOTO 10
        IF (C.EQ.'Y') C='Y'
        AFFIRM=C
        RETURN
        END
```

APPENDIX 2

Set Selection Coding

Pseudocode for a program that could be used to pair any number to form the set of size 3 and 4

```
Start
        'Type 1 pairing (set size 3, 4)
        get n;
' check ndiv3
        if ndiv3 equals 0 or 1 then
                do pairing1
        else
                cannot pair
        endif
end

procedure pairing1(n, paired, k)
  sort elements in array in ascending order
  check equality of elements and decide steps to take
  j-set size;'3 here
  n-total element;
  k- step of iteration;
 for i=1 to n, step 3
        select a_i = subst(a_i,j)
        print a_i
 next

' here change set size;
 j-set size;'4 here
 for i=1 to n, step 4
        select a_i = subst(a_i,j)
        print a_i
  next

end procedure

function subst (a)
   determine the sequencing of elements
end function
```

==========

Pseudocode for a program that could be used to pair any number to form the set of size 3, 4 and 5

```
Start
        'Type 2 pairing (set size 3, 4, 5)
  get n;
```

```
  ' check ndiv3
  if ndiv3 equals 2 or 3 then
          do pairing2
  else
          cannot pair
  endif
end


procedure pairing2(n, paired, k)
 sort elements in array in ascending order
 check equality of elements and decide steps to take
 j-set size;'3 here
 n-total element;
 k- step of iteration;
 for i=1 to n, step 3
          select ai = subst(ai,j)
          print ai
 next

' here change set size;
 j-set size;'4 here
 for i=1 to n, step 4
          select ai = subst(ai,j)
          print ai
 next

 ' here change set size;
 j-set size;' here
 for i=1 to n, step 5
          select ai = subst(ai,j)
          print ai
  next
end procedure
```