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Measuring Economic Mobility in India Using Noisy Data: A Partial Identification Approach^{*}

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Abstract

We examine economic mobility in India while accounting for misclassification to better understand the welfare effects of the rise in inequality. To proceed, we extend recently developed methods on the partial identification of transition matrices. Allowing for modest misclassification, we find overall mobility has been remarkably low: at least 65 percent of poor households remained poor or at-risk of being poor between 2005 and 2012. We also find Muslims, lower caste groups, and rural households are in a more disadvantageous position compared to Hindus, upper caste groups, and urban households. These findings cast doubt on the conventional wisdom that marginalized households in India are catching up.

Keywords: Mobility, India, Measurement Error, Partial Identification, Poverty

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1 Introduction

There has been a phenomenal rise in economic inequality in India over the past few decades. A 2018 Oxfam study reports a significant increase in the consumption Gini index in both rural and urban areas in India from 1993-94 to 2011-12.^[1] According to Global Wealth Report (GWR) 2017, the share of wealth owned by the bottom 50% of the population declined from 8.1% to only 4.2% between 2002 and 2012, while that of top 1% increased from 15.7% to 25.7%.^[2] Among the countries for which GWR gives the share of wealth held by the top 1%, only Indonesia and the US have higher values. In a recent study Chancel and Piketty (2019) find that current inequality in India is at its highest level in 96 years. The authors (p. S60) state: “India in fact comes out as a country with one of the highest increase in top 1% income share concentration over the past thirty years.”

Given this dramatic rise in inequality, it is imperative to accurately measure the extent of economic mobility in India. Mobility is salient because the welfare effects of rising inequality depend crucially on the level of economic mobility. Economic mobility (or a lack thereof) can attenuate (or accentuate) the adverse effects of inequality. Ceteris paribus, economic mobility – whereby households move more freely throughout the income/consumption distribution – results in a more equal distribution of lifetime income/consumption relative to the distribution of income/consumption in a given period. Conversely, *intragenerational* mobility, leading to less variation in lifetime income/consumption, may be undesirable if it implies a decoupling between ability and/or effort and income/consumption (Jäntti and Jenkins 2015). Moreover, as discussed in Glewwe (2012) and Dang et al. (2014), the nexus of inequality and mobility has crucial implications for policy design. If inequality is high and mobility is low, then low socioeconomic status (SES) households will often find themselves in a poverty trap and policies should perhaps target the acquisition of assets by such households. However, if inequality and mobility are both high, but households are unable to smooth consumption during periods of low income, then policies that target the sources of consumption volatility or expand access to credit and insurance markets may be warranted.

Here, we seek to analyze the degree of economic mobility in India while overcoming a major data challenge: measurement error in income or consumption data. It is widely accepted that income and consumption data from surveys suffer from measurement error. Vanneman and Dubey (2013, p. 441) argue that measurement error is particularly problematic in the Indian context as “most Indian households receive income from more than one source” and this “variety of income

¹https://www.oxfamindia.org/sites/default/files/WideningGaps_IndiaInequalityReport2018.pdf

²<https://www.credit-suisse.com/about-us/en/reports-research/global-wealth-report.html>

sources and household economic strategies presents a much greater challenge for income measurement in India than is typical in rich-country data.” Focusing on household consumption instead of income does not ameliorate these issues (Glewwe 1991). Vanneman and Dubey (2013, p. 443) state that “survey measures of [consumption] expenditures have their own measurement problems (for example, respondent fatigue) and volatility (marriages, debts, and health crises can create unrepresentative spikes for some households).” Meyer and Sullivan (2003, p. 1182) conclude: “In practice, survey income, expenditure and consumption are all measured with significant error.”

Overcoming measurement error in income or consumption data is not trivial as previous research demonstrates that such errors are nonclassical (i.e., the errors are mean-reverting and serially correlated) (Duncan and Hill 1985; Bound and Krueger 1991; Bound et al. 1994; Pischke 1995; Bound et al. 2001; Kapteyn and Ypma 2007; Gottschalk and Huynh 2010; Jäntti and Jenkins, 2015). This introduces added complications in measuring economic mobility. Pavlopoulos et al. (2012, p. 750) conclude that “ignoring [measurement error] can cause ‘enormous bias’ in the estimation of income/consumption dynamics.” Moreover, discretizing income/consumption into bins and analyzing transition probabilities, as we do here, does not ameliorate the problem; *measurement error* in the continuous data leads to *misclassification* in the categorical data.

In this paper, we address the problem of misclassification while examining intragenerational economic mobility in India. To do so, we extend recent work in Millimet et al. (2020) on the partial identification of transition matrices and apply our methodology to panel data on household consumption from the India Human Development Survey (IHDS). Compared with Millimet et al. (2020), our approach bounds consumption transition probabilities under different (but overlapping) assumptions concerning misclassification errors and the underlying consumption dynamics. First, under minimal assumptions concerning the misclassification process, we derive sharp bounds on transition probabilities. Second, we narrow the bounds by imposing more structure via shape restrictions, monotone instrumental variable (MIV) restrictions that assume monotonic relationships between the true consumption expenditure and certain observed covariates (Manski and Pepper 2000), and assumptions concerning the temporal properties of misclassification. Specifically, given the time structure of the data, we consider two assumptions absent in the prior work on the time series properties of misclassification; we refer to these as temporal independence and temporal invariance. Our approach leads to meaningful bounds on the rates of mobility even when we have no information about the misclassification probabilities in the form of validation data (gold standard measurements) or repeated measurements (which is the case in the present context).

Our analysis is focused on consumption mobility for two reasons. First, India determines a household’s official poverty status using monthly per capita consumption expenditure (Government of India Planning Commission 2014). As such, we are able to examine economic mobility as it relates to poverty (i.e., poverty mobility). Thus, our focus is on absolute mobility (consumption relative to the poverty line) rather than relative mobility within the overall distribution, allowing us to distinguish between transitions into and out of poverty. Second, consumption is conventionally viewed as the preferred welfare indicator in developing countries because it is thought to better capture long-run welfare levels than current income. For example, Meyer and Sullivan (2003, p. 1210) note, “conceptual arguments as to whether income or consumption is a better measure of material well-being of the poor almost always favor consumption...[since it] captures permanent income, reflects the insurance value of government programs and credit markets, better accommodates illegal activity and price changes, and is more likely to reflect private and government transfers.”

Our analysis yields several striking findings. First, we show that a modest amount of misclassification leads to bounds on mobility rates that can be quite wide and almost uninformative in the absence of other information or restrictions. This is an incredibly valuable result in its own right as it indicates that mobility estimates that do not account for misclassification produce a false sense of certitude; self-observed mobility rates in survey data can be highly misleading. Second, the restrictions considered here contain significant identifying power as the bounds can be severely narrowed.

Third, under our most restrictive set of assumptions, but still allowing for misclassification of up to 20% of the sample, we find that the probability of being in poverty in 2012 conditional on being in poverty in 2005 is at least 22%, the probability of being in an insecure nonpoor state (i.e., monthly per capita household consumption expenditure is between the poverty line and twice the poverty line) in 2012 conditional on being in poverty in 2005 is at least 42%, and the probability of being in a secure nonpoor state (i.e., monthly per capita household consumption expenditure is at least twice the poverty line) in 2012 conditional on being in poverty in 2005 is at most 30%. Under the same set of assumptions, we also find that the probability of being in poverty in 2012 conditional on being in an insecure nonpoor state in 2005 is at least 6% and at most 21%, while the probability of being in poverty in 2012 conditional on being in a secure nonpoor state in 2005 is at most 19%. These figures indicate that mobility has been remarkably low in India.

Finally, even upon imposition of the strongest, yet arguably plausible, set of assumptions, we

cannot rule out that one could be underestimating the probability of remaining impoverished over the sample period and overestimating the probability of escaping poverty by ignoring misclassification error. Thus, not only do misclassification errors mask the true level immobility in India, but the conventional wisdom that misclassification causes one to overstate mobility is not justified in the absence of stronger assumptions than considered here.

We also compare the mobility rates of various subpopulations, finding evidence of substantial heterogeneity. First, Muslims do better than Hindus or other religious groups in terms of escaping poverty; however, they are at a disadvantage compared to Hindus and other religious groups in terms of transitioning to secure nonpoor or remaining secure nonpoor. Second, compared to Brahmin and Non-Brahmin Upper Caste groups as well as Other Backward Classes (OBCs), Scheduled Castes (SCs) and Scheduled Tribes (STs) are less likely to escape poverty and more likely to move into poverty. Finally, rural households, compared to urban households, are more likely to remain in poverty. They are also less likely to escape poverty and more likely to enter into poverty than the urban households.

Overall, our analysis suggests inequality in India is paired with relatively low economic mobility. Our findings also challenge the conventional wisdom that marginalized households – those belonging to minority religious groups, lower castes, or living in rural regions – are catching up on average.

We are not the first to examine economic mobility in India. However, early studies primarily rely on unrepresentative panel data collected for relatively small samples from rural India and do not address measurement error in income or consumption (see Fields (2007) for a brief review of this literature). Subsequent studies utilize more representative data but continue to ignore measurement error (Krishna and Shariff 2011; Gautam et al. 2012; Thorat et al. 2017). Recently, a few studies analyze economic mobility in India while accounting for measurement error (Pradhan and Mukherjee 2015; Barrientos Q. et al. 2018; Azam 2016; Arunachalam and Shenoy 2017; Dang and Lanjouw 2018). While admirable, these studies differ from ours in that they often rely on instrumental variables (IV) estimation, strong functional form or distributional assumptions, or assume that measurement error is only of a certain type (e.g., rank-preserving measurement error).

We contribute to this growing literature by discerning what can be learned about mobility in India in the presence of noisy survey data by employing an approach that offers several distinct advantages relative to the existing literature. First, our approach is suitable to address a wide class of misclassification errors such as non-classical or non-rank preserving errors. Second, our approach does not rely on functional form or distributional assumptions. Rather it is based on

assumptions that are fairly weak, transparent, easily understood by policymakers, and easy to impose or not impose depending on one’s beliefs. Third, our approach focuses on estimation of transition matrices, thus examining mobility throughout the distribution. At the very least, our approach provides complementary information to the existing literature.

Our work, methodologically, is related to at least three strands of literature. First, it is related to the econometric literature on economic mobility (see, e.g., McGarry 1995; Pavlopoulos et al. 2012; Dang et al. 2014; Lee et al. 2017). Second, it is related to the general statistical literature on misclassification (see, e.g., Kuha 1997; Kuha and Skinner 1997; Bassi et al. 2000; Rosychuk and Thompson 2001, 2003; Breen and Moisio 2004; Kuha et al. 2005; Shlomo and Skinner 2010). Finally, it is related to the literature on partial identification (see, e.g., Horowitz and Manski 1995; Manski and Pepper 2000; Kreider and Pepper 2007; Kreider et al. 2012).

The rest of the paper unfolds as follows. Section 2 describes the data. Section 3 presents our empirical approach. Section 4 presents the results. The final section concludes. Detailed derivation of the non-parametric bounds is relegated to Supplemental Appendix A, a more complete review of the literature is provided in Supplemental Appendix B, and information pertaining to state-level poverty lines for India can be found in Supplemental Appendix C.

2 Data

2.1 India Human Development Survey (IHDS)

The data come from the IHDS. IHDS is a nationally representative multi-topic panel household survey conducted by NCAER in New Delhi and University of Maryland (Desai et al. 2010; Desai et al. 2015). It was designed to complement existing Indian household surveys by bringing together a wide range of socioeconomic topics in a single survey. The sample was drawn using stratified random sampling with survey weights provided.

The first wave was conducted in 2004-05 and covered 41,554 households in 1,503 villages and 971 urban neighborhoods across India. The second wave was conducted in 2011-12 and covered 42,152 households. Not all households in the first wave could be surveyed in the second wave; in fact, 6,911 households interviewed in the first wave are not available in the second wave. However, our sample size only declines to 38,737 as some households in the first wave split into multiple households in the second wave. Rather than exclude these households, we map each ‘split’ household in the second wave back to their corresponding (combined) household in the first wave. Since our measure of

household well-being is per capita consumption, the change in household size is incorporated into the analysis.

As shown in Table C3 in the Appendix, households that dropped out of the sample tend to be relatively well-off in terms of consumption and education in the first wave. They are also more likely to reside in urban areas. Since the attrition rate for impoverished households in the first wave is extremely low, it is likely that our estimates of transitions out of poverty do not suffer from sample selection bias. However, some caution should be exercised when interpreting the dynamics of households with the highest level of initial consumption.

Both waves are based on interviews with a knowledgeable informant from the household. The interviews covered health, education, employment, economic status, marriage, fertility, gender relations, and social capital. The survey instruments were translated into 13 Indian languages and were administered by local interviewers. Both waves are now publicly available through the Inter-university Consortium for Political and Social Research (ICPSR).

The IHDS is well-suited to our purpose for several reasons. First, the IHDS is the most recent household panel survey conducted in India. Second, it is a nationwide panel – following rural and urban households – with a larger sample size than other panel surveys conducted in India. The closest alternative is the ARIS/REDS panel study of Indian households. However, the sample size is much smaller and only includes rural households. Third, Bartik et al. (2018) note that the IHDS produces comparable data, including the poverty rate, to other national data sources in India.

2.2 Analytic Sample

We examine economic mobility using per capita monthly consumption expenditure. It is derived from total annual household consumption expenditure. In the 2004-05 wave, this is aggregated from information on forty-seven different consumption categories. In the 2011-12 wave, this is aggregated from information on fifty-two different consumption categories.

Our outcome variable is based on the poverty ratio (*POVRATIO*) of the household, defined as the ratio of a household’s per capita monthly consumption expenditure to the corresponding poverty line. The poverty line is based on the official Indian poverty line for per capita monthly consumption as recommended by the Suresh Tendulkar committee in 2012 (Government of India Planning Commission, 2009). The poverty line is state-, year-, and urban-/rural-specific (see Table B1 in Appendix B). As noted by the former Deputy Chairman of the Planning Commission of India, Montek Singh Ahluwalia, the Tendulkar poverty line is to be used as a relevant reference point “to

see how development is helping to take more and more individuals above a fixed line over time and across states.”³ According to a report in *The Hindu* (18 March 2016), the NITI Ayog, the policy think tank of the Government of India that replaced the Planning Commission in 2015, also favors the use of the Tendulkar poverty line for tracking progress in combating extreme poverty.⁴ Because we assess the dynamics of consumption relative to the poverty line in each period and location, we need not convert nominal consumption into real terms.

Using *POVRATIO*, we classify households in each wave into three partitions. The first partition consists of households with *POVRATIO* less than one; these are the households officially classified as *poor*. The second partition consists of households with *POVRATIO* at least one but less than two (households between 100% and 200% of the official poverty line); we refer to these households as *insecure nonpoor* – a term used by the World Bank, USAID, and others – since these households are at-risk of becoming impoverished. Dang and Lanjouw (2018) create a category of non-poor households that are ‘vulnerable’ to poverty, defined as having a 15% probability of falling below the poverty line in the next period. The authors’ vulnerability line for India over the same time period as our sample is “close to the twice the national poverty line” and “offers qualitatively similar results” (p. 157). The third partition consists of the households with *POVRATIO* at least two (households with consumption exceeding 200% of the official poverty line); we refer to these as *secure nonpoor*. Using these three partitions, we estimate 3×3 consumption transition matrices for the full sample as well as for different subsamples.

Our sample consists of 38,737 households from across India: these are the households who were interviewed in both the waves of the survey, who have no missing (or invalid) information on consumption, income, household size, and other demographic characteristics of the household head (e.g., education, gender, caste, religion), annual total household income and annual total consumption expenditure are non-negative, and age of the household head in the first round is at least 18. Summary statistics are presented in Table 1. Table C2 in the Appendix provides unweighted summary statistics for comparison.

As can be observed in Table 1, between 2005 and 2012, average household size declined from 5.83 to 4.76. We feel this decline deserves some explanation as it might appear somewhat counter-intuitive. According to the IHDS, a household is defined as all those who live under the same roof

³<https://www.thehindu.com/news/national/tendulkar-poverty-line-will-remain-reference-point/article2509910.ece>.

⁴<https://www.thehindu.com/business/Industry/niti-aayog-task-force-backs-tendulkar-poverty-line/article8371390.ece>.

and share the same kitchen for 6+ months during the prior year. Therefore, a potential explanation for this decline is the splitting of households between the two waves as noted above (specifically, of the 40,018 households in wave 1, the number of households who continue living as a single unit in the second survey round is 30,462; the remainder of our final sample are split households). If households split from a single unit to multiple units over time, a decline in average household size is possible. Additionally, there are three reasons for a decline in household size: out-migration of household members, women moving out of the household after marriage, and deaths. It is worth emphasizing here the decline in household size could potentially affect our estimate of poverty mobility (since some of the factors affecting migration may be related to poverty). However, we do not think that is a cause of concern. Our objective is to examine mobility using a specific measure of household well-being: per capita consumption. If households move up or down in the per capita consumption distribution over time due to changes in household composition, then that is a part of what we seek to understand. We are agnostic about *why* they move up or down. Finally, since our well-being measure is per capita consumption expenditure, it is not clear in what direction changes in household composition will impact mobility.

3 Empirical Framework

3.1 Setup

The setup is similar to previous work in Millimet et al. (2020). However, given the Indian context as well as the time structure of the data, we consider different (but overlapping) assumptions than in the prior paper. In what follows, we present the setup and discuss the main assumptions. We relegate the formal derivations to Appendix A.

To begin, let y_{it}^* , denote the true consumption for household i , $i = 1, \dots, N$, in period t , $t = 0, 1$. Define the true $K \times K$ transition matrix as $P_{0,1}^*$, given by

$$P_{0,1}^* = \begin{bmatrix} p_{11}^* & \cdots & \cdots & p_{1K}^* \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p_{K1}^* & \cdots & \cdots & p_{KK}^* \end{bmatrix}. \quad (1)$$

Elements of this matrix have the following form

$$\begin{aligned} p_{kl}^* &= \frac{\Pr(\zeta_{k-1}^0 \leq y_0^* < \zeta_k^0, \zeta_{l-1}^1 \leq y_1^* < \zeta_l^1)}{\Pr(\zeta_{k-1}^0 \leq y_0^* < \zeta_k^0)} \\ &= \frac{\Pr(y_0^* \in k, y_1^* \in l)}{\Pr(y_0^* \in k)} \quad k, l = 1, \dots, K, \end{aligned} \quad (2)$$

where the ζ s are cutoff points between the K partitions such that $0 = \zeta_0^t < \zeta_1^t < \zeta_2^t < \dots < \zeta_{K-1}^t < \zeta_K^t < \infty$, $t = 0, 1$. Thus, p_{kl}^* is a conditional probability. A complete lack of mobility implies p_{kl}^* equals unity if $k = l$ and zero otherwise.

We can define *conditional* transition matrices, conditioned upon $X = x$, where X denotes a vector of observed attributes. Denote the conditional transition matrix as $P_{0,1}^*(x)$, with elements given by

$$\begin{aligned} p_{kl}^*(x) &= \frac{\Pr(\zeta_{k-1}^0 \leq y_0^* < \zeta_k^0, \zeta_{l-1}^1 \leq y_1^* < \zeta_l^1 | X = x)}{\Pr(\zeta_{k-1}^0 \leq y_0^* < \zeta_k^0 | X = x)} \\ &= \frac{\Pr(y_0^* \in k, y_1^* \in l | X = x)}{\Pr(y_0^* \in k | X = x)} \quad k, l = 1, \dots, K. \end{aligned} \quad (3)$$

Implicit in this definition is the assumption that X includes only time invariant attributes. Moreover, while the probabilities are conditional on X , the cutoff points ζ are not. Thus, we are capturing movements within the *overall* distribution among those with $X = x$.

It should be noted that the cutoff points in (2) and (3) discretize the underlying continuous variables, y_t^* . Despite the popularity of this approach, Bulli (2001) cautions against it as the discretized data may not represent a proper Markov chain even if the underlying data originate from a continuous state space Markov process. With this caveat in mind, our analysis is useful nonetheless as it conveys the loss of information about (1), which is used by researchers and policymakers alike (e.g., Lee et al. 2017), due to misclassification. Moreover, the solution used in Bulli (2001) is not feasible here as it requires data on many time periods even in the absence of measurement error. That said, extending the analysis in Bulli (2001) to allow for measurement error in the underlying continuous variable should be the subject of future research.

As discussed in more detail in Section 3, in this study we set $K = 3$. The outcome, y^* , denotes *POVRATIO*. In each period t , the partitions are set as $\zeta_0^t = 0$, $\zeta_1^t = 1$, $\zeta_2^t = 2$, and $\zeta_3^t \rightarrow \infty$, $t = 0, 1$. Thus, partition one includes households with *POVRATIO* $\in (-\infty, 1)$ in period t (poor households). Partition 2 includes households with *POVRATIO* $\in [1, 2)$ in period t (insecure non-poor households). Partition 3 includes all households with *POVRATIO* $\in [2, \infty)$ in period t

(secure non-poor households). In the terminology of Millimet et al. (2020), $P_{0,1}^*$ and $P_{0,1}^*(x)$ utilize unequal-sized partitions since the sample is not equally split among the partitions. This means that mobility is not zero-sum; for example, a household may move up to a new partition without another household necessarily moving down. Note, we focus on estimation of transition matrices with relatively few partitions (as opposed to many partitions) since this is what policymakers are likely to care about. This is because policymakers primarily base resource allocation decisions on only a few cutoffs such as the poverty line and the near-poverty (vulnerability) line. Moreover, increasing the number of partitions will exacerbate misclassification (Millimet et al. 2020).

Our objective is to learn about the elements of $P_{0,1}^*$ or $P_{0,1}^*(x)$. With a random sample $\{y_{it}^*, x_i\}$, the transition probabilities are nonparametrically identified; consistent estimates are given by the empirical transition probabilities. However, as stated previously, ample evidence indicates that consumption is measured with error. Let y_{it} denote the observed consumption for household i in period t . With data $\{y_{it}, x_i\}$, the empirical transition probabilities are inconsistent for p_{kl}^* and $p_{kl}^*(x)$. Rather than invoking overly strong, and likely implausible assumptions, to point identify the transition probabilities, our goal is to bound the probabilities given in (2) and (3).

To proceed, we characterize the relationships between the true partitions of $\{y_{it}^*\}_{t=0}^1$ and the observed partitions of $\{y_{it}\}_{t=0}^1$ using the following joint probabilities:

$$\theta_{(k,l)}^{(k',l')} = \Pr(y_0 \in k', y_1 \in l', y_0^* \in k, y_1^* \in l). \quad (4)$$

While conditional misclassification probabilities are more intuitive, these joint probabilities are easier to work with (e.g., Kreider et al. 2012).

In (4) the subscript (k, l) indexes the true partitions in periods 0 and 1 and the superscript (k', l') indicates the observed partitions. With this notation, we can now rewrite the elements of $P_{0,1}^*$ as

$$\begin{aligned} p_{kl}^* &= \frac{\Pr(y_0^* \in k, y_1^* \in l)}{\Pr(y_0^* \in k)} \\ &= \frac{\Pr(y_0 \in k, y_1 \in l) + \sum_{\substack{k', l' = 1, 2, \dots, K \\ (k', l') \neq (k, l)}} \theta_{(k,l)}^{(k', l')} - \sum_{\substack{k', l' = 1, 2, \dots, K \\ (k', l') \neq (k, l)}} \theta_{(k', l')}^{(k, l)}}{\Pr(y_0 \in k) + \sum_{\substack{k', l', \tilde{l} = 1, 2, \dots, K \\ k' \neq k}} \theta_{(k, \tilde{l})}^{(k', l')} - \sum_{\substack{k', l', \tilde{l} = 1, 2, \dots, K \\ k' \neq k}} \theta_{(k', \tilde{l})}^{(k, l')}}} \\ &\equiv \frac{r_{kl} + Q_{1,kl} - Q_{2,kl}}{p_k + Q_{3,k} - Q_{4,k}}. \end{aligned} \quad (5)$$

$Q_{1,kl}$ measures the proportion of false *negatives* associated with partition kl (i.e., the probability of being misclassified conditional on kl being the true partition). $Q_{2,kl}$ measures the proportion of false *positives* associated with partition kl (i.e., the probability of being misclassified conditional on kl being the observed partition). Similarly, $Q_{3,k}$ and $Q_{4,k}$ measure the proportion of false *negatives* and *positives* associated with partition k , respectively.

The data identify r_{kl} and p_k (and, hence, $p_{kl} \equiv r_{kl}/p_k$), but not the misclassification parameters, θ . One can compute sharp bounds by searching across the unknown misclassification parameters. However, absent further restrictions, obtaining informative bounds on the transition probabilities is not possible. In the Section 3.2, we introduce assumptions on the θ s to potentially yield informative bounds. Some of these assumptions are considered in Millimet et al. (2020), while some are new. Section 3.3 considers restrictions on the underlying mobility process and are identical to those considered in Millimet et al. (2020).

3.2 Misclassification

3.2.1 Assumptions

Allowing for misclassification, we obtain bounds on the elements of $P_{0,1}^*$, given in (5). In the interest of brevity, we focus attention from here primarily on the unconditional transition matrix. We return to the conditional transition matrix in Section 3.3. We begin by considering the following two misclassification assumptions from Millimet et al. (2020).

Assumption 1 (No Missclassification). *Misreporting does not alter an observation's partition in the consumption distribution in either period. Formally, $\sum_{k,l} \theta_{(k,l)}^{(k,l)} = 1$ or, equivalently,*

$$\sum_{\substack{k,k',l,l'=1,2,\dots,K \\ (k',l') \neq (k,l)}} \theta_{(k,l)}^{(k',l')} = 0.$$

Assumption 2 (Maximum Arbitrary Misclassification Rate). *The total misclassification rate in the data is bounded from above by $Q \in (0, 1)$. Formally, $1 - \sum_{k,l} \theta_{(k,l)}^{(k,l)} \leq Q$ or, equivalently,*

$$\sum_{\substack{k,k',l,l'=1,2,\dots,K \\ (k',l') \neq (k,l)}} \theta_{(k,l)}^{(k',l')} \leq Q.$$

Assumption 1 is quite strong, but is simply used as a benchmark. Under this assumption, measurement error in the underlying continuous data is allowed as long as it does not lead to

misclassification in the discretized data. Under Assumption 2, the number of θ s defined by (4) is $K^2(K^2 - 1)$; which, in our case, equals 72. Assumption 2 limits the sum of these parameters, but not the number of non-zero parameters. As discussed in Millimet et al. (2020), the amount of misclassification is unknown and dependent on the choice of K . We consider sensitivity to the choice of Q , setting Q to 0.10, 0.20, and 0.30. Thus, we allow for fairly rampant misclassification as we allow for up to 30% of the sample to be misclassified.

Finally, we consider the following two additional assumptions not considered in Millimet et al. (2020) in light of the structure of the Indian data. They are imposed in combination with Assumption 2.

Assumption 3 (Temporal Independence). *Misclassification probabilities are independent across time periods. Formally,*

$$\theta_{(k,l)}^{(k',l')} = \alpha_k^{k'} \cdot \beta_l^{l'},$$

where $\alpha_k^{k'}$ ($\beta_l^{l'}$) is the probability of being observed in partition k' (l') in the initial (terminal) period when the true partition is k (l). That is,

$$\begin{aligned}\alpha_k^{k'} &= P(y_0 \in k' | y_0^* \in k) \\ \beta_l^{l'} &= P(y_1 \in l' | y_1^* \in l).\end{aligned}$$

Assumption 4 (Temporal Invariance). *Misclassification probabilities are independent across time periods as well as temporally invariant. Formally,*

$$\theta_{(k,l)}^{(k',l')} = \alpha_k^{k'} \cdot \alpha_l^{l'}.$$

In other words, $\alpha_k^{k'} = \beta_k^{k'} \quad \forall k, k'$.

Assumption 3 restricts misreporting behavior such that the decision to misreport is independent across periods. This rules out a household's consumption history affecting its propensity to misreport its current consumption. However, it does allow misclassification to be mean-reverting (or not) within each period as the misclassification probabilities are allowed to differ depending on the true partition. While there is some prior evidence that measurement error is serially correlated, which would invalidate this assumption, this evidence is typically derived from annual data. As our data are collected seven years apart, we believe the assumption is worth exploring in our application.

That said, long-term persistence in misclassification may arise if households believe their survey responses will be used to determine eligibility for means-tested social welfare programs. However, the assumption likely has significant identifying power as it reduces the number of misclassification parameters to $2K(K - 1)$; or, in our case, to 12.

Assumption 4 further restricts the probability of misreporting in particular directions to be constant over the sample period. This assumes that data accuracy and other sources of misclassification such as stigma do not change across time periods. Again, we believe this worth exploring in our application given the seven-year gap between periods in our data; we do not expect significant changes in reporting norms over this span. As with Assumption 3, this restriction is likely to have significant identifying power as it further reduces the number of parameters to $K(K - 1)$; or, in our case, to six.

3.2.2 Bounds

Under Assumption 1 consistent estimates are given by the empirical transition probabilities (Proposition 1 in Millimet et al. (2020)):

$$\hat{p}_{kl} = \frac{\sum_i \omega_i \mathbf{I}(y_{0i} \in k, y_{1i} \in l)}{\sum_i \omega_i \mathbf{I}(y_{0i} \in k)},$$

where ω_i is the survey weight attached to observation i . Absent this assumption, the transition probabilities are no longer nonparametrically identified. Bounds under combinations of Assumptions 2–4 are detailed in Appendix A and entail searching over values of θ to determine the set of values of the transition probabilities that are consistent with the data and the assumptions.

3.3 Mobility

3.3.1 Assumptions

The preceding section provides bounds on the transition probabilities considering only restrictions on the misclassification process. Here, we introduce restrictions on the mobility process that may further serve to tighten the bounds. The restrictions may be imposed alone or in combination.

First, we consider shape restrictions which place inequality constraints on the population transition probabilities. Specifically, we assume that large transitions are less likely than smaller ones.

Assumption 5 (Shape Restrictions). *The transition probabilities are weakly decreasing in the size*

of the transition. Formally, p_{kl}^* is weakly decreasing in $|k - l|$, the absolute difference between k and l .

This assumption implies that within each row or each column of the transition matrix, the diagonal element (i.e., the conditional staying probability) is the largest. The remaining elements decline weakly monotonically moving away from the diagonal element. Note, this assumption relates to the size of real mobility in absolute value; it makes no claims about the direction of misclassification.

This assumption seems plausible in light of prior studies. For example, Lee et al. (2017) use a structural model to simulate the consumption expenditure distribution uncontaminated by measurement error in South Korea for the period 2002-2007. The authors then examine transition probabilities over this period, which is similar in length to the period we examine here. They find that the conditional staying probability is the largest and the remaining elements decline weakly monotonically as one moves away from the diagonal element. A similar pattern is found in Glewwe (2012). Using data from Vietnam to compute expenditure mobility for the period 1992-1997, the author finds the conditional staying probabilities to be the highest and the other transition probabilities to decline with the size of the transition when measurement error is ignored. Correcting for measurement error, the author finds that expenditure mobility is lower, thereby reinforcing the shape restrictions considered here. Chetty et al. (2014) also find a similar pattern when examining intergenerational mobility in the US using administrative data. Aside from the second quintile, the conditional staying probabilities are the largest and the remaining elements decline weakly monotonically moving away from the diagonal.

Second, we consider a monotonicity restriction which places inequality constraints on population transition probabilities across observations with different observed attributes (Manski and Pepper 2000; Chetverikov et al. 2018).

Assumption 6 (Monotonicity). *The conditional probability of upward mobility is weakly increasing in a vector of attributes, u , and the conditional probability of downward mobility is weakly decreasing in the same vector of attributes. Formally, if $u_2 \geq u_1$, then*

$$\begin{aligned} p_{11}^*(u_1) &\geq p_{11}^*(u_2) \\ p_{KK}^*(u_1) &\leq p_{KK}^*(u_2) \\ p_{kl}^*(u_1) &\leq p_{kl}^*(u_2) \quad \forall l > k \\ p_{kl}^*(u_1) &\geq p_{kl}^*(u_2) \quad \forall l < k. \end{aligned}$$

The variable U is referred as a monotone instrumental variable (MIV). Note, the monotonicity assumption provides no information on the conditional staying probabilities, $p_{kk}^*(u)$, for $k = 2, \dots, K - 1$.

In our application, we use the education level of the household head in the initial period as the MIV. Here, households are grouped into four bins based on years of completed schooling: zero, 1-5, 6-10, and 11-15.

Given our MIV, Assumption 6 restricts the probability of upward (downward) mobility to be no lower (higher) for households where the head belongs to a higher educational group. This assumption is consistent with the widely held belief that “for most people, gaining an education and thus improving one’s chances in the job market is the key to becoming upwardly mobile” (Johnson et al., 2018, p. 4). Using US data, Schirmer (1990) finds that, among workers starting in the same occupation, better educated workers are more likely to move to a higher-level occupation. Using Swiss data, Li et al. (2000) show that education and training increases chances for upward occupational mobility. Using Indonesian data, Akresh et al. (2018) find that those who benefited from Indonesia’s large school building programme in the 1970s engage in more migration, more likely to be formal workers, and more likely to work in non- agricultural employment. This is consistent with enhanced upward mobility as prior research finds that migration and formal sector employment are both positively associated with upward mobility (Beegle 2011; Ferraira et al. 2013; McCaig and Pavnick 2015). That said, it is possible that education affects only the level of income/consumption, not its dynamics. As Assumption 6 is stated in terms of weak inequalities, it does not rule out this possibility. Instead, it only rules out the possibility that education leads to a strictly lower (higher) probability of upward (downward) mobility. There is no empirical evidence, to our knowledge, suggesting that is the case.

Implementing the monotonicity restriction requires us to first bound the transition probabilities conditional on u . From (3) and (5), we have

$$\begin{aligned}
p_{kl}^*(u) &= \frac{\Pr(y_0 \in k, y_1 \in l | U = u) + \sum_{\substack{k', l' = 1, 2, \dots, K \\ (k', l') \neq (k, l)}} \theta_{(k, l)}^{(k', l')}(u) - \sum_{\substack{k', l' = 1, 2, \dots, K \\ (k', l') \neq (k, l)}} \theta_{(k', l')}^{(k, l)}(u)}{\Pr(y_0 \in k | X = u) + \sum_{\substack{k', l' = 1, 2, \dots, K \\ k' \neq k}} \theta_{(k, l)}^{(k', l')}(u) - \sum_{\substack{k', l' = 1, 2, \dots, K \\ k' \neq k}} \theta_{(k', l')}^{(k, l)}(u)} \\
&\equiv \frac{r_{kl}(u) + Q_{1,kl}(u) - Q_{2,kl}(u)}{p_k(u) + Q_{3,k}(u) - Q_{4,k}(u)} \tag{6}
\end{aligned}$$

where now $Q_{j,\cdot}(u)$, $j = 1, \dots, 4$, represent the proportions of false positives and negatives conditional on u .

3.3.2 Bounds

The bounds under various combinations of Assumptions 5–6 are relegated to Appendix A, where survey weights, ω , continue to be used. However, two salient issues must be discussed. First, estimates of the bounds under these assumptions suffer from finite sample bias as they rely on infima and suprema. To circumvent this issue, a bootstrap bias correction procedure is used. As presented in Kreider and Pepper (2007), the bias correction adjusts the sample estimates of the bounds by the difference between the mean of the nonparametric bootstrap estimates of the end points and the corresponding sample estimate. Formally, letting LB_{kl} and UB_{kl} represent the population values of the end points and \widehat{LB}_{kl} and \widehat{UB}_{kl} the corresponding sample estimates, the bias-corrected estimates are given by

$$\begin{aligned} LB_{kl}^{bc} &= 2\widehat{LB}_{kl} - E^*[\widehat{LB}_{kl}] \\ UB_{kl}^{bc} &= 2\widehat{UB}_{kl} - E^*[\widehat{UB}_{kl}], \end{aligned}$$

where $E^*[\cdot]$ denotes the expected value with respect to the bootstrap distribution. We follow the same procedure here except the resampling procedure must account for the survey weights. To do so, we use the approach in Kolenikov (2010), which is based on the rescaling bootstrap procedure developed in Rao et al. (1992).

Second, inference is conducted by using the same bootstrap approach to account for survey weights along with the Imbens-Manski (2004) correction to obtain 90% confidence intervals (CIs). As discussed in Millimet et al. (2020), while the literature on inference in partially identified models is expanding rapidly, the Imbens-Manski (2004) approach is preferable in the current context. We do not combine bias correction with inference as this requires a double bootstrap procedure with complex survey data which is beyond the scope of the current paper. Instead, we report three intervals: the point estimates of the bounds with no bias correction, the point estimates of the bounds using the bootstrap bias correction, and the Imbens-Manski (2004) CIs for the partially identified parameter with no bias correction. This turns out to have little impact qualitatively.

4 Results

4.1 Full Sample Analysis

Results for the 3×3 transition matrix based on the full sample are presented in Tables 2-4. In all tables, partitions for poor, insecure nonpoor, and secure nonpoor are represented by the numbers 1, 2, and 3, respectively. For each cell of the transition matrix, three intervals are reported: point estimates of the bounds using the survey weights with no bias correction (shown in square brackets), the point estimates of the bounds using the survey weights and the bootstrap bias correction accounting for the weights (shown in curly brackets), and the Imbens-Manski CI for the partially identified parameter using the survey weights with no bias correction (shown in parentheses). Note, throughout the discussion of the results, unless otherwise noted, we focus on the point estimates for simplicity. Finite sample bias and sampling uncertainty do not qualitatively affect our findings.

Overall, between 2005 and 2012, the observed poverty rate declined from 35% to 17%, the proportion of insecure nonpoor households remained unchanged at 44%, and the proportion of secure nonpoor households increased from 21% to 41% (see Table 1). Turning to mobility, Table 2 presents our baseline results under the strong assumption of No Missclassification (Assumption [1](#)). Recall, households are assumed to self-report consumption accurately enough so as to be classified into the correct partitions. In this case, the probability of a household remaining in poverty across the initial and terminal periods is 0.292, the probability of remaining insecure nonpoor is 0.480, and the probability of remaining secure nonpoor is 0.652. Furthermore, we find that the probabilities of observing larger transitions in the self-reported consumption distribution are less likely than smaller movements. For example, the probability of moving from impoverished to insecure nonpoor is 0.507; the probability of moving from impoverished to secure nonpoor is 0.201.

Table 3 presents the worst case bounds obtained only under Assumption [2](#) for different values of the maximum amount of misclassification, Q . Specifically, we vary Q from 0.10 to 0.20 to 0.30 across Panels I–III, respectively. This choice is guided by the findings of Millimet et al. (2020). Specifically, Millimet et al. (2020) use US data and employ a simulation-based approach that relies on parameter values obtained from the previous literature on measurement error in US income and consumption data to quantify Q . The authors find that the misclassification rate is roughly 20% when the data is discretized into three partitions. Thus, considering a range from 0.10 to 0.30 seems reasonable.

In Panels II and III the bounds are nearly uninformative. This means a relatively moderate

amount of arbitrary misclassification ($Q = 0.20$), in the absence of other information, results in an inability to say much more about mobility over this time period. The importance of this finding cannot be overstated as it indicates that mistakenly believing mobility estimates that do not account for misclassification gives a false sense of certitude. For example, while based on the assumption of no misclassification error, one may be tempted to believe that probability of a household remaining in poverty across the initial and terminal period is 29%, in reality the true estimate of the probability of a household remaining in poverty across the initial and terminal period could be anything between zero and 86% in Panel II; the bounds are completely uninformative in Panel III with $Q = 0.30$.

Even in Panel I where we assume that at most only 10% of the sample is misclassified we are limited in what we can learn. The conditional staying probability in poverty can be anywhere from 0 to 58%, in the insecure poor state from 25% to 71%, and in the secure nonpoor state from 18% to 100%. Similarly, with even 10% misclassification, the probability of moving from impoverished to secure nonpoor over the sample period can be anywhere from 0% to 49% and the probability of moving from secure nonpoor to impoverished from 0% to 52%. On the other hand, assuming the misclassification rate is at most 10%, we do learn that probability of moving from insecure nonpoor to secure nonpoor is no lower than 16% over the sample period.

To see what can be learned if one is willing to invoke stronger assumptions, we turn to Table 4. Again, we impose Assumption 2 and vary Q from 0.10 to 0.20 to 0.30 across Panels I-III, respectively. In addition, we add Assumptions 5 and 6. Column A adds no further assumptions, whereas Column B adds Assumption 3 and Column C adds Assumption 4. Importantly, under this richer set of assumptions, the bounds do not necessarily widen with Q . This arises because of the shape restrictions. With small Q , the data may reject the shape restrictions (due to the bounds for the off-diagonal probabilities being strictly larger than the bounds for the conditional staying probabilities). In this case, the shape restrictions are not imposed. However, as Q increases, this is less likely to occur, leading to vastly different and narrower bounds due to now imposing the shape restrictions.

Turning to Column A, we find that Assumptions 5 and 6 along have little identifying power. The only substantive change relative to the worst case bounds in Table 3 occurs in Panel I. Now we find that the conditional staying probability for poverty is at least 22% assuming the misclassification rate is no more than 10%. As the sample spans a seven-year interval, this suggests a considerable amount of persistence among the poor.

When we add the assumption of temporal independence in Column B, we see that the bounds tighten considerably. Now, even when the misclassification rate is allowed to be as high as 30%, we are able learn much about mobility. For example, the probability of remaining impoverished is at least 22%, of transitioning from impoverished to insecure nonpoor is at least 29%, and of transitioning from impoverished to secure nonpoor is no more than 49%. For households initially insecure nonpoor, their chances of remaining insecure nonpoor is at least 27%, and of transitioning from insecure nonpoor to secure nonpoor (impoverished) is at least (at most) 16% (36%).

Adding the stronger assumption of temporal invariance in Column C, the bounds are further tightened. For example, for the poor, the bounds on probability remaining in poverty over the sample period narrows from $[0.221, 0.541]$ in Column B of Panel III to $[0.359, 0.395]$. Thus, even with a 30% misclassification rate, we are able to state that more than one-third of all impoverished households in 2005 continue to be impoverished seven years later. The bounds on the remaining conditional staying probabilities are also quite narrow: $[0.394, 0.523]$ for the insecure poor and $[0.575, 0.643]$ for the secure poor. We also find that the probability of moving from poverty to insecure (secure) nonpoor is $[0.387, 0.390]$ ($[0.216, 0.255]$). We also find that the probability of moving from secure nonpoor to insecure nonpoor (poor) is no greater than 43% (24%).

In sum, under our strongest set of assumptions but allowing for 30% of the sample to be misclassified, 36–40% of impoverished households remain so, 39% escape poverty but remain at-risk, and 22–26% completely escape poverty over the seven-year sample period. If we relax the assumption of temporal independence and invariance, but continue to maintain the other assumptions, the data are essentially uninformative about consumption mobility unless we assume that misclassification is no more than 10%.

These results have two implications. First, mobility out of poverty over the sample period is remarkably low. This is consistent with Dang and Lanjouw’s (2018, p. 133-134) findings that in India “aggregate trends in poverty reduction mask a considerable degree of entry into—and to a larger extent, exit out of—poverty and vulnerability but that a substantial core of the poor have remained poor over the duration of the study period [early 2000s].” Second, if we naïvely ignore misclassification error, we *underestimate* the probability of remaining impoverished and *overestimate* the probability of escaping poverty. Thus, misclassification errors appear to mask the true level immobility in India. Ignoring misclassification, the *observed* probability of staying impoverished is 29%. Allowing for a maximum misclassification rate of 30%, the probability of staying impoverished is *at least* 36%. This result confirms the suspicion in Arunachalam and

Shenoy (2017) that measurement error is likely to mask a poverty trap in India.

4.2 Heterogeneity Analysis

In this section, we analyze heterogeneity in economic mobility across different population subgroups. In Table 5, we report the results for the 3×3 transition matrix by religion. Table 6 contains the results by caste. Table 7 presents the results by geographic area (urban or rural). For brevity, for all the cases, we only present results under our strongest set of assumptions, Assumption [2](#), [4](#), [5](#), and [6](#). In addition, we continue to vary Q from 0.10 to 0.20 to 0.30 across Panels I-III in each table.

Religion India contains multiple religious groups; namely, Hindus and various minority groups including Muslims, Christian, Sikh, Buddhist, Jain, Tribal, and others. We divide the minority groups into two groups, Muslims and others. Results are shown in Table 5. Comparing these groups, we are unable to unambiguously rank them in terms of mobility even under our strongest set of assumptions and assuming a maximum misclassification rate of 10%. This arises because of overlap in the bounds across the groups in most cases.

In terms of the conditional staying probability in poverty, the bounds are $[0.232, 0.331]$ for Hindus, $[0.204, 0.306]$ for Muslims, and $[0.240, 0.323]$ for others when $Q = 0.10$. Thus, while it is possible that Muslims have significantly lower persistence in poverty than the other religions, we cannot rule out the reverse as well. With $Q = 0.30$ we find that the conditional staying probability is at least 35% for Hindus, while it could be as low as 34% (31%) for Muslims (others).

In terms of the conditional staying probability for secure nonpoor, we find that Muslims are at a distinct disadvantage even when $Q = 0.30$. In this case, the bounds are $[0.577, 0.643]$ for Hindus, $[0.285, 0.604]$ for Muslims, and $[0.709, 0.813]$ for others. Thus, Muslims are at the greatest risk of losing their status as secure nonpoor, while others enjoy the highest persistence as secure nonpoor.

In terms of mobility, some strong conclusions can be drawn when $Q = 0.10$. For example, the probability of transitioning from insecure nonpoor to secure nonpoor is definitely higher for others than Muslims as the bounds are $[0.400, 0.439]$ and $[0.298, 0.374]$, respectively. Similarly, the probability of transitioning from secure nonpoor to insecure poor is definitely lower for others than Muslims as the bounds are $[0.210, 0.254]$ and $[0.294, 0.446]$, respectively. For Hindus, the bounds overlap both of the other groups preventing conclusions from being drawn. That said, the probability of moving up from insecure nonpoor to secure nonpoor is at least 35% for Hindus, whereas it could be as low as 30% for Muslims. Similarly, the probability of moving down from

secure nonpoor to insecure nonpoor is no more than 36% for Hindus, whereas it could as high as 45% for Muslims.

In sum, the results make it clear that definitive comparisons across religious groups are difficult even under our strongest set of assumptions and a relatively modest amount of misclassification. However, there is some evidence that Muslims may be doing better than Hindus or other religious groups in terms of escaping poverty, but are at a disadvantage compared to Hindus and other religious groups in terms of transitioning to secure nonpoor or remaining secure nonpoor.

Caste Next, we explore heterogeneity in economic mobility across castes; namely, Brahmins and non-Brahmin Upper Castes (UCs), Scheduled Castes and Scheduled Tribes (SCs/STs), and Other Backward Classes (OBCs). Results are shown in Table 6.

In terms of the conditional staying probability in poverty, the bounds are $[0.089, 0.268]$ for UCs, $[0.328, 0.387]$ for SCs/STs, and $[0.173, 0.296]$ for OBCs when $Q = 0.10$. Thus, SCs/STs have unambiguously higher persistence in poverty than the other castes; the bounds for UCs and OBCs overlap and therefore cannot persistence cannot be ranked among these castes. SCs/STs also have higher persistence as insecure poor than UCs, but not necessarily OBCs. Similarly, SCs/STs have unambiguously lower persistence as secure nonpoor; the bounds are $[0.723, 0.744]$ for UCs, $[0.427, 0.551]$ for SCs/STs, and $[0.606, 0.606]$ for OBCs. With $Q = 0.30$ we continue to find strong evidence of worse outcomes for SCs/STs.

In terms of mobility, strong conclusions continue to emerge when $Q = 0.10$. For example, the probability of transitioning from impoverished to secure nonpoor is definitely lower for SCs/STs than for the Hindus and also possibly OBCs; the bounds are $[0.232, 0.282]$ for UCs, $[0.121, 0.188]$ for SCs/STs, and $[0.183, 0.257]$ for OBCs. The probability of transitioning from insecure nonpoor to secure nonpoor is also lowest for SCs/STs as the bounds are $[0.422, 0.465]$ for UCs, $[0.268, 0.346]$ for the SCs/ST and $[0.353, 0.440]$ for the OBCs, respectively. Finally, the probability of transitioning from insecure nonpoor to impoverished is higher for SCs/STs than UCs as the bounds are $[0.157, 0.225]$ and $[0.077, 0.132]$, respectively. The corresponding bounds for OBCs are predominantly lower than that for SCs/STs as well, although there is some overlap. With $Q = 0.30$ the results continue to paint a picture of lower upward mobility and higher downward mobility for SCs/STs, although the bounds always partially overlap.

In sum, under the assumption of modest misclassification ($Q = 0.10$), SCs/STs have the lowest probability of escaping poverty and highest probability of downward mobility over the sample

period. Absent stronger assumptions, conclusions cannot be drawn on how mobility compares across UCs and OBCs.

Geographic Location Lastly, we explore heterogeneity in economic mobility across households living in rural and urban areas. Table 7 displays the results. In general, most bounds overlap across the two groups making unambiguous statements not possible. The two exceptions is the conditional staying probability as insecure nonpoor and secure nonpoor when $Q = 0.10$. For the insecure (secure) nonpoor, the bounds are $[0.450, 0.464]$ ($[0.749, 0.781]$) for urban households and $[0.471, 0.509]$ ($[0.561, 0.586]$) for rural households. This result is complemented by a higher probability of transitioning from secure nonpoor to insecure poor for the urban households than the rural households; the bounds are $[0.200, 0.251]$ for urban households and $[0.261, 0.439]$ for rural households. With $Q = 0.30$, the conditional staying probability as secure nonpoor continues to be strictly lower for rural households although we cannot make any unambiguous claim regarding the conditional staying probability as insecure nonpoor. Thus, under our most stringent set of assumptions, but even allowing for a 30% misclassification rate, the probability of downward mobility from the secure nonpoor state is definitely higher in rural areas.

In all other cases, the bounds overlap. That said, the results suggest that rural households are worse off. For example, with $Q = 0.30$ the conditional staying probability in poverty is at least 36% for rural households, while it could be as low as 34% for urban households. Similarly, the probability of transitioning from secure nonpoor to poor (insecure nonpoor) could be as high as 27% (51%) in rural areas, whereas it is no more than 17% (27%) in urban areas.

In sum, our findings indicate that, compared to urban households, rural households have a distinctly higher probability of downward mobility. In particular, rural households have a significantly more difficult time remaining secure nonpoor.

5 Conclusion

In this paper, we provide bounds on the extent of economic mobility in India over the period 2005 to 2012 using IHDS panel data on household consumption while rigorously accounting for misclassification in a transparent manner. Methodologically, we extend recent work in Millimet et al. (2020) on the partial identification of transition matrices by considering the identifying power of additional assumptions on the misclassification process: temporal independence and temporal invariance. In the application, we reveal how little can be learned about poverty dynamics under

relatively small amounts of misclassification absent additional information. We then show the identifying power that accompanies additional information via assumptions on the nature of the misclassification as well as restrictions on consumption dynamics.

We find that, under reasonable assumptions, for the population as a whole, mobility in India is remarkably low: allowing for misclassification errors in up to 30% of the sample, at least 3 in 10 poor households remain poor between 2005 and 2012, nearly 4 in 10 households manage to escape poverty but remain at-risk, and fewer than 3 in 10 poor households manage to attain the status of secure nonpoor. Further, we show that if we mistakenly believe that there is no misclassification error, we might be underestimating the probability of the poor remaining poor, and over estimating the probability of the poor escaping poverty and becoming insecure nonpoor. Under stronger assumptions, we also find clear rankings among different population subgroups in terms of economic status. Among religious groups, Muslims are at a disadvantage compared to Hindus or other religious groups. Among castes, SCs/STs are the worst off. Finally, rural households are at a distinct disadvantage relative to urban households.

Our results for the population as a whole suggest that inequality in India can be characterized as chronic as households belonging to the lower rungs of the economic ladder are likely to find themselves caught in a poverty trap. As a result, our findings suggest that poverty reduction efforts should focus on ways to improve the permanent economic status of households, possibly through acquisition of assets and capabilities, rather than on ways to deal with temporary volatility. Our findings also cast doubt on the conventional wisdom that marginalized groups in India – households belonging to minority communities, lower castes, or living in rural regions – are catching up on average. This calls for a re-evaluation of existing social policies that are designed to improve the economic status of the marginalized groups in India.

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Table 1. Summary Statistics (Weighted)

	2005 (Wave 1)		2012 (Wave 2)	
	Mean	SD	Mean	SD
Household Consumption				
Annual Total Consumption (in Rs.)	50,109.17	45,922.68	106,700.10	107,396.90
Monthly Per Capita Consumption (in Rs.)	792.67	772.54	2,081.59	2,422.13
Poverty Status				
Poor (POVRATIO < 1)	0.35	0.48	0.17	0.37
Insecure Nonpoor (POVRATIO >= 1, < 2)	0.44	0.50	0.44	0.50
Secure Nonpoor (POVRATIO >= 2)	0.21	0.41	0.41	0.49
Household Size	5.83	2.99	4.76	2.29
Education (Household Head): Years of Schooling				
0 (Illiterate)	0.39	0.49		
1-5 years	0.21	0.41		
6-10 years	0.29	0.46		
11-15 years	0.11	0.31		
Caste (Household Head)				
Brahmin and Others	0.27	0.44		
OBC	0.42	0.49		
SC/ST	0.31	0.46		
Religion				
Hindu	0.83	0.38		
Muslims	0.12	0.32		
Other Religions	0.06	0.23		
Percentage of Males (Household Head)	0.91	0.29		
Percentage of Urban Residents (Household Head)	0.24	0.43		
Number of Observations	38,737		38,737	

Notes: POVRATIO is defined as the ratio of household per capita monthly household consumption expenditure to the poverty line per capita monthly consumption expenditure. In our analysis we use information pertaining to education level, caste, religion and the living region of household heads only from the first wave; hence we report summary statistics of these variables only for the first wave. Appropriate survey weights from the IHDS dataset are used.

Table 2. Full Sample Transition Matrices: No Misclassification

	1	2	3
1	[0.292,0.292] {0.292,0.292} (0.286,0.299)	[0.507,0.507] {0.507,0.507} (0.499,0.514)	[0.201,0.201] {0.201,0.201} (0.195,0.207)
2	[0.133,0.133] {0.133,0.133} (0.128,0.139)	[0.480,0.480] {0.480,0.480} (0.473,0.488)	[0.386,0.386] {0.386,0.386} (0.379,0.394)
3	[0.054,0.054] {0.054,0.054} (0.050,0.058)	[0.294,0.294] {0.294,0.294} (0.285,0.303)	[0.652,0.652] {0.652,0.652} (0.643,0.661)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table 3. Full Sample Transition Matrices: Arbitrary Misclassification (Worst Case Bounds)**I. Q = 0.10**

	1	2	3
1	[0.006,0.578] {0.006,0.578} (0.001,0.585)	[0.221,0.793] {0.221,0.793} (0.214,0.799)	[0.000,0.487] {0.000,0.487} (0.000,0.492)
2	[0.000,0.362] {0.000,0.362} (0.000,0.367)	[0.251,0.709] {0.251,0.709} (0.245,0.716)	[0.157,0.616] {0.157,0.616} (0.151,0.622)
3	[0.000,0.522] {0.000,0.522} (0.000,0.529)	[0.000,0.762] {0.000,0.762} (0.000,0.771)	[0.184,1.000] {0.184,1.000} (0.175,1.000)

II. Q = 0.20

	1	2	3
1	[0.000,0.864] {0.000,0.864} (0.000,0.873)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,0.773] {0.000,0.773} (0.000,0.780)
2	[0.000,0.591] {0.000,0.591} (0.000,0.597)	[0.022,0.939] {0.022,0.939} (0.015,0.945)	[0.000,0.845] {0.000,0.845} (0.000,0.852)
3	[0.000,0.989] {0.000,0.989} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)

III. Q = 0.30

	1	2	3
1	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)
2	[0.000,0.820] {0.000,0.820} (0.000,0.827)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)
3	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Q = maximum misclassification rate. Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table 4. Full Sample Transition Matrices: Misclassification Assumptions (Arbitrary + Shape + Monotonicity)

A. Without Temporal Independence/Invariance				B. With Temporal Independence			C. With Temporal Invariance					
I. Q = 0.10												
	1	2	3		1	2	3		1	2	3	
1	[0.221,0.541] {0.220,0.541} (0.213,0.549)	[0.292,0.578] {0.292,0.579} (0.283,0.585)	[0.000,0.487] {0.000,0.487} (0.000,0.492)	1	[0.243,0.378] {0.229,0.378} (0.192,0.384)	[0.435,0.461] {0.435,0.475} (0.427,0.500)	[0.161,0.296] {0.147,0.296} (0.109,0.301)	1	[0.229,0.330] {0.229,0.330} (0.223,0.334)	[0.458,0.593] {0.458,0.592} (0.452,0.599)	[0.164,0.249] {0.163,0.249} (0.158,0.253)	
2	[0.000,0.360] {0.000,0.360} (0.000,0.365)	[0.269,0.709] {0.269,0.709} (0.263,0.716)	[0.157,0.598] {0.159,0.600} (0.151,0.604)	2	[0.061,0.209] {0.061,0.209} (0.057,0.213)	[0.411,0.556] {0.404,0.556} (0.406,0.561)	[0.311,0.463] {0.312,0.463} (0.305,0.469)	2	[0.095,0.171] {0.095,0.171} (0.091,0.175)	[0.456,0.498] {0.456,0.498} (0.449,0.503)	[0.347,0.437] {0.347,0.437} (0.342,0.444)	
3	[0.000,0.350] {0.000,0.351} (0.000,0.355)	[0.000,0.670] {0.000,0.670} (0.000,0.676)	[0.184,1.000] {0.185,1.000} (0.175,1.000)	3	[0.000,0.203] {0.000,0.203} (0.000,0.207)	[0.161,0.450] {0.158,0.450} (0.151,0.457)	[0.496,0.839] {0.496,0.842} (0.489,0.849)	3	[0.000,0.113] {0.000,0.115} (0.000,0.132)	[0.242,0.355] {0.239,0.354} (0.231,0.363)	[0.645,0.654] {0.645,0.654} (0.637,0.659)	
II. Q = 0.20												
	1	2	3		1	2	3		1	2	3	
1	[0.029,0.756] {0.029,0.755} (0.025,0.766)	[0.076,0.831] {0.077,0.830} (0.066,0.837)	[0.000,0.745] {0.000,0.745} (0.000,0.751)	1	[0.316,0.469] {0.283,0.471} (0.309,0.476)	[0.364,0.483] {0.364,0.484} (0.355,0.489)	[0.062,0.320] {0.057,0.353} (0.048,0.336)	1	[0.221,0.367] {0.207,0.367} (0.178,0.371)	[0.419,0.657] {0.419,0.676} (0.412,0.713)	[0.122,0.297] {0.117,0.297} (0.109,0.301)	
2	[0.000,0.583] {0.000,0.584} (0.000,0.589)	[0.038,0.939] {0.038,0.938} (0.031,0.945)	[0.009,0.829] {0.008,0.830} (0.004,0.836)	2	[0.012,0.285] {0.012,0.285} (0.008,0.290)	[0.346,0.633] {0.346,0.633} (0.340,0.639)	[0.234,0.516] {0.235,0.517} (0.228,0.524)	2	[0.060,0.210] {0.060,0.209} (0.056,0.214)	[0.421,0.512] {0.421,0.512} (0.413,0.517)	[0.310,0.488] {0.310,0.505} (0.304,0.511)	
3	[0.000,0.572] {0.000,0.573} (0.000,0.578)	[0.000,0.896] {0.000,0.897} (0.000,0.903)	[0.050,1.000] {0.050,1.000} (0.048,1.000)	3	[0.000,0.276] {0.000,0.277} (0.000,0.281)	[0.042,0.581] {0.041,0.579} (0.033,0.589)	[0.340,0.958] {0.340,0.959} (0.332,0.967)	3	[0.000,0.194] {0.000,0.198} (0.000,0.207)	[0.177,0.371] {0.173,0.371} (0.168,0.384)	[0.629,0.633] {0.629,0.634} (0.616,0.639)	
III. Q = 0.30												
	1	2	3		1	2	3		1	2	3	
1	[0.000,0.971] {0.000,0.970} (0.000,0.983)	[0.000,0.989] {0.000,0.988} (0.000,0.994)	[0.000,0.923] {0.000,0.922} (0.000,0.928)	1	[0.221,0.541] {0.199,0.541} (0.213,0.549)	[0.292,0.578] {0.292,0.579} (0.283,0.585)	[0.000,0.487] {0.000,0.487} (0.000,0.492)	1	[0.359,0.395] {0.331,0.395} (0.354,0.399)	[0.387,0.390] {0.387,0.390} (0.380,0.397)	[0.216,0.255] {0.215,0.283} (0.200,0.266)	
2	[0.000,0.807] {0.000,0.808} (0.000,0.814)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	2	[0.000,0.360] {0.000,0.360} (0.000,0.365)	[0.269,0.803] {0.269,0.803} (0.263,0.817)	[0.157,0.598] {0.158,0.600} (0.151,0.605)	2	[0.080,0.249] {0.078,0.248} (0.060,0.252)	[0.394,0.523] {0.394,0.523} (0.385,0.527)	[0.271,0.460] {0.271,0.460} (0.265,0.471)	
3	[0.000,0.793] {0.000,0.794} (0.000,0.801)	[0.000,0.970] {0.000,0.970} (0.000,0.972)	[0.028,1.000] {0.028,1.000} (0.027,1.000)	3	[0.000,0.350] {0.000,0.351} (0.000,0.355)	[0.000,0.752] {0.000,0.752} (0.000,0.764)	[0.184,1.000] {0.184,1.000} (0.175,1.000)	3	[0.000,0.240] {0.000,0.241} (0.000,0.245)	[0.142,0.425] {0.140,0.424} (0.133,0.448)	[0.575,0.643] {0.575,0.643} (0.552,0.648)	

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table 5. Subsample Transition Matrices by Religion: Arbitrary + Shape + Monotonicity + TIV

A. Hindu			B. Muslim			C. Others					
I. Q = 0.10											
	1	2	3		1	2	3		1	2	3
1	[0.232,0.331] {0.232,0.331} (0.225,0.337)	[0.448,0.587] {0.445,0.586} (0.440,0.595)	[0.166,0.251] {0.167,0.251} (0.161,0.256)	1	[0.204,0.306] {0.204,0.309} (0.189,0.320)	[0.511,0.640] {0.509,0.640} (0.494,0.656)	[0.151,0.223] {0.151,0.223} (0.138,0.233)	1	[0.240,0.323] {0.239,0.333} (0.214,0.369)	[0.487,0.561] {0.486,0.568} (0.456,0.596)	[0.189,0.269] {0.181,0.266} (0.165,0.290)
2	[0.093,0.169] {0.093,0.169} (0.089,0.174)	[0.454,0.496] {0.454,0.496} (0.446,0.503)	[0.350,0.441] {0.350,0.441} (0.344,0.449)	2	[0.109,0.181] {0.108,0.180} (0.098,0.190)	[0.503,0.535] {0.503,0.535} (0.489,0.548)	[0.298,0.374] {0.298,0.375} (0.284,0.390)	2	[0.122,0.175] {0.131,0.177} (0.096,0.191)	[0.411,0.439] {0.392,0.438} (0.382,0.457)	[0.400,0.439] {0.400,0.431} (0.383,0.474)
3	[0.000,0.110] {0.000,0.110} (0.000,0.132)	[0.245,0.355] {0.245,0.355} (0.233,0.364)	[0.645,0.653] {0.645,0.653} (0.635,0.659)	3	[0.004,0.152] {0.000,0.152} (0.000,0.162)	[0.294,0.446] {0.290,0.446} (0.271,0.469)	[0.538,0.569] {0.538,0.571} (0.509,0.588)	3	[0.017,0.061] {0.016,0.063} (0.000,0.106)	[0.210,0.254] {0.208,0.255} (0.181,0.286)	[0.729,0.757] {0.729,0.757} (0.713,0.780)
II. Q = 0.20											
	1	2	3		1	2	3		1	2	3
1	[0.243,0.368] {0.227,0.368} (0.195,0.373)	[0.409,0.631] {0.408,0.654} (0.401,0.694)	[0.125,0.299] {0.118,0.300} (0.110,0.305)	1	[0.177,0.338] {0.149,0.340} (0.100,0.351)	[0.466,0.719] {0.466,0.752} (0.451,0.777)	[0.103,0.266] {0.099,0.266} (0.089,0.277)	1	[0.363,0.385] {0.368,0.375} (0.241,0.404)	[0.446,0.705] {0.445,0.708} (0.417,0.743)	[0.214,0.275] {0.238,0.264} (0.160,0.312)
2	[0.058,0.208] {0.058,0.208} (0.054,0.212)	[0.419,0.511] {0.419,0.511} (0.410,0.517)	[0.313,0.496] {0.313,0.509} (0.306,0.515)	2	[0.086,0.218] {0.092,0.217} (0.072,0.228)	[0.483,0.546] {0.483,0.546} (0.464,0.557)	[0.261,0.405] {0.261,0.399} (0.247,0.426)	2	[0.080,0.216] {0.082,0.217} (0.053,0.232)	[0.363,0.460] {0.327,0.459} (0.349,0.476)	[0.362,0.460] {0.362,0.459} (0.347,0.476)
3	[0.000,0.191] {0.000,0.193} (0.000,0.205)	[0.181,0.372] {0.178,0.372} (0.171,0.385)	[0.628,0.633] {0.628,0.633} (0.615,0.640)	3	[0.000,0.218] {0.000,0.217} (0.000,0.228)	[0.199,0.477] {0.190,0.478} (0.177,0.497)	[0.423,0.591] {0.423,0.593} (0.369,0.607)	3	[0.000,0.136] {0.000,0.143} (0.000,0.172)	[0.145,0.282] {0.139,0.252} (0.118,0.295)	[0.718,0.778] {0.718,0.778} (0.705,0.807)
III. Q = 0.30											
	1	2	3		1	2	3		1	2	3
1	[0.354,0.396] {0.326,0.397} (0.349,0.401)	[0.378,0.387] {0.377,0.387} (0.370,0.395)	[0.216,0.268] {0.216,0.292} (0.198,0.281)	1	[0.337,0.364] {0.039,0.365} (0.189,0.376)	[0.430,0.554] {0.430,0.840} (0.416,0.734)	[0.109,0.232] {0.121,0.275} (0.077,0.307)	1	[0.314,0.409] {0.286,0.401} (0.288,0.428)	[0.412,0.420] {0.411,0.420} (0.386,0.436)	[0.179,0.274] {0.188,0.303} (0.136,0.326)
2	[0.081,0.247] {0.084,0.247} (0.059,0.251)	[0.392,0.522] {0.392,0.522} (0.382,0.528)	[0.274,0.459] {0.274,0.458} (0.268,0.472)	2	[0.051,0.255] {0.044,0.255} (0.035,0.265)	[0.448,0.554] {0.448,0.554} (0.426,0.564)	[0.223,0.458] {0.223,0.462} (0.209,0.483)	2	[0.153,0.243] {0.188,0.235} (0.109,0.265)	[0.385,0.476] {0.398,0.475} (0.320,0.491)	[0.326,0.424] {0.326,0.366} (0.307,0.491)
3	[0.000,0.238] {0.000,0.238} (0.000,0.243)	[0.146,0.423] {0.145,0.423} (0.136,0.449)	[0.577,0.643] {0.577,0.643} (0.551,0.649)	3	[0.000,0.255] {0.000,0.254} (0.000,0.265)	[0.165,0.504] {0.156,0.507} (0.142,0.528)	[0.285,0.604] {0.286,0.606} (0.258,0.619)	3	[0.000,0.209] {0.000,0.169} (0.000,0.236)	[0.082,0.291] {0.071,0.204} (0.056,0.303)	[0.709,0.813] {0.709,0.813} (0.697,0.854)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). TIV = Temporal Invariance. Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table 6. Subsample Transition Matrices by Caste: Arbitrary + Shape + Monotonicity + TIV

A. Brahmin/Upper Caste			B. SC/ST			C. OBC					
I. Q = 0.10											
	1	2	3		1	2	3		1	2	3
1	[0.089,0.268] {0.089,0.267} (0.074,0.302)	[0.501,0.671] {0.501,0.671} (0.483,0.688)	[0.232,0.282] {0.231,0.284} (0.205,0.308)	1	[0.328,0.387] {0.328,0.388} (0.318,0.395)	[0.443,0.540] {0.442,0.540} (0.433,0.551)	[0.121,0.188] {0.119,0.187} (0.113,0.195)	1	[0.173,0.296] {0.173,0.297} (0.163,0.303)	[0.472,0.621] {0.467,0.620} (0.462,0.632)	[0.183,0.257] {0.182,0.255} (0.174,0.271)
2	[0.077,0.132] {0.082,0.132} (0.066,0.138)	[0.426,0.458] {0.414,0.458} (0.417,0.466)	[0.422,0.465] {0.422,0.465} (0.415,0.473)	2	[0.157,0.225] {0.160,0.224} (0.142,0.233)	[0.489,0.525] {0.489,0.525} (0.475,0.535)	[0.268,0.346] {0.268,0.345} (0.255,0.360)	2	[0.086,0.159] {0.086,0.158} (0.080,0.164)	[0.464,0.503] {0.464,0.503} (0.454,0.512)	[0.353,0.440] {0.353,0.440} (0.346,0.449)
3	[0.000,0.063] {0.000,0.064} (0.000,0.083)	[0.214,0.277] {0.213,0.277} (0.202,0.285)	[0.723,0.744] {0.723,0.744} (0.715,0.755)	3	[0.000,0.224] {0.000,0.224} (0.000,0.232)	[0.289,0.490] {0.289,0.491} (0.260,0.506)	[0.427,0.551] {0.427,0.554} (0.399,0.566)	3	[0.000,0.121] {0.000,0.123} (0.000,0.141)	[0.274,0.394] {0.271,0.394} (0.257,0.410)	[0.606,0.606] {0.606,0.609} (0.590,0.618)
II. Q = 0.20											
	1	2	3		1	2	3		1	2	3
1	[0.272,0.337] {0.271,0.337} (0.249,0.350)	[0.447,0.513] {0.448,0.514} (0.431,0.554)	[0.216,0.281] {0.215,0.282} (0.197,0.319)	1	[0.401,0.410] {0.396,0.412} (0.350,0.418)	[0.406,0.410] {0.405,0.411} (0.397,0.462)	[0.180,0.193] {0.178,0.193} (0.133,0.214)	1	[0.277,0.338] {0.273,0.339} (0.191,0.344)	[0.428,0.579] {0.425,0.583} (0.418,0.675)	[0.144,0.295] {0.144,0.302} (0.133,0.313)
2	[0.049,0.171] {0.048,0.171} (0.038,0.177)	[0.376,0.477] {0.374,0.477} (0.369,0.484)	[0.381,0.482] {0.380,0.482} (0.373,0.489)	2	[0.117,0.267] {0.114,0.266} (0.099,0.275)	[0.456,0.537] {0.456,0.537} (0.440,0.547)	[0.226,0.397] {0.226,0.400} (0.213,0.412)	2	[0.077,0.195] {0.082,0.195} (0.057,0.201)	[0.440,0.517] {0.440,0.517} (0.426,0.525)	[0.318,0.453] {0.318,0.448} (0.310,0.473)
3	[0.000,0.129] {0.000,0.132} (0.000,0.139)	[0.156,0.285] {0.153,0.285} (0.146,0.293)	[0.715,0.757] {0.715,0.757} (0.707,0.771)	3	[0.000,0.267] {0.000,0.266} (0.000,0.275)	[0.224,0.515] {0.225,0.517} (0.202,0.530)	[0.233,0.578] {0.231,0.582} (0.214,0.591)	3	[0.000,0.192] {0.000,0.192} (0.000,0.198)	[0.205,0.431] {0.202,0.430} (0.191,0.454)	[0.569,0.621] {0.569,0.623} (0.546,0.631)
III. Q = 0.30											
	1	2	3		1	2	3		1	2	3
1	[0.295,0.368] {0.243,0.368} (0.281,0.380)	[0.405,0.929] {0.406,0.932} (0.391,0.944)	[0.215,0.300] {0.216,0.352} (0.201,0.329)	1	[0.371,0.429] {0.374,0.430} (0.364,0.436)	[0.377,0.400] {0.375,0.393} (0.367,0.422)	[0.171,0.250] {0.177,0.247} (0.124,0.263)	1	[0.368,0.369] {0.364,0.372} (0.096,0.376)	[0.393,0.890] {0.403,0.430} (0.383,0.904)	[0.255,0.264] {0.206,0.225} (0.126,0.336)
2	[0.025,0.208] {0.017,0.207} (0.013,0.215)	[0.345,0.491] {0.319,0.491} (0.338,0.498)	[0.345,0.495] {0.345,0.495} (0.337,0.502)	2	[0.079,0.309] {0.076,0.308} (0.060,0.318)	[0.409,0.547] {0.409,0.547} (0.390,0.555)	[0.183,0.454] {0.183,0.457} (0.170,0.469)	2	[0.043,0.232] {0.044,0.232} (0.023,0.238)	[0.399,0.527] {0.399,0.527} (0.384,0.534)	[0.281,0.484] {0.281,0.481} (0.273,0.503)
3	[0.000,0.179] {0.000,0.181} (0.000,0.191)	[0.114,0.293] {0.112,0.293} (0.105,0.300)	[0.707,0.714] {0.707,0.714} (0.700,0.726)	3	[0.000,0.309] {0.000,0.308} (0.000,0.319)	[0.178,0.527] {0.180,0.528} (0.158,0.540)	[0.183,0.593] {0.164,0.595} (0.170,0.603)	3	[0.000,0.229] {0.000,0.229} (0.000,0.235)	[0.163,0.458] {0.159,0.459} (0.150,0.473)	[0.455,0.630] {0.455,0.632} (0.410,0.639)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). TIV = Temporal Invariance. Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table 7. Subsample Transition Matrices by Region: Arbitrary + Shape + Monotonicity + TIV

A. Urban			B. Rural				
I. Q = 0.10							
	1	2	3	1	2	3	
1	[0.234,0.286]	[0.500,0.573]	[0.194,0.266]	1	[0.250,0.341]	[0.452,0.577]	[0.158,0.238]
	{0.229,0.287}	{0.501,0.578}	{0.193,0.270}		{0.250,0.342}	{0.451,0.577}	{0.157,0.238}
	(0.209,0.293)	(0.487,0.606)	(0.185,0.295)		(0.243,0.348)	(0.444,0.585)	(0.152,0.243)
2	[0.068,0.111]	[0.450,0.464]	[0.436,0.468]	2	[0.128,0.190]	[0.471,0.509]	[0.318,0.393]
	{0.068,0.110}	{0.432,0.464}	{0.436,0.468}		{0.136,0.190}	{0.471,0.508}	{0.318,0.389}
	(0.052,0.115)	(0.442,0.471)	(0.429,0.474)		(0.119,0.196)	(0.462,0.516)	(0.311,0.404)
3	[0.000,0.051]	[0.200,0.251]	[0.749,0.781]	3	[0.000,0.161]	[0.261,0.439]	[0.561,0.586]
	{0.000,0.054}	{0.197,0.251}	{0.749,0.780}		{0.000,0.161}	{0.256,0.439}	{0.561,0.586}
	(0.000,0.069)	(0.189,0.258)	(0.742,0.791)		(0.000,0.167)	(0.249,0.452)	(0.548,0.593)
II. Q = 0.20							
	1	2	3	1	2	3	
1	[0.123,0.332]	[0.455,0.752]	[0.125,0.347]	1	[0.324,0.375]	[0.414,0.554]	[0.122,0.262]
	{0.117,0.334}	{0.456,0.761}	{0.122,0.348}		{0.359,0.376}	{0.413,0.529}	{0.112,0.228}
	(0.067,0.340)	(0.443,0.819)	(0.114,0.354)		(0.218,0.381)	(0.406,0.677)	(0.105,0.287)
2	[0.064,0.147]	[0.402,0.482]	[0.393,0.485]	2	[0.088,0.229]	[0.439,0.522]	[0.280,0.442]
	{0.071,0.148}	{0.402,0.482}	{0.393,0.485}		{0.086,0.229}	{0.439,0.522}	{0.280,0.443}
	(0.044,0.153)	(0.388,0.488)	(0.387,0.491)		(0.075,0.234)	(0.429,0.528)	(0.272,0.452)
3	[0.000,0.111]	[0.152,0.263]	[0.737,0.804]	3	[0.000,0.226]	[0.206,0.479]	[0.487,0.603]
	{0.000,0.111}	{0.152,0.263}	{0.737,0.803}		{0.000,0.226}	{0.205,0.479}	{0.487,0.603}
	(0.000,0.127)	(0.142,0.269)	(0.731,0.816)		(0.000,0.232)	(0.194,0.488)	(0.464,0.609)
III. Q = 0.30							
	1	2	3	1	2	3	
1	[0.337,0.355]	[0.418,0.909]	[0.226,0.245]	1	[0.363,0.401]	[0.382,0.386]	[0.213,0.255]
	{0.287,0.362}	{0.419,0.908}	{0.226,0.294}		{0.351,0.402}	{0.382,0.387}	{0.211,0.267}
	(0.327,0.370)	(0.407,0.926)	(0.216,0.266)		(0.358,0.406)	(0.375,0.395)	(0.187,0.268)
2	[0.041,0.184]	[0.362,0.496]	[0.362,0.498]	2	[0.051,0.268]	[0.395,0.532]	[0.241,0.488]
	{0.035,0.184}	{0.348,0.496}	{0.362,0.498}		{0.049,0.267}	{0.395,0.532}	{0.241,0.490}
	(0.009,0.189)	(0.357,0.501)	(0.356,0.503)		(0.041,0.273)	(0.384,0.538)	(0.233,0.500)
3	[0.000,0.166]	[0.108,0.274]	[0.726,0.752]	3	[0.000,0.265]	[0.169,0.505]	[0.291,0.613]
	{0.000,0.165}	{0.109,0.274}	{0.726,0.752}		{0.000,0.265}	{0.168,0.508}	{0.292,0.613}
	(0.000,0.174)	(0.099,0.279)	(0.721,0.764)		(0.000,0.271)	(0.158,0.514)	(0.278,0.618)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). TIV = Temporal Invariance. Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Supplemental Appendix

*Measuring Economic Mobility in India Using Noisy Data:
A Partial Identification Approach*

A Derivation of Bounds

This appendix is organized as follows. Section [A.1](#) begins by presenting the set of misclassification parameters, θ , under Assumptions 2-4. Then, using this information, each element of the transition matrix is considered and the corresponding bounds are derived. Note, the proofs under Assumption 2 alone are provided in Supplemental Appendix C in Millimet et al. (2020). Section [A.2](#) then considers how the bounds may be tightened under any combination of Assumptions 2-4 plus Assumptions 5 and/or 6. The proofs in these cases are also identical to Millimet et al. (2020).

A.1 Misclassification Assumptions

- Maximum Arbitrary Misclassification Rate (Assumption 2)

$$\theta_{kl}^{k'l'} = \Pr(y_o \in k', y_1 \in l', y_o^* \in k, y_1^* \in l)$$

– 72 elements

* General: # elements = $K^2(K^2 - 1)$

– Assumption 2 implies

$$\sum \theta_{kl}^{k'l'} \leq Q$$

- Add Temporal Independence (Assumption 3)

$$\begin{aligned}
\theta_{kl}^{k'l'} &= \alpha_k^{k'} \beta_l^{l'} \\
\alpha_k^{k'} &= \Pr(y_o \in k', y_o^* \in k) \\
\alpha_k^k &= \Pr(y_o \in k, y_o^* \in k) = 1 - \sum_{k' \neq k} \alpha_k^{k'} \\
\beta_l^{l'} &= \Pr(y_1 \in l', y_1^* \in l) \\
\beta_l^l &= \Pr(y_1 \in l, y_1^* \in l) = 1 - \sum_{l' \neq l} \beta_l^{l'}
\end{aligned}$$

– Now only 12 elements

* General: # elements = $2K(K-1)$

– Implies

$$\begin{array}{lllll}
\theta_{11}^{12} = (1 - \alpha_1^2 - \alpha_1^3) \beta_1^2 & \theta_{12}^{11} = (1 - \alpha_1^2 - \alpha_1^3) \beta_2^1 & \theta_{13}^{11} = (1 - \alpha_1^2 - \alpha_1^3) \beta_3^1 & \theta_{21}^{11} = \alpha_2^1 (1 - \beta_1^2 - \beta_1^3) & \theta_{22}^{11} = \alpha_2^1 \beta_2^1 \\
\theta_{11}^{13} = (1 - \alpha_1^2 - \alpha_1^3) \beta_1^3 & \theta_{12}^{13} = (1 - \alpha_1^2 - \alpha_1^3) \beta_2^3 & \theta_{13}^{12} = (1 - \alpha_1^2 - \alpha_1^3) \beta_3^2 & \theta_{21}^{12} = \alpha_2^1 \beta_1^2 & \theta_{22}^{12} = \alpha_2^1 (1 - \beta_2^1 - \beta_3^1) \\
\theta_{11}^{21} = \alpha_1^2 (1 - \beta_1^2 - \beta_1^3) & \theta_{12}^{21} = \alpha_1^2 \beta_2^1 & \theta_{13}^{21} = \alpha_1^2 \beta_3^1 & \theta_{21}^{13} = \alpha_2^1 \beta_1^3 & \theta_{22}^{13} = \alpha_2^1 \beta_2^3 \\
\theta_{11}^{22} = \alpha_1^2 \beta_1^2 & \theta_{12}^{22} = \alpha_1^2 (1 - \beta_2^1 - \beta_2^3) & \theta_{13}^{22} = \alpha_1^2 \beta_3^2 & \theta_{21}^{22} = (1 - \alpha_2^1 - \alpha_2^3) \beta_1^2 & \theta_{22}^{21} = (1 - \alpha_2^1 - \alpha_2^3) \beta_2^1 \\
\theta_{11}^{23} = \alpha_1^2 \beta_1^3 & \theta_{12}^{23} = \alpha_1^2 \beta_2^3 & \theta_{13}^{23} = \alpha_1^2 (1 - \beta_3^1 - \beta_3^3) & \theta_{21}^{23} = (1 - \alpha_2^1 - \alpha_2^3) \beta_1^3 & \theta_{22}^{22} = (1 - \alpha_2^1 - \alpha_2^3) \beta_2^3 \\
\theta_{11}^{31} = \alpha_1^3 (1 - \beta_1^2 - \beta_1^3) & \theta_{12}^{31} = \alpha_1^3 \beta_2^1 & \theta_{13}^{31} = \alpha_1^3 \beta_3^1 & \theta_{21}^{31} = \alpha_2^3 (1 - \beta_1^2 - \beta_1^3) & \theta_{22}^{31} = \alpha_2^3 \beta_2^1 \\
\theta_{11}^{32} = \alpha_1^3 \beta_1^2 & \theta_{12}^{32} = \alpha_1^3 (1 - \beta_2^1 - \beta_2^3) & \theta_{13}^{32} = \alpha_1^3 \beta_3^2 & \theta_{21}^{32} = \alpha_2^3 \beta_1^2 & \theta_{22}^{32} = \alpha_2^3 (1 - \beta_2^1 - \beta_2^3) \\
\theta_{11}^{33} = \alpha_1^3 \beta_1^3 & \theta_{12}^{33} = \alpha_1^3 \beta_2^3 & \theta_{13}^{33} = \alpha_1^3 (1 - \beta_3^1 - \beta_3^3) & \theta_{21}^{33} = \alpha_2^3 \beta_1^3 & \theta_{22}^{33} = \alpha_2^3 \beta_2^3 \\
\theta_{23}^{11} = \alpha_2^1 \beta_3^1 & \theta_{31}^{11} = \alpha_3^1 (1 - \beta_1^2 - \beta_1^3) & \theta_{32}^{11} = \alpha_3^1 \beta_2^1 & \theta_{33}^{11} = \alpha_3^1 \beta_3^1 & \\
\theta_{23}^{12} = \alpha_2^1 \beta_3^2 & \theta_{31}^{12} = \alpha_3^1 \beta_1^2 & \theta_{32}^{12} = \alpha_3^1 (1 - \beta_2^1 - \beta_2^3) & \theta_{33}^{12} = \alpha_3^1 \beta_3^2 & \\
\theta_{23}^{13} = \alpha_2^1 (1 - \beta_3^1 - \beta_3^3) & \theta_{31}^{13} = \alpha_3^1 \beta_1^3 & \theta_{32}^{13} = \alpha_3^1 \beta_2^3 & \theta_{33}^{13} = \alpha_3^1 (1 - \beta_3^1 - \beta_3^3) & \\
\theta_{23}^{21} = (1 - \alpha_2^1 - \alpha_2^3) \beta_3^1 & \theta_{31}^{21} = \alpha_3^2 (1 - \beta_1^2 - \beta_1^3) & \theta_{32}^{21} = \alpha_3^2 \beta_2^1 & \theta_{33}^{21} = \alpha_3^2 \beta_3^1 & \\
\theta_{23}^{22} = (1 - \alpha_2^1 - \alpha_2^3) \beta_3^2 & \theta_{31}^{22} = \alpha_3^2 \beta_1^2 & \theta_{32}^{22} = \alpha_3^2 (1 - \beta_2^1 - \beta_2^3) & \theta_{33}^{22} = \alpha_3^2 \beta_3^2 & \\
\theta_{23}^{31} = \alpha_2^3 \beta_3^1 & \theta_{31}^{23} = \alpha_3^2 \beta_1^3 & \theta_{32}^{23} = \alpha_3^2 \beta_2^3 & \theta_{33}^{23} = \alpha_3^2 (1 - \beta_3^1 - \beta_3^3) & \\
\theta_{23}^{32} = \alpha_2^3 \beta_3^2 & \theta_{31}^{32} = (1 - \alpha_3^1 - \alpha_3^3) \beta_2^1 & \theta_{32}^{31} = (1 - \alpha_3^1 - \alpha_3^3) \beta_2^2 & \theta_{33}^{31} = (1 - \alpha_3^1 - \alpha_3^3) \beta_3^1 & \\
\theta_{23}^{33} = \alpha_2^3 (1 - \beta_3^1 - \beta_3^3) & \theta_{31}^{33} = (1 - \alpha_3^1 - \alpha_3^3) \beta_2^3 & \theta_{32}^{32} = (1 - \alpha_3^1 - \alpha_3^3) \beta_2^3 & \theta_{33}^{32} = (1 - \alpha_3^1 - \alpha_3^3) \beta_3^3 &
\end{array}$$

* Under Assumption 2

$$\begin{aligned}
\sum \theta_{kl}^{k'l'} &= (1 - \alpha_1^2 - \alpha_1^3) (\beta_1^2 + \beta_1^3) + (\alpha_1^2 + \alpha_1^3) \\
&\quad + (1 - \alpha_1^2 - \alpha_1^3) (\beta_2^1 + \beta_2^3) + (\alpha_1^2 + \alpha_1^3) \\
&\quad + (1 - \alpha_1^2 - \alpha_1^3) (\beta_3^1 + \beta_3^3) + (\alpha_1^2 + \alpha_1^3) \\
&\quad + (1 - \alpha_2^1 - \alpha_2^3) (\beta_1^2 + \beta_1^3) + (\alpha_2^1 + \alpha_2^3) \\
&\quad + (1 - \alpha_2^1 - \alpha_2^3) (\beta_2^1 + \beta_2^3) + (\alpha_2^1 + \alpha_2^3) \\
&\quad + (1 - \alpha_2^1 - \alpha_2^3) (\beta_3^1 + \beta_3^3) + (\alpha_2^1 + \alpha_2^3) \\
&\quad + (1 - \alpha_3^1 - \alpha_3^3) (\beta_1^2 + \beta_1^3) + (\alpha_3^1 + \alpha_3^3) \\
&\quad + (1 - \alpha_3^1 - \alpha_3^3) (\beta_2^1 + \beta_2^3) + (\alpha_3^1 + \alpha_3^3) \\
&\quad + (1 - \alpha_3^1 - \alpha_3^3) (\beta_3^1 + \beta_3^3) + (\alpha_3^1 + \alpha_3^3) \\
&= (3 - \alpha_1^2 - \alpha_1^3 - \alpha_2^1 - \alpha_2^3 - \alpha_3^1 - \alpha_3^3) (\beta_1^2 + \beta_1^3 + \beta_2^1 + \beta_2^3 + \beta_3^1 + \beta_3^3) \\
&\quad + 3 (\alpha_1^2 + \alpha_1^3 + \alpha_2^1 + \alpha_2^3 + \alpha_3^1 + \alpha_3^3) \leq Q \\
&\Rightarrow \alpha_1^2, \alpha_1^3, \alpha_2^1, \alpha_2^3, \alpha_3^1, \alpha_3^3, \beta_1^2, \beta_1^3, \beta_2^1, \beta_2^3, \beta_3^1, \beta_3^3 \leq Q/3 \\
&\Rightarrow \alpha, \beta \leq Q/K \quad (\text{for generic } K)
\end{aligned}$$

- Add Temporal Invariance (Assumption 4)

$$\begin{aligned}
\theta_{kl}^{k'l'} &= \theta_k^{k'} \theta_l^{l'} \\
\theta_k^{k'} &= \Pr(y_o \in k', y_o^* \in k) = \Pr(y_1 \in k', y_1^* \in k) \\
\theta_k^k &= \Pr(y_o \in k, y_o^* \in k) = \Pr(y_1 \in k, y_1^* \in k) = 1 - \theta_k^{k'}
\end{aligned}$$

– Now only 6 elements

* General: # elements = $K(K - 1)$

– Implies

$$\begin{aligned}
\theta_{11}^{12} &= \theta_1^2 (1 - \theta_1^2) & \theta_{12}^{11} &= \theta_2^1 (1 - \theta_1^2) & \theta_{21}^{11} &= \theta_2^1 (1 - \theta_1^2) & \theta_{22}^{11} &= (\theta_2^1)^2 \\
\theta_{11}^{21} &= \theta_1^2 (1 - \theta_1^2) & \theta_{12}^{21} &= \theta_1^2 \theta_2^1 & \theta_{21}^{12} &= \theta_2^1 \theta_1^2 & \theta_{22}^{12} &= \theta_2^1 (1 - \theta_2^1) \\
\theta_{11}^{22} &= (\theta_1^2)^2 & \theta_{12}^{22} &= \theta_1^2 (1 - \theta_2^1) & \theta_{21}^{22} &= \theta_1^2 (1 - \theta_2^1) & \theta_{22}^{22} &= \theta_2^1 (1 - \theta_2^1)
\end{aligned}$$

* Under Assumption 2 (solution: set all θ s but one to zero, solve using quadratic formula)

$$\begin{aligned}
\sum \theta_{kl}^{k'l'} &= (6 - \theta_1^2 - \theta_1^3 - \theta_2^1 - \theta_2^3 - \theta_3^1 - \theta_3^2) (\theta_1^2 + \theta_1^3 + \theta_2^1 + \theta_2^3 + \theta_3^1 + \theta_3^2) \leq Q \\
\Rightarrow &\theta_1^2, \theta_1^3, \theta_2^1, \theta_2^3, \theta_3^1, \theta_3^2 \leq 3 - \sqrt{9 - Q} \\
\Rightarrow &\theta \leq K - \sqrt{K^2 - Q} \quad (\text{for generic } K)
\end{aligned}$$

A.1.1 p_{11}^*

$$p_{11}^* = \frac{r_{11} + \overbrace{\left[\theta_{11}^{12} + \theta_{11}^{13} + \theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} \right]}^{Q_{1,11}} - \overbrace{\left[\theta_{12}^{11} + \theta_{13}^{11} + \theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11} \right]}^{Q_{2,11}}}{p_1 + \underbrace{\left[\theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} + \theta_{12}^{21} + \theta_{12}^{22} + \theta_{12}^{23} + \theta_{12}^{31} + \theta_{12}^{32} + \theta_{12}^{33} + \theta_{13}^{21} + \theta_{13}^{22} + \theta_{13}^{23} + \theta_{13}^{31} + \theta_{13}^{32} + \theta_{13}^{33} \right]}_{Q_{3,1}} - \underbrace{\left[\theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \theta_{21}^{12} + \theta_{22}^{12} + \theta_{23}^{12} + \theta_{21}^{13} + \theta_{22}^{13} + \theta_{23}^{13} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11} + \theta_{31}^{12} + \theta_{32}^{12} + \theta_{33}^{12} + \theta_{31}^{13} + \theta_{32}^{13} + \theta_{33}^{13} \right]}_{Q_{4,1}}}$$

- $\theta_{kl}^{k'l'}$ = unique element

Maximum Arbitrary Errors: Assumption 2

$$LB_{11} = \frac{r_{11} - Q}{p_1} \geq 0$$

$$UB_{11} = \frac{r_{11} + Q}{p_1} \leq 1$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{11}^* = \frac{r_{11} + \overbrace{\left[\alpha_1^1 \beta_1^2 + \alpha_1^1 \beta_1^3 + \alpha_1^2 \beta_1^1 + \alpha_1^2 \beta_1^2 + \alpha_1^2 \beta_1^3 + \alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_1^3 \right]}^{Q_{1,11}} - \overbrace{\left[\alpha_1^1 \beta_2^1 + \alpha_1^1 \beta_3^1 + \alpha_2^1 \beta_1^1 + \alpha_2^1 \beta_2^1 + \alpha_2^1 \beta_3^1 + \alpha_3^1 \beta_1^1 + \alpha_3^1 \beta_2^1 + \alpha_3^1 \beta_3^1 \right]}^{Q_{2,11}}}{p_1 + \underbrace{\left[\alpha_1^2 \beta_1^1 + \alpha_1^2 \beta_1^2 + \alpha_1^2 \beta_1^3 + \alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_1^3 + \alpha_1^2 \beta_2^1 + \alpha_1^2 \beta_2^2 + \alpha_1^2 \beta_2^3 + \alpha_1^3 \beta_2^1 + \alpha_1^3 \beta_2^2 + \alpha_1^3 \beta_2^3 + \alpha_1^2 \beta_3^1 + \alpha_1^2 \beta_3^2 + \alpha_1^2 \beta_3^3 + \alpha_1^3 \beta_3^1 + \alpha_1^3 \beta_3^2 + \alpha_1^3 \beta_3^3 \right]}_{Q_{3,1}} - \underbrace{\left[\alpha_2^1 \beta_1^1 + \alpha_2^1 \beta_2^1 + \alpha_2^1 \beta_3^1 + \alpha_2^2 \beta_1^1 + \alpha_2^2 \beta_2^1 + \alpha_2^2 \beta_3^1 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_2^2 + \alpha_2^2 \beta_3^2 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_2^1 + \alpha_2^3 \beta_3^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_2^2 + \alpha_2^3 \beta_3^2 + \alpha_2^3 \beta_1^3 + \alpha_2^3 \beta_2^3 + \alpha_2^3 \beta_3^3 \right]}_{Q_{4,1}}}$$

- Simplifying

$$Q_{1,11} = (\alpha_1^2 + \alpha_1^3) + (\beta_1^2 + \beta_1^3) (1 - \alpha_1^2 - \alpha_1^3) \quad (\text{TI})$$

$$= 2 (\theta_1^2 + \theta_1^3) - (\theta_1^2 + \theta_1^3)^2 \quad (\text{TIV})$$

$$Q_{2,11} = (\alpha_2^1 + \alpha_3^1) (1 + \beta_2^1 + \beta_3^1 - \beta_1^2 - \beta_1^3) + (\beta_2^1 + \beta_3^1) (1 - \alpha_1^2 - \alpha_1^3) \quad (\text{TI})$$

$$= 2 (\theta_2^1 + \theta_3^1) (1 - \theta_1^2 - \theta_1^3) + (\theta_2^1 + \theta_3^1)^2 \quad (\text{TIV})$$

$$Q_{3,1} = 3 (\alpha_1^2 + \alpha_1^3) \quad (\text{TI})$$

$$= 3 (\theta_1^2 + \theta_1^3) \quad (\text{TIV})$$

$$Q_{4,1} = 3 (\alpha_2^1 + \alpha_3^1) \quad (\text{TI})$$

$$= 3 (\theta_2^1 + \theta_3^1) \quad (\text{TIV})$$

- Under Temporal Independence (Assumption 3)

$$p_{11}^* = \frac{r_{11} + (\alpha_1^2 + \alpha_1^3 - \alpha_2^1 - \alpha_3^1) + (\beta_1^2 + \beta_1^3 - \beta_2^1 - \beta_3^1) (1 + \alpha_2^1 + \alpha_3^1 - \alpha_1^2 - \alpha_1^3)}{p_1 + 3(\alpha_1^2 + \alpha_1^3 - \alpha_2^1 - \alpha_3^1)}$$

– Yields

$$\begin{aligned} LB_{11}^{TI} &= \min \left\{ \frac{r_{11} - Q/3}{p_1}, \frac{r_{11} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}}, \frac{r_{11} - \hat{Q}}{p_1 - 3\hat{Q}} \right\} \geq 0 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{11}, (1 - p_1)/3, Q/3\}, \quad \hat{Q} = \min \{r_{11}, p_1/3, Q/3\} \\ UB_{11}^{TI} &= \max \left\{ \frac{r_{11} + Q/3}{p_1}, \frac{r_{11} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}}, \frac{r_{11} - \hat{Q}}{p_1 - 3\hat{Q}} \right\} \leq 1 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{11}, (1 - p_1)/3, Q/3\}, \quad \hat{Q} = \min \{r_{11}, p_1/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{11} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}} \right) / \partial \tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{11} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) &= \operatorname{sgn} \left(\left(p_1 + 3\tilde{\tilde{Q}} \right) - 3 \left(r_{11} + \tilde{\tilde{Q}} \right) \right) \\ &= \operatorname{sgn} (p_1 - 3r_{11}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{11} - \hat{Q}}{p_1 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{11} - \hat{Q}}{p_1 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_1 - 3\hat{Q} \right) + 3 \left(r_{11} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{11} - p_1) \end{aligned}$$

- Under Temporal Invariance (Assumption 4)

$$p_{11}^* = \frac{r_{11} + 2(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1) - (\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1)^2}{p_1 + 3(\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1)}$$

– Yields

$$\begin{aligned} LB_{11}^{TIV} &= \min \left\{ \frac{r_{11} + 2\hat{Q} - \hat{Q}^2}{p_1 + 3\hat{Q}}, \frac{r_{11} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_1 - 3\tilde{\tilde{Q}}} \right\} \geq 0 \\ \hat{Q} &= \min \left\{ (1 - p_1)/3, 3 - \sqrt{9 - Q} \right\}, \\ \tilde{\tilde{Q}} &= \begin{cases} 0 & r_{11} \geq 2p_1/3 \\ \min \left\{ \frac{(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (-1 + \sqrt{1 + r_{11}}), p_1/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases} \\ UB_{11}^{TIV} &= \max \left\{ \frac{r_{11} + 2\hat{Q} - \hat{Q}^2}{p_1 + 3\hat{Q}}, \frac{r_{11} - 2\tilde{\tilde{Q}} - \tilde{\tilde{Q}}^2}{p_1 - 3\tilde{\tilde{Q}}} \right\} \geq 0 \\ \hat{Q} &= \begin{cases} 0 & r_{11} \geq 2p_1/3 \\ \min \left\{ \frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (1 - p_1)/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases}, \\ \tilde{\tilde{Q}} &= \begin{cases} 0 & r_{11} < 2p_1/3 \\ \min \left\{ \frac{(2/3)p_1 - \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}, (-1 + \sqrt{1 + r_{11}}), p_1/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{11}-2\tilde{Q}-\tilde{Q}^2}{p_1-3\tilde{Q}} \right) / \partial \tilde{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
& \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{11}-2\tilde{Q}-\tilde{Q}^2}{p_1-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) = \operatorname{sgn} \left(\left(-2 - 2\tilde{Q} \right) \left(p_1 - 3\tilde{Q} \right) + 3 \left(r_{11} - 2\tilde{Q} - \tilde{Q}^2 \right) \right) \\
& = \operatorname{sgn} \left(-(2/3)p_1 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{11} \right) \\
& \Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{11}-2\tilde{Q}-\tilde{Q}^2}{p_1-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) \Bigg|_{\tilde{Q}=0} = \operatorname{sgn} \left(-(2/3)p_1 + r_{11} \right) \geq 0 \\
& \Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{11}-2\tilde{Q}-\tilde{Q}^2}{p_1-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) \Bigg|_{\tilde{Q}=1} = \operatorname{sgn} \left(-(4/3)p_1 + 1 + r_{11} \right) \geq 0
\end{aligned}$$

2. Ensure $r_{11} - 2\tilde{Q} - \tilde{Q}^2 \geq 0$

$$\begin{aligned}
& r_{11} - 2\tilde{Q} - \tilde{Q}^2 \geq 0 \\
& \Rightarrow \tilde{Q}^2 + 2\tilde{Q} - r_{11} \leq 0 \\
& \Rightarrow \tilde{Q} \leq \frac{-2 + \sqrt{4 + 4r_{11}}}{2} \\
& \Rightarrow \tilde{Q} \leq -1 + \sqrt{1 + r_{11}}
\end{aligned}$$

3. Minimize $\frac{r_{11}-2\tilde{Q}-\tilde{Q}^2}{p_1-3\tilde{Q}}$ s.t. \tilde{Q} being feasible and $r_{11} < 2p_1/3$

$$\begin{aligned}
& \frac{\partial \left(\frac{r_{11}-2\tilde{Q}-\tilde{Q}^2}{p_1-3\tilde{Q}} \right)}{\partial \tilde{Q}} \propto -(2/3)p_1 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{11} = 0 \\
& \Rightarrow \tilde{Q}^* = \frac{(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}
\end{aligned}$$

4. Maximize $\frac{r_{11}-2\tilde{Q}-\tilde{Q}^2}{p_1-3\tilde{Q}}$ s.t. \tilde{Q} being feasible and $r_{11} > 2p_1/3$

$$\begin{aligned}
& \frac{\partial \left(\frac{r_{11}-2\tilde{Q}-\tilde{Q}^2}{p_1-3\tilde{Q}} \right)}{\partial \tilde{Q}} \propto -(2/3)p_1 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{11} = 0 \\
& \Rightarrow \tilde{Q}^* = \frac{(2/3)p_1 - \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}
\end{aligned}$$

Note: If $\sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]} = .$, then maximize \tilde{Q} .

5. Evaluate $\partial \left(\frac{r_{11}+2\hat{Q}-\hat{Q}^2}{p_1+3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
\operatorname{sgn} \left(\frac{\partial \left(\frac{r_{11}+2\hat{Q}-\hat{Q}^2}{p_1+3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left((2-2\hat{Q})(p_1+3\hat{Q}) - 3(r_{11}+2\hat{Q}-\hat{Q}^2) \right) \\
&= \operatorname{sgn} \left((2/3)p_1(1-\hat{Q}) - \hat{Q}^2 - r_{11} \right) \\
\Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{11}+2\hat{Q}-\hat{Q}^2}{p_1+3\hat{Q}} \right)}{\partial \hat{Q}} \right) \Big|_{\hat{Q}=0} &= \operatorname{sgn}((2/3)p_1 - r_{11}) \geq 0 \\
\Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{11}+2\hat{Q}-\hat{Q}^2}{p_1+3\hat{Q}} \right)}{\partial \hat{Q}} \right) \Big|_{\hat{Q}=1} &= \operatorname{sgn}(-1 - r_{11}) < 0
\end{aligned}$$

6. Maximize $\frac{r_{11}+2\hat{Q}-\hat{Q}^2}{p_1+3\hat{Q}}$ s.t. \hat{Q} being feasible and $r_{11} < 2p_1/3$

$$\begin{aligned}
\frac{\partial \left(\frac{r_{11}+2\hat{Q}-\hat{Q}^2}{p_1+3\hat{Q}} \right)}{\partial \hat{Q}} &\propto (2/3)p_1(1-\hat{Q}) - \hat{Q}^2 - r_{11} = 0 \\
\Rightarrow \hat{Q}^* &= \frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4[(2/3)p_1 - r_{11}]}}{2}
\end{aligned}$$

7. Minimize $\frac{r_{11}+2\hat{Q}-\hat{Q}^2}{p_1+3\hat{Q}} \Rightarrow \hat{Q} = 0$ or maximize \hat{Q} . However, if the minimum occurs when $\hat{Q} = 0$, then $\frac{r_{11}-2\hat{Q}-\hat{Q}^2}{p_1-3\hat{Q}} < \frac{r_{11}}{p_1}$ and this will be the binding *LB*.

A.1.2 p_{12}^*

$$p_{12}^* = \frac{r_{12} + \overbrace{\left[\theta_{\bullet 12}^{11} + \theta_{\bullet 12}^{13} + \theta_{12}^{21} + \theta_{12}^{22} + \theta_{12}^{23} + \theta_{12}^{31} + \theta_{12}^{32} + \theta_{12}^{33} \right]}^{Q_{1,12}} - \overbrace{\left[\theta_{\bullet 11}^{12} + \theta_{\bullet 13}^{12} + \theta_{21}^{12} + \theta_{22}^{12} + \theta_{23}^{12} + \theta_{31}^{12} + \theta_{32}^{12} + \theta_{33}^{12} \right]}^{Q_{2,12}}}{p_1 + \underbrace{\left[\theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} + \theta_{\bullet 12}^{21} + \theta_{\bullet 12}^{22} + \theta_{\bullet 12}^{23} + \theta_{\bullet 12}^{31} + \theta_{\bullet 12}^{32} + \theta_{\bullet 12}^{33} + \theta_{13}^{21} + \theta_{13}^{22} + \theta_{13}^{23} + \theta_{13}^{31} + \theta_{13}^{32} + \theta_{13}^{33} \right]}_{Q_{3,1}}} - \underbrace{\left[\theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \theta_{\bullet 21}^{12} + \theta_{\bullet 22}^{12} + \theta_{\bullet 23}^{12} + \theta_{\bullet 21}^{13} + \theta_{\bullet 22}^{13} + \theta_{\bullet 23}^{13} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11} + \theta_{\bullet 31}^{12} + \theta_{\bullet 32}^{12} + \theta_{\bullet 33}^{12} + \theta_{\bullet 31}^{13} + \theta_{\bullet 32}^{13} + \theta_{\bullet 33}^{13} \right]}_{Q_{4,1}}}$$

- $\theta_{kl}^{k'l'}$ = unique element

Maximum Arbitrary Errors: Assumption 2

$$\begin{aligned} LB_{12} &= \frac{r_{12} - Q}{p_1} \geq 0 \\ UB_{12} &= \frac{r_{12} + Q}{p_1} \leq 1 \end{aligned}$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{12}^* = \frac{r_{12} + \overbrace{\left[\alpha_1^1 \beta_2^1 + \alpha_1^1 \beta_2^3 + \alpha_1^2 \beta_2^1 + \alpha_1^2 \beta_2^2 + \alpha_1^2 \beta_2^3 + \alpha_1^3 \beta_2^1 + \alpha_1^3 \beta_2^2 + \alpha_1^3 \beta_2^3 \right]}^{Q_{1,12}} - \overbrace{\left[\alpha_1^1 \beta_1^2 + \alpha_1^1 \beta_1^3 + \alpha_2^1 \beta_1^2 + \alpha_2^1 \beta_1^3 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_1^3 + \alpha_3^1 \beta_1^2 + \alpha_3^1 \beta_1^3 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_1^3 + \alpha_3^3 \beta_1^2 + \alpha_3^3 \beta_1^3 \right]}^{Q_{2,12}}}{p_1 + \underbrace{\left[\alpha_1^2 \beta_1^1 + \alpha_1^2 \beta_1^2 + \alpha_1^2 \beta_1^3 + \alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_1^3 + \alpha_1^2 \beta_2^1 + \alpha_1^2 \beta_2^2 + \alpha_1^2 \beta_2^3 + \alpha_1^3 \beta_2^1 + \alpha_1^3 \beta_2^2 + \alpha_1^3 \beta_2^3 + \alpha_2^2 \beta_3^1 + \alpha_2^2 \beta_3^2 + \alpha_2^2 \beta_3^3 + \alpha_2^3 \beta_3^1 + \alpha_2^3 \beta_3^2 + \alpha_2^3 \beta_3^3 + \alpha_3^2 \beta_3^1 + \alpha_3^2 \beta_3^2 + \alpha_3^2 \beta_3^3 + \alpha_3^3 \beta_3^1 + \alpha_3^3 \beta_3^2 + \alpha_3^3 \beta_3^3 \right]}_{Q_{3,1}}} - \underbrace{\left[\alpha_2^1 \beta_1^1 + \alpha_2^1 \beta_1^2 + \alpha_2^1 \beta_1^3 + \alpha_2^2 \beta_1^1 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_1^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_1^3 + \alpha_3^2 \beta_1^1 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_1^3 + \alpha_3^3 \beta_1^1 + \alpha_3^3 \beta_1^2 + \alpha_3^3 \beta_1^3 \right]}_{Q_{4,1}}}$$

- Simplifying

$$\begin{aligned} Q_{1,12} &= (\alpha_1^2 + \alpha_1^3) + (\beta_2^1 + \beta_2^3) (1 - \alpha_1^2 - \alpha_1^3) & (TI) \\ &= (\theta_1^2 + \theta_1^3) + (\theta_2^1 + \theta_2^3) (1 - \theta_1^2 - \theta_1^3) & (TIV) \\ Q_{2,12} &= (\alpha_2^1 + \alpha_3^1) (1 + \beta_1^2 + \beta_3^2 - \beta_2^1 - \beta_2^3) + (\beta_1^2 + \beta_3^2) (1 - \alpha_1^2 - \alpha_1^3) & (TI) \\ &= (\theta_2^1 + \theta_3^1) (1 + \theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^3) + (\theta_1^2 + \theta_3^2) (1 - \theta_1^2 - \theta_1^3) & (TIV) \\ Q_{3,1} &= 3 (\alpha_1^2 + \alpha_1^3) & (TI) \\ &= 3 (\theta_1^2 + \theta_1^3) & (TIV) \\ Q_{4,1} &= 3 (\alpha_2^1 + \alpha_3^1) & (TI) \\ &= 3 (\theta_2^1 + \theta_3^1) & (TIV) \end{aligned}$$

- Under Temporal Independence (Assumption 3)

$$p_{12}^* = \frac{r_{12} + (\beta_2^1 + \beta_2^3 - \beta_1^2 - \beta_3^2) + (\alpha_1^2 + \alpha_1^3 - \alpha_2^1 - \alpha_3^1) (1 + \beta_1^2 + \beta_3^2 - \beta_2^1 - \beta_2^3)}{p_1 + 3(\alpha_1^2 + \alpha_1^3 - \alpha_2^1 - \alpha_3^1)}$$

– Yields

$$\begin{aligned} LB_{12}^{TI} &= \min \left\{ \frac{r_{12} - Q/3}{p_1}, \frac{r_{12} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}}, \frac{r_{12} - \hat{Q}}{p_1 - 3\hat{Q}} \right\} \geq 0 \\ &\quad \tilde{\tilde{Q}} < \min \{1 - r_{12}, (1 - p_1)/3, Q/3\}, \quad \hat{Q} < \min \{r_{12}, p_1/3, Q/3\} \\ UB_{12}^{TI} &= \max \left\{ \frac{r_{12} + Q/3}{p_1}, \frac{r_{12} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}}, \frac{r_{12} - \hat{Q}}{p_1 - 3\hat{Q}} \right\} \leq 1 \\ &\quad \tilde{\tilde{Q}} < \min \{1 - r_{12}, (1 - p_1)/3, Q/3\}, \quad \hat{Q} < \min \{r_{12}, p_1/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{12} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}} \right) / \partial \tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{12} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) &= \operatorname{sgn} \left(\left(p_1 + 3\tilde{\tilde{Q}} \right) - 3 \left(r_{12} + \tilde{\tilde{Q}} \right) \right) \\ &= \operatorname{sgn} (p_1 - 3r_{12}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{12} - \hat{Q}}{p_1 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{12} - \hat{Q}}{p_1 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_1 - 3\hat{Q} \right) + 3 \left(r_{12} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{12} - p_1) \end{aligned}$$

Since both derivatives can take either sign, it is possible either could be the LB , UB ..

- Under Temporal Invariance (Assumption 4)

$$p_{12}^* = \frac{r_{12} + (\theta_1^3 + \theta_2^3 - \theta_3^1 - \theta_3^2) + (\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1) (\theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^3)}{p_1 + 3 (\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1)}$$

– Yields

$$\begin{aligned} LB_{12}^{TIV} &= \min \left\{ \frac{r_{12} - (3 - \sqrt{9 - Q})}{p_1}, \frac{r_{12} + \tilde{Q}^2}{p_1 + 3\tilde{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \left\{ \frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4r_{12}}}{2}, \sqrt{1 - r_{12}}, (1 - p_1)/3, 3 - \sqrt{9 - Q} \right\} \\ UB_{12}^{TIV} &= \max \left\{ \frac{r_{12} + (3 - \sqrt{9 - Q})}{p_1}, \frac{r_{12} + \tilde{Q}^2}{p_1 - 3\tilde{Q}} \right\} \leq 1 \quad \tilde{Q} = \min \left\{ \sqrt{1 - r_{12}}, p_1/3, 3 - \sqrt{9 - Q} \right\} \end{aligned}$$

– Proof:

1. Evaluate $\partial LB_{12}^{TIV} / \partial \tilde{Q}$ and see when the sign is positive/negative.

$$\begin{aligned} \text{sgn} \left(\frac{\partial LB_{12}^{TIV}}{\partial \tilde{Q}} \right) &= \text{sgn} \left(2\tilde{Q} \left(p_1 + 3\tilde{Q} \right) - 3 \left(r_{12} + \tilde{Q}^2 \right) \right) \\ &= \text{sgn} \left(\tilde{Q} \left((2/3)p_1 + \tilde{Q} \right) - r_{12} \right) \\ \Rightarrow \text{sgn} \left(\frac{\partial LB_{12}^{TIV}}{\partial \tilde{Q}} \right) \Big|_{\tilde{Q}=0} &= \text{sgn}(-r_{12}) < 0 \\ &\Rightarrow \tilde{Q} > 0 \end{aligned}$$

2. Minimize LB_{12}^{TIV} s.t. \tilde{Q} being feasible

$$\begin{aligned} \frac{\partial LB_{12}^{TIV}}{\partial \tilde{Q}} &\propto \tilde{Q} \left((2/3)p_1 + \tilde{Q} \right) - r_{12} = 0 \\ \Rightarrow \tilde{Q}^* &= \frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4r_{12}}}{2} \end{aligned}$$

So, derivative starts off negative and then reaches zero at \tilde{Q}^* . Thus, $\frac{r_{12} + \tilde{Q}^2}{p_1 + 3\tilde{Q}}$ is minimized at \tilde{Q}^* .

A.1.3 p_{13}^*

$$p_{13}^* = \frac{r_{13} + \overbrace{\left[\theta_{13}^{11} + \theta_{13}^{12} + \theta_{13}^{21} + \theta_{13}^{22} + \theta_{13}^{23} + \theta_{13}^{31} + \theta_{13}^{32} + \theta_{13}^{33} \right]}^{Q_{1,13}} - \overbrace{\left[\theta_{11}^{13} + \theta_{12}^{13} + \theta_{21}^{13} + \theta_{22}^{13} + \theta_{23}^{13} + \theta_{31}^{13} + \theta_{32}^{13} + \theta_{33}^{13} \right]}^{Q_{2,13}}}{p_1 + \underbrace{\left[\theta_{11}^{21} + \theta_{11}^{22} + \theta_{11}^{23} + \theta_{11}^{31} + \theta_{11}^{32} + \theta_{11}^{33} + \theta_{12}^{21} + \theta_{12}^{22} + \theta_{12}^{23} + \theta_{12}^{31} + \theta_{12}^{32} + \theta_{12}^{33} + \theta_{13}^{21} + \theta_{13}^{22} + \theta_{13}^{23} + \theta_{13}^{31} + \theta_{13}^{32} + \theta_{13}^{33} \right]}_{Q_{3,1}}}$$

$$- \underbrace{\left[\theta_{21}^{11} + \theta_{22}^{11} + \theta_{23}^{11} + \theta_{21}^{12} + \theta_{22}^{12} + \theta_{23}^{12} + \theta_{21}^{13} + \theta_{22}^{13} + \theta_{23}^{13} + \theta_{31}^{11} + \theta_{32}^{11} + \theta_{33}^{11} + \theta_{31}^{12} + \theta_{32}^{12} + \theta_{33}^{12} + \theta_{31}^{13} + \theta_{32}^{13} + \theta_{33}^{13} \right]}_{Q_{4,1}}$$

- $\theta_{kl}^{k'l'}$ = unique element

Maximum Arbitrary Errors: Assumption 2

$$LB_{13} = \frac{r_{13} - Q}{p_1} \geq 0$$

$$UB_{13} = \frac{r_{13} + Q}{p_1} \leq 1$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{13}^* = \frac{r_{13} + \overbrace{\left[\alpha_1^1 \beta_3^1 + \alpha_1^1 \beta_3^2 + \alpha_1^2 \beta_3^1 + \alpha_1^2 \beta_3^2 + \alpha_1^2 \beta_3^3 + \alpha_1^3 \beta_3^1 + \alpha_1^3 \beta_3^2 + \alpha_1^3 \beta_3^3 \right]}^{Q_{1,13}} - \overbrace{\left[\alpha_1^1 \beta_1^3 + \alpha_1^1 \beta_2^3 + \alpha_2^1 \beta_1^3 + \alpha_2^1 \beta_2^3 + \alpha_2^2 \beta_3^3 + \alpha_3^1 \beta_1^3 + \alpha_3^1 \beta_2^3 + \alpha_3^1 \beta_3^3 \right]}^{Q_{2,13}}}{p_1 + \underbrace{\left[\alpha_1^2 \beta_1^1 + \alpha_1^2 \beta_1^2 + \alpha_1^2 \beta_1^3 + \alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_1^3 + \alpha_1^2 \beta_2^1 + \alpha_1^2 \beta_2^2 + \alpha_1^2 \beta_2^3 + \alpha_1^3 \beta_2^1 + \alpha_1^3 \beta_2^2 + \alpha_1^3 \beta_2^3 + \alpha_1^2 \beta_3^1 + \alpha_1^2 \beta_3^2 + \alpha_1^2 \beta_3^3 + \alpha_1^3 \beta_3^1 + \alpha_1^3 \beta_3^2 + \alpha_1^3 \beta_3^3 \right]}_{Q_{3,1}}}$$

$$- \underbrace{\left[\alpha_2^1 \beta_1^1 + \alpha_2^1 \beta_2^1 + \alpha_2^1 \beta_3^1 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_2^2 + \alpha_2^2 \beta_3^2 + \alpha_2^3 \beta_1^3 + \alpha_2^3 \beta_2^3 + \alpha_2^3 \beta_3^3 + \alpha_3^1 \beta_1^1 + \alpha_3^1 \beta_2^1 + \alpha_3^1 \beta_3^1 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_2^2 + \alpha_3^2 \beta_3^2 + \alpha_3^3 \beta_1^3 + \alpha_3^3 \beta_2^3 + \alpha_3^3 \beta_3^3 \right]}_{Q_{4,1}}}$$

- Simplifying

$$Q_{1,13} = (\alpha_1^2 + \alpha_1^3) + (\beta_3^1 + \beta_3^2) (1 - \alpha_1^2 - \alpha_1^3) \quad (\text{TI})$$

$$= (\theta_1^2 + \theta_1^3) + (\theta_3^1 + \theta_3^2) (1 - \theta_1^2 - \theta_1^3) \quad (\text{TIV})$$

$$Q_{2,13} = (\alpha_2^1 + \alpha_3^1) (1 + \beta_1^3 + \beta_2^3 - \beta_3^1 - \beta_3^2) + (\beta_1^3 + \beta_2^3) (1 - \alpha_1^2 - \alpha_1^3) \quad (\text{TI})$$

$$= (\theta_2^1 + \theta_3^1) (1 + \theta_1^3 + \theta_2^3 - \theta_3^1 - \theta_3^2) + (\theta_1^3 + \theta_2^3) (1 - \theta_1^2 - \theta_1^3) \quad (\text{TIV})$$

$$Q_{3,1} = 3 (\alpha_1^2 + \alpha_1^3) \quad (\text{TI})$$

$$= 3 (\theta_1^2 + \theta_1^3) \quad (\text{TIV})$$

$$Q_{4,1} = 3 (\alpha_2^1 + \alpha_3^1) \quad (\text{TI})$$

$$= 3 (\theta_2^1 + \theta_3^1) \quad (\text{TIV})$$

- Under Temporal Independence (Assumption 3)

$$p_{13}^* = \frac{r_{13} + (\beta_3^1 + \beta_3^2 - \beta_1^3 - \beta_2^3) + (\alpha_1^2 + \alpha_1^3 - \alpha_2^1 - \alpha_3^1) (1 + \beta_1^3 + \beta_2^3 - \beta_3^1 - \beta_3^2)}{p_1 + 3(\alpha_1^2 + \alpha_1^3 - \alpha_2^1 - \alpha_3^1)}$$

– Yields

$$\begin{aligned} LB_{13}^{TI} &= \min \left\{ \frac{r_{13} - Q/3}{p_1}, \frac{r_{13} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}}, \frac{r_{13} - \hat{Q}}{p_1 - 3\hat{Q}} \right\} \geq 0 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{13}, (1 - p_1)/3, Q/3\}, \quad \hat{Q} = \min \{r_{13}, p_1/3, Q/3\} \\ UB_{13}^{TI} &= \max \left\{ \frac{r_{13} + Q/3}{p_1}, \frac{r_{13} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}}, \frac{r_{13} - \hat{Q}}{p_1 - 3\hat{Q}} \right\} \leq 1 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{13}, (1 - p_1)/3, Q/3\}, \quad \hat{Q} = \min \{r_{13}, p_1/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{13} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}} \right) / \partial \tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{13} + \tilde{\tilde{Q}}}{p_1 + 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) &= \operatorname{sgn} \left(\left(p_1 + 3\tilde{\tilde{Q}} \right) - 3 \left(r_{13} + \tilde{\tilde{Q}} \right) \right) \\ &= \operatorname{sgn} (p_1 - 3r_{13}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{13} - \hat{Q}}{p_1 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{13} - \hat{Q}}{p_1 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_1 - 3\hat{Q} \right) + 3 \left(r_{13} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{13} - p_1) \end{aligned}$$

- Under Temporal Invariance (Assumption 4)

$$p_{13}^* = \frac{r_{13} + (\theta_3^2 + \theta_1^2 - \theta_2^1 - \theta_2^3) + (\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1) (\theta_1^3 + \theta_2^3 - \theta_3^1 - \theta_3^2)}{p_1 + 3 (\theta_1^2 + \theta_1^3 - \theta_2^1 - \theta_3^1)}$$

– Yields

$$\begin{aligned} LB_{13}^{TIV} &= \min \left\{ \frac{r_{13} - (3 - \sqrt{9 - Q})}{p_1}, \frac{r_{13} + \tilde{Q}^2}{p_1 + 3\tilde{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \left\{ \frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4r_{13}}}{2}, \sqrt{1 - r_{13}}, (1 - p_1)/3, 3 - \sqrt{9 - Q} \right\} \\ UB_{13}^{TIV} &= \max \left\{ \frac{r_{13} + (3 - \sqrt{9 - Q})}{p_1}, \frac{r_{13} + \tilde{Q}^2}{p_1 - 3\tilde{Q}} \right\} \leq 1 \quad \tilde{Q} = \min \left\{ \sqrt{1 - r_{13}}, p_1/3, 3 - \sqrt{9 - Q} \right\} \end{aligned}$$

– Proof:

1. Evaluate $\partial LB_{13}^{TIV} / \partial \tilde{Q}$ and see when the sign is positive/negative.

$$\begin{aligned} \text{sgn} \left(\frac{\partial LB_{13}^{TIV}}{\partial \tilde{Q}} \right) &= \text{sgn} \left(2\tilde{Q} \left(p_1 + 3\tilde{Q} \right) - 3 \left(r_{13} + \tilde{Q}^2 \right) \right) \\ &= \text{sgn} \left(\tilde{Q} \left((2/3)p_1 + \tilde{Q} \right) - r_{13} \right) \\ \Rightarrow \text{sgn} \left(\frac{\partial LB_{13}^{TIV}}{\partial \tilde{Q}} \right) \Big|_{\tilde{Q}=0} &= \text{sgn}(-r_{13}) < 0 \\ &\Rightarrow \tilde{Q} > 0 \end{aligned}$$

2. Minimize LB_{13}^{TIV} s.t. \tilde{Q} being feasible

$$\begin{aligned} \frac{\partial LB_{13}^{TIV}}{\partial \tilde{Q}} &\propto \tilde{Q} \left((2/3)p_1 + \tilde{Q} \right) - r_{13} = 0 \\ \Rightarrow \tilde{Q}^* &= \frac{-(2/3)p_1 + \sqrt{(4/9)p_1^2 + 4r_{13}}}{2} \end{aligned}$$

So, derivative starts off negative and then reaches zero at \tilde{Q}^* . Thus, $\frac{r_{13} + \tilde{Q}^2}{p_1 + 3\tilde{Q}}$ is minimized at \tilde{Q}^* .

A.1.4 p_{21}^*

$$p_{21}^* = \frac{r_{21} + \overbrace{\left[\theta_{21}^{11} + \theta_{21}^{12} + \theta_{21}^{13} + \theta_{21}^{22} + \theta_{21}^{23} + \theta_{21}^{31} + \theta_{21}^{32} + \theta_{21}^{33} \right]}^{Q_{1,21}} - \overbrace{\left[\theta_{11}^{21} + \theta_{12}^{21} + \theta_{13}^{21} + \theta_{22}^{21} + \theta_{23}^{21} + \theta_{31}^{21} + \theta_{32}^{21} + \theta_{33}^{21} \right]}^{Q_{2,21}}}{p_2 + \underbrace{\left[\theta_{21}^{11} + \theta_{21}^{12} + \theta_{21}^{13} + \theta_{21}^{31} + \theta_{21}^{32} + \theta_{21}^{33} + \theta_{22}^{11} + \theta_{22}^{12} + \theta_{22}^{13} + \theta_{22}^{31} + \theta_{22}^{32} + \theta_{22}^{33} + \theta_{23}^{11} + \theta_{23}^{12} + \theta_{23}^{13} + \theta_{23}^{31} + \theta_{23}^{32} + \theta_{23}^{33} \right]}_{Q_{3,2}} - \underbrace{\left[\theta_{11}^{21} + \theta_{12}^{21} + \theta_{13}^{21} + \theta_{22}^{21} + \theta_{23}^{21} + \theta_{31}^{21} + \theta_{32}^{21} + \theta_{33}^{21} + \theta_{22}^{22} + \theta_{23}^{22} + \theta_{31}^{22} + \theta_{32}^{22} + \theta_{33}^{22} + \theta_{22}^{23} + \theta_{23}^{23} + \theta_{31}^{23} + \theta_{32}^{23} + \theta_{33}^{23} \right]}_{Q_{4,2}}}$$

- $\theta_{kl}^{k'l'} = \text{unique element}$

Maximum Arbitrary Errors: Assumption 2

$$\begin{aligned} LB_{21} &= \frac{r_{21} - Q}{p_2} \geq 0 \\ UB_{21} &= \frac{r_{21} + Q}{p_2} \leq 1 \end{aligned}$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{21}^* = \frac{r_{21} + \overbrace{\left[\alpha_2^1 \beta_1^1 + \alpha_2^1 \beta_1^2 + \alpha_2^1 \beta_1^3 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_1^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_1^3 \right]}^{Q_{1,21}} - \overbrace{\left[\alpha_1^2 \beta_1^1 + \alpha_1^2 \beta_2^1 + \alpha_1^2 \beta_3^1 + \alpha_2^2 \beta_2^1 + \alpha_2^2 \beta_3^1 + \alpha_3^2 \beta_1^1 + \alpha_3^2 \beta_2^1 + \alpha_3^2 \beta_3^1 \right]}^{Q_{2,21}}}{p_2 + \underbrace{\left[\alpha_2^1 \beta_1^1 + \alpha_2^1 \beta_1^2 + \alpha_2^1 \beta_1^3 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_1^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_1^3 + \alpha_2^1 \beta_2^1 + \alpha_2^1 \beta_2^2 + \alpha_2^1 \beta_2^3 + \alpha_2^2 \beta_2^1 + \alpha_2^2 \beta_2^2 + \alpha_2^2 \beta_2^3 + \alpha_2^3 \beta_2^1 + \alpha_2^3 \beta_2^2 + \alpha_2^3 \beta_2^3 + \alpha_2^1 \beta_3^1 + \alpha_2^1 \beta_3^2 + \alpha_2^1 \beta_3^3 + \alpha_2^2 \beta_3^1 + \alpha_2^2 \beta_3^2 + \alpha_2^2 \beta_3^3 + \alpha_2^3 \beta_3^1 + \alpha_2^3 \beta_3^2 + \alpha_2^3 \beta_3^3 \right]}_{Q_{4,2}} - \underbrace{\left[\alpha_1^2 \beta_1^1 + \alpha_1^2 \beta_2^1 + \alpha_1^2 \beta_3^1 + \alpha_1^2 \beta_2^2 + \alpha_1^2 \beta_3^2 + \alpha_1^2 \beta_3^3 + \alpha_1^2 \beta_1^3 + \alpha_1^2 \beta_2^3 + \alpha_1^2 \beta_3^3 + \alpha_3^2 \beta_1^1 + \alpha_3^2 \beta_2^1 + \alpha_3^2 \beta_3^1 + \alpha_3^2 \beta_2^2 + \alpha_3^2 \beta_3^2 + \alpha_3^2 \beta_1^3 + \alpha_3^2 \beta_2^3 + \alpha_3^2 \beta_3^3 \right]}_{Q_{3,2}}}$$

- Simplifying

$$\begin{aligned} Q_{1,21} &= (\alpha_2^1 + \alpha_2^3) + (\beta_1^2 + \beta_1^3) (1 - \alpha_2^1 - \alpha_2^3) & \text{(TI)} \\ &= (\theta_2^1 + \theta_2^3) + (\theta_1^2 + \theta_1^3) (1 - \theta_2^1 - \theta_2^3) & \text{(TIV)} \\ Q_{2,21} &= (\alpha_1^2 + \alpha_3^2) (1 + \beta_2^1 + \beta_3^1 - \beta_1^2 - \beta_1^3) + (\beta_2^1 + \beta_3^1) (1 - \alpha_2^1 - \alpha_2^3) & \text{(TI)} \\ &= (\theta_1^2 + \theta_3^2) (1 + \theta_2^1 + \theta_3^1 - \theta_1^2 - \theta_1^3) + (\theta_2^1 + \theta_3^1) (1 - \theta_2^1 - \theta_2^3) & \text{(TIV)} \\ Q_{3,2} &= 3 (\alpha_2^1 + \alpha_2^3) & \text{(TI)} \\ &= 3 (\theta_2^1 + \theta_2^3) & \text{(TIV)} \\ Q_{4,2} &= 3 (\alpha_1^2 + \alpha_3^2) & \text{(TI)} \\ &= 3 (\theta_1^2 + \theta_3^2) & \text{(TIV)} \end{aligned}$$

- Under Temporal Independence (Assumption 3)

$$p_{21}^* = \frac{r_{21} + (\beta_1^2 + \beta_1^3 - \beta_2^1 - \beta_3^1) + (\alpha_2^1 + \alpha_2^3 - \alpha_1^2 - \alpha_3^2)(1 + \beta_2^1 + \beta_3^1 - \beta_1^2 - \beta_1^3)}{p_2 + 3(\alpha_2^1 + \alpha_2^3 - \alpha_1^2 - \alpha_3^2)}$$

– Yields

$$\begin{aligned} LB_{21}^{TI} &= \min \left\{ \frac{r_{21} - Q/3}{p_2}, \frac{r_{21} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}, \frac{r_{21} - \hat{Q}}{p_2 - 3\hat{Q}} \right\} \geq 0 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{21}, (1 - p_2)/3, Q/3\}, \quad \hat{Q} = \min \{r_{21}, p_2/3, Q/3\} \\ UB_{21}^{TI} &= \max \left\{ \frac{r_{21} + Q/3}{p_2}, \frac{r_{21} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}, \frac{r_{21} - \hat{Q}}{p_2 - 3\hat{Q}} \right\} \leq 1 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{21}, (1 - p_2)/3, Q/3\}, \quad \hat{Q} = \min \{r_{21}, p_2/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{21} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}} \right) / \partial \tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{21} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) &= \operatorname{sgn} \left(\left(p_2 + 3\tilde{\tilde{Q}} \right) - 3 \left(r_{21} + \tilde{\tilde{Q}} \right) \right) \\ &= \operatorname{sgn} (p_2 - 3r_{21}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{21} - \hat{Q}}{p_2 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{21} - \hat{Q}}{p_2 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_2 - 3\hat{Q} \right) + 3 \left(r_{21} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{21} - p_2) \end{aligned}$$

- Under Temporal Invariance (Assumption 3)

$$p_{21}^* = \frac{r_{21} + (\theta_1^3 + \theta_2^3 - \theta_3^1 - \theta_3^2) + (\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2) (\theta_2^1 + \theta_3^1 - \theta_1^2 - \theta_1^3)}{p_2 + 3 (\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2)}$$

– Yields

$$\begin{aligned} LB_{21}^{TIV} &= \min \left\{ \frac{r_{21} - (3 - \sqrt{9 - Q})}{p_2}, \frac{r_{21} + \tilde{Q}^2}{p_1 + 3\tilde{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \left\{ \frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4r_{21}}}{2}, \sqrt{1 - r_{21}}, (1 - p_2)/3, 3 - \sqrt{9 - Q} \right\} \\ UB_{21}^{TIV} &= \max \left\{ \frac{r_{21} + (3 - \sqrt{9 - Q})}{p_2}, \frac{r_{21} + \tilde{Q}^2}{p_2 - 3\tilde{Q}} \right\} \leq 1 \quad \tilde{Q} = \min \left\{ \sqrt{1 - r_{21}}, p_2/3, 3 - \sqrt{9 - Q} \right\} \end{aligned}$$

– Proof:

1. Evaluate $\partial LB_{21}^{TIV} / \partial \tilde{Q}$ and see when the sign is positive/negative.

$$\begin{aligned} \text{sgn} \left(\frac{\partial LB_{21}^{TIV}}{\partial \tilde{Q}} \right) &= \text{sgn} \left(2\tilde{Q} \left(p_2 + 3\tilde{Q} \right) - 3 \left(r_{21} + \tilde{Q}^2 \right) \right) \\ &= \text{sgn} \left(\tilde{Q} \left((2/3)p_2 + \tilde{Q} \right) - r_{21} \right) \\ \Rightarrow \text{sgn} \left(\frac{\partial LB_{21}^{TIV}}{\partial \tilde{Q}} \right) \Big|_{\tilde{Q}=0} &= \text{sgn}(-r_{21}) < 0 \\ &\Rightarrow \tilde{Q} > 0 \end{aligned}$$

2. Minimize LB_{21}^{TIV} s.t. \tilde{Q} being feasible

$$\begin{aligned} \frac{\partial LB_{21}^{TIV}}{\partial \tilde{Q}} &\propto \tilde{Q} \left((2/3)p_2 + \tilde{Q} \right) - r_{21} = 0 \\ \Rightarrow \tilde{Q}^* &= \frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4r_{21}}}{2} \end{aligned}$$

So, derivative starts off negative and then reaches zero at \tilde{Q}^* . Thus, $\frac{r_{21} + \tilde{Q}^2}{p_2 + 3\tilde{Q}}$ is minimized at \tilde{Q}^* .

A.1.5 p_{22}^*

$$p_{22}^* = \frac{r_{22} + \overbrace{\left[\theta_{22}^{11} + \theta_{22}^{12} + \theta_{22}^{13} + \theta_{22}^{21} + \theta_{22}^{23} + \theta_{22}^{31} + \theta_{22}^{32} + \theta_{22}^{33} \right]}^{Q_{1,22}} - \overbrace{\left[\theta_{11}^{22} + \theta_{12}^{22} + \theta_{13}^{22} + \theta_{21}^{22} + \theta_{23}^{22} + \theta_{31}^{22} + \theta_{32}^{22} + \theta_{33}^{22} \right]}^{Q_{2,22}}}{p_2 + \underbrace{\left[\theta_{21}^{11} + \theta_{21}^{12} + \theta_{21}^{13} + \theta_{21}^{31} + \theta_{21}^{32} + \theta_{21}^{33} + \theta_{22}^{11} + \theta_{22}^{12} + \theta_{22}^{13} + \theta_{22}^{31} + \theta_{22}^{32} + \theta_{22}^{33} + \theta_{23}^{11} + \theta_{23}^{12} + \theta_{23}^{13} + \theta_{23}^{31} + \theta_{23}^{32} + \theta_{23}^{33} \right]}_{Q_{3,2}}}$$

$$- \underbrace{\left[\theta_{11}^{21} + \theta_{12}^{21} + \theta_{13}^{21} + \theta_{11}^{22} + \theta_{12}^{22} + \theta_{13}^{22} + \theta_{11}^{23} + \theta_{12}^{23} + \theta_{13}^{23} + \theta_{31}^{21} + \theta_{32}^{21} + \theta_{33}^{21} + \theta_{31}^{22} + \theta_{32}^{22} + \theta_{33}^{22} + \theta_{31}^{23} + \theta_{32}^{23} + \theta_{33}^{23} \right]}_{Q_{4,2}}$$

- $\theta_{kl}^{k'l'}$ = unique element

Maximum Arbitrary Errors: Assumption 2

$$LB_{22} = \frac{r_{22} - Q}{p_2} \geq 0$$

$$UB_{22} = \frac{r_{22} + Q}{p_2} \leq 1$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{22}^* = \frac{r_{22} + \overbrace{\left[\alpha_2^1 \beta_2^1 + \alpha_2^1 \beta_2^2 + \alpha_2^1 \beta_2^3 + \alpha_2^2 \beta_2^1 + \alpha_2^2 \beta_2^2 + \alpha_2^2 \beta_2^3 + \alpha_2^3 \beta_2^1 + \alpha_2^3 \beta_2^2 + \alpha_2^3 \beta_2^3 \right]}^{Q_{1,22}} - \overbrace{\left[\alpha_1^2 \beta_1^2 + \alpha_1^2 \beta_2^2 + \alpha_1^2 \beta_3^2 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_2^2 + \alpha_2^2 \beta_3^2 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_2^2 + \alpha_3^2 \beta_3^2 \right]}^{Q_{2,22}}}{p_2 + \underbrace{\left[\alpha_2^1 \beta_1^1 + \alpha_2^1 \beta_1^2 + \alpha_2^1 \beta_1^3 + \alpha_2^2 \beta_1^1 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_1^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_1^3 + \alpha_2^1 \beta_2^1 + \alpha_2^1 \beta_2^2 + \alpha_2^1 \beta_2^3 + \alpha_2^2 \beta_2^1 + \alpha_2^2 \beta_2^2 + \alpha_2^2 \beta_2^3 + \alpha_2^3 \beta_2^1 + \alpha_2^3 \beta_2^2 + \alpha_2^3 \beta_2^3 + \alpha_2^1 \beta_3^1 + \alpha_2^1 \beta_3^2 + \alpha_2^1 \beta_3^3 + \alpha_2^2 \beta_3^1 + \alpha_2^2 \beta_3^2 + \alpha_2^2 \beta_3^3 + \alpha_2^3 \beta_3^1 + \alpha_2^3 \beta_3^2 + \alpha_2^3 \beta_3^3 \right]}_{Q_{3,2}}}$$

$$- \underbrace{\left[\alpha_1^2 \beta_1^1 + \alpha_1^2 \beta_1^2 + \alpha_1^2 \beta_1^3 + \alpha_1^2 \beta_2^1 + \alpha_1^2 \beta_2^2 + \alpha_1^2 \beta_2^3 + \alpha_1^2 \beta_3^1 + \alpha_1^2 \beta_3^2 + \alpha_1^2 \beta_3^3 + \alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_1^3 + \alpha_1^3 \beta_2^1 + \alpha_1^3 \beta_2^2 + \alpha_1^3 \beta_2^3 + \alpha_1^3 \beta_3^1 + \alpha_1^3 \beta_3^2 + \alpha_1^3 \beta_3^3 \right]}_{Q_{4,2}}}$$

- Simplifying

$$Q_{1,22} = (\alpha_2^1 + \alpha_2^3) + (\beta_2^1 + \beta_2^3) (1 - \alpha_2^1 - \alpha_2^3) \quad (\text{TI})$$

$$= 2 (\theta_2^1 + \theta_2^3) - (\theta_2^1 + \theta_2^3)^2 \quad (\text{TIV})$$

$$Q_{2,22} = (\alpha_1^2 + \alpha_3^2) (1 + \beta_1^2 + \beta_3^2 - \beta_2^1 - \beta_2^3) + (\beta_1^2 + \beta_3^2) (1 - \alpha_2^1 - \alpha_2^3) \quad (\text{TI})$$

$$= 2 (\theta_1^2 + \theta_3^2) (1 - \theta_2^1 - \theta_2^3) + (\theta_1^2 + \theta_3^2)^2 \quad (\text{TIV})$$

$$Q_{3,2} = 3 (\alpha_2^1 + \alpha_2^3) \quad (\text{TI})$$

$$= 3 (\theta_2^1 + \theta_2^3) \quad (\text{TIV})$$

$$Q_{4,2} = 3 (\alpha_1^2 + \alpha_3^2) \quad (\text{TI})$$

$$= 3 (\theta_1^2 + \theta_3^2) \quad (\text{TIV})$$

- Under Temporal Independence (Assumption 3)

$$p_{22}^* = \frac{r_{22} + (\alpha_2^1 + \alpha_2^3 - \alpha_1^2 - \alpha_3^2) + (\beta_2^1 + \beta_2^3 - \beta_1^2 - \beta_3^2) (1 + \alpha_1^2 + \alpha_3^2 - \alpha_2^1 - \alpha_2^3)}{p_2 + 3(\alpha_2^1 + \alpha_2^3 - \alpha_1^2 - \alpha_3^2)}$$

– Yields

$$\begin{aligned} LB_{22}^{TI} &= \min \left\{ \frac{r_{22} - Q/3}{p_2}, \frac{r_{22} + \tilde{Q}}{p_2 + 3\tilde{Q}}, \frac{r_{22} - \hat{Q}}{p_2 - 3\hat{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \{1 - r_{22}, (1 - p_2)/3, Q/3\}, \quad \hat{Q} = \min \{r_{22}, p_2/3, Q/3\} \\ UB_{22}^{TI} &= \max \left\{ \frac{r_{22} + Q/3}{p_2}, \frac{r_{22} + \tilde{Q}}{p_2 + 3\tilde{Q}}, \frac{r_{22} - \hat{Q}}{p_2 - 3\hat{Q}} \right\} \leq 1 \\ \tilde{Q} &= \min \{1 - r_{22}, (1 - p_2)/3, Q/3\}, \quad \hat{Q} = \min \{r_{22}, p_2/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{22} + \tilde{Q}}{p_2 + 3\tilde{Q}} \right) / \partial \tilde{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22} + \tilde{Q}}{p_2 + 3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) &= \operatorname{sgn} \left(\left(p_2 + 3\tilde{Q} \right) - 3 \left(r_{22} + \tilde{Q} \right) \right) \\ &= \operatorname{sgn} (p_2 - 3r_{22}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{22} - \hat{Q}}{p_2 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22} - \hat{Q}}{p_2 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_2 - 3\hat{Q} \right) + 3 \left(r_{22} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{22} - p_2) \end{aligned}$$

- Under Temporal Invariance (Assumption 4)

$$p_{22}^* = \frac{r_{22} + 2(\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2) - (\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2)^2}{p_2 + 3(\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2)}$$

– Yields

$$\begin{aligned}
LB_{22}^{TIV} &= \min \left\{ \frac{r_{22} + 2\widehat{Q} - \widehat{Q}^2}{p_2 + 3\widehat{Q}}, \frac{r_{22} - 2\widetilde{Q} - \widetilde{Q}^2}{p_2 - 3\widetilde{Q}} \right\} \geq 0 \\
\widehat{Q} &= \min \left\{ (1 - p_2)/3, 3 - \sqrt{9 - Q} \right\}, \\
\widetilde{Q} &= \begin{cases} 0 & r_{22} \geq 2p_2/3 \\ \min \left\{ \frac{(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]}}{2}, (-1 + \sqrt{1 + r_{22}}), p_2/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases} \\
UB_{22}^{TIV} &= \min \left\{ \frac{r_{22} + 2\widehat{Q} - \widehat{Q}^2}{p_2 + 3\widehat{Q}}, \frac{r_{22} - 2\widetilde{Q} - \widetilde{Q}^2}{p_2 - 3\widetilde{Q}} \right\} \geq 0 \\
\widehat{Q} &= \begin{cases} 0 & r_{22} \geq 2p_2/3 \\ \min \left\{ \frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]}}{2}, (1 - p_2)/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases}, \\
\widetilde{Q} &= \begin{cases} 0 & r_{22} < 2p_2/3 \\ \min \left\{ \frac{(2/3)p_2 - \sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]}}{2}, (-1 + \sqrt{1 + r_{22}}), p_2/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases}
\end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{22}-2\tilde{Q}-\tilde{Q}^2}{p_2-3\tilde{Q}} \right) / \partial \tilde{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
& \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22}-2\tilde{Q}-\tilde{Q}^2}{p_2-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) = \operatorname{sgn} \left(\left(-2 - 2\tilde{Q} \right) \left(p_2 - 3\tilde{Q} \right) + 3 \left(r_{22} - 2\tilde{Q} - \tilde{Q}^2 \right) \right) \\
& = \operatorname{sgn} \left(-(2/3)p_2 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{22} \right) \\
& \Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22}-2\tilde{Q}-\tilde{Q}^2}{p_2-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) \bigg|_{\tilde{Q}=0} = \operatorname{sgn} \left(-(2/3)p_2 + r_{22} \right) \geq 0 \\
& \Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22}-2\tilde{Q}-\tilde{Q}^2}{p_2-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) \bigg|_{\tilde{Q}=1} = \operatorname{sgn} \left(-(4/3)p_2 + 1 + r_{22} \right) \geq 0
\end{aligned}$$

2. Ensure $r_{22} - 2\tilde{Q} - \tilde{Q}^2 \geq 0$

$$\begin{aligned}
& r_{22} - 2\tilde{Q} - \tilde{Q}^2 \geq 0 \\
& \Rightarrow \tilde{Q}^2 + 2\tilde{Q} - r_{22} \leq 0 \\
& \Rightarrow \tilde{Q} \leq \frac{-2 + \sqrt{4 + 4r_{22}}}{2} \\
& \Rightarrow \tilde{Q} \leq -1 + \sqrt{1 + r_{22}}
\end{aligned}$$

3. Minimize $\frac{r_{22}-2\tilde{Q}-\tilde{Q}^2}{p_2-3\tilde{Q}}$ s.t. \tilde{Q} being feasible and $r_{22} < 2p_2/3$

$$\begin{aligned}
& \frac{\partial \left(\frac{r_{22}-2\tilde{Q}-\tilde{Q}^2}{p_2-3\tilde{Q}} \right)}{\partial \tilde{Q}} \propto -(2/3)p_2 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{22} = 0 \\
& \Rightarrow \tilde{Q}^* = \frac{(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]}}{2}
\end{aligned}$$

4. Maximize $\frac{r_{22}-2\tilde{Q}-\tilde{Q}^2}{p_2-3\tilde{Q}}$ s.t. \tilde{Q} being feasible and $r_{22} \geq 2p_2/3$

$$\begin{aligned}
& \frac{\partial \left(\frac{r_{22}-2\tilde{Q}-\tilde{Q}^2}{p_2-3\tilde{Q}} \right)}{\partial \tilde{Q}} \propto -(2/3)p_2 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{22} = 0 \\
& \Rightarrow \tilde{Q}^* = \frac{(2/3)p_2 - \sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]}}{2}
\end{aligned}$$

Note: If $\sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]} = .$, then maximize \tilde{Q} .

5. Evaluate $\partial \left(\frac{r_{22}+2\hat{Q}-\hat{Q}^2}{p_2+3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
\operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22}+2\hat{Q}-\hat{Q}^2}{p_2+3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left((2-2\hat{Q})(p_2+3\hat{Q}) - 3(r_{22}+2\hat{Q}-\hat{Q}^2) \right) \\
&= \operatorname{sgn} \left((2/3)p_2(1-\hat{Q}) - \hat{Q}^2 - r_{22} \right) \\
\Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22}+2\hat{Q}-\hat{Q}^2}{p_2+3\hat{Q}} \right)}{\partial \hat{Q}} \right) \Big|_{\hat{Q}=0} &= \operatorname{sgn}((2/3)p_2 - r_{22}) \geq 0 \\
\Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22}+2\hat{Q}-\hat{Q}^2}{p_2+3\hat{Q}} \right)}{\partial \hat{Q}} \right) \Big|_{\hat{Q}=1} &= \operatorname{sgn}(-1 - r_{22}) < 0
\end{aligned}$$

6. Maximize $\frac{r_{22}+2\hat{Q}-\hat{Q}^2}{p_2+3\hat{Q}}$ s.t. \hat{Q} being feasible and $r_{22} < 2p_2/3$

$$\begin{aligned}
\operatorname{sgn} \left(\frac{\partial \left(\frac{r_{22}+2\hat{Q}-\hat{Q}^2}{p_2+3\hat{Q}} \right)}{\partial \hat{Q}} \right) &\propto (2/3)p_2(1-\hat{Q}) - \hat{Q}^2 - r_{22} = 0 \\
\Rightarrow \hat{Q}^* &= \frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4[(2/3)p_2 - r_{22}]}}{2}
\end{aligned}$$

7. Minimize $\frac{r_{22}+2\hat{Q}-\hat{Q}^2}{p_2+3\hat{Q}} \Rightarrow \hat{Q} = 0$ or maximize \hat{Q} . However, if the minimum occurs when $\hat{Q} = 0$, then $\frac{r_{22}-2\hat{Q}-\hat{Q}^2}{p_2-3\hat{Q}} < \frac{r_{22}}{p_2}$ and this will be the binding *LB*.

A.1.6 p_{23}^*

$$p_{23}^* = \frac{r_{23} + \overbrace{\left[\theta_{23}^{11} + \theta_{23}^{12} + \theta_{23}^{13} + \theta_{23}^{21} + \theta_{23}^{22} + \theta_{23}^{31} + \theta_{23}^{32} + \theta_{23}^{33} \right]}^{Q_{1,23}} - \overbrace{\left[\theta_{11}^{23} + \theta_{12}^{23} + \theta_{13}^{23} + \theta_{21}^{23} + \theta_{22}^{23} + \theta_{31}^{23} + \theta_{32}^{23} + \theta_{33}^{23} \right]}^{Q_{2,23}}}{p_2 + \underbrace{\left[\theta_{21}^{11} + \theta_{21}^{12} + \theta_{21}^{13} + \theta_{21}^{31} + \theta_{21}^{32} + \theta_{21}^{33} + \theta_{22}^{11} + \theta_{22}^{12} + \theta_{22}^{13} + \theta_{22}^{31} + \theta_{22}^{32} + \theta_{22}^{33} + \theta_{23}^{11} + \theta_{23}^{12} + \theta_{23}^{13} + \theta_{23}^{31} + \theta_{23}^{32} + \theta_{23}^{33} \right]}_{Q_{3,2}} - \underbrace{\left[\theta_{11}^{21} + \theta_{12}^{21} + \theta_{13}^{21} + \theta_{11}^{22} + \theta_{12}^{22} + \theta_{13}^{22} + \theta_{11}^{23} + \theta_{12}^{23} + \theta_{13}^{23} + \theta_{31}^{21} + \theta_{32}^{21} + \theta_{33}^{21} + \theta_{31}^{22} + \theta_{32}^{22} + \theta_{33}^{22} + \theta_{31}^{23} + \theta_{32}^{23} + \theta_{33}^{23} \right]}_{Q_{4,2}}}$$

- $\theta_{kl}^{k'l'} = \text{unique element}$

Maximum Arbitrary Errors: Assumption 2

$$\begin{aligned} LB_{23} &= \frac{r_{23} - \tilde{Q}}{p_2} \geq 0 & \tilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{23} &= \frac{r_{23} + \tilde{Q}}{p_2 - \tilde{Q}} \leq 1 & \tilde{Q} &= \begin{cases} 0 & \text{AE} \\ \min\{p_2, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{23}^* = \frac{r_{23} + \overbrace{\left[\alpha_2^1 \beta_3^1 + \alpha_2^1 \beta_3^2 + \alpha_2^1 \beta_3^3 + \alpha_2^2 \beta_3^1 + \alpha_2^2 \beta_3^2 + \alpha_2^2 \beta_3^3 + \alpha_2^3 \beta_3^1 + \alpha_2^3 \beta_3^2 + \alpha_2^3 \beta_3^3 \right]}^{Q_{1,23}} - \overbrace{\left[\alpha_1^2 \beta_1^3 + \alpha_1^2 \beta_2^3 + \alpha_1^2 \beta_3^3 + \alpha_2^2 \beta_1^3 + \alpha_2^2 \beta_2^3 + \alpha_2^2 \beta_3^3 + \alpha_3^2 \beta_1^3 + \alpha_3^2 \beta_2^3 + \alpha_3^2 \beta_3^3 \right]}^{Q_{2,23}}}{p_2 + \underbrace{\left[\alpha_2^1 \beta_1^1 + \alpha_2^1 \beta_1^2 + \alpha_2^1 \beta_1^3 + \alpha_2^2 \beta_1^1 + \alpha_2^2 \beta_1^2 + \alpha_2^2 \beta_1^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_1^3 + \alpha_2^1 \beta_2^1 + \alpha_2^1 \beta_2^2 + \alpha_2^1 \beta_2^3 + \alpha_2^2 \beta_2^1 + \alpha_2^2 \beta_2^2 + \alpha_2^2 \beta_2^3 + \alpha_2^3 \beta_2^1 + \alpha_2^3 \beta_2^2 + \alpha_2^3 \beta_2^3 + \alpha_2^1 \beta_3^1 + \alpha_2^1 \beta_3^2 + \alpha_2^1 \beta_3^3 + \alpha_2^2 \beta_3^1 + \alpha_2^2 \beta_3^2 + \alpha_2^2 \beta_3^3 + \alpha_2^3 \beta_3^1 + \alpha_2^3 \beta_3^2 + \alpha_2^3 \beta_3^3 \right]}_{Q_{4,2}} - \underbrace{\left[\alpha_1^2 \beta_1^1 + \alpha_1^2 \beta_2^1 + \alpha_1^2 \beta_3^1 + \alpha_1^2 \beta_1^2 + \alpha_1^2 \beta_2^2 + \alpha_1^2 \beta_3^2 + \alpha_1^2 \beta_1^3 + \alpha_1^2 \beta_2^3 + \alpha_1^2 \beta_3^3 + \alpha_3^2 \beta_1^1 + \alpha_3^2 \beta_2^1 + \alpha_3^2 \beta_3^1 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_2^2 + \alpha_3^2 \beta_3^2 + \alpha_3^2 \beta_1^3 + \alpha_3^2 \beta_2^3 + \alpha_3^2 \beta_3^3 \right]}_{Q_{3,2}}}$$

- Simplifying

$$\begin{aligned} Q_{1,23} &= (\alpha_2^1 + \alpha_2^3) + (\beta_3^1 + \beta_3^2) (1 - \alpha_2^1 - \alpha_2^3) & (\text{TI}) \\ &= (\theta_2^1 + \theta_2^3) + (\theta_3^1 + \theta_3^2) (1 - \theta_2^1 - \theta_2^3) & (\text{TIV}) \\ Q_{2,23} &= (\alpha_1^2 + \alpha_3^2) (1 + \beta_1^3 + \beta_2^3 - \beta_3^1 - \beta_3^2) + (\beta_1^3 + \beta_2^3) (1 - \alpha_2^1 - \alpha_2^3) & (\text{TI}) \\ &= (\theta_1^2 + \theta_3^2) (1 + \theta_1^3 + \theta_2^3 - \theta_3^1 - \theta_3^2) + (\theta_1^3 + \theta_2^3) (1 - \theta_2^1 - \theta_2^3) & (\text{TIV}) \\ Q_{3,2} &= 3 (\alpha_2^1 + \alpha_2^3) & (\text{TI}) \\ &= 3 (\theta_2^1 + \theta_2^3) & (\text{TIV}) \\ Q_{4,2} &= 3 (\alpha_1^2 + \alpha_3^2) & (\text{TI}) \\ &= 3 (\theta_1^2 + \theta_3^2) & (\text{TIV}) \end{aligned}$$

- Under Temporal Independence (Assumption 3)

$$p_{23}^* = \frac{r_{23} + (\beta_3^1 + \beta_3^2 - \beta_1^3 - \beta_2^3) + (\alpha_2^1 + \alpha_2^3 - \alpha_1^2 - \alpha_3^2) (1 + \beta_1^3 + \beta_2^3 - \beta_3^1 - \beta_3^2)}{p_2 + 3(\alpha_2^1 + \alpha_2^3 - \alpha_1^2 - \alpha_3^2)}$$

– Yields

$$\begin{aligned} LB_{23}^{TI} &= \min \left\{ \frac{r_{23} - Q/3}{p_1}, \frac{r_{23} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}, \frac{r_{23} - \hat{Q}}{p_2 - 3\hat{Q}} \right\} \geq 0 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{23}, (1 - p_2)/3, Q/3\}, \quad \hat{Q} = \min \{r_{23}, p_2/3, Q/3\} \\ UB_{23}^{TI} &= \max \left\{ \frac{r_{23} + Q/3}{p_2}, \frac{r_{23} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}}, \frac{r_{23} - \hat{Q}}{p_2 - 3\hat{Q}} \right\} \leq 1 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{23}, (1 - p_2)/3, Q/3\}, \quad \hat{Q} = \min \{r_{23}, p_2/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{23} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}} \right) / \partial \tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{23} + \tilde{\tilde{Q}}}{p_2 + 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) &= \operatorname{sgn} \left(\left(p_2 + 3\tilde{\tilde{Q}} \right) - 3 \left(r_{23} + \tilde{\tilde{Q}} \right) \right) \\ &= \operatorname{sgn} (p_2 - 3r_{23}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{23} - \hat{Q}}{p_2 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{23} - \hat{Q}}{p_2 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_2 - 3\hat{Q} \right) + 3 \left(r_{23} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{23} - p_2) \end{aligned}$$

- Under Temporal Invariance (Assumption 4)

$$p_{23}^* = \frac{r_{23} + (\theta_2^1 + \theta_3^1 - \theta_1^2 - \theta_1^3) + (\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2) (\theta_1^3 + \theta_2^3 - \theta_3^1 - \theta_3^2)}{p_2 + 3 (\theta_2^1 + \theta_2^3 - \theta_1^2 - \theta_3^2)}$$

– Yields

$$\begin{aligned} LB_{23}^{TIV} &= \min \left\{ \frac{r_{23} - (3 - \sqrt{9 - Q})}{p_2}, \frac{r_{23} + \tilde{Q}^2}{p_2 + 3\tilde{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \left\{ \frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4r_{23}}}{2}, \sqrt{1 - r_{23}}, (1 - p_2)/3, 3 - \sqrt{9 - Q} \right\} \\ UB_{23}^{TIV} &= \max \left\{ \frac{r_{23} + (3 - \sqrt{9 - Q})}{p_2}, \frac{r_{23} + \tilde{Q}^2}{p_2 - 3\tilde{Q}} \right\} \leq 1 \quad \tilde{Q} = \min \left\{ \sqrt{1 - r_{23}}, p_2/3, 3 - \sqrt{9 - Q} \right\} \end{aligned}$$

– Proof:

1. Evaluate $\partial LB_{23}^{TIV} / \partial \tilde{Q}$ and see when the sign is positive/negative.

$$\begin{aligned} \text{sgn} \left(\frac{\partial LB_{23}^{TIV}}{\partial \tilde{Q}} \right) &= \text{sgn} \left(2\tilde{Q} \left(p_2 + 3\tilde{Q} \right) - 3 \left(r_{23} + \tilde{Q}^2 \right) \right) \\ &= \text{sgn} \left(\tilde{Q} \left((2/3)p_2 + \tilde{Q} \right) - r_{23} \right) \\ \Rightarrow \text{sgn} \left(\frac{\partial LB_{23}^{TIV}}{\partial \tilde{Q}} \right) \Big|_{\tilde{Q}=0} &= \text{sgn}(-r_{23}) < 0 \\ &\Rightarrow \tilde{Q} > 0 \end{aligned}$$

2. Minimize LB_{23}^{TIV} s.t. \tilde{Q} being feasible

$$\begin{aligned} \frac{\partial LB_{23}^{TIV}}{\partial \tilde{Q}} &\propto \tilde{Q} \left((2/3)p_2 + \tilde{Q} \right) - r_{23} = 0 \\ \Rightarrow \tilde{Q}^* &= \frac{-(2/3)p_2 + \sqrt{(4/9)p_2^2 + 4r_{23}}}{2} \end{aligned}$$

So, derivative starts off negative and then reaches zero at \tilde{Q}^* . Thus, $\frac{r_{23} + \tilde{Q}^2}{p_2 + 3\tilde{Q}}$ is minimized at \tilde{Q}^* .

A.1.7 p_{31}^*

$$p_{31}^* = \frac{r_{31} + \overbrace{\left[\theta_{31}^{11} + \theta_{31}^{12} + \theta_{31}^{13} + \theta_{31}^{21} + \theta_{31}^{22} + \theta_{31}^{23} + \theta_{31}^{32} + \theta_{31}^{33} \right]}^{Q_{1,31}} - \overbrace{\left[\theta_{11}^{31} + \theta_{12}^{31} + \theta_{13}^{31} + \theta_{21}^{31} + \theta_{22}^{31} + \theta_{23}^{31} + \theta_{32}^{31} + \theta_{33}^{31} \right]}^{Q_{2,31}}}{p_3 + \underbrace{\left[\theta_{31}^{11} + \theta_{31}^{12} + \theta_{31}^{13} + \theta_{31}^{21} + \theta_{31}^{22} + \theta_{31}^{23} + \theta_{32}^{11} + \theta_{32}^{12} + \theta_{32}^{13} + \theta_{32}^{21} + \theta_{32}^{22} + \theta_{32}^{23} + \theta_{33}^{11} + \theta_{33}^{12} + \theta_{33}^{13} + \theta_{33}^{21} + \theta_{33}^{22} + \theta_{33}^{23} \right]}_{Q_{3,3}} - \underbrace{\left[\theta_{11}^{31} + \theta_{12}^{31} + \theta_{13}^{31} + \theta_{21}^{32} + \theta_{22}^{32} + \theta_{23}^{32} + \theta_{31}^{33} + \theta_{32}^{33} + \theta_{33}^{33} + \theta_{21}^{31} + \theta_{22}^{31} + \theta_{23}^{31} + \theta_{21}^{32} + \theta_{22}^{32} + \theta_{23}^{32} + \theta_{21}^{33} + \theta_{22}^{33} + \theta_{23}^{33} \right]}_{Q_{4,3}}}$$

- $\theta_{kl}^{k'l'} = \text{unique element}$

Maximum Arbitrary Errors: Assumption 2

$$\begin{aligned} LB_{31} &= \frac{r_{31} - \tilde{Q}}{p_3} \geq 0 & \tilde{Q} &= \begin{cases} Q & \text{AE} \\ Q/3 & \text{UE} \end{cases} \\ UB_{31} &= \frac{r_{31} + \tilde{Q}}{p_3 - \tilde{Q}} \leq 1 & \tilde{Q} &= \begin{cases} 0 & \text{AE} \\ \min\{p_3, Q/3\} & \text{UE} \end{cases} \end{aligned}$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{31}^* = \frac{r_{31} + \overbrace{\left[\alpha_3^1 \beta_1^1 + \alpha_3^1 \beta_1^2 + \alpha_3^1 \beta_1^3 + \alpha_3^2 \beta_1^1 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_1^3 + \alpha_3^3 \beta_1^1 + \alpha_3^3 \beta_1^2 + \alpha_3^3 \beta_1^3 \right]}^{Q_{1,31}} - \overbrace{\left[\alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_1^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_1^3 + \alpha_3^3 \beta_2^1 + \alpha_3^3 \beta_2^2 + \alpha_3^3 \beta_2^3 \right]}^{Q_{2,31}}}{p_3 + \underbrace{\left[\alpha_3^1 \beta_1^1 + \alpha_3^1 \beta_1^2 + \alpha_3^1 \beta_1^3 + \alpha_3^2 \beta_1^1 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_1^3 + \alpha_3^3 \beta_1^1 + \alpha_3^3 \beta_1^2 + \alpha_3^3 \beta_1^3 + \alpha_3^1 \beta_2^1 + \alpha_3^1 \beta_2^2 + \alpha_3^1 \beta_2^3 + \alpha_3^2 \beta_2^1 + \alpha_3^2 \beta_2^2 + \alpha_3^2 \beta_2^3 + \alpha_3^3 \beta_2^1 + \alpha_3^3 \beta_2^2 + \alpha_3^3 \beta_2^3 + \alpha_3^1 \beta_3^1 + \alpha_3^1 \beta_3^2 + \alpha_3^1 \beta_3^3 + \alpha_3^2 \beta_3^1 + \alpha_3^2 \beta_3^2 + \alpha_3^2 \beta_3^3 + \alpha_3^3 \beta_3^1 + \alpha_3^3 \beta_3^2 + \alpha_3^3 \beta_3^3 \right]}_{Q_{3,3}} - \underbrace{\left[\alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_1^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_1^3 + \alpha_3^3 \beta_2^1 + \alpha_3^3 \beta_2^2 + \alpha_3^3 \beta_2^3 + \alpha_3^1 \beta_3^1 + \alpha_3^1 \beta_3^2 + \alpha_3^1 \beta_3^3 + \alpha_3^2 \beta_3^1 + \alpha_3^2 \beta_3^2 + \alpha_3^2 \beta_3^3 + \alpha_3^3 \beta_3^1 + \alpha_3^3 \beta_3^2 + \alpha_3^3 \beta_3^3 \right]}_{Q_{4,3}}}$$

- Simplifying

$$\begin{aligned} Q_{1,31} &= (\alpha_3^1 + \alpha_3^2) + (\beta_1^2 + \beta_1^3) (1 - \alpha_3^1 - \alpha_3^2) & (\text{TI}) \\ &= (\theta_3^1 + \theta_3^2) + (\theta_1^2 + \theta_1^3) (1 - \theta_3^1 - \theta_3^2) & (\text{TIV}) \\ Q_{2,31} &= (\alpha_1^3 + \alpha_2^3) (1 + \beta_2^1 + \beta_3^1 - \beta_1^2 - \beta_1^3) + (\beta_2^1 + \beta_3^1) (1 - \alpha_3^1 - \alpha_3^2) & (\text{TI}) \\ &= (\theta_1^3 + \theta_2^3) (1 + \theta_2^1 + \theta_3^1 - \theta_1^2 - \theta_1^3) + (\theta_2^1 + \theta_3^1) (1 - \theta_3^1 - \theta_3^2) & (\text{TIV}) \\ Q_{3,3} &= 3 (\alpha_3^1 + \alpha_3^2) & (\text{TI}) \\ &= 3 (\theta_3^1 + \theta_3^2) & (\text{TIV}) \\ Q_{4,3} &= 3 (\alpha_1^3 + \alpha_2^3) & (\text{TI}) \\ &= 3 (\theta_1^3 + \theta_2^3) & (\text{TIV}) \end{aligned}$$

- Under Temporal Independence (Assumption 3)

$$p_{31}^* = \frac{r_{31} + (\beta_1^2 + \beta_1^3 - \beta_2^1 - \beta_3^1) + (\alpha_3^1 + \alpha_3^2 - \alpha_1^3 - \alpha_2^3) (1 + \beta_2^1 + \beta_3^1 - \beta_1^2 - \beta_1^3)}{p_3 + 3(\alpha_3^1 + \alpha_3^2 - \alpha_1^3 - \alpha_2^3)}$$

– Yields

$$\begin{aligned} LB_{31}^{TI} &= \min \left\{ \frac{r_{31} - Q/3}{p_3}, \frac{r_{31} + \tilde{\tilde{Q}}}{p_3 + 3\tilde{\tilde{Q}}}, \frac{r_{31} - \hat{Q}}{p_3 - 3\hat{Q}} \right\} \geq 0 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{31}, (1 - p_3)/3, Q/3\}, \quad \hat{Q} = \min \{r_{31}, p_3/3, Q/3\} \\ UB_{31}^{TI} &= \max \left\{ \frac{r_{31} + Q/3}{p_3}, \frac{r_{31} + \tilde{\tilde{Q}}}{p_3 + 3\tilde{\tilde{Q}}}, \frac{r_{31} - \hat{Q}}{p_3 - 3\hat{Q}} \right\} \leq 1 \\ \tilde{\tilde{Q}} &= \min \{1 - r_{31}, (1 - p_3)/3, Q/3\}, \quad \hat{Q} = \min \{r_{31}, p_3/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{31} + \tilde{\tilde{Q}}}{p_3 + 3\tilde{\tilde{Q}}} \right) / \partial \tilde{\tilde{Q}}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{31} + \tilde{\tilde{Q}}}{p_3 + 3\tilde{\tilde{Q}}} \right)}{\partial \tilde{\tilde{Q}}} \right) &= \operatorname{sgn} \left(\left(p_3 + 3\tilde{\tilde{Q}} \right) - 3 \left(r_{31} + \tilde{\tilde{Q}} \right) \right) \\ &= \operatorname{sgn} (p_3 - 3r_{31}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{31} - \hat{Q}}{p_3 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{31} - \hat{Q}}{p_3 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_3 - 3\hat{Q} \right) + 3 \left(r_{31} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{31} - p_3) \end{aligned}$$

- Under Temporal Invariance (Assumption 4)

$$p_{31}^* = \frac{r_{31} + (\theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^3) + (\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3) (\theta_2^1 + \theta_3^1 - \theta_1^2 - \theta_1^3)}{p_3 + 3 (\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3)}$$

– Yields

$$\begin{aligned} LB_{31}^{TIV} &= \min \left\{ \frac{r_{31} - (3 - \sqrt{9 - Q})}{p_3}, \frac{r_{31} + \tilde{Q}^2}{p_3 + 3\tilde{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \left\{ \frac{-(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4r_{31}}}{2}, \sqrt{1 - r_{31}}, (1 - p_3)/3, 3 - \sqrt{9 - Q} \right\} \\ UB_{31}^{TIV} &= \max \left\{ \frac{r_{31} + (3 - \sqrt{9 - Q})}{p_3}, \frac{r_{31} + \tilde{Q}^2}{p_3 - 3\tilde{Q}} \right\} \leq 1 \quad \tilde{Q} = \min \left\{ \sqrt{1 - r_{31}}, p_3/3, 3 - \sqrt{9 - Q} \right\} \end{aligned}$$

– Proof:

1. Evaluate $\partial LB_{31}^{TIV} / \partial \tilde{Q}$ and see when the sign is positive/negative.

$$\begin{aligned} \text{sgn} \left(\frac{\partial LB_{31}^{TIV}}{\partial \tilde{Q}} \right) &= \text{sgn} \left(2\tilde{Q} \left(p_3 + 3\tilde{Q} \right) - 3 \left(r_{31} + \tilde{Q}^2 \right) \right) \\ &= \text{sgn} \left(\tilde{Q} \left((2/3)p_3 + \tilde{Q} \right) - r_{31} \right) \\ \Rightarrow \text{sgn} \left(\frac{\partial LB_{31}^{TIV}}{\partial \tilde{Q}} \right) \Big|_{\tilde{Q}=0} &= \text{sgn}(-r_{31}) < 0 \\ &\Rightarrow \tilde{Q} > 0 \end{aligned}$$

2. Minimize LB_{31}^{TIV} s.t. \tilde{Q} being feasible

$$\begin{aligned} \frac{\partial LB_{31}^{TIV}}{\partial \tilde{Q}} &\propto \tilde{Q} \left((2/3)p_3 + \tilde{Q} \right) - r_{31} = 0 \\ \Rightarrow \tilde{Q}^* &= \frac{-(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4r_{31}}}{2} \end{aligned}$$

So, derivative starts off negative and then reaches zero at \tilde{Q}^* . Thus, $\frac{r_{31} + \tilde{Q}^2}{p_3 + 3\tilde{Q}}$ is minimized at \tilde{Q}^* .

A.1.8 p_{32}^*

$$p_{32}^* = \frac{r_{32} + \overbrace{\left[\theta_{32}^{11} + \theta_{32}^{12} + \theta_{32}^{13} + \theta_{32}^{21} + \theta_{32}^{22} + \theta_{32}^{23} + \theta_{32}^{31} + \theta_{32}^{33} \right]}^{Q_{1,32}} - \overbrace{\left[\theta_{11}^{32} + \theta_{12}^{32} + \theta_{13}^{32} + \theta_{21}^{32} + \theta_{22}^{32} + \theta_{23}^{32} + \theta_{31}^{32} + \theta_{33}^{32} \right]}^{Q_{2,32}}}{p_3 + \underbrace{\left[\theta_{31}^{11} + \theta_{31}^{12} + \theta_{31}^{13} + \theta_{31}^{21} + \theta_{31}^{22} + \theta_{31}^{23} + \theta_{32}^{11} + \theta_{32}^{12} + \theta_{32}^{13} + \theta_{32}^{21} + \theta_{32}^{22} + \theta_{32}^{23} + \theta_{33}^{11} + \theta_{33}^{12} + \theta_{33}^{13} + \theta_{33}^{21} + \theta_{33}^{22} + \theta_{33}^{23} \right]}_{Q_{3,3}}} - \underbrace{\left[\theta_{11}^{31} + \theta_{12}^{31} + \theta_{13}^{31} + \theta_{11}^{32} + \theta_{12}^{32} + \theta_{13}^{32} + \theta_{11}^{33} + \theta_{12}^{33} + \theta_{13}^{33} + \theta_{21}^{31} + \theta_{22}^{31} + \theta_{23}^{31} + \theta_{21}^{32} + \theta_{22}^{32} + \theta_{23}^{32} + \theta_{21}^{33} + \theta_{22}^{33} + \theta_{23}^{33} \right]}_{Q_{4,3}}$$

- $\theta_{kl}^{k'l'}$ = unique element

Maximum Arbitrary Errors: Assumption 2

$$LB_{32} = \frac{r_{32} - Q}{p_3} \geq 0$$

$$UB_{32} = \frac{r_{32} + Q}{p_3} \leq 1$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{32}^* = \frac{r_{32} + \overbrace{\left[\alpha_3^1 \beta_2^1 + \alpha_3^1 \beta_2^2 + \alpha_3^1 \beta_2^3 + \alpha_3^2 \beta_2^1 + \alpha_3^2 \beta_2^2 + \alpha_3^2 \beta_2^3 + \alpha_3^3 \beta_2^1 + \alpha_3^3 \beta_2^2 + \alpha_3^3 \beta_2^3 \right]}^{Q_{1,32}} - \overbrace{\left[\alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_2^2 + \alpha_1^3 \beta_3^2 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_2^2 + \alpha_2^3 \beta_3^2 + \alpha_3^3 \beta_1^2 + \alpha_3^3 \beta_2^2 + \alpha_3^3 \beta_3^2 \right]}^{Q_{2,32}}}{p_3 + \underbrace{\left[\alpha_3^1 \beta_1^1 + \alpha_3^1 \beta_1^2 + \alpha_3^1 \beta_1^3 + \alpha_3^2 \beta_1^1 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_1^3 + \alpha_3^3 \beta_1^1 + \alpha_3^3 \beta_1^2 + \alpha_3^3 \beta_1^3 + \alpha_3^2 \beta_2^1 + \alpha_3^2 \beta_2^2 + \alpha_3^2 \beta_2^3 + \alpha_3^3 \beta_2^1 + \alpha_3^3 \beta_2^2 + \alpha_3^3 \beta_2^3 + \alpha_3^1 \beta_3^1 + \alpha_3^1 \beta_3^2 + \alpha_3^1 \beta_3^3 + \alpha_3^2 \beta_3^1 + \alpha_3^2 \beta_3^2 + \alpha_3^2 \beta_3^3 + \alpha_3^3 \beta_3^1 + \alpha_3^3 \beta_3^2 + \alpha_3^3 \beta_3^3 \right]}_{Q_{3,3}}} - \underbrace{\left[\alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_2^1 + \alpha_1^3 \beta_3^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_2^2 + \alpha_1^3 \beta_3^2 + \alpha_1^3 \beta_1^3 + \alpha_1^3 \beta_2^3 + \alpha_1^3 \beta_3^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_2^1 + \alpha_2^3 \beta_3^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_2^2 + \alpha_2^3 \beta_3^2 + \alpha_2^3 \beta_1^3 + \alpha_2^3 \beta_2^3 + \alpha_2^3 \beta_3^3 \right]}_{Q_{4,3}}}$$

- Simplifying

$$\begin{aligned} Q_{1,32} &= (\alpha_3^1 + \alpha_3^2) + (\beta_2^1 + \beta_2^2) (1 - \alpha_3^1 - \alpha_3^2) & \text{(TI)} \\ &= (\theta_3^1 + \theta_3^2) + (\theta_2^2 + \theta_2^3) (1 - \theta_3^1 - \theta_3^2) & \text{(TIV)} \\ Q_{2,32} &= (\alpha_1^3 + \alpha_2^3) (1 + \beta_1^2 + \beta_3^2 - \beta_2^1 - \beta_2^2) + (\beta_1^2 + \beta_3^2) (1 - \alpha_3^1 - \alpha_3^2) & \text{(TI)} \\ &= (\theta_1^3 + \theta_2^3) (1 + \theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^2) + (\theta_1^2 + \theta_3^2) (1 - \theta_3^1 - \theta_3^2) & \text{(TIV)} \\ Q_{3,3} &= 3 (\alpha_3^1 + \alpha_3^2) & \text{(TI)} \\ &= 3 (\theta_3^1 + \theta_3^2) & \text{(TIV)} \\ Q_{4,3} &= 3 (\alpha_1^3 + \alpha_2^3) & \text{(TI)} \\ &= 3 (\theta_1^3 + \theta_2^3) & \text{(TIV)} \end{aligned}$$

- Under Temporal Independence (Assumption 3)

$$p_{32}^* = \frac{r_{32} + (\beta_2^1 + \beta_2^3 - \beta_1^2 - \beta_3^2) + (\alpha_3^1 + \alpha_3^2 - \alpha_1^3 - \alpha_2^3) (1 + \beta_1^2 + \beta_3^2 - \beta_2^1 - \beta_2^3)}{p_3 + 3(\alpha_3^1 + \alpha_3^2 - \alpha_1^3 - \alpha_2^3)}$$

– Yields

$$\begin{aligned} LB_{32}^{TI} &= \min \left\{ \frac{r_{32} - Q/3}{p_3}, \frac{r_{32} + \tilde{Q}}{p_3 + 3\tilde{Q}}, \frac{r_{32} - \hat{Q}}{p_3 - 3\hat{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \{1 - r_{32}, (1 - p_3)/3, Q/3\}, \quad \hat{Q} = \min \{r_{32}, p_3/3, Q/3\} \\ UB_{32}^{TI} &= \max \left\{ \frac{r_{32} + Q/3}{p_3}, \frac{r_{32} + \tilde{Q}}{p_3 + 3\tilde{Q}}, \frac{r_{32} - \hat{Q}}{p_3 - 3\hat{Q}} \right\} \leq 1 \\ \tilde{Q} &= \min \{1 - r_{32}, (1 - p_3)/3, Q/3\}, \quad \hat{Q} = \min \{r_{32}, p_3/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{32} + \tilde{Q}}{p_3 + 3\tilde{Q}} \right) / \partial \tilde{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{32} + \tilde{Q}}{p_3 + 3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) &= \operatorname{sgn} \left(\left(p_3 + 3\tilde{Q} \right) - 3 \left(r_{32} + \tilde{Q} \right) \right) \\ &= \operatorname{sgn} (p_3 - 3r_{32}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{32} - \hat{Q}}{p_3 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{32} - \hat{Q}}{p_3 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_3 - 3\hat{Q} \right) + 3 \left(r_{32} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{32} - p_3) \end{aligned}$$

- Under Temporal Invariance (Assumption 4)

$$p_{32}^* = \frac{r_{32} + (\theta_2^1 + \theta_3^1 - \theta_1^2 - \theta_1^3) + (\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3) (\theta_1^2 + \theta_3^2 - \theta_2^1 - \theta_2^3)}{p_3 + 3 (\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3)}$$

– Yields

$$\begin{aligned} LB_{32}^{TIV} &= \min \left\{ \frac{r_{32} - (3 - \sqrt{9 - Q})}{p_3}, \frac{r_{32} + \tilde{Q}^2}{p_3 + 3\tilde{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \left\{ \frac{-(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4r_{32}}}{2}, \sqrt{1 - r_{32}}, (1 - p_3)/3, 3 - \sqrt{9 - Q} \right\} \\ UB_{32}^{TIV} &= \max \left\{ \frac{r_{32} + (3 - \sqrt{9 - Q})}{p_3}, \frac{r_{32} + \tilde{Q}^2}{p_3 - 3\tilde{Q}} \right\} \leq 1 \quad \tilde{Q} = \min \left\{ \sqrt{1 - r_{32}}, p_3/3, 3 - \sqrt{9 - Q} \right\} \end{aligned}$$

– Proof:

1. Evaluate $\partial LB_{32}^{TIV} / \partial \tilde{Q}$ and see when the sign is positive/negative.

$$\begin{aligned} \text{sgn} \left(\frac{\partial LB_{32}^{TIV}}{\partial \tilde{Q}} \right) &= \text{sgn} \left(2\tilde{Q} \left(p_3 + 3\tilde{Q} \right) - 3 \left(r_{32} + \tilde{Q}^2 \right) \right) \\ &= \text{sgn} \left(\tilde{Q} \left((2/3)p_3 + \tilde{Q} \right) - r_{32} \right) \\ \Rightarrow \text{sgn} \left(\frac{\partial LB_{32}^{TIV}}{\partial \tilde{Q}} \right) \Big|_{\tilde{Q}=0} &= \text{sgn}(-r_{32}) < 0 \\ &\Rightarrow \tilde{Q} > 0 \end{aligned}$$

2. Minimize LB_{32}^{TIV} s.t. \tilde{Q} being feasible

$$\begin{aligned} \frac{\partial LB_{32}^{TIV}}{\partial \tilde{Q}} &\propto \tilde{Q} \left((2/3)p_3 + \tilde{Q} \right) - r_{32} = 0 \\ \Rightarrow \tilde{Q}^* &= \frac{-(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4r_{32}}}{2} \end{aligned}$$

So, derivative starts off negative and then reaches zero at \tilde{Q}^* . Thus, $\frac{r_{32} + \tilde{Q}^2}{p_3 + 3\tilde{Q}}$ is minimized at \tilde{Q}^* .

A.1.9 p_{33}^*

$$p_{33}^* = \frac{r_{33} + \overbrace{\left[\theta_{33}^{11} + \theta_{33}^{12} + \theta_{33}^{13} + \theta_{33}^{21} + \theta_{33}^{22} + \theta_{33}^{23} + \theta_{33}^{31} + \theta_{33}^{32} \right]}^{Q_{1,33}} - \overbrace{\left[\theta_{11}^{33} + \theta_{12}^{33} + \theta_{13}^{33} + \theta_{21}^{33} + \theta_{22}^{33} + \theta_{23}^{33} + \theta_{31}^{33} + \theta_{32}^{33} \right]}^{Q_{2,33}}}{p_3 + \underbrace{\left[\theta_{31}^{11} + \theta_{31}^{12} + \theta_{31}^{13} + \theta_{31}^{21} + \theta_{31}^{22} + \theta_{31}^{23} + \theta_{32}^{11} + \theta_{32}^{12} + \theta_{32}^{13} + \theta_{32}^{21} + \theta_{32}^{22} + \theta_{32}^{23} + \theta_{33}^{11} + \theta_{33}^{12} + \theta_{33}^{13} + \theta_{33}^{21} + \theta_{33}^{22} + \theta_{33}^{23} \right]}_{Q_{3,3}} - \underbrace{\left[\theta_{11}^{31} + \theta_{12}^{31} + \theta_{13}^{31} + \theta_{11}^{32} + \theta_{12}^{32} + \theta_{13}^{32} + \theta_{11}^{33} + \theta_{12}^{33} + \theta_{13}^{33} + \theta_{21}^{31} + \theta_{22}^{31} + \theta_{23}^{31} + \theta_{21}^{32} + \theta_{22}^{32} + \theta_{23}^{32} + \theta_{21}^{33} + \theta_{22}^{33} + \theta_{23}^{33} \right]}_{Q_{4,3}}}$$

- $\theta_{kl}^{k'l'} = \text{unique element}$

Maximum Arbitrary Errors: Assumption 2

$$LB_{33} = \frac{r_{33} - Q}{p_3} \geq 0$$

$$UB_{33} = \frac{r_{33} + Q}{p_3} \leq 1$$

Temporal Independence, Temporal Invariance: Assumptions 3 & 4

- Implies

$$p_{33}^* = \frac{r_{33} + \overbrace{\left[\alpha_3^1 \beta_3^1 + \alpha_3^1 \beta_3^2 + \alpha_3^1 \beta_3^3 + \alpha_3^2 \beta_3^1 + \alpha_3^2 \beta_3^2 + \alpha_3^2 \beta_3^3 + \alpha_3^3 \beta_3^1 + \alpha_3^3 \beta_3^2 + \alpha_3^3 \beta_3^3 \right]}^{Q_{1,33}} - \overbrace{\left[\alpha_1^3 \beta_1^3 + \alpha_1^3 \beta_2^3 + \alpha_1^3 \beta_3^3 + \alpha_2^3 \beta_1^3 + \alpha_2^3 \beta_2^3 + \alpha_2^3 \beta_3^3 + \alpha_3^3 \beta_1^3 + \alpha_3^3 \beta_2^3 + \alpha_3^3 \beta_3^3 \right]}^{Q_{2,33}}}{p_3 + \underbrace{\left[\alpha_3^1 \beta_1^1 + \alpha_3^1 \beta_1^2 + \alpha_3^1 \beta_1^3 + \alpha_3^2 \beta_1^1 + \alpha_3^2 \beta_1^2 + \alpha_3^2 \beta_1^3 + \alpha_3^3 \beta_1^1 + \alpha_3^3 \beta_1^2 + \alpha_3^3 \beta_1^3 + \alpha_3^1 \beta_2^1 + \alpha_3^1 \beta_2^2 + \alpha_3^1 \beta_2^3 + \alpha_3^2 \beta_2^1 + \alpha_3^2 \beta_2^2 + \alpha_3^2 \beta_2^3 + \alpha_3^3 \beta_2^1 + \alpha_3^3 \beta_2^2 + \alpha_3^3 \beta_2^3 + \alpha_3^1 \beta_3^1 + \alpha_3^1 \beta_3^2 + \alpha_3^1 \beta_3^3 + \alpha_3^2 \beta_3^1 + \alpha_3^2 \beta_3^2 + \alpha_3^2 \beta_3^3 + \alpha_3^3 \beta_3^1 + \alpha_3^3 \beta_3^2 + \alpha_3^3 \beta_3^3 \right]}_{Q_{3,3}} - \underbrace{\left[\alpha_1^3 \beta_1^1 + \alpha_1^3 \beta_2^1 + \alpha_1^3 \beta_3^1 + \alpha_1^3 \beta_1^2 + \alpha_1^3 \beta_2^2 + \alpha_1^3 \beta_3^2 + \alpha_1^3 \beta_1^3 + \alpha_1^3 \beta_2^3 + \alpha_1^3 \beta_3^3 + \alpha_2^3 \beta_1^1 + \alpha_2^3 \beta_2^1 + \alpha_2^3 \beta_3^1 + \alpha_2^3 \beta_1^2 + \alpha_2^3 \beta_2^2 + \alpha_2^3 \beta_3^2 + \alpha_2^3 \beta_1^3 + \alpha_2^3 \beta_2^3 + \alpha_2^3 \beta_3^3 + \alpha_3^3 \beta_1^1 + \alpha_3^3 \beta_2^1 + \alpha_3^3 \beta_3^1 + \alpha_3^3 \beta_1^2 + \alpha_3^3 \beta_2^2 + \alpha_3^3 \beta_3^2 + \alpha_3^3 \beta_1^3 + \alpha_3^3 \beta_2^3 + \alpha_3^3 \beta_3^3 \right]}_{Q_{4,3}}}$$

- Simplifying

$$Q_{1,33} = (\alpha_3^1 + \alpha_3^2) + (\beta_3^1 + \beta_3^2) (1 - \alpha_3^1 - \alpha_3^2) \quad (\text{TI})$$

$$= 2 (\theta_3^1 + \theta_3^2) - (\theta_3^1 + \theta_3^2)^2 \quad (\text{TIV})$$

$$Q_{2,33} = (\alpha_1^3 + \alpha_2^3) (1 + \beta_1^3 + \beta_2^3 - \beta_3^1 - \beta_3^2) + (\beta_1^3 + \beta_2^3) (1 - \alpha_3^1 - \alpha_3^2) \quad (\text{TI})$$

$$= 2 (\theta_1^3 + \theta_2^3) (1 - \theta_3^1 - \theta_3^2) + (\theta_1^3 + \theta_2^3)^2 \quad (\text{TIV})$$

$$Q_{3,3} = 3 (\alpha_3^1 + \alpha_3^2) \quad (\text{TI})$$

$$= 3 (\theta_3^1 + \theta_3^2) \quad (\text{TIV})$$

$$Q_{4,3} = 3 (\alpha_1^3 + \alpha_2^3) \quad (\text{TI})$$

$$= 3 (\theta_1^3 + \theta_2^3) \quad (\text{TIV})$$

- Under Temporal Independence (Assumption 3)

$$p_{33}^* = \frac{r_{33} + (\alpha_3^1 + \alpha_3^2 - \alpha_1^3 - \alpha_2^3) + (\beta_3^1 + \beta_3^2 - \beta_1^3 - \beta_2^3) (1 + \alpha_1^3 + \alpha_2^3 - \alpha_3^1 - \alpha_3^2)}{p_2 + 3(\alpha_3^1 + \alpha_3^2 - \alpha_1^3 - \alpha_2^3)}$$

– Yields

$$\begin{aligned} LB_{33}^{TI} &= \min \left\{ \frac{r_{33} - Q/3}{p_3}, \frac{r_{33} + \tilde{Q}}{p_3 + 3\tilde{Q}}, \frac{r_{33} - \hat{Q}}{p_3 - 3\hat{Q}} \right\} \geq 0 \\ \tilde{Q} &= \min \{1 - r_{33}, (1 - p_3)/3, Q/3\}, \quad \hat{Q} = \min \{r_{33}, p_3/3, Q/3\} \\ UB_{33}^{TI} &= \max \left\{ \frac{r_{33} + Q/3}{p_3}, \frac{r_{33} + \tilde{Q}}{p_3 + 3\tilde{Q}}, \frac{r_{33} - \hat{Q}}{p_3 - 3\hat{Q}} \right\} \leq 1 \\ \tilde{Q} &= \min \{1 - r_{33}, (1 - p_3)/3, Q/3\}, \quad \hat{Q} = \min \{r_{33}, p_3/3, Q/3\} \end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{33} + \tilde{Q}}{p_3 + 3\tilde{Q}} \right) / \partial \tilde{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{33} + \tilde{Q}}{p_3 + 3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) &= \operatorname{sgn} \left(\left(p_3 + 3\tilde{Q} \right) - 3 \left(r_{33} + \tilde{Q} \right) \right) \\ &= \operatorname{sgn} (p_3 - 3r_{33}) \end{aligned}$$

2. Evaluate $\partial \left(\frac{r_{33} - \hat{Q}}{p_3 - 3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned} \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{33} - \hat{Q}}{p_3 - 3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left(- \left(p_3 - 3\hat{Q} \right) + 3 \left(r_{33} - \hat{Q} \right) \right) \\ &= \operatorname{sgn} (3r_{33} - p_3) \end{aligned}$$

- Under Temporal Invariance (Assumption 4)

$$p_{33}^* = \frac{r_{33} + 2(\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3) - (\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3)^2}{p_2 + 3(\theta_3^1 + \theta_3^2 - \theta_1^3 - \theta_2^3)}$$

– Yields

$$\begin{aligned}
LB_{33}^{TIV} &= \min \left\{ \frac{r_{33} + 2\widehat{Q} - \widehat{Q}^2}{p_3 + 3\widehat{Q}}, \frac{r_{33} - 2\widetilde{Q} - \widetilde{Q}^2}{p_3 - 3\widetilde{Q}} \right\} \geq 0 \\
\widehat{Q} &= \min \left\{ (1 - p_3)/3, 3 - \sqrt{9 - Q} \right\}, \\
\widetilde{Q} &= \begin{cases} 0 & r_{33} \geq 2p_3/3 \\ \min \left\{ \frac{(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}, (-1 + \sqrt{1 + r_{33}}), p_3/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases} \\
UB_{33}^{TIV} &= \min \left\{ \frac{r_{33} + 2\widehat{Q} - \widehat{Q}^2}{p_3 + 3\widehat{Q}}, \frac{r_{33} - 2\widetilde{Q} - \widetilde{Q}^2}{p_3 - 3\widetilde{Q}} \right\} \geq 0 \\
\widehat{Q} &= \begin{cases} 0 & r_{33} \geq 2p_3/3 \\ \min \left\{ \frac{-(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}, (1 - p_3)/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases}, \\
\widetilde{Q} &= \begin{cases} 0 & r_{33} < 2p_3/3 \\ \min \left\{ \frac{(2/3)p_3 - \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}, (-1 + \sqrt{1 + r_{33}}), p_3/3, 3 - \sqrt{9 - Q} \right\} & \text{otherwise} \end{cases}
\end{aligned}$$

– Proof:

1. Evaluate $\partial \left(\frac{r_{33}-2\tilde{Q}-\tilde{Q}^2}{p_3-3\tilde{Q}} \right) / \partial \tilde{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
& \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{33}-2\tilde{Q}-\tilde{Q}^2}{p_3-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) = \operatorname{sgn} \left(\left(-2 - 2\tilde{Q} \right) \left(p_3 - 3\tilde{Q} \right) + 3 \left(r_{33} - 2\tilde{Q} - \tilde{Q}^2 \right) \right) \\
& = \operatorname{sgn} \left(-(2/3)p_3 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{33} \right) \\
& \Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{33}-2\tilde{Q}-\tilde{Q}^2}{p_3-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) \Bigg|_{\tilde{Q}=0} = \operatorname{sgn} \left(-(2/3)p_3 + r_{33} \right) \geq 0 \\
& \Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{33}-2\tilde{Q}-\tilde{Q}^2}{p_3-3\tilde{Q}} \right)}{\partial \tilde{Q}} \right) \Bigg|_{\tilde{Q}=1} = \operatorname{sgn} \left(-(4/3)p_3 + 1 + r_{33} \right) \geq 0
\end{aligned}$$

2. Ensure $r_{33} - 2\tilde{Q} - \tilde{Q}^2 \geq 0$

$$\begin{aligned}
& r_{33} - 2\tilde{Q} - \tilde{Q}^2 \geq 0 \\
& \Rightarrow \tilde{Q}^2 + 2\tilde{Q} - r_{33} \leq 0 \\
& \Rightarrow \tilde{Q} \leq \frac{-2 + \sqrt{4 + 4r_{33}}}{2} \\
& \Rightarrow \tilde{Q} \leq -1 + \sqrt{1 + r_{33}}
\end{aligned}$$

3. Minimize $\frac{r_{33}-2\tilde{Q}-\tilde{Q}^2}{p_3-3\tilde{Q}}$ s.t. \tilde{Q} being feasible and $r_{33} < 2p_3/3$

$$\begin{aligned}
& \frac{\partial \left(\frac{r_{33}-2\tilde{Q}-\tilde{Q}^2}{p_3-3\tilde{Q}} \right)}{\partial \tilde{Q}} \propto -(2/3)p_3 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{33} = 0 \\
& \Rightarrow \tilde{Q}^* = \frac{(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}
\end{aligned}$$

4. Maximize $\frac{r_{33}-2\tilde{Q}-\tilde{Q}^2}{p_3-3\tilde{Q}}$ s.t. \tilde{Q} being feasible and $r_{33} > 2p_3/3$

$$\begin{aligned}
& \frac{\partial \left(\frac{r_{33}-2\tilde{Q}-\tilde{Q}^2}{p_3-3\tilde{Q}} \right)}{\partial \tilde{Q}} \propto -(2/3)p_3 \left(1 + \tilde{Q} \right) + \tilde{Q}^2 + r_{33} = 0 \\
& \Rightarrow \tilde{Q}^* = \frac{(2/3)p_3 - \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}
\end{aligned}$$

Note: If $\sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]} = .$, then maximize \tilde{Q} .

5. Evaluate $\partial \left(\frac{r_{33}+2\hat{Q}-\hat{Q}^2}{p_3+3\hat{Q}} \right) / \partial \hat{Q}$ and see when the sign is positive/negative. Both are possible.

$$\begin{aligned}
\operatorname{sgn} \left(\frac{\partial \left(\frac{r_{33}+2\hat{Q}-\hat{Q}^2}{p_3+3\hat{Q}} \right)}{\partial \hat{Q}} \right) &= \operatorname{sgn} \left((2-2\hat{Q})(p_3+3\hat{Q}) - 3(r_{33}+2\hat{Q}-\hat{Q}^2) \right) \\
&= \operatorname{sgn} \left((2/3)p_3(1-\hat{Q}) - \hat{Q}^2 - r_{33} \right) \\
\Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{33}+2\hat{Q}-\hat{Q}^2}{p_3+3\hat{Q}} \right)}{\partial \hat{Q}} \right) \Big|_{\hat{Q}=0} &= \operatorname{sgn}((2/3)p_3 - r_{33}) \geq 0 \\
\Rightarrow \operatorname{sgn} \left(\frac{\partial \left(\frac{r_{33}+2\hat{Q}-\hat{Q}^2}{p_3+3\hat{Q}} \right)}{\partial \hat{Q}} \right) \Big|_{\hat{Q}=1} &= \operatorname{sgn}(-1 - r_{33}) < 0
\end{aligned}$$

6. Maximize $\frac{r_{33}+2\hat{Q}-\hat{Q}^2}{p_3+3\hat{Q}}$ s.t. \hat{Q} being feasible and $r_{33} < 2p_3/3$

$$\begin{aligned}
\frac{\partial \left(\frac{r_{33}+2\hat{Q}-\hat{Q}^2}{p_3+3\hat{Q}} \right)}{\partial \hat{Q}} &\propto (2/3)p_3(1-\hat{Q}) - \hat{Q}^2 - r_{33} = 0 \\
\Rightarrow \hat{Q}^* &= \frac{-(2/3)p_3 + \sqrt{(4/9)p_3^2 + 4[(2/3)p_3 - r_{33}]}}{2}
\end{aligned}$$

7. Minimize $\frac{r_{33}+2\hat{Q}-\hat{Q}^2}{p_3+3\hat{Q}} \Rightarrow \hat{Q} = 0$ or maximize \hat{Q} . However, if the minimum occurs when $\hat{Q} = 0$, then $\frac{r_{33}-2\tilde{\tilde{Q}}-\tilde{\tilde{Q}}^2}{p_3-3\tilde{\tilde{Q}}} < \frac{r_{33}}{p_3}$ and this will be the binding *LB*.

A.2 Tightening the Bounds

A.2.1 Shape Restrictions: Assumption 5

$$p_{11}^*, p_{22}^*, p_{33}^*$$

$$\begin{aligned} LB_{kk}^S &= \max \left\{ \sup_{k' \neq k} LB_{kk'}, \sup_{k' \neq k} LB_{k'k} \right\} \\ UB_{kk}^S &= UB_{kk} \end{aligned}$$

$$p_{12}^*, p_{13}^*, p_{23}^*$$

$$\begin{aligned} LB_{kk}^S &= LB_{kk} \\ UB_{kk}^S &= \min \{UB_{11}, UB_{12}, UB_{22}\} \end{aligned}$$

$$p_{21}^*, p_{31}^*, p_{32}^*$$

$$\begin{aligned} LB_{kk}^S &= LB_{kk} \\ UB_{kk}^S &= \min \{UB_{11}, UB_{21}, UB_{22}\} \end{aligned}$$

A.2.2 Monotonicity Restrictions: Assumption 6

$$p_{kl}^*(u) = \frac{r_{kl}(u) + Q_{1,kl}(u) - Q_{2,kl}(u)}{p_k(u) + Q_{3,k}(u) - Q_{4,k}(u)}$$

- Let $Q(u)$ be probability of misclassification conditional on $U = u$. Then

$$\sum_u p_u Q(u) \leq Q$$

- Implies

$$Q(u) = \begin{cases} Q/p_u & \text{No Independence} \\ Q & \text{Independence} \end{cases}$$

- Bounds

- Bounds on $p_{kl}^*(u)$ are identical to baseline with Q replaced by $Q(u)$
- After bounding $P_{01}^*(u)$, impose shape if desired
- Derive bounds on P_{01}^*
- Impose shape if desired

B Literature Review

Measurement error in income and consumption is widely acknowledged. Despite this, the literature on income and consumption mobility that focuses on India has only recently begun to take this issue seriously. Barrientos Q. et al. (2018) use panel data on rural households collected by the National Council of Applied Economic Research (NCAER) in 1994 and 2005. The authors estimate a bivariate probit model with poverty status in 1994 and 2005 as outcomes. Poverty status in 1994 is an included covariate in the equation for poverty status in 2005 and is instrumented for using land ownership in 1994. However, misclassification in poverty status in 2005 is not addressed, nor is the fact that IV is not generally a valid solution to misclassification of binary outcomes (Black et al. 2000). Pradhan and Mukherjee (2015) use the ARIS/REDS data spanning three decades (1982-2006) to assess income mobility. The authors employ an IV strategy proposed in Glewwe (2012) to estimate the correlation between ‘true’ initial and final incomes. Initial income is instrumented for using the dependency ratio (i.e., the ratio of family size to the number of income earners), land ownership, land reform (a dummy that captures the effect of implementation of land reforms in the village), and rainfall shocks. In contrast to Barrientos Q. et al. (2018), Pradhan and Mukherjee (2015) find evidence of low income mobility.

Generally speaking, IV is a useful tool to overcome measurement error in the absence of validation data or repeated measurements; it is an alternative mechanism by which additional information can be brought to bear on the analysis. However, the consistency of these IV studies rests on the validity of the chosen instruments. As noted by Lee et al. (2017, p. 39), “the plausibility of these instruments, as is often the case, can be debated.” Specifically, one can argue that the chosen instruments may be correlated with the error term due to the nonclassical nature of the measurement error or other omitted sources of heterogeneity. Moreover, IV techniques – as implemented in this literature – estimate a *single parameter* to characterize mobility. Heterogeneity in mobility patterns across the full income or consumption distribution is ignored.

In contrast, Azam (2016) examines economic mobility by calculating directional rank mobility (in addition to the traditional transition probabilities) following a novel approach de-

veloped in Bhattacharya and Mazumder (2011), Mazumder (2014), and Corak et al. (2014). This approach defines upward (downward) directional rank mobility as the probability that a household’s position in the income distribution in the final period surpasses (falls below) by a given amount the household’s position in the income distribution in the initial period, conditional on the household’s initial position in the income distribution. The author uses longitudinal household survey data collected by the NCAER to examine rural households from 1994 to 2012, as well as data from the IHDS to examine urban households from 2005 to 2012. By focusing on ranks, rather than actual incomes, rank-preserving measurement errors – but not other types of errors – are allowed, as acknowledged in Bhattacharya and Mazumder (2011). This seems like an untenable assumption.

Arunachalam and Shenoy (2017) develop a new method to detect household poverty traps and apply it to Indian data. Their method exploits the fact that a household just inside the threshold of a poverty trap is likely to suffer negative income growth as the trap pulls the household towards the impoverished steady state. In contrast, a household just above the threshold of a poverty trap is propelled to a higher steady state. Thus, the existence of poverty trap implies that the probability of negative income growth is a decreasing function of current household income. By contrast, if there are no poverty traps and households are converging to a single steady state, the probability of negative income growth is always rising. Using the ARIS-REDS data (1969-1999), the authors find no evidence of poverty traps. However, as noted in Arunachalam and Shenoy (2017, p. 221), “measurement error ... may mask a poverty trap.” To address this, the authors propose a consistent measure of household income across survey waves instead of using self-reported income. Nonetheless, Arunachalam and Shenoy (2017, p. 223) state that “given the complexity of a poor household’s balance sheet, it is not clear what the ideal measure of income is, let alone whether our definition matches it” and that “[even] these precautions may not remove all measurement error.”

Finally, Dang and Lanjouw (2018) use three cross-sectional rounds of data from the National Sample Survey (NSS) to compute rates of economic mobility using a synthetic panel approach developed in Dang et al. (2014). While the authors do not explicitly address measurement error, their synthetic panel approach treats subsequent consumption as missing data which is an extreme form of measurement error. Specifically, the authors posit a static

model of consumption using only covariates that are collected in one survey round but whose values can be inferred for other rounds (e.g., time invariant variables). The model estimates, along with various assumptions concerning how unobserved determinants of consumption are correlated over time, are used to impute future consumption and then estimate a poverty transition matrix. However, measurement error in observed consumption used to estimate the static model upon which future consumption is imputed is not addressed.

In addition to the studies discussed above, a separate literature assesses intergenerational educational mobility (e.g., Azam and Bhatt 2015; Asher et al. 2018; Mohammed 2019) and intergenerational occupational mobility (Hnatkovska et al. 2013) in India. However, given our focus on intragenerational consumption mobility, we refrain from discussing these studies further.

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C Supplemental Tables

Table C1. State specific Poverty Lines (Tendulkar Committee estimates): Monthly Per Capita Expenditure (Rs.)

ID	State/Union Territories	2005		2012	
		Rural	Urban	Rural	Urban
1	Jammu & Kashmir	522	603	891	988
2	Himachal Pradesh	520	606	913	1,064
3	Punjab	544	643	1,054	1,155
4	Chandigarh	643	643	1,155	1,155
5	Uttarkhand/Uttaranchal	486	602	880	1,082
6	Haryana	529	626	1,015	1,169
7	Delhi	541	642	1,145	1,134
8	Rajasthan	478	568	905	1,002
9	Uttar Pradesh	435	532	768	941
10	Bihar	433	526	778	923
11	Sikkim	532	742	930	1,226
12	Arunachal Pradesh	547	618	930	1,060
13	Nagaland	687	783	1,270	1,302
14	Manipur	578	641	1,118	1,170
15	Mizoram	639	700	1,066	1,155
16	Tripura	450	556	798	920
17	Meghalaya	503	746	888	1,154
18	Assam	478	600	828	1,008
19	West Bengal	445	573	783	981
20	Jharkhand	405	531	748	974
21	Orissa	408	497	695	861
22	Chhattisgarh	399	514	738	849
23	Madhya Pradesh	408	532	771	897
24	Gujarat	502	659	932	1,152
25	Daman & Diu	609	671	1,090	1,134
26	Dadra & Nagar Haveli	485	632	967	1,126
27	Maharashtra	485	632	967	1,126
28	Andhra Pradesh	433	563	860	1,009
29	Karnataka	418	588	902	1,089
30	Goa	609	671	1,090	1,134
32	Kerala	537	585	1,018	987
33	Tamil Nadu	442	560	880	937
34	Puducherry	385	506	1,301	1,309

Source: Planning Commission (Available at <http://niti.gov.in/state-statistics>)

Table C2. Summary Statistics (Unweighted)

	2005 (Wave 1)		2012 (Wave 2)	
	Mean	SD	Mean	SD
Household Consumption				
Total Consumption (in Rs.)	54,493.00	52,225.89	115,364.00	119,271.50
Per Capita Consumption (in Rs.)	863.08	896.88	2,200.60	2,584.49
Poverty Status				
Poor (POVRATIO < 1)	0.33	0.47	0.15	0.36
Insecure Nonpoor (POVRATIO >= 1, < 2)	0.43	0.49	0.44	0.50
Secure Nonpoor (POVRATIO >= 2)	0.24	0.43	0.41	0.49
Household Size	5.85	3.02	4.87	2.33
Education (Household Head): Years of Schooling				
0 (Illiterate)	0.36	0.48		
1-5 years	0.21	0.41		
6-10 years	0.31	0.46		
11-15 years	0.12	0.32		
Caste (Household Head)				
Brahmin and Others	0.30	0.46		
OBC	0.40	0.49		
SC/ST	0.30	0.46		
Religion				
Hindu	0.81	0.39		
Muslims	0.12	0.32		
Other Religions	0.07	0.26		
Percentage of Males (Household Head)	0.91	0.29		
Percentage of Urban Residents (Household Head)	0.30	0.46		
Number of Observations	38,737		38,737	

Notes: POVRATIO is defined as the ratio of household per capita monthly household consumption expenditure to the poverty line per capita monthly consumption expenditure. In our analysis we use information pertaining to education level, caste, religion and the living region of household heads only from the first wave; hence we report summary statistics of these variables only for the first wave.

Table C3. Summary Statistics: Households Present in 2005 but not 2012

	Weighted		Unweighted	
	Mean	SD	Mean	SD
Household Consumption				
Total Consumption (in Rs.)	95,441.80	93,876.55	106,603.80	101,802.80
Per Capita Consumption (in Rs.)	2,099.87	2,285.81	2,323.71	2,542.57
Poverty Status				
Poor (POVRATIO < 1)	0.03	0.16	0.02	0.15
Insecure Nonpoor (POVRATIO >= 1, < 2)	0.25	0.44	0.22	0.41
Secure Nonpoor (POVRATIO >= 2)	0.72	0.45	0.76	0.43
Household Size	4.20	1.98	4.24	1.96
Education (Household Head): Years of Schooling				
0 (Illiterate)	0.28	0.45	0.25	0.43
1-5 years	0.17	0.37	0.15	0.36
6-10 years	0.32	0.47	0.34	0.47
11-15 years	0.22	0.42	0.26	0.44
Caste (Household Head)				
Brahmin and Others	0.45	0.50	0.46	0.50
OBC	0.18	0.38	0.16	0.37
SC/ST	0.38	0.48	0.38	0.49
Religion				
Hindu	0.79	0.41	0.79	0.40
Muslims	0.14	0.35	0.14	0.34
Other Religions	0.07	0.25	0.07	0.25
Percentage of Males (Household Head)	0.89	0.32	0.89	0.31
Percentage of Urban Residents (Household Head)	0.52	0.50	0.61	0.49
Number of Observations	6,534		6,534	

Notes: The sample size of households who were present in the 2005 wave but not in 2012 is 6911. However, we have valid information on demographics for only 6534 households out of the 6911 households. POVRATIO is defined as the ratio of household per capita monthly household consumption expenditure to the poverty line per capita monthly consumption expenditure. In our analysis we use information pertaining to education level, caste, religion and the living region of household heads only from the first wave; hence we report summary statistics of these variables only for the first wave.

Table C4. Subsample Transition Matrices by Religion: Arbitrary + Shape + Monotonicity

A. Hindu			B. Muslim			C. Others					
I. Q = 0.10											
1	2	3	1	2	3	1	2	3			
1	[0.214,0.546] {0.213,0.546} (0.205,0.556)	[0.280,0.583] {0.280,0.583} (0.269,0.591)	[0.000,0.491] {0.000,0.490} (0.000,0.497)	1	[0.295,0.493] {0.295,0.495} (0.279,0.510)	[0.349,0.525] {0.349,0.526} (0.329,0.540)	[0.015,0.356] {0.001,0.356} (0.000,0.392)	1	[0.155,0.558] {0.160,0.562} (0.127,0.587)	[0.327,0.585] {0.326,0.593} (0.291,0.610)	[0.000,0.517] {0.000,0.514} (0.000,0.560)
2	[0.000,0.357] {0.000,0.358} (0.000,0.363)	[0.268,0.708] {0.268,0.708} (0.260,0.716)	[0.160,0.601] {0.160,0.601} (0.153,0.608)	2	[0.000,0.365] {0.000,0.364} (0.000,0.375)	[0.308,0.742] {0.308,0.742} (0.294,0.758)	[0.114,0.546] {0.114,0.551} (0.099,0.563)	2	[0.000,0.375] {0.000,0.376} (0.000,0.393)	[0.275,0.654] {0.275,0.653} (0.253,0.675)	[0.206,0.586] {0.207,0.593} (0.186,0.610)
3	[0.000,0.348] {0.000,0.348} (0.000,0.353)	[0.000,0.666] {0.000,0.667} (0.000,0.674)	[0.190,1.000] {0.190,1.000} (0.180,1.000)	3	[0.000,0.365] {0.000,0.364} (0.000,0.375)	[0.013,0.721] {0.000,0.723} (0.000,0.738)	[0.114,0.981] {0.114,0.993} (0.099,1.000)	3	[0.000,0.358] {0.000,0.359} (0.000,0.374)	[0.000,0.583] {0.000,0.580} (0.000,0.605)	[0.376,1.000] {0.379,1.000} (0.353,1.000)
II. Q = 0.20											
1	2	3	1	2	3	1	2	3			
1	[0.025,0.762] {0.024,0.760} (0.020,0.773)	[0.065,0.839] {0.065,0.838} (0.053,0.846)	[0.000,0.747] {0.000,0.748} (0.000,0.755)	1	[0.079,0.717] {0.081,0.721} (0.065,0.737)	[0.132,0.770] {0.134,0.777} (0.109,0.791)	[0.000,0.689] {0.000,0.688} (0.000,0.703)	1	[0.035,0.769] {0.035,0.768} (0.022,0.807)	[0.116,0.816] {0.116,0.817} (0.074,0.840)	[0.000,0.825] {0.000,0.833} (0.000,0.840)
2	[0.000,0.581] {0.000,0.581} (0.000,0.587)	[0.036,0.938] {0.036,0.938} (0.028,0.946)	[0.007,0.833] {0.006,0.832} (0.003,0.841)	2	[0.000,0.585] {0.000,0.585} (0.000,0.598)	[0.087,0.963] {0.087,0.963} (0.071,0.978)	[0.007,0.763] {0.004,0.767} (0.001,0.783)	2	[0.000,0.610] {0.000,0.608} (0.000,0.631)	[0.038,0.893] {0.038,0.893} (0.021,0.913)	[0.051,0.825] {0.051,0.833} (0.033,0.850)
3	[0.000,0.569] {0.000,0.570} (0.000,0.576)	[0.000,0.894] {0.000,0.896} (0.000,0.903)	[0.052,1.000] {0.052,1.000} (0.050,1.000)	3	[0.000,0.585] {0.000,0.585} (0.000,0.598)	[0.000,0.945] {0.000,0.948} (0.000,0.961)	[0.023,1.000] {0.024,1.000} (0.020,1.000)	3	[0.000,0.584] {0.000,0.584} (0.000,0.604)	[0.000,0.827] {0.000,0.833} (0.000,0.848)	[0.094,1.000] {0.094,1.000} (0.078,1.000)
III. Q = 0.30											
1	2	3	1	2	3	1	2	3			
1	[0.000,0.977] {0.000,0.975} (0.000,0.990)	[0.000,0.991] {0.000,0.991} (0.000,0.996)	[0.000,0.926] {0.000,0.927} (0.000,0.932)	1	[0.000,0.934] {0.000,0.935} (0.000,0.959)	[0.000,0.961] {0.000,0.962} (0.000,0.977)	[0.000,0.888] {0.000,0.885} (0.000,0.902)	1	[0.000,0.980] {0.000,0.988} (0.000,1.000)	[0.000,0.994] {0.000,0.998} (0.000,1.000)	[0.000,0.925] {0.000,0.925} (0.000,0.937)
2	[0.000,0.804] {0.000,0.805} (0.000,0.812)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	2	[0.000,0.806] {0.000,0.806} (0.000,0.822)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,0.967] {0.000,0.972} (0.000,0.989)	2	[0.000,0.845] {0.000,0.844} (0.000,0.868)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)
3	[0.000,0.791] {0.000,0.791} (0.000,0.798)	[0.000,0.969] {0.000,0.969} (0.000,0.971)	[0.029,1.000] {0.029,1.000} (0.027,1.000)	3	[0.000,0.806] {0.000,0.806} (0.000,0.822)	[0.000,0.993] {0.000,0.993} (0.000,0.997)	[0.006,1.000] {0.006,1.000} (0.002,1.000)	3	[0.000,0.811] {0.000,0.816} (0.000,0.832)	[0.000,0.946] {0.000,0.947} (0.000,0.951)	[0.052,1.000] {0.052,1.000} (0.048,1.000)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table C5. Subsample Transition Matrices by Religion: Arbitrary + Shape + Monotonicity + TID

A. Hindu			B. Muslim			C. Others					
I. Q = 0.10											
1	2	3	1	2	3	1	2	3			
1	[0.255,0.380] {0.247,0.380} (0.197,0.387)	[0.424,0.446] {0.422,0.454} (0.414,0.481)	[0.174,0.299] {0.166,0.299} (0.111,0.304)	1	[0.178,0.341] {0.214,0.343} (0.165,0.357)	[0.494,0.643] {0.492,0.520} (0.474,0.657)	[0.099,0.266] {0.137,0.266} (0.087,0.277)	1	[0.369,0.396] {0.348,0.387} (0.296,0.428)	[0.468,0.583] {0.466,0.583} (0.434,0.606)	[0.169,0.262] {0.182,0.211} (0.123,0.308)
2	[0.059,0.208] {0.059,0.208} (0.054,0.212)	[0.405,0.554] {0.403,0.554} (0.400,0.562)	[0.315,0.466] {0.315,0.466} (0.308,0.473)	2	[0.075,0.217] {0.074,0.215} (0.064,0.227)	[0.469,0.595] {0.456,0.595} (0.455,0.610)	[0.261,0.399] {0.261,0.397} (0.247,0.415)	2	[0.074,0.215] {0.070,0.217} (0.054,0.232)	[0.369,0.493] {0.332,0.492} (0.350,0.513)	[0.367,0.493] {0.367,0.494} (0.348,0.510)
3	[0.000,0.200] {0.000,0.201} (0.000,0.205)	[0.162,0.447] {0.159,0.447} (0.151,0.454)	[0.497,0.838] {0.497,0.841} (0.490,0.849)	3	[0.000,0.217] {0.000,0.217} (0.000,0.227)	[0.189,0.547] {0.175,0.548} (0.161,0.565)	[0.376,0.811] {0.376,0.825} (0.355,0.839)	3	[0.000,0.162] {0.000,0.162} (0.000,0.172)	[0.146,0.340] {0.140,0.337} (0.118,0.357)	[0.620,0.820] {0.620,0.820} (0.601,0.870)
II. Q = 0.20											
1	2	3	1	2	3	1	2	3			
1	[0.309,0.475] {0.274,0.476} (0.302,0.483)	[0.352,0.487] {0.351,0.488} (0.342,0.494)	[0.051,0.339] {0.048,0.374} (0.035,0.357)	1	[0.382,0.412] {0.325,0.419} (0.352,0.433)	[0.421,0.437] {0.421,0.439} (0.402,0.470)	[0.166,0.197] {0.123,0.236} (0.111,0.246)	1	[0.262,0.487] {0.230,0.499} (0.236,0.514)	[0.398,0.504] {0.396,0.520} (0.362,0.532)	[0.054,0.340] {0.035,0.374} (0.020,0.402)
2	[0.010,0.283] {0.010,0.283} (0.006,0.288)	[0.345,0.632] {0.345,0.632} (0.338,0.639)	[0.238,0.521] {0.237,0.521} (0.230,0.529)	2	[0.020,0.291] {0.017,0.291} (0.010,0.301)	[0.382,0.670] {0.382,0.668} (0.370,0.693)	[0.187,0.473] {0.187,0.477} (0.173,0.489)	2	[0.034,0.279] {0.034,0.286} (0.014,0.308)	[0.356,0.573] {0.356,0.573} (0.327,0.594)	[0.286,0.504] {0.286,0.511} (0.267,0.535)
3	[0.000,0.274] {0.000,0.274} (0.000,0.279)	[0.044,0.576] {0.043,0.575} (0.033,0.584)	[0.344,0.956] {0.344,0.957} (0.335,0.967)	3	[0.000,0.291] {0.000,0.291} (0.000,0.301)	[0.088,0.645] {0.080,0.645} (0.044,0.671)	[0.187,0.912] {0.172,0.920} (0.174,0.956)	3	[0.000,0.282] {0.000,0.284} (0.000,0.295)	[0.033,0.461] {0.026,0.458} (0.007,0.480)	[0.498,0.967] {0.498,0.974} (0.478,0.993)
III. Q = 0.30											
1	2	3	1	2	3	1	2	3			
1	[0.214,0.546] {0.191,0.546} (0.205,0.556)	[0.280,0.583] {0.280,0.581} (0.269,0.591)	[0.000,0.491] {0.000,0.490} (0.000,0.497)	1	[0.295,0.493] {0.278,0.495} (0.279,0.510)	[0.349,0.525] {0.349,0.526} (0.329,0.540)	[0.015,0.356] {0.001,0.373} (0.000,0.392)	1	[0.155,0.558] {0.135,0.562} (0.127,0.587)	[0.327,0.585] {0.326,0.592} (0.291,0.610)	[0.000,0.517] {0.000,0.531} (0.000,0.560)
2	[0.000,0.357] {0.000,0.358} (0.000,0.363)	[0.268,0.799] {0.268,0.799} (0.260,0.816)	[0.160,0.601] {0.160,0.600} (0.153,0.610)	2	[0.000,0.365] {0.000,0.364} (0.000,0.375)	[0.308,0.882] {0.308,0.882} (0.294,0.901)	[0.114,0.546] {0.114,0.550} (0.099,0.563)	2	[0.000,0.375] {0.000,0.376} (0.000,0.393)	[0.275,0.654] {0.275,0.648} (0.253,0.688)	[0.206,0.586] {0.206,0.592} (0.186,0.611)
3	[0.000,0.348] {0.000,0.348} (0.000,0.353)	[0.000,0.742] {0.000,0.744} (0.000,0.757)	[0.190,1.000] {0.190,1.000} (0.180,1.000)	3	[0.000,0.365] {0.000,0.364} (0.000,0.375)	[0.013,0.862] {0.000,0.881} (0.000,0.898)	[0.114,0.987] {0.091,1.000} (0.099,1.000)	3	[0.000,0.358] {0.000,0.359} (0.000,0.374)	[0.000,0.583] {0.000,0.580} (0.000,0.605)	[0.376,1.000] {0.376,1.000} (0.353,1.000)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio ≥ 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table C6. Subsample Transition Matrices by Caste: Arbitrary + Shape + Monotonicity

A. Brahmin/Upper Caste			B. SC/ST			C. OBC					
I. Q = 0.10											
	1	2	3		1	2	3		1	2	3
1	[0.102,0.530] {0.102,0.529} (0.094,0.550)	[0.288,0.605] {0.289,0.608} (0.267,0.618)	[0.000,0.610] {0.000,0.609} (0.000,0.639)	1	[0.274,0.564] {0.275,0.567} (0.264,0.575)	[0.294,0.568] {0.294,0.572} (0.280,0.583)	[0.000,0.362] {0.000,0.361} (0.000,0.369)	1	[0.228,0.513] {0.228,0.515} (0.217,0.524)	[0.290,0.546] {0.289,0.547} (0.276,0.556)	[0.000,0.482] {0.000,0.483} (0.000,0.507)
2	[0.000,0.312] {0.000,0.312} (0.000,0.320)	[0.286,0.665] {0.286,0.665} (0.277,0.674)	[0.242,0.617] {0.241,0.620} (0.233,0.629)	2	[0.000,0.433] {0.000,0.432} (0.000,0.443)	[0.274,0.758] {0.274,0.758} (0.265,0.772)	[0.067,0.553] {0.067,0.556} (0.055,0.566)	2	[0.000,0.339] {0.000,0.339} (0.000,0.346)	[0.277,0.705] {0.277,0.705} (0.266,0.714)	[0.173,0.600] {0.173,0.599} (0.164,0.610)
3	[0.000,0.289] {0.000,0.289} (0.000,0.296)	[0.000,0.538] {0.000,0.538} (0.000,0.548)	[0.429,0.994] {0.429,0.995} (0.419,1.000)	3	[0.000,0.433] {0.000,0.432} (0.000,0.443)	[0.000,0.743] {0.000,0.744} (0.000,0.764)	[0.059,1.000] {0.060,1.000} (0.045,1.000)	3	[0.000,0.336] {0.000,0.336} (0.000,0.342)	[0.000,0.680] {0.000,0.683} (0.000,0.690)	[0.173,1.000] {0.173,1.000} (0.164,1.000)
II. Q = 0.20											
	1	2	3		1	2	3		1	2	3
1	[0.001,0.813] {0.000,0.811} (0.000,0.838)	[0.006,0.872] {0.000,0.874} (0.000,0.884)	[0.000,0.872] {0.000,0.872} (0.000,0.881)	1	[0.069,0.745] {0.068,0.746} (0.059,0.760)	[0.113,0.782] {0.113,0.782} (0.098,0.793)	[0.000,0.570] {0.000,0.570} (0.000,0.580)	1	[0.020,0.750] {0.018,0.749} (0.013,0.766)	[0.053,0.818] {0.054,0.818} (0.035,0.828)	[0.000,0.770] {0.000,0.769} (0.000,0.781)
2	[0.000,0.523] {0.000,0.523} (0.000,0.532)	[0.055,0.891] {0.055,0.890} (0.045,0.901)	[0.036,0.851] {0.035,0.852} (0.027,0.862)	2	[0.000,0.681] {0.000,0.682} (0.000,0.692)	[0.065,0.970] {0.066,0.970} (0.054,0.983)	[0.000,0.806] {0.000,0.810} (0.000,0.823)	2	[0.000,0.554] {0.000,0.554} (0.000,0.562)	[0.057,0.923] {0.057,0.923} (0.046,0.934)	[0.015,0.820] {0.014,0.819} (0.008,0.831)
3	[0.000,0.490] {0.000,0.489} (0.000,0.497)	[0.000,0.782] {0.000,0.783} (0.000,0.792)	[0.157,1.000] {0.158,1.000} (0.150,1.000)	3	[0.000,0.681] {0.000,0.682} (0.000,0.692)	[0.000,0.957] {0.000,0.960} (0.000,0.977)	[0.010,1.000] {0.010,1.000} (0.008,1.000)	3	[0.000,0.552] {0.000,0.553} (0.000,0.560)	[0.000,0.896] {0.000,0.897} (0.000,0.908)	[0.034,1.000] {0.035,1.000} (0.032,1.000)
III. Q = 0.30											
	1	2	3		1	2	3		1	2	3
1	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,0.997] {0.000,1.000} (0.000,1.000)	1	[0.000,0.926] {0.000,0.926} (0.000,0.942)	[0.000,0.962] {0.000,0.963} (0.000,0.971)	[0.000,0.779] {0.000,0.779} (0.000,0.791)	1	[0.000,0.987] {0.000,0.988} (0.000,1.000)	[0.000,0.995] {0.000,0.995} (0.000,1.000)	[0.000,0.966] {0.000,0.966} (0.000,0.974)
2	[0.000,0.733] {0.000,0.734} (0.000,0.745)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,0.997] {0.000,1.000} (0.000,1.000)	2	[0.000,0.885] {0.000,0.887} (0.000,0.898)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	2	[0.000,0.768] {0.000,0.768} (0.000,0.779)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)
3	[0.000,0.690] {0.000,0.690} (0.000,0.699)	[0.000,0.922] {0.000,0.921} (0.000,0.925)	[0.076,1.000] {0.077,1.000} (0.073,1.000)	3	[0.000,0.885] {0.000,0.888} (0.000,0.898)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	3	[0.000,0.768] {0.000,0.769} (0.000,0.779)	[0.000,0.985] {0.000,0.985} (0.000,0.988)	[0.013,1.000] {0.013,1.000} (0.010,1.000)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table C7. Subsample Transition Matrices by Caste: Arbitrary + Shape + Monotonicity + TID

A. Brahmin/Upper Caste			B. SC/ST			C. OBC					
I. Q = 0.10											
	1	2	3		1	2	3		1	2	3
1	[0.225,0.324] {0.210,0.322} (0.195,0.358)	[0.477,0.576] {0.478,0.590} (0.457,0.631)	[0.199,0.299] {0.200,0.313} (0.175,0.348)	1	[0.413,0.427] {0.404,0.429} (0.370,0.437)	[0.420,0.434] {0.418,0.428} (0.409,0.477)	[0.139,0.167] {0.143,0.168} (0.110,0.195)	1	[0.278,0.341] {0.277,0.343} (0.207,0.349)	[0.448,0.583] {0.444,0.584} (0.436,0.636)	[0.139,0.274] {0.138,0.279} (0.128,0.318)
2	[0.032,0.171] {0.031,0.171} (0.027,0.177)	[0.383,0.511] {0.377,0.511} (0.376,0.520)	[0.396,0.517] {0.395,0.519} (0.387,0.525)	2	[0.107,0.266] {0.104,0.266} (0.093,0.274)	[0.429,0.591] {0.429,0.591} (0.417,0.603)	[0.226,0.385] {0.226,0.387} (0.213,0.397)	2	[0.053,0.194] {0.053,0.194} (0.047,0.200)	[0.427,0.559] {0.423,0.559} (0.418,0.569)	[0.318,0.454] {0.318,0.455} (0.310,0.465)
3	[0.000,0.134] {0.000,0.134} (0.000,0.138)	[0.155,0.336] {0.154,0.335} (0.144,0.345)	[0.632,0.845] {0.632,0.846} (0.622,0.856)	3	[0.000,0.266] {0.000,0.266} (0.000,0.275)	[0.196,0.571] {0.197,0.571} (0.162,0.589)	[0.254,0.804] {0.254,0.803} (0.237,0.838)	3	[0.000,0.192] {0.000,0.192} (0.000,0.198)	[0.187,0.484] {0.185,0.484} (0.169,0.496)	[0.461,0.813] {0.461,0.815} (0.449,0.831)
II. Q = 0.20											
	1	2	3		1	2	3		1	2	3
1	[0.222,0.436] {0.184,0.435} (0.207,0.454)	[0.383,0.501] {0.383,0.504} (0.362,0.514)	[0.128,0.395] {0.133,0.432} (0.090,0.431)	1	[0.344,0.496] {0.337,0.497} (0.334,0.506)	[0.354,0.479] {0.352,0.479} (0.342,0.498)	[0.030,0.292] {0.028,0.292} (0.018,0.300)	1	[0.328,0.433] {0.300,0.438} (0.318,0.444)	[0.369,0.447] {0.367,0.447} (0.356,0.455)	[0.134,0.303] {0.123,0.333} (0.113,0.326)
2	[0.003,0.227] {0.002,0.227} (0.000,0.237)	[0.363,0.588] {0.363,0.588} (0.355,0.597)	[0.319,0.527] {0.318,0.530} (0.310,0.542)	2	[0.040,0.350] {0.039,0.349} (0.026,0.359)	[0.345,0.682] {0.345,0.681} (0.335,0.706)	[0.143,0.469] {0.143,0.472} (0.129,0.481)	2	[0.009,0.268] {0.009,0.267} (0.004,0.273)	[0.350,0.632] {0.350,0.632} (0.340,0.642)	[0.246,0.527] {0.246,0.527} (0.237,0.537)
3	[0.000,0.223] {0.000,0.223} (0.000,0.228)	[0.061,0.437] {0.061,0.437} (0.049,0.446)	[0.531,0.939] {0.531,0.939} (0.521,0.951)	3	[0.000,0.350] {0.000,0.349} (0.000,0.359)	[0.039,0.682] {0.028,0.681} (0.006,0.706)	[0.143,0.961] {0.068,0.972} (0.129,0.994)	3	[0.000,0.264] {0.000,0.264} (0.000,0.270)	[0.042,0.589] {0.036,0.590} (0.023,0.599)	[0.300,0.958] {0.300,0.964} (0.286,0.977)
III. Q = 0.30											
	1	2	3		1	2	3		1	2	3
1	[0.102,0.530] {0.102,0.529} (0.094,0.550)	[0.288,0.605] {0.289,0.608} (0.267,0.619)	[0.000,0.610] {0.000,0.609} (0.000,0.639)	1	[0.274,0.564] {0.262,0.567} (0.264,0.575)	[0.294,0.568] {0.294,0.571} (0.280,0.583)	[0.000,0.362] {0.000,0.361} (0.000,0.369)	1	[0.228,0.513] {0.207,0.515} (0.217,0.524)	[0.290,0.546] {0.289,0.547} (0.276,0.556)	[0.000,0.482] {0.000,0.504} (0.000,0.507)
2	[0.000,0.312] {0.000,0.312} (0.000,0.320)	[0.286,0.667] {0.286,0.660} (0.277,0.686)	[0.242,0.617] {0.241,0.619} (0.233,0.630)	2	[0.000,0.433] {0.000,0.432} (0.000,0.443)	[0.274,0.896] {0.261,0.896} (0.265,0.928)	[0.067,0.553] {0.067,0.555} (0.055,0.567)	2	[0.000,0.339] {0.000,0.339} (0.000,0.346)	[0.277,0.778] {0.277,0.778} (0.266,0.806)	[0.173,0.602] {0.173,0.599} (0.164,0.613)
3	[0.000,0.289] {0.000,0.289} (0.000,0.296)	[0.000,0.538] {0.000,0.538} (0.000,0.548)	[0.429,1.000] {0.429,1.000} (0.419,1.000)	3	[0.000,0.433] {0.000,0.432} (0.000,0.443)	[0.000,0.864] {0.000,0.864} (0.000,0.918)	[0.059,1.000] {0.025,1.000} (0.045,1.000)	3	[0.000,0.336] {0.000,0.336} (0.000,0.342)	[0.000,0.744] {0.000,0.749} (0.000,0.771)	[0.173,1.000] {0.139,1.000} (0.164,1.000)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio >= 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). TID = Temporal Independence. Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table C8. Subsample Transition Matrices by Region: Arbitrary + Shape + Monotonicity

A. Urban				B. Rural			
I. Q = 0.10							
	1	2	3		1	2	3
1	[0.138,0.498] {0.137,0.501} (0.126,0.512)	[0.329,0.571] {0.330,0.574} (0.312,0.583)	[0.003,0.534] {0.000,0.533} (0.000,0.562)	1	[0.238,0.547] {0.238,0.547} (0.230,0.556)	[0.287,0.568] {0.286,0.568} (0.276,0.576)	[0.000,0.456] {0.000,0.456} (0.000,0.462)
2	[0.000,0.291] {0.000,0.291} (0.000,0.296)	[0.322,0.671] {0.322,0.671} (0.314,0.680)	[0.257,0.600] {0.257,0.602} (0.248,0.611)	2	[0.000,0.382] {0.000,0.381} (0.000,0.388)	[0.266,0.721] {0.266,0.720} (0.258,0.729)	[0.127,0.581] {0.128,0.581} (0.119,0.589)
3	[0.000,0.271] {0.000,0.271} (0.000,0.276)	[0.000,0.533] {0.000,0.533} (0.000,0.542)	[0.442,1.000] {0.442,1.000} (0.433,1.000)	3	[0.000,0.378] {0.000,0.378} (0.000,0.383)	[0.000,0.694] {0.000,0.694} (0.000,0.705)	[0.127,1.000] {0.128,1.000} (0.119,1.000)
II. Q = 0.20							
	1	2	3		1	2	3
1	[0.021,0.727] {0.020,0.726} (0.016,0.745)	[0.100,0.818] {0.102,0.818} (0.081,0.827)	[0.000,0.863] {0.000,0.864} (0.000,0.870)	1	[0.032,0.760] {0.032,0.761} (0.027,0.771)	[0.073,0.830] {0.073,0.831} (0.062,0.840)	[0.000,0.706] {0.000,0.706} (0.000,0.714)
2	[0.000,0.506] {0.000,0.505} (0.000,0.513)	[0.091,0.901] {0.091,0.901} (0.082,0.910)	[0.067,0.836] {0.067,0.838} (0.059,0.846)	2	[0.000,0.608] {0.000,0.608} (0.000,0.615)	[0.037,0.950] {0.037,0.949} (0.029,0.960)	[0.000,0.812] {0.000,0.813} (0.000,0.821)
3	[0.000,0.476] {0.000,0.476} (0.000,0.482)	[0.000,0.803] {0.000,0.804} (0.000,0.811)	[0.162,1.000] {0.162,1.000} (0.154,1.000)	3	[0.000,0.606] {0.000,0.605} (0.000,0.612)	[0.000,0.925] {0.000,0.925} (0.000,0.937)	[0.023,1.000] {0.024,1.000} (0.021,1.000)
III. Q = 0.30							
	1	2	3		1	2	3
1	[0.000,0.955] {0.000,0.954} (0.000,0.977)	[0.000,0.991] {0.000,0.990} (0.000,0.995)	[0.000,0.971] {0.000,0.970} (0.000,0.975)	1	[0.000,0.974] {0.000,0.974} (0.000,0.986)	[0.000,0.988] {0.000,0.989} (0.000,0.994)	[0.000,0.905] {0.000,0.905} (0.000,0.912)
2	[0.000,0.720] {0.000,0.720} (0.000,0.730)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	2	[0.000,0.835] {0.000,0.834} (0.000,0.844)	[0.000,1.000] {0.000,1.000} (0.000,1.000)	[0.000,1.000] {0.000,1.000} (0.000,1.000)
3	[0.000,0.681] {0.000,0.681} (0.000,0.689)	[0.000,0.915] {0.000,0.915} (0.000,0.918)	[0.083,1.000] {0.083,1.000} (0.080,1.000)	3	[0.000,0.833] {0.000,0.833} (0.000,0.842)	[0.000,0.991] {0.000,0.991} (0.000,0.994)	[0.007,1.000] {0.008,1.000} (0.005,1.000)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio ≥ 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.

Table C9. Subsample Transition Matrices by Region: Arbitrary + Shape + Monotonicity + TID

A. Urban				B. Rural			
I. Q = 0.10							
	1	2	3		1	2	3
1	[0.093,0.333] {0.093,0.336} (0.085,0.345)	[0.481,0.677] {0.481,0.675} (0.466,0.692)	[0.125,0.361] {0.122,0.364} (0.115,0.368)	1	[0.325,0.388] {0.333,0.390} (0.246,0.396)	[0.429,0.555] {0.428,0.556} (0.419,0.594)	[0.120,0.246] {0.111,0.238} (0.106,0.287)
2	[0.025,0.122] {0.025,0.118} (0.015,0.136)	[0.408,0.518] {0.408,0.518} (0.398,0.525)	[0.410,0.522] {0.411,0.522} (0.403,0.529)	2	[0.078,0.229] {0.078,0.228} (0.073,0.234)	[0.420,0.568] {0.420,0.567} (0.412,0.576)	[0.280,0.427] {0.280,0.427} (0.273,0.435)
3	[0.000,0.131] {0.000,0.131} (0.000,0.134)	[0.126,0.320] {0.122,0.320} (0.113,0.328)	[0.656,0.874] {0.656,0.878} (0.648,0.887)	3	[0.000,0.226] {0.000,0.226} (0.000,0.231)	[0.190,0.520] {0.190,0.520} (0.177,0.530)	[0.410,0.810] {0.410,0.810} (0.399,0.823)
II. Q = 0.20							
	1	2	3		1	2	3
1	[0.271,0.421] {0.233,0.426} (0.261,0.433)	[0.405,0.474] {0.406,0.475} (0.389,0.483)	[0.158,0.324] {0.161,0.361} (0.129,0.350)	1	[0.325,0.475] {0.311,0.478} (0.318,0.483)	[0.358,0.481] {0.358,0.480} (0.348,0.488)	[0.049,0.317] {0.045,0.331} (0.033,0.334)
2	[0.000,0.200] {0.000,0.200} (0.000,0.209)	[0.399,0.597] {0.399,0.597} (0.391,0.605)	[0.333,0.512] {0.334,0.515} (0.325,0.526)	2	[0.020,0.305] {0.020,0.305} (0.015,0.311)	[0.343,0.644] {0.343,0.644} (0.335,0.652)	[0.203,0.504] {0.203,0.504} (0.196,0.512)
3	[0.000,0.203] {0.000,0.202} (0.000,0.207)	[0.019,0.427] {0.019,0.427} (0.010,0.435)	[0.549,0.981] {0.549,0.981} (0.541,0.990)	3	[0.000,0.302] {0.000,0.302} (0.000,0.307)	[0.058,0.616] {0.052,0.616} (0.043,0.626)	[0.227,0.942] {0.227,0.948} (0.216,0.957)
III. Q = 0.30							
	1	2	3		1	2	3
1	[0.138,0.498] {0.136,0.501} (0.126,0.512)	[0.329,0.571] {0.330,0.574} (0.312,0.583)	[0.003,0.534] {0.000,0.534} (0.000,0.562)	1	[0.238,0.547] {0.218,0.547} (0.230,0.556)	[0.287,0.568] {0.286,0.568} (0.276,0.576)	[0.000,0.456] {0.000,0.456} (0.000,0.462)
2	[0.000,0.291] {0.000,0.291} (0.000,0.296)	[0.322,0.688] {0.322,0.688} (0.314,0.711)	[0.257,0.600] {0.257,0.602} (0.248,0.611)	2	[0.000,0.382] {0.000,0.381} (0.000,0.388)	[0.266,0.841] {0.266,0.838} (0.258,0.863)	[0.127,0.586] {0.127,0.586} (0.119,0.593)
3	[0.000,0.271] {0.000,0.271} (0.000,0.276)	[0.000,0.533] {0.000,0.533} (0.000,0.542)	[0.442,1.000] {0.442,1.000} (0.433,1.000)	3	[0.000,0.378] {0.000,0.378} (0.000,0.383)	[0.000,0.799] {0.000,0.803} (0.000,0.825)	[0.127,1.000] {0.093,1.000} (0.119,1.000)

Notes: Outcome = POVRATIO. 1 = poverty ratio < 1. 2 = poverty ratio is between 1 and 2. 3 = poverty ratio ≥ 2. Q = maximum misclassification rate. Education of the household head used as Monotone Instrumental Variable (MIV). TID = Temporal Independence. Point estimates for bounds provided in brackets. Bias-corrected point estimates for bounds provided in braces obtained using 100 bootstrap repetitions. 90% Imbens-Manski confidence intervals for the non-bias-corrected bounds provided in parentheses obtained using 250 bootstrap repetitions. See text for further details.