

Meier, Martin; Schipper, Burkhard

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CONDITIONAL DOMINANCE IN GAMES WITH UNAWARENESS*

Martin Meier[†]

Burkhard C. Schipper[‡]

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Abstract

Heifetz, Meier and Schipper (2013) introduced generalized extensive-form games that allow for asymmetric unawareness. Here, we study the normal form of a generalized extensive-form game. The generalized normal-form game associated to a generalized extensive-form game with unawareness may consist of a collection of normal-form games. We use it to characterize extensive-form rationalizability (resp. prudent rationalizability) in generalized extensive-form games by iterative conditional strict (resp. weak) dominance in the associated generalized normal-form. We also show that the analogue to iterated admissibility for generalized normal-form games is not independent of the extensive-form structure. This is because under unawareness, a player's information set not only determines which nodes he considers possible but also of which game tree(s) he is aware of.

Keywords: Awareness, unknown unknowns, extensive-form games, normal-form games, extensive-form rationalizability, prudent rationalizability, iterated conditional dominance, iterated admissibility.

JEL-Classifications: C72, D83.

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[†]Department of Economics, University of Bath, and Institut für Höhere Studien, Wien. Email: m.meier@bath.ac.uk

[‡]Department of Economics, University of California, Davis. Email: bcschipper@ucdavis.edu

1 Introduction

Recent years witnessed the extension of game theory to unawareness. The challenge was to invent formal tools for modeling asymmetric lack of conception, beliefs about other players' lack of conception etc. that goes beyond modeling standard asymmetric lack of information. Several frameworks have been proposed both for static games (Meier and Schipper, 2014, Feinberg, 2021, Sadzik, 2021, Perea, 2022) and dynamic games (Halpern and Rego, 2014, Heifetz, Meier, and Schipper, 2013, 2021, Feinberg, 2021, Grant and Quiggin, 2013, Li, 2008, Schipper, 2021, 2017, Guarino, 2020); see Schipper (2014) for a non-technical review. These tools have already lead to interesting applications such as in contract theory (Filiz-Ozbay, 2012, Auster, 2013, Auster and Pavoni, 2021, von Thadden and Zhao, 2012, Francetich and Schipper, 2022), political economy (Schipper and Woo, 2019), and finance (Schipper and Zhou, 2021).¹ Developing sensible solution concepts to games with unawareness has been another challenge. The problem with familiar equilibrium concepts is that a player's awareness may change strategically during the course of play making the equilibrium assumption of an ex ante ready-made behavioral convention often implausible; see Schipper (2021) of an extensive discussion. As a remedy, rationalizability concepts have been proposed (Heifetz, Meier, and Schipper, 2013, 2021) and successfully applied to applications (Schipper and Woo, 2019, Francetich and Schipper, 2022). It is well-known that rationalizability in the normal-form is equivalent to iterated elimination of strictly dominated strategies (Pearce, 1984). A similar result holds for extensive-form rationalizability and iterated elimination of conditional strictly dominated strategies (Shimoji and Watson, 1998, Chen and Micali, 2013). Such an equivalence is quite handy in applications as it is often easier to eliminate strategies by dominance arguments than eliminating beliefs. This paper studies the connection of rationalizability and dominance in games with unawareness.

We focus on generalized extensive-form games with unawareness (Heifetz, Meier, and Schipper, 2013) because as already mentioned, the challenge under unawareness is the strategic change of players' awareness throughout the play. This elevates the time structure of the strategic interaction as embodied in the generalized extensive-form. At the same time, it would make an equivalence between rationalizability notions and iterated dominance notions more surprising as latter are typically applied to the associated normal-form while former apply to the extensive-form. Arguably, the normal form “folds” the time structure. Nevertheless, Kohlberg and Mertens (1986) argued that for standard games without unawareness, the normal form contains all strategically relevant information. Could a similar claim also apply to strategic situations in which the very perceptions of the strategic situation by players change strategically in ways that ex ante are not anticipated by all players? To answer such a question in the context of extensive-form rationalizability notions, we first need to define the appropriate notion of normal form associated to a generalized extensive-form game with unawareness. For a standard

¹See <http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm> for a bibliography on unawareness.

extensive-form game without unawareness, the associated normal form features strategies of the players as primitives. A strategy assigns an action to each information set of the player. Games with unawareness consist of a forest of partially-ordered trees, each representing a partial view of the objectively feasible sequences of moves. A player’s information at a node of a tree may consist of nodes in a “less expressive” tree. A player may not be aware of all strategies *ex ante* and may discover further actions during the course of play. Yet, for any tree, there is a well-defined set of partial strategies, namely objects that assign actions to all information sets of the player in the tree and any less expressive tree. The associated normal form consists of a partially ordered set of normal forms, indexed by trees, taking the sets of partial strategies as primitives.

With the definition of associated normal form in place, we can characterize extensive-form rationalizability à la Pearce (1984) and Battigalli (1997) and extended to unawareness in Heifetz, Meier, and Schipper (2013) by iterated elimination of conditionally dominated strategies similar to Shimoji and Watson (1998) for standard games without unawareness. Yet, the twist is that (partial) strategies eliminated in less expressive normal forms must trigger also the elimination of any strategies in more expressive normal forms that they are part of. We illustrate this with a simple example in the next section.

In generalized extensive-form games with unawareness, extensive-form rationalizability may involve imprudent behavior. It may be rationalizable for a player to make an opponent aware of one of the opponent’s actions that is extremely bad for the player because the player believes that the opponent will not take this action. Heifetz, Meier, and Schipper (2021) introduced a “refinement” of extensive-form rationalizability using the idea of prudence or caution that proved to be instrumental in applications such as disclosure of verifiable information (Heifetz, Meier, and Schipper, 2013, Li and Schipper, 2020, 2021), electoral campaigning (Schipper and Woo, 2019), and screening under unawareness (Francetich and Schipper, 2022). This solution concept, called prudent rationalizability, could be viewed as an extensive-form analogue of iterated admissibility in dynamic games with unawareness. While iterated admissibility is known to be a refinement of (normal-form) rationalizability in normal-form games, prudent rationalizability is surprisingly *not* always a refinement of extensive-form rationalizability in terms of strategies (not even for standard extensive-form games without unawareness); see Heifetz, Meier, and Schipper (2021) for examples. In this paper, we characterize prudent rationalizability in generalized extensive-form games by iterated elimination of conditionally weakly dominated strategies in the associated generalized normal-form game. Again, the twist is that (partial) strategies eliminated in less expressive normal forms must trigger also the elimination of any strategies in more expressive normal forms that they are part of.

One could argue that the characterization of extensive-form rationalizability (resp. prudent rationalizability) by iterative conditional strict (resp. weak) dominance falls short of showing that one can capture sophisticated extensive-form reasoning in the normal-form. This is be-

cause we condition on normal-form information sets corresponding to information sets in the extensive-form game. In our context, a relevant normal-form information set consists of a (partial) strategy subspace corresponding to the (partial) strategies leading to the information set in the (generalized) extensive-form game. Thus, we implicitly make use of extensive-form structures in the definition of iterative conditional strict (resp. weak) dominance when conditioning on normal-form information sets. For standard games without unawareness, the remedy would be to use iterative admissibility. Indeed, Brandenburger and Friedenberg (2011) show that for standard extensive-form games with perfect recall, iterated admissibility coincides with iterative conditional weak dominance at every level. Thus, by an inductive application of Lemma 4 of Pearce (1984), in standard games without unawareness, prudent rationalizability and iterated admissibility in the associated normal-form game coincides at every level. However, as we demonstrate, for generalized extensive-form games with unawareness the appropriate definition of iterative admissibility must make use of information sets as well. This is due to the fact that information sets in generalized extensive-form games do not only model a player’s information in the standard sense but also her awareness. The player’s awareness of strategies is crucial for admissibility since at least at the first level, a player cautiously considers possible any of the opponents’ strategies that she is aware of. We defined iterated admissibility for games with unawareness and show that in extensive-form games with unawareness iterated admissibility is conceptually closer to iterated conditional weak dominance because it cannot be independent of awareness encoded in the extensive-form. We show that prudent rationalizability is characterized by iterated admissibility in games with unawareness.

The paper is organized as follows: The next section provides an introductory example. Section 3 recalls definitions of generalized extensive-form games with unawareness and extensive-form rationalizability. Section 4 provides the characterization by iterated conditional dominance. Section 5 focuses on prudent rationalizability and its characterization by conditional weak dominance and admissibility. Proofs are relegated to an appendix.

2 Introductory Example

In order to provide some intuition for the notion of normal form associated to extensive-form games with unawareness and the characterization of extensive-form rationalizability by iterated conditional dominance in games with unawareness, consider first the following standard “battle-of-the-sexes” game where Bach and Stravinsky concerts are the two available choices for each player

		II	
		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3

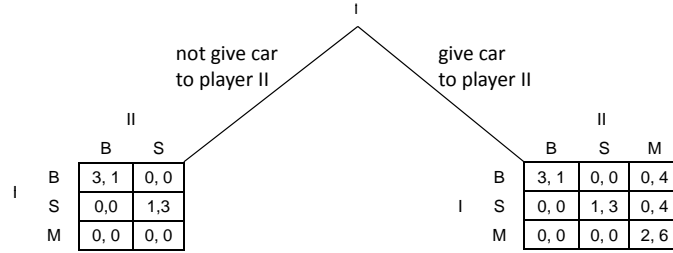
augmented by a dominant Mozart concert:

		II		
		B	S	M
I	B	3, 1	0, 0	0, 4
	S	0, 0	1, 3	0, 4
	M	0, 0	0, 0	2, 6

The new game is dominance solvable, and (M,M) is the unique Nash equilibrium.

Suppose that the Mozart concert is in a distant town, and II can go there only if player I gives him her car in the first place: Here, if player I does not give the car to player II, player II

Figure 1:



may conclude by forward induction that player I would go to the Bach concert with the hope of getting the payoff 3 (because by giving the car to II, player I could have achieved the payoff 2). The best reply of player II is to follow suit and attend the Bach concert as well. Hence, in the unique rationalizable outcome, player I is not to give the car to player II and to go to the Bach concert.²

How is this solution captured by iterated conditional strict dominance? Figure 2 shows the associated normal-form game. Each action in the normal form is a strategy in the extensive form. E.g., *nBB* assigns the action "not give the car to player II" to player I's first information set (i.e. the root of the tree in Figure 1), *B* to the left game and *B* to the right game. In Figure 2 we also indicated the three normal-form information sets with rectangles. A normal-form information set is a subspace of strategy profiles that corresponds to an information set in

²See Heifetz, Meier, and Schipper (2013) for further discussion of this example. For a discussion of forward induction in battle-of-the-sexes games, see van Damme (1989).

Figure 2:

	BB	BS	BM	SB	SS	SM	
nBB	3, 1	3, 1	3, 1	0, 0	0, 0	0, 0	
nBS	3, 1	3, 1	3, 1	0, 0	0, 0	0, 0	
nBM	3, 1	3, 1	3, 1	0, 0	0, 0	0, 0	
nSB	0, 0	0, 0	0, 0	1, 3	1, 3	1, 3	2.
nSS	0, 0	0, 0	0, 0	1, 3	1, 3	1, 3	
nSM	0, 0	0, 0	0, 0	1, 3	1, 3	1, 3	
nMB	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	1.
nMS	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	
nMM	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	
gBB	3, 1	0, 0	0, 4	3, 1	0, 0	0, 4	2.
gBS	0, 0	1, 3	0, 4	0, 0	1, 3	0, 4	
gBM	0, 0	0, 0	2, 6	0, 0	0, 0	2, 6	4.
gSB	3, 1	0, 0	0, 4	3, 1	0, 0	0, 4	2.
gSS	0, 0	1, 3	0, 4	0, 0	1, 3	0, 4	
gSM	0, 0	0, 0	2, 6	0, 0	0, 0	2, 6	4.
gMB	3, 1	0, 0	0, 4	3, 1	0, 0	0, 4	2.
gMS	0, 0	1, 3	0, 4	0, 0	1, 3	0, 4	
gMM	0, 0	0, 0	2, 6	0, 0	0, 0	2, 6	

1.
1.
3.

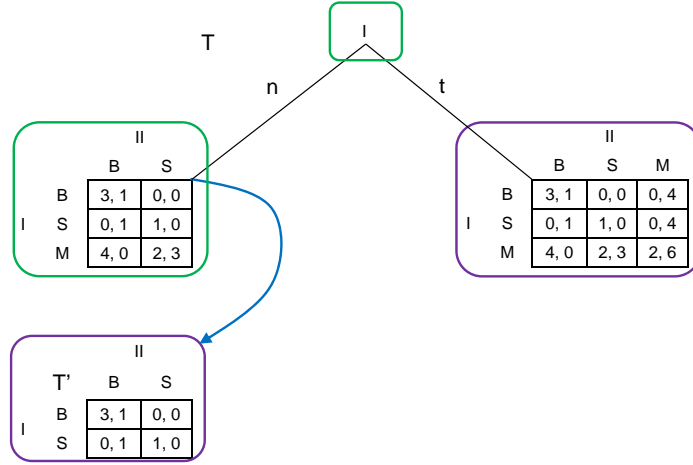
the extensive-form. The outer green rectangle corresponds to the root of the tree, an information of player I. The smaller purple rectangles correspond to the left and right subgame respectively. They represent information sets of both players.

With the dashed lines and the numbers beside them, we indicated the order of iterative elimination of conditional strictly dominated strategies. For instance, at the first level, for player II strategy *BS* is strictly dominated by strategy *BM conditional* on being in the lower normal-form information set. We emphasize that it is not strictly dominated per se but dominated conditionally on the lower normal-form information set. Thus, conditioning on normal-form information sets is crucial. As is shown in Figure 2, iterative conditional strict dominance singles out the forward induction outcome of extensive-form rationalizability. This is just an example of the general result by Shimoji and Watson (1998).

How to apply such a procedure to dynamic games with unawareness? An example of such a game is shown in Figure 3. This game is different from the Battle-of-the-Sexes with unawareness in Heifetz, Meier, and Schipper (2013). Compared to that game, we changed the payoffs in

order to demonstrate a particular feature of iterated conditional strict dominance in games with unawareness.³ There are now two trees, an upper tree T and the lower “tree” T' . This

Figure 3: Example of a Generalized Extensive-form Game



lower game T' is a partial description of the upper tree T and consists just of a 2×2 game. Any information set that belongs to both players is purple. The information sets that belong only to player I are green. When player I chooses n , then player II's information set is the purple information set in three T' . We indicate this with the blue arrow. This illustrates a feature of extensive-form games with unawareness, namely that the information set at a node in a tree may consist of nodes in a less expressive tree.

By taking action n at the root of the tree T , player I can keep player II unaware of his action M since the information set of player II is at the lower tree T' , a 2×2 game that misses the third action M . If instead player I chooses t at the root of the upper tree T , then he makes player II aware of action M and the information set of player II is now at the right side of the upper tree T . At this information set player II knows that player I told him about action M but he also realizes if player I would not tell him about M , then he would be unaware of it and view the game as in the lower tree T' . The purple information set of the lower tree T' in Figure 3 is also an information set of player 1. It models player I's state of mind as viewed

³The game is interesting yet for another reason. Heifetz, Meier, and Schipper (2013) compared a game with unavailability of an action to a close game with unawareness of an action. In these games, extensive-form rationalizability in game with unavailability of an action lead to a sharper prediction than extensive-form rationalizability in the game with unawareness of an action. The current example shows that this is not generally the case. I.e., replace the payoffs in Figure 1 with the payoffs from Figure 3. The resulting standard game is analogous to Figure 3 but with unavailability of an action rather than unawareness of an action. One can show that in this game with unavailability of an action, the extensive-form rationalizable outcome is not unique while there is a unique extensive-form rationalizable outcome in the game of Figure 3.

by anyone in the lower tree T' , i.e., anyone unaware of M . The green information sets in the upper tree T belong to player I only.

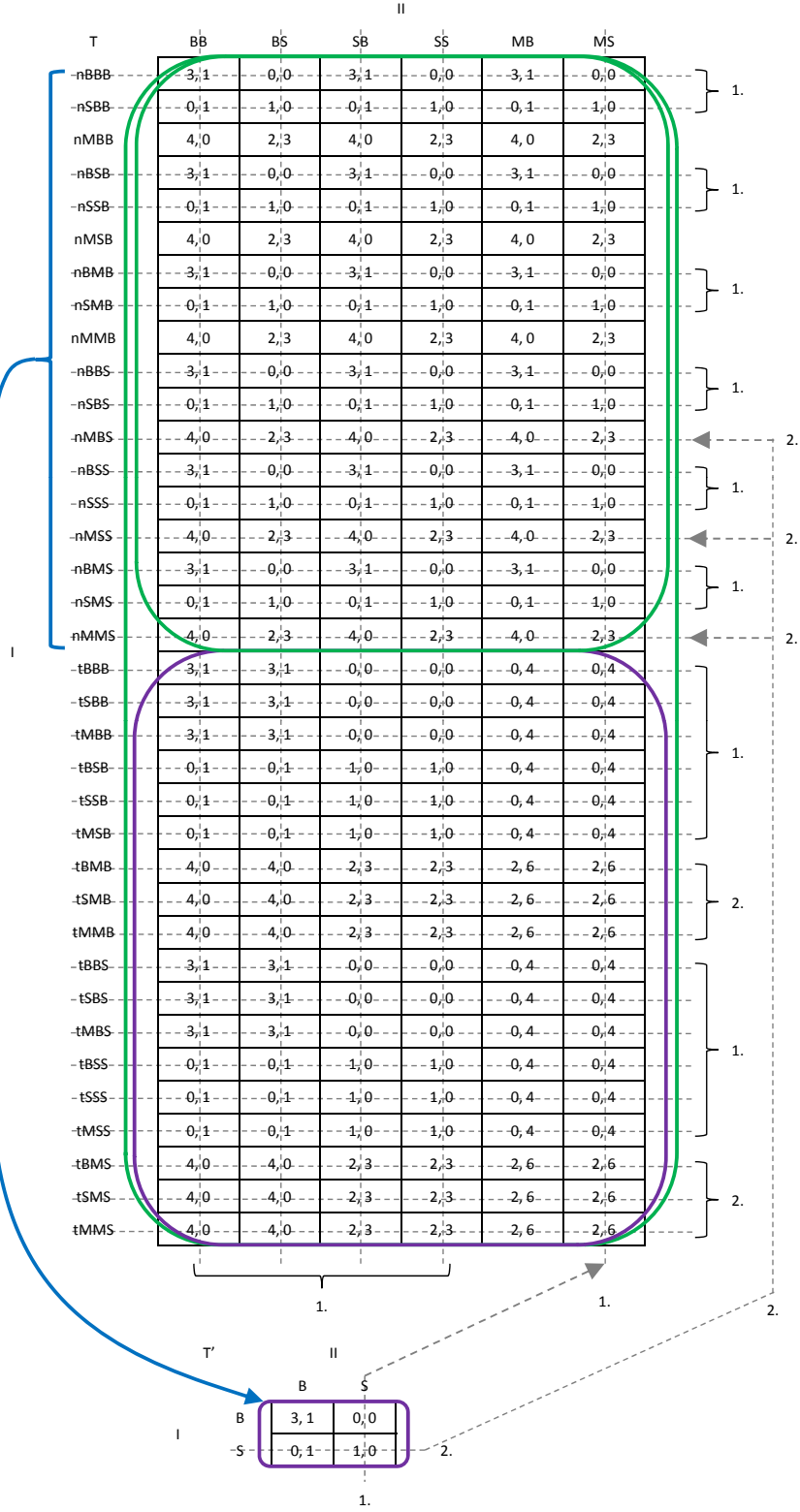
What is the associated normal form? For standard extensive-form games, the associated normal-form game is the strategic game in which the players' actions are the strategies in the extensive-form game. A strategy assigns to each of the player's information set an action. Since in a standard extensive-form game, every player is aware of all actions, in principle he can "control" his entire strategy *ex ante*. In generalized extensive-form games with unawareness, for each information set of a player her strategy specifies – from the point of view of the modeler – what the player would do if and when that information set of hers is ever reached. In this sense, a player does not necessarily 'own' her full strategy at the beginning of the game, because she might not be initially aware of all of her information sets. For instance, in Figure 3 a strategy of player II assigns an action to the right game in the upper tree T and an action to the lower game T' . A strategy for player I assigns an action to the root of the upper tree T , the left upper game, the right upper game and the lower tree T' . When players face the game in the lower tree T' , then they can only choose partial strategies, i.e. strategies restricted to information sets in T' . For instance, the set of T' -partial strategies of player I is $\{B, S\}$. With this notion of strategy, we can define the associated generalized normal-form as usual. Notice though that we get two normal-form games, one for the entire strategies and one for the T' -partial strategies. This is shown in Figure 4. Strategy $nMSB$ of player I means that he chooses n at the root of tree T , M in the left game of tree T , S in the right game of tree T , and B in the lower tree T' . Strategy BS of player II means that he chooses B in the upper right game of tree T and S in the lower tree T' . The T' -partial strategy B in the lower normal form indicates that the player chooses B in the lower tree T' .

We also indicated the normal-form information sets. As before, green belongs to player I only, purple to both players, and the blue arrow indicates that when player I chooses any strategy with action n , then the player II is unaware of M and his information set is in the lower tree T' . Here, the normal-form information set after player I chooses action n in T is in the lower normal-form associated with T' .

With the dashed lines and the numbers beside them, we indicate the order of iterative elimination of conditional strictly dominated strategies. But this algorithm is now more subtle since conditional dominance of a T' -partial strategy implies that all strategies in the game T with the same components (i.e., actions) are deleted as well. In the first round, this is the case for player II where deletion of S in the game T' implies that all strategies with S as the second component in the game T are eliminated as well. In particular, this applies to strategy MS that is not otherwise conditionally strictly dominated in the upper normal form associated to T .⁴ We indicate this with dashed arrows. A similar case arises for player II in the second

⁴This is the main reason for presenting an example that differs in payoffs from the example in Heifetz, Meier, and Schipper (2013). These different payoffs allow us to illustrate the elimination of a strategy in a normal form

Figure 4: Associated Generalized Normal-form Game of the Example



round. After two rounds, the process stops. The remaining strategies are the extensive-form rationalizable strategies. More precisely, each round of elimination of conditionally strictly dominated strategies corresponds to the corresponding level of extensive-form rationalization. In Section 3, we show that this is generally the case.

3 Generalized Extensive-Form Games with Unawareness

In this section we outline generalized extensive-form games with unawareness as introduced in Heifetz, Meier, and Schipper (2013).⁵ To define a generalized extensive-form game Γ , consider first, as a building block, a finite perfect information game with a set of players I , a set of decision nodes N_0 , active players I_n at node n with finite action sets A_n^i of player $i \in I_n$ (for $n \in N_0$), chance nodes C_0 , and terminal nodes Z_0 with a payoff vector $(p_i^z)_{i \in I} \in \mathbb{R}^I$ for the players for every $z \in Z_0$. The nodes $\bar{N}_0 = N_0 \cup C_0 \cup Z_0$ constitute a tree.

Consider now a family \mathbf{T} of subtrees of \bar{N}_0 , partially ordered (\preceq) by inclusion. One of the trees $T_1 \in \mathbf{T}$ is meant to represent the modeler's view of the paths of play that are *objectively* feasible; each other tree represents the feasible paths of play as *subjectively* viewed by some player at some node at one of the trees.

In each tree $T \in \mathbf{T}$, denote by n_T the copy in T of the node $n \in \bar{N}_0$ whenever the copy of n is part of the tree T . However, in what follows we will typically avoid the subscript T when no confusion may arise.

Denote by N_i^T the set of nodes in which player $i \in I$ is active in the tree $T \in \mathbf{T}$. We require that all the terminal nodes in each tree $T \in \mathbf{T}$ are copies of nodes in Z_0 . Moreover, if for two decision nodes $n, n' \in N_i^T$ (i.e. $i \in I_n \cap I_{n'}$) it is the case that $A_n^i \cap A_{n'}^i \neq \emptyset$, then $A_n^i = A_{n'}^i$.

Denote by N the union of all decision nodes in all trees $T \in \mathbf{T}$, by C the union of all chance nodes, by Z the union of terminal nodes, and by $\bar{N} = N \cup C \cup Z$. (Copies n_T of a given node n in different subtrees T are distinct from one another, so that \bar{N} is a disjoint union of sets of nodes.) For a node $n \in \bar{N}$ we denote by T_n the tree containing n .

For each decision node $n \in N$ and each active player $i \in I_n$, the information set is denoted by $\pi_i(n)$. It is the set of nodes that the player i considers as possible at n . $\pi_i(n)$ will be in a different tree than in the tree T_n if at n the player is unaware of some of the paths in T_n , and rather envisages the dynamic interaction as taking place in the tree containing $\pi_i(n)$. We impose properties analogous to standard extensive-form games with perfect recall (see Heifetz, Meier, and Schipper, 2013, Properties I0 to I6).

We denote by H_i the set of i 's information sets in all trees. For an information set $h_i \in H_i$, we denote by T_{h_i} the tree containing h_i . For two information sets h_i, h'_i in a given tree T , we

because it is conditionally dominated in a less expressive normal form.

⁵See also Heifetz, Meier, and Schipper (2021), Schipper (2021), and Schipper (2019).

say that h_i precedes h'_i (or that h'_i succeeds h_i) if for every $n' \in h'_i$ there is a path n, \dots, n' such that $n \in h_i$. We denote the precedence relation by $h_i \rightsquigarrow h'_i$.

Standard properties imply that if $n', n'' \in h_i$ where $h_i = \pi_i(n)$ is an information set, then $A_{n'}^i = A_{n''}^i$ (see Heifetz, Meier, and Schipper, 2013, Remark 1, for details). Thus, if $n \in h_i$ we write also A_{h_i} for A_n^i .

Perfect recall property guarantees that with the precedence relation \rightsquigarrow player i 's information sets H_i form an *arborescence*: For every information set $h'_i \in H_i$, the information sets preceding it $\{h_i \in H_i : h_i \rightsquigarrow h'_i\}$ are totally ordered by \rightsquigarrow .

For trees $T, T' \in \mathbf{T}$ we denote $T \succ T'$ whenever for some node $n \in T$ and some player $i \in I_n$ it is the case that $\pi_i(n) \subseteq T'$. Denote by \hookrightarrow the transitive closure of \succ . That is, $T \hookrightarrow T''$ if and only if there is a sequence of trees $T, T', \dots, T'' \in \mathbf{T}$ satisfying $T \succ T' \succ \dots \succ T''$.

A generalized extensive-form game Γ consists of a partially ordered set \mathbf{T} of subtrees of a tree \bar{N}_0 along with information sets $\pi_i(n)$ for every $n \in T$, $T \in \mathbf{T}$ and $i \in I_n$, satisfying all properties imposed in Heifetz, Meier, and Schipper (2013).⁶

For every tree $T \in \mathbf{T}$, the T -partial game is the partially ordered set of trees including T and all trees T' in Γ satisfying $T \hookrightarrow T'$, with information sets as defined in Γ . A T -partial game is a generalized game, i.e. it satisfies the same properties.

We denote by H_i^T the set of i 's information sets in the T -partial game.

A (pure) strategy

$$s_i \in S_i \equiv \prod_{h_i \in H_i} A_{h_i}$$

for player i specifies an action of player i at each of her information sets $h_i \in H_i$. Denote by

$$S = \prod_{j \in I} S_j$$

the set of strategy profiles in the generalized extensive-form game.

If $s_i = (a_{h_i})_{h_i \in H_i} \in S_i$, we denote by

$$s_i(h_i) = a_{h_i}$$

the player's action at the information set h_i . If player i is active at node n , we say that at node n the strategy prescribes to her the action $s_i(\pi_i(n))$.

In generalized extensive-form games, a strategy cannot be conceived as an ex ante plan of action. If $h_i \subseteq T$ but $T \not\hookrightarrow T'$, then at h_i player i may be interpreted as being unaware of her information sets in $H_i^{T'} \setminus H_i^T$.

⁶To save space, we refer to Heifetz, Meier, and Schipper (2013) for the exact statement and discussions of these properties.

Thus, a strategy of player i should rather be viewed as a list of answers to the hypothetical questions “what would the player do if h_i were the set of nodes she considered as possible?”, for $h_i \in H_i$. However, there is no guarantee that such a question about the information set $h'_i \in H_i^{T'}$ would even be meaningful to the player if it were asked at a different information set $h_i \in H_i^T$ when $T \not\rightarrow T'$. The answer should therefore be interpreted as given by the modeler, as part of the description of the situation.

For a strategy $s_i \in S_i$ and a tree $T \in \mathbf{T}$, we denote by s_i^T the strategy in the T -partial game induced by s_i . If $R_i \subseteq S_i$ is a set of strategies of player i , denote by R_i^T the set of strategies induced by R_i in the T -partial game. The set of i 's strategies in the T -partial game is thus denoted by S_i^T . Denote by $S^T = \prod_{j \in I} S_j^T$ the set of strategy profiles in the T -partial game.

We say that a strategy profile $s \in S$ *reaches* the information set $h_i \in H_i$ if the players' actions and nature's moves (if there are any) in T_{h_i} lead to h_i with a positive probability. (Notice that unlike in standard games, an information set $\pi_i(n)$ may be contained in tree $T' \neq T_n$. In such a case, $s_i(\pi_i(n))$ induces an action to player i also in n and not only in the nodes of $\pi_i(n)$.)

We say that the strategy $s_i \in S_i$ *reaches* the information set h_i if there is a strategy profile $s_{-i} \in S_{-i}$ of the other players such that the strategy profile (s_i, s_{-i}) reaches h_i . Otherwise, we say that the information set h_i is excluded by the strategy s_i .

Similarly, we say that the strategy profile $s_{-i} \in S_{-i}$ *reaches* the information set h_i if there exists a strategy $s_i \in S_i$ such that the strategy profile (s_i, s_{-i}) reaches h_i .

A strategy profile $(s_j)_{j \in I}$ *reaches a node* $n \in T$ if the players' actions $s_j(\pi_j(n'))_{j \in I}$ and nature's moves in the nodes $n' \in T$ lead to n with a positive probability. Since we consider only finite trees, $(s_j)_{j \in I}$ reaches an information set $h_i \in H_i$ if and if there is a node $n \in h_i$ such that $(s_j)_{j \in I}$ reaches n .

As it is the case also in standard games, for every given node, a given strategy profile of the players induces a distribution over terminal nodes in each tree, and hence an expected payoff for each player in the tree.

For an information set h_i , let $s_i/\tilde{s}_i^{h_i}$ denote the strategy that is obtained by replacing actions prescribed by s_i at the information set h_i and its successors by actions prescribed by \tilde{s}_i . The strategy $s_i/\tilde{s}_i^{h_i}$ is called an h_i -replacement of s_i .

3.1 Extensive-Form Rationalizability

Extensive-form rationalizability is an iterative procedure that for each player at each level rationalizes strategies with appropriate beliefs on the previous level rationalizable strategies of opponents. First, we need to define belief systems.

A *belief system* of player i

$$b_i = (b_i(h_i))_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta(S_{-i}^{T_{h_i}})$$

is a profile of beliefs - a belief $b_i(h_i) \in \Delta(S_{-i}^{T_{h_i}})$ about the other players' strategies in the T_{h_i} -partial game, for each information set $h_i \in H_i$, with the following properties

- $b_i(h_i)$ reaches h_i , i.e. $b_i(h_i)$ assigns probability 1 to the set of strategy profiles of the other players in the T_{h_i} -partial game that reach h_i .
- If h_i precedes h'_i ($h_i \rightsquigarrow h'_i$), then $b_i(h'_i)$ is derived from $b_i(h_i)$ by conditioning whenever possible.

Denote by B_i the set of player i 's belief systems.

For a belief system $b_i \in B_i$, a strategy $s_i \in S_i$ and an information set $h_i \in H_i$, define player i 's expected payoff **at** h_i to be the expected payoff for player i in T_{h_i} given $b_i(h_i)$, the actions prescribed by s_i at h_i and its successors, and conditional on the fact that h_i has been reached.⁷

We say that with the belief system b_i and the strategy s_i player i is *rational* at the information set $h_i \in H_i$, if either s_i doesn't reach h_i in the tree T_{h_i} , or if s_i does reach h_i in the tree T_{h_i} then there exists no h_i -replacement of s_i which yields player i a higher expected payoff in T_{h_i} given the belief $b_i(h_i)$ on the other players' strategies $S_{-i}^{T_{h_i}}$.

The generalization of Pearce's (1984) notion of extensive-form correlated rationalizable strategies is stated next. The inductive definition below generalizes Definition 2 in Battigalli (1997) to unawareness, which he proved to be equivalent to Pearce's original definition when correlation is allowed. This is the definition of extensive-form rationalizability used in Heifetz, Meier, and Schipper (2013).

Definition 1 (Extensive-form rationalizable strategies) *Define, inductively, the follow-*

⁷Even if this condition is counterfactual due to the fact that the strategy s_i does not reach h_i . The conditioning is thus on the event that nature's moves, if there are any, have led to the information set h_i , and assuming that player i 's past actions (in the information sets preceding h_i) have led to h_i *even if these actions are distinct than those prescribed by s_i* .

ing sequence of belief systems and strategies of player i .

$$\begin{aligned}
B_i^1 &= B_i \\
S_i^1 &= \left\{ s_i \in S_i : \begin{array}{l} \text{There exists a belief system } b_i \in B_i^1 \text{ with which for every information set} \\ h_i \in H_i, \text{ player } i \text{ is rational at } h_i. \end{array} \right\} \\
&\vdots \\
B_i^k &= \left\{ b_i \in B_i^{k-1} : \begin{array}{l} \text{For every information set } h_i, \text{ if there exists some profile of the other} \\ \text{players' strategies } s_{-i} \in S_{-i}^{k-1} \text{ such that } s_{-i} \text{ reaches } h_i \text{ in the tree } T_{h_i}, \\ \text{then } b_i(h_i) \text{ assigns probability 1 to } S_{-i}^{k-1, T_{h_i}}. \end{array} \right\} \\
S_i^k &= \left\{ s_i \in S_i : \begin{array}{l} \text{There exists a belief system } b_i \in B_i^k \text{ with which for every information set} \\ h_i \in H_i, \text{ player } i \text{ is rational at } h_i. \end{array} \right\}
\end{aligned}$$

The set of player i 's **extensive-form rationalizable strategies** is

$$S_i^\infty = \bigcap_{k=1}^{\infty} S_i^k.$$

Heifetz, Meier, and Schipper (2013) proved that $S_i^k \subseteq S_i^{k-1}$ for every $k > 1$. They also proved that for every finite generalized extensive-form game, the set of extensive-form correlated rationalizable strategies is non-empty.

Battigalli and Siniscalchi (2002) characterized extensive-form rationalizability by common strong belief in rationality. This has been extended to extensive-form games with unawareness by Guarino (2020).

4 Characterization of Extensive-Form Rationalizability by Conditional Dominance

4.1 Associated Normal-Form Games

Consider a generalized extensive-form game Γ with a partially ordered set of trees \mathbf{T} . The *associated normal-form game* G is defined by $\langle I, \langle (S_i^T)_{i \in I}, (u_i^T)_{i \in I} \rangle_{T \in \mathbf{T}} \rangle$, where I is the set of players in Γ and S_i^T is player i 's set of T -partial strategies. Recall that if player i is active at node $n \in T$, then at node n the strategy $s_i \in S_i^T$ prescribes to her the action $s_i(\pi_i(n))$. Hence, each profile of strategies in S^T induces a distribution over terminal nodes in T (even if there is a player active in T with no information set in T). $u_i^T(s)$ is the expected value of the payoffs associated with the terminal nodes in T reached by $s \in S^T$ weighted by the probabilities associated to the moves of nature. (Note that while strategy profiles in S^T reach terminal nodes also in trees $T' \in \mathbf{T}$, $T \hookrightarrow T'$, u_i^T concerns payoffs in the tree T only.)

Recall that H_i^T denotes player i 's set of extensive-form information sets in the T -partial game. For each $h_i \in H_i^T$, let $S^T(h_i) \subseteq S^T$ be the subset of the T -partial strategy space containing T -partial strategy profiles that reach the information set h_i . Define also $S_i^T(h_i) \subseteq S_i^T$ and $S_{-i}^T(h_i) \subseteq S_{-i}^T$ to be the set of player i 's T -partial strategies reaching h_i and the set of profiles of the other players' T -partial strategies reaching h_i respectively. For the entire game denote by $S(h_i) \subseteq S$ the set of strategy profiles that reach h_i . Similarly, $S_i(h_i) \subseteq S_i$ and $S_{-i}(h_i) \subseteq S_{-i}$ are the set of player i 's strategies reaching h_i and the set of profiles of the other players' strategies reaching h_i respectively.

Given Γ and its associated normal-form game G , define player i 's set of *normal-form information sets*⁸ by

$$\mathcal{X}_i = \{S^{T_{h_i}}(h_i) : h_i \in H_i\}.$$

These are the “normal-form versions” of information sets in the generalized extensive-form game.

For $T \in \mathbf{T}$, any set $Y \subseteq S^T$ is called a *restriction* for player i (or an i -product set) of T -partial strategies if $Y = Y_i \times Y_{-i}$ for some $Y_i \subseteq S_i^T$ and $Y_{-i} \subseteq S_{-i}^T$. Clearly, a player's normal-form information set is a restriction. I.e., if $S^{T_{h_i}}(h_i)$ is a normal-form information set of player i , then it is a restriction for player i of T_{h_i} -partial strategy profiles.

4.2 Iterated Conditional Strict Dominance and Extensive-Form Rationalizability

We say that $s_i \in S_i^T$ is *strictly dominated* in a restriction $Y \subseteq S^T$ if $s_i \in Y_i$, $Y_{-i} \neq \emptyset$, and there exists a mixed strategy $\sigma_i \in \Delta(Y_i)$ such that $u_i^T(\sigma_i, s_{-i}) > u_i^T(s_i, s_{-i})$ for all $s_{-i} \in Y_{-i}$.

Denote by $\mathbf{S} = \bigcup_{T \in \mathbf{T}} S^T$ and $\mathbf{S}_i = \bigcup_{T \in \mathbf{T}} S_i^T$.

For $T \hookrightarrow T'$ and a T -partial strategy $s_i \in S_i^T$, we denote the T' -partial strategy $s_i^{T'} \in S_i^{T'}$ induced by s_i . For $\tilde{s}_i \in S_i^{T'}$, define

$$[\tilde{s}_i] := \bigcup_{T \hookrightarrow T'} \{s_i \in S_i^T : s_i^{T'} = \tilde{s}_i\}.$$

That is, $[\tilde{s}_i]$ is the set of strategies in \mathbf{S}_i which at information sets $h_i \in H_i^{T'}$ prescribe the same actions as strategy \tilde{s}_i .

Let $(Y^T)_{T \in \mathbf{T}}$ be a collection of i -product sets, one for each $T \in \mathbf{T}$. Define $\mathbf{Y} = \bigcup_{T \in \mathbf{T}} Y^T$. Given such a \mathbf{Y} , we say that $s_i \in S_i^T$ is *conditionally strictly dominated on* $(\mathcal{X}_i, \mathbf{Y})$ if for

⁸We abuse here slightly existing terminology. In the literature on standard games, normal-form information sets refer more generally to subsets of the strategy space of a pure strategy reduced normal-form game for which there exists an extensive-form game with corresponding information sets (see Mailath, Samuelson, and Swinkels, 1993). For our characterization, we are just interested in the normal-form versions of information sets of a *given* generalized extensive-form game.

some $\tilde{s}_i \in S_i^{T'}$, $T \hookrightarrow T'$, $s_i \in [\tilde{s}_i]$, we have that \tilde{s}_i is strictly dominated in $X \cap Y^{T'}$ for some normal-form information set $X \in \mathcal{X}_i$, $X \subseteq S^{T'}$.

Note that this definition implies as a special case that $s_i \in S_i^T$ is conditionally strictly dominated on $(\mathcal{X}_i, \mathbf{Y})$ if there exists a normal-form information set $X \in \mathcal{X}_i$, $X \subseteq S^T$ such that s_i is strictly dominated in $X \cap Y^T$. Yet, the domination “across” normal-forms makes the definition a non-trivial generalization of conditional strict dominance in standard games.

For \mathbf{Y} define

$$U_i(\mathbf{Y}) = \{s_i \in \mathbf{S}_i : s_i \text{ is not conditionally strictly dominated on } (\mathcal{X}_i, \mathbf{Y})\},$$

$$U(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{i \in I} (U_i(\mathbf{Y}) \cap S_i^T),$$

and

$$U_{-i}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{j \in I \setminus \{i\}} (U_j(\mathbf{Y}) \cap S_j^T).$$

Define inductively

$$U^0(\mathbf{S}) = \mathbf{S},$$

$$U^{k+1}(\mathbf{S}) = U(U^k(\mathbf{S})) \text{ for } k \geq 0,$$

$$U^\infty(\mathbf{S}) = \bigcap_{k=0}^\infty U^k(\mathbf{S}),$$

and similarly for $U_i^k(\mathbf{S})$ and $U_{-i}^k(\mathbf{S})$.

This procedure formalizes iterated elimination of conditionally strictly dominated strategies.

Example (continuation). We will illustrate the definitions with the introductory example of Section 2. In this strategic situation, player I may deceive player II by hiding player II’s dominant action M . Recall that player I is the row player, while player II is the column player. For the row player in the upper normal form, the first component of her strategy refers to actions at the root of the upper tree, the second to her action in the upper left subgame, the third to the upper right subgame, and the last component to the action in the lower game. For the column player, the first component of his strategy refers to the action taken in the upper information set while the second is the action taken in the lower information set.

The entire upper normal form is the normal-form information set (marked green) of player I (but not player II) associated with player I’s information set at the beginning of the T -partial game (but not in the T' -partial game). We denote this information set by $X_I(\emptyset^T)$. The upper green rectangle in the upper normal form is the normal-form information set of player I (but not of player II) corresponding to her extensive-form information set after the history n in the T -partial game (but not in the T' -partial game). We denote it by $X_I(n)$. The lower purple rectangle in the upper normal-form game is the normal-form information set for both player I

and II corresponding to the information sets after history t in the T -partial game (but not in the T' -partial game). We denote it by $X_i(t)$.

Finally, the lower normal-form game is a normal-form information set (marked purple) for both player I and II both for corresponding information sets in the T' -partial normal form and in the T -partial normal-form game. It is also the normal-form information set for player II corresponding to his information set $\pi_{II}(n)$ in the T -partial game. We indicate this with the blue arrow. We denote it by $X_i(\emptyset^{T'}) = X_{II}(n)$.

The definition of \mathbf{S}_i is illustrated by the example $\mathbf{S}_{II} = \{BB, BS, SB, SS, MB, MS, B, S\}$, while the definition $[\tilde{s}_i]$ can be illustrated by $[\text{"S"}] = \{BS, SS, MS, S\}$. These are all the strategies of player II that prescribe the action “S” (“Stravinsky”) at the information set $\pi_{II}(n)$.

Iterated elimination of conditionally strictly dominated strategies proceeds as follows:

$$\begin{aligned} U_i^0(\mathbf{S}) &= \mathbf{S}_i, i = I, II, \\ U_I^1(\mathbf{S}) &= \{nMBB, nMSB, nMMB, nMBS, nMSS, nMMS, \\ &\quad tBMB, tSMB, tMMB, tBMS, tSMS, tMMS, \\ &\quad B, S\}, \\ U_{II}^1(\mathbf{S}) &= \{MB, B\}. \end{aligned}$$

For instance, strategy $nSBB$ is conditionally strictly dominated by $nMBB$ in the normal-form information set $X_I(\emptyset^T)$ or $X_I(n)$. More interestingly, MS is conditionally strictly dominated on $(\mathcal{X}_{II}, \mathbf{S})$ because $MS \in [\text{"S"}]$ and S is strictly dominated by B in $X_{II}(n)$. So this example demonstrates that *an action in the upper normal form may be deleted because of strict dominance in the lower normal form*. This is one reason why we chose this game to demonstrate iterated conditional strict dominance.

Applying the definitions iteratively yields

$$\begin{aligned} U_I^2(\mathbf{S}) &= \{nMBB, nMSB, nMMB, B\}, \\ &= U_I^k(\mathbf{S}) \text{ for } k \geq 2, \\ U_{II}^2(\mathbf{S}) &= U_{II}^2(\mathbf{S}) = \{MB, B\}, \\ &= U_{II}^k(\mathbf{S}) \text{ for } k \geq 1. \end{aligned}$$

Note that $U_i^\infty(\mathbf{S}) \cap S_i = \hat{S}_i^\infty$. That is, the set of strategies remaining after iterated elimination of conditionally strictly dominated strategies coincides with the set of extensive-form correlated rationalizable strategies. Both solution concepts predict that player I will not give the car to player II and attend the Mozart concert, while player II will attend the Bach concert. \square

Our main result is that iterated elimination of conditionally strictly dominated strategies characterizes extensive-form rationalizability in generalized extensive-form games.

Theorem 1 *For every finite generalized extensive-form game, $U_i^k(\mathbf{S}) \cap S_i = S_i^k$, $k \geq 1$. Consequently, $U_i^\infty(\mathbf{S}) \cap S_i = S_i^\infty$.*

The proof is contained in the appendix. It proceeds by induction and makes use of Lemma 3 by Pearce (1984). Theorem 1 generalizes Shimoji and Watson (1998) to extensive-form games with unawareness.

5 Prudent Rationalizability and Iterated Conditional Weak Dominance

Heifetz, Meier, and Schipper (2021) defined an outcome refinement of extensive-form rationalizability called prudent rationalizability.

Definition 2 (Prudent rationalizability) *For every player $i \in I$, let*

$$\bar{S}_i^0 = S_i.$$

For $k \geq 1$ define inductively

$$\begin{aligned} \bar{B}_i^k &= \left\{ b_i \in B_i : \begin{array}{l} \text{For every information set } h_i, \text{ if there exists some profile } s_{-i} \in \bar{S}_{-i}^{k-1} \text{ of the} \\ \text{other players' strategies such that } s_{-i} \text{ reaches } h_i \text{ in the tree } T_{h_i}, \text{ then the} \\ \text{support of } b_i(h_i) \text{ is the set of strategy profiles } s_{-i} \in \bar{S}_{-i}^{k-1, T_{h_i}} \text{ that reach } h_i. \end{array} \right\} \\ \bar{S}_i^k &= \left\{ s_i \in \bar{S}_i^{k-1} : \begin{array}{l} \text{There exists a belief system } b_i \in \bar{B}_i^k \text{ with which for every information set} \\ h_i \in H_i, \text{ player } i \text{ is rational at } h_i. \end{array} \right\} \end{aligned}$$

*The set of **prudent rationalizable strategies** of player i is*

$$\bar{S}_i^\infty = \bigcap_{k=1}^{\infty} \bar{S}_i^k$$

Heifetz, Meier, and Schipper (2021) proved that the set of prudent rationalizable strategies is non-empty and discussed various properties and examples.

5.1 Iterated Conditional Weak Dominance

We say that $s_i \in S_i^T$ is *weakly dominated* in a restriction $Y \subseteq S^T$ if $s_i \in Y_i$, $Y_{-i} \neq \emptyset$, and there exists a mixed strategy $\sigma_i \in \Delta(Y_i)$ such that $u_i^T(\sigma_i, s_{-i}) \geq u_i^T(s_i, s_{-i})$ for all $s_{-i} \in Y_{-i}$ and $u_i^T(\sigma_i, s_{-i}) > u_i^T(s_i, s_{-i})$ for some $s_{-i} \in Y_{-i}$.

Given a \mathbf{Y} , we say that $s_i \in S_i^T$ is *conditionally weakly dominated on* $(\mathcal{X}_i, \mathbf{Y})$ if for some $\tilde{s}_i \in S_i^{T'}$, $T \hookrightarrow T'$, $s_i \in [\tilde{s}_i]$, we have that \tilde{s}_i is weakly dominated in $X \cap Y^{T'}$ for some normal-form information set $X \in \mathcal{X}_i$, $X \subseteq S^{T'}$.

Note that this definition implies as a special case that $s_i \in S_i^T$ is conditionally weakly dominated on $(\mathcal{X}_i, \mathbf{Y})$ if there exists a normal-form information set $X \in \mathcal{X}_i$, $X \subseteq S^T$ such that s_i is weakly dominated in $X \cap Y^T$. Yet, the weak domination “across” normal forms makes this definition a non-trivial generalization of conditional weak dominance.

For \mathbf{Y} define

$$W_i(\mathbf{Y}) = \{s_i \in \mathbf{S}_i : s_i \text{ is not conditionally weakly dominated on } (\mathcal{X}_i, \mathbf{Y})\},$$

$$W(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{i \in I} (W_i(\mathbf{Y}) \cap S_i^T),$$

and

$$W_{-i}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{j \in I \setminus \{i\}} (W_j(\mathbf{Y}) \cap S_j^T).$$

Define inductively

$$W^0(\mathbf{S}) = \mathbf{S},$$

$$W^{k+1}(\mathbf{S}) = W(W^k(\mathbf{S})) \text{ for } k \geq 0,$$

$$W^\infty(\mathbf{S}) = \bigcap_{k=0}^\infty W^k(\mathbf{S}),$$

and similarly for $W_i^k(\mathbf{S})$ and $W_{-i}^k(\mathbf{S})$.

This procedure formalizes iterated elimination of conditionally weakly dominated strategies. We characterize prudent rationalizability by iterated elimination of conditionally weakly dominated strategies.

Theorem 2 *For every finite generalized extensive-form game, $W_i^k(\mathbf{S}) \cap S_i = \bar{S}_i^k$, $k \geq 1$. Consequently, $W_i^\infty(\mathbf{S}) \cap S_i = \bar{S}_i^\infty$.*

Proof. The proof is analogous to Theorem 1 in the appendix. Instead of using Lemma 3 in Pearce (1984), we now use Lemma 4 in Pearce (1984). \square

5.2 Iterated Admissibility

For standard extensive-form games with perfect recall and without unawareness, Brandenburger and Friedenberg (2011, Proposition 3.1) showed that iterated elimination of conditionally weakly dominated strategies coincides with iterated admissibility at each level of iteration in the associated normal-form game.

How to define iterated admissibility in generalized normal-form games with unawareness? Consider the example in Section 2. A first straightforward approach could be to apply iterated admissibility to each of the normal-form games separately. At the first level, the set of admissible strategies coincides with all the set of strategies that are not conditionally strict dominated except for MS of player II. (Recall that under conditional strict dominance, we were able to delete strategy MS because S was strictly dominated in the lower game T' .) Notice that at the second level, strategy MB becomes now weakly dominated by MS . But MB is the only extensive-form rationalizable strategy and prudent rationalizable strategy of player II in the T -partial game. What is wrong with this straightforward approach is that it does not eliminate a strategy in the T -partial game when it is weakly dominated by another strategy in a lower game. Strategies MB and MS differ in the second component only, the action of player II in the lower game T' . It is in this lower game T' that S is dominated by B , and hence any strategy prescribing S at the lower game T' should be eliminated. This motivates us to define iterated admissibility as a procedure that conditions more coarsely on a normal-form instead of any normal-form information set. But not any normal-form game will do. We also need to insure that for each information set, we condition on the “correct” normal-form game, namely the normal-form game that represents the player’s awareness at this information set.

More formally, let $\mathcal{S}_i = \{S^{T_{h_i}} : h_i \in H_i\}$. Given a \mathbf{Y} , we say that $s_i \in S_i^T$ is *NF-conditionally weakly dominated on $(\mathcal{S}_i, \mathbf{Y})$* if for some $\tilde{s}_i \in S_i^{T'}$, $T \hookrightarrow T'$, $s_i \in [\tilde{s}_i]$, we have that \tilde{s}_i is weakly dominated in $S^{T'} \cap Y^{T'}$ for some normal form $S^{T'} \in \mathcal{S}_i$.

Note that this definition implies as a special case that $s_i \in S_i^T$ is NF-conditionally weakly dominated on $(\mathcal{S}_i, \mathbf{Y})$ if there exists a normal form $S^T \in \mathcal{S}_i$, such that s_i is weakly dominated in $S^T \cap Y^T$. Again, the weak domination “across” normal forms makes this definition a non-trivial generalization of weak dominance.

For \mathbf{Y} define

$$\tilde{W}_i(\mathbf{Y}) = \{s_i \in \mathbf{S}_i : s_i \text{ is not NF-conditionally weakly dominated on } (\mathcal{S}_i, \mathbf{Y})\},$$

$$\tilde{W}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{i \in I} \left(\tilde{W}_i(\mathbf{Y}) \cap S_i^T \right),$$

and

$$\tilde{W}_{-i}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{j \in I \setminus \{i\}} \left(\tilde{W}_j(\mathbf{Y}) \cap S_j^T \right).$$

Define inductively

$$\tilde{W}^0(\mathbf{S}) = \mathbf{S},$$

$$\tilde{W}^{k+1}(\mathbf{S}) = \tilde{W}(\tilde{W}^k(\mathbf{S})) \text{ for } k \geq 0,$$

$$\tilde{W}^\infty(\mathbf{S}) = \bigcap_{k=0}^\infty \tilde{W}^k(\mathbf{S}),$$

and similarly for $\tilde{W}_i^k(\mathbf{S})$ and $\tilde{W}_{-i}^k(\mathbf{S})$.

This procedure defines iterated admissibility. We call any strategy of player i in $\tilde{W}_i(\mathbf{Y})$ admissible on $(\mathcal{S}_i, \mathbf{Y})$, and any strategy of player i in $\tilde{W}_i^\infty(\mathbf{S})$ as iterative admissible. While these definitions are identical to admissibility (resp., iterative admissibility) when restricted to standard games without unawareness, this terminology might be somewhat misleading. In games with unawareness, iterated admissibility is conceptually closer to iterated conditional weak dominance because it makes explicit use of information sets (for selecting appropriately among the normal-form games). This is due to the fact that information sets in generalized extensive-form games do not only model a player's information but also his awareness. The player's awareness of strategies is crucial for admissibility since it does not rule out any opponents' strategy from being played among the opponents' strategies the player is aware of.

Theorem 3 *For every finite generalized extensive-form game, $\tilde{W}_i^k(\mathbf{S}) = \bar{S}_i^k$, $k \geq 1$. Consequently, $\tilde{W}_i^\infty(\mathbf{S}) = \bar{S}_i^\infty$.*

The proof is contained in the appendix. There, we first define a version of prudent rationalizability that weakens dynamic properties of belief systems. We then show inductively the equivalence between that version of prudent rationalizability and iterated admissibility making use again of Lemma 4 in Pearce (1984). Finally, we show inductively the equivalence between that weakened version of prudent rationalizability and Definition 2.

As a corollary, iterated admissibility is equivalent to iterated conditional weak dominance at every level in games with unawareness. This result generalizes a result by Brandenburger and Friedenberg (2011, Proposition 3.1) to generalized extensive-form games with unawareness.

Corollary 1 *For every finite generalized extensive-form game, $\tilde{W}_i^k(\mathbf{S}) = W_i^k(\mathbf{S})$, $k \geq 1$. Consequently, $\tilde{W}_i^\infty(\mathbf{S}) = W_i^\infty(\mathbf{S})$.*

A Proofs

A.1 Proof of Theorem 1

A *general belief system* of player i

$$\tilde{b}_i = (\tilde{b}_i(h_i))_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta(S_{-i}^{T_{h_i}})$$

is a profile of beliefs – a belief $\tilde{b}_i(h_i) \in \Delta(S_{-i}^{T_{h_i}})$ about the other players' strategies in the T_{h_i} -partial extensive-form game, for each information set $h_i \in H_i$, such that $\tilde{b}_i(h_i)$ reaches h_i , i.e., $\tilde{b}_i(h_i)$ assigns probability 1 to the set of strategy profiles of the other players that reach h_i . The difference between a belief system and a general belief system is that in the latter *we do not impose conditioning*.

For $k \geq 1$ let \tilde{B}_i^k and \tilde{S}_i^k be defined inductively like \hat{B}_i^k and \hat{S}_i^k in Definition 1, respectively, the only change being that belief systems are replaced by generalized belief systems.

Lemma 1 $U_i^k(\mathbf{S}) \cap S_i = \tilde{S}_i^k$ for $k \geq 1$. Consequently, $U_i^\infty(\mathbf{S}) \cap S_i = \tilde{S}_i^\infty$.

Proof of the Lemma 1. We proceed by induction. The case $k = 0$ is straightforward since $U_i^0(\mathbf{S}) \cap S_i = S_i = \tilde{S}_i^0$ for all $i \in I$.

Suppose now that we have shown $U_i^k(\mathbf{S}) \cap S_i = \tilde{S}_i^k$ for all $i \in I$. We want to show that $U_i^{k+1}(\mathbf{S}) \cap S_i = \tilde{S}_i^{k+1}$ for all $i \in I$.

“ \subseteq ”: First we show, if $s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$ then $s_i \in \tilde{S}_i^{k+1}$.

$s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$ if $s_i \in S_i$ is not conditionally strictly dominated on $(\mathcal{X}_i, U^k(\mathbf{S}))$.

$s_i \in S_i$ is not conditionally strictly dominated on $(\mathcal{X}_i, U^k(\mathbf{S}))$ if for all $T' \in \mathbf{T}$ with $T_1 \hookrightarrow T'$ and all $\tilde{s}_i \in S_i^{T'}$ such that $s_i \in [\tilde{s}_i]$, we have that there does not exist a normal-form information set $X \in \mathcal{X}_i$ with $X \subseteq S^{T'}$ such that \tilde{s}_i is strictly dominated in $X \cap U^k(\mathbf{S})$.

For any information set $h_i \in H_i$, if $\tilde{s}_i \in S_i^{T_{h_i}}$ is not strictly dominated in $S^{T_{h_i}}(h_i) \cap U^k(\mathbf{S})$, then

- (i) either \tilde{s}_i does not reach h_i , in which case \tilde{s}_i is trivially rational at h_i ; or
- (ii) by Lemma 3 in Pearce (1984) there exists a belief $\tilde{b}_i(h_i) \in \Delta(S_{-i}^{T_{h_i}}(h_i) \cap U_{-i}^k(\mathbf{S}))$ for which \tilde{s}_i is rational at h_i . Since by the induction hypothesis $U^k(\mathbf{S}) \cap S = \tilde{S}^k$, we have in this case that there exists a belief at h_i with $\tilde{b}_i(h_i)(\tilde{S}_{-i}^{k, T_{h_i}}) = 1$ for which \tilde{s}_i is rational at h_i .

By the definitions of $[\tilde{s}_i]$ and “reach”, if \tilde{s}_i reaches h_i in the tree T_{h_i} and $s_i \in [\tilde{s}_i]$, then s_i reaches h_i in the tree T_{h_i} . Hence, if $\tilde{s}_i \in S_i^{T_{h_i}}$ is rational at h_i given $\tilde{b}_i(h_i)$, then $s_i \in [\tilde{s}_i]$ is rational at h_i given $\tilde{b}_i(h_i)$.

We need to show that beliefs in (ii) define a generalized belief system in \tilde{B}_i^{k+1} . Consider any $\tilde{b}'_i = (\tilde{b}'_i(h_i))_{h_i \in H_i} \in \tilde{B}_i^{k+1}$. For all $h_i \in H_i$ for which there exists a profile of player i 's opponents' strategies $s_{-i} \in \tilde{S}_{-i}^k$ that reach h_i , replace $\tilde{b}'_i(h_i)$ by $\tilde{b}_i(h_i)$ as defined in (ii). Call the new belief system \tilde{b}_i . Then this is a generalized belief system. Moreover, $\tilde{b}_i \in \tilde{B}_i^{k+1}$.

Hence, if s_i is not conditionally strictly dominated on $(\mathcal{X}_i, U^k(\mathbf{S}))$ then there exists a generalized belief system $\tilde{b}_i \in \tilde{B}_i^{k+1}$ for which s_i is rational at every $h_i \in H_i$. Thus $s_i \in \tilde{S}_i^{k+1}$.

“ \supseteq ”: We show next, if $s_i \in \tilde{S}_i^{k+1}$ then $s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$.

If $s_i \in \tilde{S}_i^{k+1}$ then there exists a generalized belief system $\tilde{b}_i \in \tilde{B}_i^{k+1}$ such that for all $h_i \in H_i$ the strategy s_i is rational given $\tilde{b}_i(h_i)$. That is, either

- (I) s_i does not reach h_i , or

(II) s_i reaches h_i and there does not exist an h_i -replacement of s_i which yields a higher expected payoff in T_{h_i} given $\tilde{b}_i(h_i)$ that assigns probability 1 to T_{h_i} -partial strategies of player i 's opponents in $\tilde{S}_{-i}^{k, T_{h_i}}$ that reach h_i in T_{h_i} . By the induction hypothesis, $\tilde{S}_{-i}^k = U_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}$. Hence $\tilde{b}_i(h_i) \in \Delta(U_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}(h_i))$.

If $s_i \in [\tilde{s}_i]$ with $\tilde{s}_i \in S_i^{T_{h_i}}$ and s_i reaches h_i in the tree T_{h_i} , then \tilde{s}_i reaches h_i in the tree T_{h_i} . Hence, if $s_i \in [\tilde{s}_i]$ with $\tilde{s}_i \in S_i^{T_{h_i}}$ is rational at h_i given $\tilde{b}_i(h_i)$, then \tilde{s}_i is rational at h_i given $\tilde{b}_i(h_i)$.

Thus, if s_i is rational at h_i given $\tilde{b}_i(h_i)$, then $\tilde{s}_i \in S_i^{T_{h_i}}$ with $s_i \in [\tilde{s}_i]$ is not strictly dominated in $U_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}(h_i)$ either because s_i does not reach h_i (case (I)), or because of Lemma 3 in Pearce (1984) (in case (II)).

It then follows that if the strategy s_i is rational at all $h_i \in H_i$ given \tilde{b}_i then s_i is not conditionally strictly dominated on $(\mathcal{X}_i, U^k(\mathbf{S}))$. Hence $s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$. \square

Lemma 2 $\tilde{S}_i^k = \hat{S}_i^k$ for $k \geq 1$. Consequently, $\tilde{S}_i^\infty = \hat{S}_i^\infty$.

Proof of the Lemma 2. $\hat{S}_i^k \subseteq \tilde{S}_i^k$ for $k \geq 1$ since if s_i is rational at each information set $h_i \in H_i$ given the belief system $b_i \in B_i$ then there exists a generalized belief system $\tilde{b}_i \in \tilde{B}_i^k$, namely $\tilde{b}_i = b_i$, such that s_i is rational at each information set $h_i \in H_i$ given \tilde{b}_i .

We need to show the reverse inclusion, $\tilde{S}_i^k \subseteq \hat{S}_i^k$ for $k \geq 1$. The first step is to show how to construct a (consistent) belief system from a generalized belief system. Let s_i be rational given $\tilde{b}_i \in \tilde{B}_i^1$, i.e., $s_i \in \tilde{S}_i^1$. Consider an information set $h_i^0 \in H_i$ such that in T_{h_i} there does not exist an information set h_i that precedes h_i^0 . Define $b_i(h_i^0) \equiv \tilde{b}_i(h_i^0)$.

Assume, inductively, that we have already defined b_i for a subset of information sets $H'_i \subseteq H_i$ such that for each $h'_i \in H'_i$ all the predecessors of h'_i are also in H'_i . For each successor information set h''_i of each information set $h'_i \in H'_i$ such that $h''_i \notin H'_i$ define $b_i(h''_i)$ as follows:

- If $b_i(h'_i)$ reaches h''_i define $b_i(h''_i)$ by using conditioning, i.e. if $s_{-i}^{T_{h'_i}} \in S_{-i}(h''_i)$

$$b_i(h''_i)(s_{-i}^{T_{h'_i}}) = \frac{b_i(h'_i)(s_{-i}^{T_{h'_i}})}{\sum_{\tilde{s}_{-i}^{T_{h'_i}} \in S_{-i}(h''_i)} b_i(h'_i)(\tilde{s}_{-i}^{T_{h'_i}})}$$

and $b_i(h''_i)(s_{-i}^{T_{h'_i}}) = 0$ else.

- If $b_i(h'_i)$ does not reach h''_i let $b_i(h''_i) \equiv \tilde{b}_i(h''_i)$.

Since there are finitely many information sets in H_i , this inductive definition will be concluded in a finite number of steps.

Next, assuming that s_i is rational at each information set $h_i \in H_i$ with the generalized belief system \tilde{b}_i , we will show that s_i is also rational at each information set $h_i \in H_i$ according to the belief system b_i .

Consider again $h_i^0 \in H_i$ with no predecessors in $T_{h_i^0}$. Since $b_i(h_i^0) = \tilde{b}_i(h_i^0)$ and s_i is rational at h_i^0 given $\tilde{b}_i(h_i^0)$, s_i is also rational at h_i^0 given $b_i(h_i^0)$.

Assume, inductively, that we have already shown the claim for a subset of information sets $H'_i \subseteq H_i$ such that for each $h'_i \in H'_i$ all the predecessors of h'_i are also in H'_i . Consider a successor information set h''_i of an information set $h'_i \in H'_i$ such that $h''_i \notin H'_i$. Notice that each h''_i -replacement is also an h'_i -replacement. Therefore,

- If $b_i(h'_i)$ reaches h''_i , $b_i(h''_i)$ is derived from $b_i(h'_i)$ by conditioning, and hence any h''_i -replacement improving player i 's expected payoff according to $b_i(h''_i)$ would improve player i 's payoff also according to $b_i(h'_i)$, contradicting the induction hypothesis. Hence s_i is rational at h''_i given $b_i(h''_i)$.
- If $b_i(h'_i)$ does not reach h''_i , then $b_i(h''_i) = \tilde{b}_i(h''_i)$. Hence, s_i is rational at h''_i also given $b_i(h''_i)$.

Applying the same argument inductively yields $\tilde{S}_i^k = \hat{S}_i^k \forall k \geq 1$. This concludes the proof of the lemma. \square

Lemmata 1 and 2 together yield $U_i^k(\mathbf{S}) \cap S_i = \hat{S}_i^k$ for $k \geq 1$. Since it applies for all $k \geq 1$ and $i \in I$, this completes the proof of the proposition. \square

A.2 Proof of Theorem 3

The proof proceeds in several steps. First, we show that iterated admissibility is characterized by a variant of prudent rationalizability.

A *relaxed belief system* of player i

$$b_i = (b_i(h_i))_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta(S_{-i}^{T_{h_i}})$$

is a profile of beliefs - a belief $b_i(h_i) \in \Delta(S_{-i}^{T_{h_i}})$ about the other players' strategies in the T_{h_i} -partial game, for each information set $h_i \in H_i$. Compared to belief systems, for relaxed belief systems we neither require that $b_i(h_i)$ reaches h_i nor conditioning.

Denote by \ddot{B}_i the set of player i 's relaxed belief systems.

We say that with the relaxed belief system b_i and the strategy s_i player i is *relaxed rational* at the information set $h_i \in H_i$ if there does not exist an alternative strategy s'_i which yields player i a higher expected in T_{h_i} given the belief $b_i(h_i)$ on the other players' strategies $S_{-i}^{T_{h_i}}$.

Definition 3 (Prudent relaxed rationalizability) *Let*

$$\ddot{S}_i^0 = S_i$$

For $k \geq 1$ define inductively

$$\begin{aligned} \ddot{B}_i^k &= \left\{ b_i \in \ddot{B}_i : \begin{array}{l} \text{For every information set } h_i, \text{ if there exists some profile } s_{-i} \in \ddot{S}_{-i}^{k-1} \text{ of the} \\ \text{other players' strategies such that } s_{-i} \text{ reaches } h_i \text{ in the tree } T_{h_i}, \text{ then the} \\ \text{support of } b_i(h_i) \text{ is the set of strategy profiles } s_{-i} \in \ddot{S}_{-i}^{k-1, T_{h_i}}. \end{array} \right\} \\ \ddot{S}_i^k &= \left\{ s_i \in \ddot{S}_i^{k-1} : \begin{array}{l} \text{There exists a belief system } b_i \in \ddot{B}_i^k \text{ with which for every information set} \\ h_i \in H_i, \text{ player } i \text{ is } \mathbf{relaxed\ rational} \text{ at } h_i. \end{array} \right\} \end{aligned}$$

The set of prudent relaxed rationalizable strategies of player i is

$$\ddot{S}_i^\infty = \bigcap_{k=1}^{\infty} \ddot{S}_i^k$$

Proposition 1 *For every finite generalized extensive-form game, $\tilde{W}_i^k(\mathbf{S}) \cap S_i = \ddot{S}_i^k$, $k \geq 1$. Consequently, $\tilde{W}_i^\infty(\mathbf{S}) \cap S_i = \ddot{S}_i^\infty$.*

Proof of Proposition 1: We proceed by induction. The case $k = 0$ is straightforward since $\tilde{W}_i^0(\mathbf{S}) \cap S_i = S_i = \ddot{S}_i^0$ for all $i \in I$.

Suppose now that we have shown $\tilde{W}_i^k(\mathbf{S}) \cap S_i = \ddot{S}_i^k$ for all $i \in I$. We want to show that $\tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i = \ddot{S}_i^{k+1}$ for all $i \in I$.

“ \subseteq ”: First we show, if $s_i \in \tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i$ then $s_i \in \ddot{S}_i^{k+1}$.

$s_i \in \tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i$ if $s_i \in S_i$ is not NF-conditionally weakly dominated on $(S_i, \tilde{W}^k(\mathbf{S}))$.

$s_i \in S_i$ is not NF-conditionally weakly dominated on $(S_i, \tilde{W}^k(\mathbf{S}))$ if for all $T' \in \mathbf{T}$ with $T_1 \hookrightarrow T'$ and all $\tilde{s}_i \in S_i^{T'}$ such that $s_i \in [\tilde{s}_i]$, we have that \tilde{s}_i is not weakly dominated in $S_i^{T'} \cap \tilde{W}^k(\mathbf{S})$.

For any information set $h_i \in H_i$, if $\tilde{s}_i \in S_i^{T_{h_i}}$ is not weakly dominated in $S_i^{T_{h_i}} \cap \tilde{W}^k(\mathbf{S})$, then by Lemma 4 in Pearce (1984) there exists a full support belief $\ddot{b}_i(h_i) \in \Delta(S_{-i}^{T_{h_i}} \cap \tilde{W}_{-i}^k(\mathbf{S}))$ for which \tilde{s}_i is relaxed rational at h_i . Since by the induction hypothesis $\tilde{W}^k(\mathbf{S}) \cap S = \ddot{S}^k$, we have in this case that there exists a full support belief at h_i with $\ddot{b}_i(h_i) \in \Delta(\ddot{S}_{-i}^{k, T_{h_i}})$ for which \tilde{s}_i is relaxed rational at h_i .

By definition of $[\tilde{s}_i]$, if $\tilde{s}_i \in S_i^{T_{h_i}}$ is relaxed rational at h_i given $\ddot{b}_i(h_i)$, then $s_i \in [\tilde{s}_i]$ is relaxed rational at h_i given $\ddot{b}_i(h_i)$.

Note that the profile of beliefs $(\ddot{b}_i(h_i))_{h_i \in H_i}$ define a relaxed belief system in \ddot{B}_i^{k+1} .

Hence, if s_i is not NF-conditionally weakly dominated on $(S_i, \tilde{W}^k(\mathbf{S}))$ then there exists a relaxed belief system $\ddot{b}_i \in \ddot{B}_i^{k+1}$ for which s_i is relaxed rational at every $h_i \in H_i$. Thus $s_i \in \ddot{S}_i^{k+1}$.

“ \supseteq ”: We show next, if $s_i \in \check{S}_i^{k+1}$ then $s_i \in \tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i$.

If $s_i \in \check{S}_i^{k+1}$ then there exists a relaxed belief system $\check{b}_i \in \check{B}_i^{k+1}$ such that for all $h_i \in H_i$ the strategy s_i is relaxed rational given $\check{b}_i(h_i)$. That is, there does not exist an alternative strategy s'_i which yields a strictly higher expected payoff in T_{h_i} given $\check{b}_i(h_i)$ that has full support on $\check{S}_{-i}^{k, T_{h_i}}$.

By the induction hypothesis, $\check{S}_{-i}^k = \tilde{W}_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}$. Hence $\check{b}_i(h_i)$ has full support on $\tilde{W}_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}$.

If s_i is relaxed rational at h_i with $\check{b}_i(h_i)$, then for any $\tilde{s}_i \in S_i^{T_{h_i}}$ with $s_i \in [\tilde{s}_i]$, \tilde{s}_i is relaxed rational at h_i with $\check{b}_i(h_i)$. By Lemma 4 in Pearce (1984), \tilde{s}_i is not weakly dominated in $\tilde{W}_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}$.

It then follows that if the strategy s_i is rational at all $h_i \in H_i$ given \check{b}_i then s_i is not NF-conditionally weakly dominated on $(S_i, \tilde{W}^k(\mathbf{S}))$. Hence $s_i \in \tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i$. This completes the proof of Proposition 1. \square

Next, we show that prudent rationalizability is equivalent to prudent relaxed rationalizability.

Proposition 2 *For every finite generalized extensive-form game and every $i \in I$, $\bar{S}_i^k = \check{S}_i^k$, $k \geq 0$. Consequently, $\bar{S}_i^\infty = \check{S}_i^\infty$.*

Proof of Proposition 2 The proof proceeds by induction. By definition $\bar{S}_i^0 = \check{S}_i^0$.

Assume $\bar{S}_i^k = \check{S}_i^k$. We show that $\bar{S}_i^{k+1} = \check{S}_i^{k+1}$.

“ \supseteq ”: Fix $s_i \in \check{S}_i^{k+1}$ and $h_i \in H_i$ such that s_i reaches h_i (otherwise s_i is trivially rational at h_i). There exists a relaxed belief system $\check{b}_i \in \check{B}_i^{k+1}$ such that s_i is relaxed rational at h_i . By definition of \check{B}_i^{k+1} , $\check{b}_i(h_i)$ has full support on $\check{S}_{-i}^{k, T_{h_i}}$. Assume $\check{S}_{-i}^{k, T_{h_i}} \cap S_{-i}^{T_{h_i}}(h_i) \neq \emptyset$. Then $\check{b}_i(h_i) \left(\check{S}_{-i}^{k, T_{h_i}} \cap S_{-i}^{T_{h_i}}(h_i) \right) > 0$. Thus, we can consider $\check{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right)$. Note that $\check{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right) \in \bar{B}_i^{k+1}$. (Otherwise, if $\check{S}_{-i}^{k, T_{h_i}} \cap S_{-i}^{T_{h_i}}(h_i) = \emptyset$, then s_i is trivially rational at h_i .)

Suppose by contradiction that s_i is not rational at h_i with $\check{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right)$. There exists a h_i -replacement of s_i , call it $s_i/\tilde{s}_i^{h_i}$, such that $s_i/\tilde{s}_i^{h_i}$ yields a strictly higher expected payoff in T_{h_i} given the belief $\check{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right)$. That is, $u_i^{T_{h_i}} \left(s_i/\tilde{s}_i^{h_i}, \check{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right) \right) > u_i^{T_{h_i}} \left(s_i, \check{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right) \right)$. Since $u_i^{T_{h_i}} \left(s_i, \check{b}_i(h_i) \right) \geq u_i^{T_{h_i}} \left(s_i/\tilde{s}_i^{h_i}, \check{b}_i(h_i) \right)$, we must have

$\ddot{b}_i(h_i) \left(S_{-i}^{T_{h_i}}(h_i) \right) < 1$. Now

$$\begin{aligned}
u_i^{T_{h_i}} \left(s_i, \ddot{b}_i(h_i) \right) &\geq u_i^{T_{h_i}} \left(s_i / \tilde{s}_i^{h_i}, \ddot{b}_i(h_i) \right) \\
&= \ddot{b}_i(h_i) \left(S_{-i}^{T_{h_i}}(h_i) \right) u_i^{T_{h_i}} \left(s_i / \tilde{s}_i^{h_i}, \ddot{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right) \right) \\
&\quad + \left(1 - \ddot{b}_i(h_i) \left(S_{-i}^{T_{h_i}}(h_i) \right) \right) u_i^{T_{h_i}} \left(s_i / \tilde{s}_i^{h_i}, \ddot{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right) \right) \\
&> \ddot{b}_i(h_i) \left(S_{-i}^{T_{h_i}}(h_i) \right) u_i^{T_{h_i}} \left(s_i, \ddot{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right) \right) \\
&\quad + \left(1 - \ddot{b}_i(h_i) \left(S_{-i}^{T_{h_i}}(h_i) \right) \right) u_i^{T_{h_i}} \left(s_i, \ddot{b}_i(h_i) \left(\cdot \mid S_{-i}^{T_{h_i}}(h_i) \right) \right) \\
&= u_i^{T_{h_i}} \left(s_i, \ddot{b}_i(h_i) \right)
\end{aligned}$$

a contradiction.

“ \subseteq ”: Let $\bar{S}_{-i}^{k, T_{h_i}}(h_i)$ denote the set of opponents' level k -prudent rationalizable T_{h_i} -partial strategy profiles that reach h_i . For each $T \in \mathbf{T}$ for which H_i^T is nonempty, let $\bar{S}_{-i}^{k, T}(H_i^T) = \bigcup_{h_i \in H_i^T} \bar{S}_{-i}^{k, T}(h_i)$. Note that if $\bar{S}_{-i}^{k, T}(H_i^T) \subsetneq \bar{S}_{-i}^{k, T}$ then there is a terminal history in T that is not reached by any information set $h_i \in H_i^T$. (In general, $\bar{S}_{-i}^{k, T}(H_i^T)$ may not be a cross-product set).

We claim that there exists a nonempty subset of information sets $G_i^T \subseteq H_i^T$ such that $\left\{ \bar{S}_{-i}^{k, T}(h_i) \right\}_{h_i \in G_i^T}$ forms a partition of $\bar{S}_{-i}^{k, T}(H_i^T)$. To define G_i^T , we first define the rank of an information set $h_i \in H_i^T$ as the number of information sets in H_i^T required to pass in order to reach a terminal node in T . Note that if $h_i \in H_i^T$ with $n \in h_i$ there is path from n to n' in T with $i \in I_{n'}$, then property I3 (No divining of currently unimaginable paths, no expectation to forget currently conceivable paths) insures that there exists an information set $h'_i \in H_i^T$ with $n' \in h'_i$. Note further that there may be terminal nodes in T that are not reached by any information set in H_i^T .

Using the definition of rank of an information set, we construct a subset of information sets G_i^T as follows: For any information set $h_i \in H_i^T$ of lowest rank, let $h_i \in G_i^T$ if there is no information set $h'_i \in H_i^T$ that precedes h_i . Otherwise, let $h'_i \in G_i^T$ if there is no information set $h''_i \in H_i^T$ that precedes h'_i , etc. Since T is finite, H_i^T is finite, and the procedure terminates after finite steps. (In particular, if player i moves at the root of T , G_i^T is a singleton whose only information set contains the root.)

By construction, $\bigcup_{h_i \in G_i^T} \bar{S}_{-i}^{k, T}(h_i)$ covers $\bar{S}_{-i}^{k, T}(H_i^T)$. Moreover, from property I6 (Perfect recall) follows that for any two $h_i, h'_i \in G_i^T$ with $h_i \neq h'_i$ there is no profile of opponents' T -partial strategies $s_{-i}^T \in \bar{S}_{-i}^{k, T}(h_i) \cap \bar{S}_{-i}^{k, T}(h'_i)$.

Fix $s_i \in \bar{S}_i^{k+1}$ and $h_i \in H_i$ such that s_i reaches h_i (otherwise s_i is trivially rational at h_i). There exists a belief system $\bar{b}_i \in \bar{B}_i^{k+1}$ such that s_i is rational at h_i . By definition of \bar{B}_i^{k+1} , the support of $\bar{b}_i(h_i)$ is the set $\bar{S}_{-i}^{k, T_{h_i}}(h_i)$.

Construct a belief system \ddot{b}_i by setting for arbitrary $\varepsilon \in (0, 1)$,

$$\ddot{b}_i(h_i)(s_{-i}) := \begin{cases} \frac{1-\varepsilon}{|G_i^{T_{h_i}}|} \bar{b}_i(h_i)(s_{-i}) & \text{if } s_{-i}^T \in \bar{S}_{-i}^{k, T_{h_i}}(h_i), \text{ and} \\ \frac{\varepsilon}{|\bar{S}_{-i}^{k, T_{h_i}} \setminus \bar{S}_{-i}^{k, T_{h_i}}(H_i^{T_{h_i}})|} & \text{if } s_{-i}^T \in \bar{S}_{-i}^{k, T_{h_i}} \setminus \bar{S}_{-i}^{k, T_{h_i}}(H_i^{T_{h_i}}). \end{cases}$$

This defines a full support probability measure $\ddot{b}_i(h_i) \in \Delta(\bar{S}_{-i}^{k, T_{h_i}})$ for each information set $h_i \in H_i$.

\ddot{B}_i^{k+1} is the set of all belief systems defined as above from any $\bar{b}_i \in \bar{B}_i^{k+1}$ and $\varepsilon \in (0, 1)$.

Finally, note that any strategy $s_i \in \bar{S}_i^{k+1}$ that with $\bar{b}_i \in \bar{B}_i^{k+1}$ is rational at every information set $h_i \in H_i$ continues to be rational with a belief system $\ddot{b}_i \in \ddot{B}_i^{k+1}$ at every information set $h_i \in H_i$. Thus $s_i \in \ddot{S}_i^{k+1}$. This completes the proof of Proposition 2 \square

The proof of the theorem now follows now as a direct corollary. \square

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