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Input-Output Tables and Some Theory of Defective Matrices

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July 16, 2022

Abstract

Recent developments in the theory of production networks offer interesting applications and revival of input-output analysis. Some recent papers have studied the propagation of a temporary, negative shock through an input-output network. Such analyses of shock propagation relies on eigendecomposition of relevant input-output matrices. It is well known that only diagonalizable matrices can be eigendecomposed; those that are not diagonalizable, are known as defective matrices. In this paper, we provide necessary and sufficient conditions for diagonalizability of any square matrix using its rank and eigenvalues. To apply our results, we offer examples of input-output tables from India in the 1950s that were not diagonalizable and were hence, defective.

Keywords: input-output tables, diagonalizability

JEL Codes: C60

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1 Introduction

The use of input-output tables to study sectoral interconnectedness has a long tradition in economics. Wassily Leontief, one of the pioneers in this field, while studying the structure of American economy in mid-20th century remarked that “layman and professional economist alike, practical planner and the subjects of his regulative activities, all are equally aware of the existence of some kind of interconnection between even the remotest parts of a national economy” (Leontief, 1941). Even before Leontief, the classical political economy tradition has had a rich engagement with input-output tables. The French Physiocrat, François Quesnay in his *Tableau Économique (1765)* represented the interdependence between agriculture and manufacturing in France in the form of a table. Karl Marx in Volume II of Capital divides the economy into two departments to study interdependence between the sectors, where department I produces consumption goods and department II produces investment goods (Kurz, 2011; Kurz and Salvadori, 2000; Thomas, 2021).

There has been a revival of interest in the use of input-output tables to study the theory of production networks. The theory of production networks uses input-output linkages to study how idiosyncratic shocks to one sector propagate throughout the network creating macroeconomic implications (Carvalho and Tahbaz-Salehi, 2019). Recent papers in the production networks framework have made interesting use of input-output tables. For instance, Liu and Tsyvinski (2020) study a dynamical production network with N production sectors where changing the inputs of production are costly. In this setting, input-output linkages matter for the propagation of shocks. As an example, consider a temporary negative shock to some sector. Although there is instantaneous recovery of output in the sector that was hit by the shock, the output of other sectors in the network recover gradually because there are adjustment costs associated with expansion of input in those sectors. Thus, temporary shocks can have long lasting impacts on the economy. Liu

and Tsyvinski (2020) apply their model to study a historical episode—target selection for bombing Axis countries by the Allied powers during World War II. Bombing is used as a temporary negative shock which causes damage to the sectors that are hit. The damage is propagated throughout the economy because of inter-sectoral linkages and the recovery from this damage is only gradual due to the existence of adjustment costs. To understand which sectors were more vulnerable to such negative shocks, Liu and Tsyvinski (2020) make use of pre-World War II input-output tables of Germany and Japan.

The methodology used by Liu and Tsyvinski (2020) to analyse shock propagations relies on eigendecomposition of the input-output expenditure share matrix. An eigendecomposition involves factoring the expenditure share matrix into the product of S , Λ and S^{-1} , where Λ is the diagonal matrix of eigenvalues and S is the matrix of corresponding eigenvectors. The eigenvalues and eigenvectors of the input-output expenditure share matrix determine the dynamical evolution of the system after it is hit by a shock and hence they can be used to study shock propagations. The rate of decay (movement towards the pre-shocked value) is governed by the magnitude of the eigenvalue and hence sectors having larger eigenvalues would take longer to recover after they are hit by a negative shock.

Liu and Tsyvinski (2020) do not point out that the eigendecomposition analysis is only possible if the matrix of eigenvectors associated with the expenditure share matrix is invertible. Such matrices are known as *diagonalizable*. Matrices which cannot be diagonalized, and are known as *defective*, will not allow the eigendecomposition analysis. Hence, it is important to know if and when any matrix is diagonalizable. The main contribution of our paper is to provide simple conditions—a sufficient condition and a separate necessary condition—to check diagonalizability of matrices using eigenvalues and the rank of the matrix. These tests can be used by researchers before using eigendecomposition methods in production network and other related analyses.

This paper was motivated by our attempt to replicate the historical approach of [Liu and Tsyvinski \(2020\)](#) to the 1950s input-output tables for India. In carrying out this analysis, we realized that the input-output expenditure share matrices of those tables were not diagonalizable (i.e. those were defective matrices).¹ This led us to investigate necessary and sufficient conditions for diagonalizability.

The results on diagonalizability that we present in this paper lead us to flag two issues. First, some results which hold only for diagonalizable matrices are often assumed, incorrectly, to be true of all matrices. For instance, the mathematical appendix of a popular textbook in Classical Political Economy suggests that “More generally, if \mathbf{A} is a square matrix of order n and rank k , where $k < n$, then \mathbf{A} must have $n - k$ eigenvalues equal to zero” ([Pasinetti, 1977](#), p. 258). This statement is true only for diagonalizable matrices; it is not true for defective matrices. We provide two interesting results about zero as a repeated eigenvalue of a matrix that addresses this confusion.

Second, there seems to be some confusion regarding full rank (invertibility) and diagonalizability of matrices. [Liu and Tsyvinski \(2020, p. 24–25\)](#) assume that the input-output expenditure share matrices are full rank when they introduce the eigendecomposition exercise. Using examples of simple 2×2 matrices, we show that there is no connection between invertibility and diagonalizability, a fact that is well recognised in linear algebra ([Strang, 2006](#), p. 246). Thus, the invertibility (full rank) of the input-output expenditure share matrices are irrelevant for the methodology proposed in [Liu and Tsyvinski \(2020\)](#).

A reader might wonder whether it is possible to directly use the definition for diagonalizability of a matrix, i.e. that its eigenvector matrix should be invertible by studying if it has a non-zero determinant. We show from an

¹An inter-industry unit was set up in the Planning division of the Indian Statistical Institute at Calcutta to prepare input-output tables for the newly independent country. This unit prepared three 36 sector input-output tables at market prices for the years 1951-52, 1953-54, 1955-56 respectively ([Saluja, 1980](#)).

analysis of various input-output tables that it is difficult to say anything definitive about diagonalizability by just relying on this condition. That is why we recommend using the sufficient or necessary conditions that we present in this paper.

The rest of the paper is organised as follows - Section 2 presents the main results and in Section 3, we apply our results to various input-output tables. We conclude the paper in Section 4 with two observations.

2 Main Results

2.1 Some Results We Use

We will use three well-known results from linear algebra that we state here for easy reference.

Lemma 1. (*Rank-Nullity Theorem*). *Let A be a $n \times n$ matrix of real numbers. Then,*

$$\text{rank}(A) + \text{nullity}(A) = n,$$

where $\text{rank}(A)$ is the number of independent columns of A , and $\text{nullity}(A)$ is the dimension of the null space of A , i.e. the dimension of the space $\{x \in R^n | Ax = 0\}$.

Proof. For a discussion of this fundamental result, see Strang (2006, section 2.4). □

The next two result relates to diagonalizability and multiplicities of eigenvalues of a square matrix. We begin by reminding the reader of some basic definitions.

- Let (λ, x) be an eigenpair of A , i.e. λ is an eigenvalue of A and $x \neq 0$ (x cannot be the zero vector) is a corresponding right eigenvector which satisfies $Ax = \lambda x$.

- The $n \times n$ matrix A is said to be *diagonalizable* when it has a set of n linearly independent eigenvectors. This implies that

$$P^{-1}AP = D, \tag{1}$$

where D is the diagonal matrix of eigenvalues of A and P is the matrix such that its columns are the corresponding eigenvectors (Strang, 2006, p. 245).

- Suppose λ is an eigenvalue of A . The *algebraic multiplicity* of λ , denoted as $AM_A(\lambda)$, is the number of times λ is repeated as an eigenvalue of A .
- Suppose λ is an eigenvalue of A . The *geometric multiplicity* of λ , denoted as $GM_A(\lambda)$, is the dimension of the null space of $A - \lambda I$. It is also known as the eigenspace of A .

Lemma 2. (*Multiplicity Inequality*). Let A be a $n \times n$ matrix of real numbers. Then, $0 < GM_A(\lambda) \leq AM_A(\lambda)$, for each eigenvalue, λ , of A .

Proof. Since $(A - \lambda I)x = 0$ and $x \neq 0$, the null space of $A - \lambda I$ has at least one nonzero vector. That ensures that $0 < GM_A(\lambda)$. For the other inequality see Meyer (2000, p. 512). \square

Lemma 3. (*Diagonalizability and Multiplicities*). Let A be a $n \times n$ matrix of real numbers. A is diagonalizable if and only if $GM_A(\lambda) = AM_A(\lambda)$ for each eigenvalue, λ , of A .

Proof. See Meyer (2000, p. 512). \square

2.2 Conditions for Diagonalizability

We want to derive a necessary condition and a separate sufficient condition for diagonalizability of a matrix using its eigenvalues and its rank. The

first result shows that the rank of a matrix being equal to the number of nonzero eigenvalues is a necessary condition for diagonalizability. The second result shows that the rank of a matrix being equal to the number of nonzero eigenvalues and all the nonzero eigenvalues being distinct together provide a sufficient condition for diagonalizability.

Theorem 1. *(Necessary condition for diagonalizability). Let A be a $n \times n$ matrix of real numbers. If A is diagonalizable, then rank of A is equal to the number of nonzero eigenvalues A .*

Proof. According to the supposition of the theorem, the matrix A is diagonalizable. Let D be the diagonal matrix of eigenvalues of A , and let P be the matrix of corresponding eigenvectors. Since A is diagonalizable, the matrix of eigenvectors, P , is nonsingular. Thus, we have $\text{rank}(A) = \text{rank}(P^{-1}AP) = \text{rank}(D) =$ number of nonzero eigenvalues, where we have used (1) in the second step and we have used the well known result that multiplication by nonsingular matrices does not change the rank of a matrix (Meyer, 2000, p. 210). \square

Theorem 2. *(Sufficient condition for diagonalizability). Let A be a $n \times n$ matrix of real numbers. If*

1. $\text{rank}(A) =$ number of nonzero eigenvalues, and
2. all the nonzero eigenvalues are distinct,

then A is diagonalizable.

Proof. Suppose A has $n - k$ nonzero eigenvalues and 0 is repeated k times as an eigenvalue of A .

Let us first consider the zero eigenvalues. Since 0 is repeated k times as an eigenvalue of A , $AM_A(0) = k$. By the supposition of this theorem, $\text{rank}(A) = n - k$. We know from Lemma 1 that $\text{rank}(A) + \text{nullity}(A) = n$.

Thus, $nullity(A) = n - rank(A) = n - (n - k) = k$. Thus, $GM_A(0) = nullity(A) = k = AM_A(0)$.

Let us now consider the nonzero eigenvalues. By the supposition of the theorem, each of the nonzero eigenvalues are distinct. Let $\lambda \neq 0$ be an eigenvalue of A . By the supposition of the theorem, $AM_A(\lambda) = 1$. Thus, using Lemma 2, we see that $0 < GM_A(\lambda) \leq AM_A(\lambda) = 1$. This implies that $GM_A(\lambda) = AM_A(\lambda) = 1$, because $GM_A(\lambda)$ is an integer.

Thus, we have proved that for each eigenvalue of A , whether it is zero or nonzero, geometric and algebraic multiplicity is equal. Hence, using Lemma 3 we can conclude that A is diagonalizable. \square

2.3 Suggested Two-Step Method

Using the results presented in the previous sub-section, we recommend that a researcher use the following two-step method to ascertain diagonalizability before conducting the eigendecomposition analysis.

In the first step, the researcher should check the necessary condition given in Theorem 1: is the rank of the matrix equal to the number of non-zero eigenvalues? If this condition is violated, the researcher can conclude that the matrix is not diagonalizable. Hence, she should not proceed with the eigendecomposition analysis.

If the necessary condition is satisfied, i.e. the rank of the matrix is equal to the number of non-zero eigenvalues, then the researcher can proceed to the second step and check whether the sufficient conditions given in Theorem 2 is satisfied. Since the rank of the matrix is equal to the number of non-zero eigenvalues, she only needs to check whether all the nonzero eigenvalues are distinct. If they are all distinct, the researcher can conclude that the matrix is diagonalizable. If at least one nonzero eigenvalue is repeated, then the researcher cannot draw any conclusion about diagonalizability based on the results of this paper. She will need to work with the eigenvector matrix.

2.4 Full Rank and Diagonalizability are Not Related

Since there seems to be some confusion regarding the relationship between invertibility (full rank) and diagonalizability, we briefly discuss this issue here. We provide four examples of 2×2 matrices to highlight the fact that full rank and diagonalizability are not related to each other.²

Example 1: A diagonalizable matrix with full rank. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The rank of this matrix is 2, i.e. it is of full rank because the two columns are linearly independent. The eigenvalues of this matrix = $\{1, 1\}$. Hence, the number of non-zero eigenvalues is $2 = \text{rank of } A$. In this case, the necessary condition for diagonalizability following Theorem 1 is satisfied. However, the non-zero eigenvalues are non-distinct in this case and so the sufficient condition of Theorem 2 is not satisfied. But in this case, it is easy to see that any vector in \mathbb{R}^2 is an eigenvector of A . This is because an eigenvector, $x = (x_1, x_2)$, associated with the eigenvalue $\lambda = 1$ must satisfy

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Any real numbers x_1, x_2 will satisfy the above equation. But since x is an eigenvector, we must not allow $x_1 = x_2 = 0$. The upshot is that we can choose two linearly independent eigenvectors, which will ensure that the eigenvector matrix is invertible (it has full rank). Hence, we can say that the matrix A is diagonalizable.

²This fact is well known in mathematics; for instance, see [Strang \(2006, p. 245\)](#).

Example 2: A non-diagonalizable matrix with full rank. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The rank of this matrix is 2, i.e. it is of full rank because the two columns are linearly independent. The eigenvalues of this matrix = $\{1, 1\}$. Hence, the number of non-zero eigenvalues is $2 = \text{rank of } A$. In this case, the necessary condition for diagonalizability following Theorem 1 is satisfied. However, the non-zero eigenvalues are non-distinct in this case and so the sufficient condition of Theorem 2 is not satisfied.

Let $x = (x_1, x_2)$ be an eigenvector associated with the eigenvalue $\lambda = 1$. Hence, it must satisfy

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Any multiple of $x = (1, 0)$ will satisfy the above equation. In this case, we have only one independent eigenvector. Hence, the eigenvector matrix is not invertible (it does not have full rank). Hence, we can say that the matrix A is not diagonalizable.

Example 3: A diagonalizable matrix without full rank. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The rank of this matrix is 1. The eigenvalues of this matrix = $\{0, 1\}$. Hence, the number of non-zero eigenvalues is $1 = \text{rank of } A$. In this case, the necessary condition for diagonalizability following Theorem 1 is satisfied. Similarly, the non-zero eigenvalues are distinct. So, the sufficient condition of Theorem 2 is also satisfied. Hence, A is diagonalizable.

To check this directly, let $x = (x_1, x_2)$ be an eigenvector associated with

the eigenvalue $\lambda = 1$. Hence, it must satisfy

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Any multiple of $x = (1, 0)$ will once again satisfy the above equation.

Turning to the other eigenvalue, let $x = (x_1, x_2)$ be an eigenvector associated with the eigenvalue $\lambda = 0$. Repeating the argument from Example 1, we see that any vector in \mathbb{R}^2 will be an eigenvector associated with the eigenvalue $\lambda = 0$. Thus, one such eigenvector is $x = (0, 1)$. In this case we have two linearly independent eigenvectors. Hence, the eigenvector matrix is invertible (it is of full rank). Hence, we can say that the matrix A is diagonalizable.

Example 4: A non-diagonalizable matrix without full rank. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The rank of this matrix is 1. The eigenvalues of this matrix = $\{0, 0\}$. Hence, the number of non-zero eigenvalues is 0 is not equal to the rank of A (which is 1). In this case, the necessary condition for diagonalizability following Theorem 1 is not satisfied. Hence, A is not diagonalizable.

To check this directly, let $x = (x_1, x_2)$ be an eigenvector associated with the eigenvalue $\lambda = 0$. Hence, it must satisfy

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Any multiple of $x = (1, 0)$ will satisfy the above equation. In this case, we have only one independent eigenvector. Hence, the eigenvector matrix is not invertible (it does not have full rank). Hence, we can say that the matrix A is not diagonalizable.

2.5 Zero as a Repeated Eigenvalue

When zero is an eigenvalue of a matrix A , whether it is repeated or not makes an interesting difference. The next two results highlight this. The first result shows that if zero is repeated as an eigenvalue, then we can only derive a lower bound for the rank of the matrix using the number of nonzero eigenvalues. On the other hand, if zero has algebraic multiplicity of one, we can pin down the rank of the matrix unambiguously.

Theorem 3. *Let A be a $n \times n$ matrix of real numbers. Suppose 0 is repeated k times as an eigenvalue of A . Then, $\text{rank}(A) \geq n - k$.*

Proof. Suppose 0 is repeated k times as eigenvalues of A , and all the other eigenvalues are nonzero. Hence, $\lambda_1, \lambda_2, \dots, \lambda_{n-k}, 0, 0, \dots, 0$ are the eigenvalues of A , with $\lambda_i \neq 0$. Then, $AM_A(0) = k$. Hence, by Lemma 2, $0 < GM_A(\lambda) \leq k$.

Consider the null space of $A - \lambda I$, when $\lambda = 0$. This is just the null space of A . Hence $GM_A(0) = \text{nullity}(A)$. Thus, we get the following: $0 \leq \text{nullity}(A) \leq k$. Using Lemma 1, we get, $\text{rank}(A) = n - \text{nullity}(A) \geq n - k$. \square

The implication of this result is interesting. If a matrix has repeated zero eigenvalues, then rank of A is weakly greater than the dimension of the matrix less the number of times zero is repeated as an eigenvalue of A . Thus, in this case, we cannot pin down the rank of A without more information. We can only ascertain a lower bound for the rank.

This result throws light on a confusion in the literature: “More generally, if A is a square matrix of order n and rank k , where $k < n$, then A must have $n - k$ eigenvalues equal to zero” (Pasinetti, 1977, p. 258). In example 4 above, we have seen a matrix where $n = 2$ and rank $k = 1$. The matrix has 2 eigenvalues equal to 0. This contradicts Pasinetti’s claim that the matrix should have 1 ($= n - k$) eigenvalue equal to zero. That is because the claim is only valid for diagonalizable matrices (see Theorem 1 above) and

Table 1: Eigendecomposition of Input-Output Tables

Country-Year	Dimension of Matrix	Rank of Matrix	Non-zero eigenvalues (number)	Non-zero eigenvalues (distinct?)
Japan 1935	23	22	22	Yes
Germany 1936	40	39	39	Yes
India 1951-52	36	32	31	Yes
India 1953-54	36	32	31	Yes
India 1955-56	36	34	33	Yes

example 4 has a defective matrix. The general relationship between rank and the algebraic multiplicity of 0 is given in Theorem 3.

Theorem 4. *Let A be a $n \times n$ matrix of real numbers. Suppose 0 is repeated only once as an eigenvalue of A . Then, $\text{rank}(A) = n - 1$.*

Proof. Consider the special case when $k = 1$, i.e. the matrix A has only one eigenvalue of 0. Thus the $AM_A(0) = 1$. In this case, $0 \leq \text{nullity}(A) = GM_A(0) \leq 1$. Hence, $\text{nullity}(A) = 1$, because $\text{nullity}(A)$ is an integer. Thus, $\text{rank}(A) = n - 1$. \square

In this case, of course, there is no ambiguity. If the algebraic multiplicity of 0 is 1, then the rank of the matrix is exactly equal to $n - 1$.

3 Application of the Results to Various Input-Output Tables

Table 1 compares the results derived from the eigendecomposition of the input-output expenditure share matrix of Japan and Germany that were used by [Liu and Tsyvinski \(2020\)](#) with the same exercise applied to the

historical input-output tables from India.³ Applying our results from the previous section, we see that: (a) the input-output tables used by [Liu and Tsyvinski \(2020\)](#) have diagonalizable expenditure share matrices, and (b) the expenditure share matrices for the Indian tables are defective. This is because the necessary and sufficient conditions of [Theorem 1](#) and [Theorem 2](#) are satisfied for Japan's (1935) and Germany's (1936) input-output tables. For both these input-output tables, the number of non-zero eigenvalues of the expenditure share matrix is equal to its rank, and all the non-zero eigenvalues are distinct.

On the other hand input-output expenditure share matrices in the Indian input-output tables from the 1950s do not even satisfy the necessary condition mentioned in [Theorem 1](#). For the Indian input-output tables from the 1950s, the rank of the expenditure share matrix is different from the number of nonzero eigenvalues. Hence, the necessary condition given in [Theorem 1](#) is violated. This implies that the expenditure share matrix is not diagonalizable. Thus, one would not be able to carry out the eigendecomposition analysis proposed by [Liu and Tsyvinski \(2020\)](#) for the Indian input-output tables from the 1950s. The conclusion to be drawn is that before proceeding with the eigendecomposition exercise, it is important to check whether the matrix in question meets the criteria for diagonalizability, either directly by computing the eigenvector matrix or by using the necessary and sufficient conditions given in [Theorem 1](#) and [2](#) of this paper.

³The input-output table for Japan in 1935 was made available by [Liu and Tsyvinski \(2020\)](#) in Appendix C of their paper while we found the German 1936 input-output table from [Fremdling and Staeglin \(2014\)](#) following [Liu and Tsyvinski \(2020\)](#)'s references. The Indian input-output tables for 1951-52, 1953-54 and 1955-56 were found in [IARNIW \(1960\)](#), [UN \(1961\)](#) and [Chakraverti \(1968\)](#) respectively. These input-output tables are available upon request.

4 Conclusion

In the emerging literature on production networks, an interesting approach to studying dynamic effects of shocks has been developed by [Liu and Tsyvinski \(2020\)](#). The methodology relies on eigendecomposition of input-output expenditure share matrices. Since eigendecomposition is only possible for diagonalizable matrices, it is important to check for diagonalizability before applying the methodology of [Liu and Tsyvinski \(2020\)](#). In this paper we have presented necessary and sufficient conditions for diagonalizability of any matrix that can be easily checked using its eigenvalues and its rank.

We conclude the paper with two observations. First, the identification of diagonalizability of a matrix is easier to do with the necessary and sufficient conditions we have provided in this paper than by using the magnitude of the determinant of the eigenvector matrix that is generated by computer packages like Matlab or R. Even with the input-output tables used by [Liu and Tsyvinski \(2020\)](#), when we only rely on the determinant value of the eigenvector matrix, it is difficult to arrive at a firm decision about the diagonalizability of the matrix. This is because the determinant values are very close to, but different from, 0 (the real part of the complex number being zero and the imaginary part being a very small number).⁴ The second point to note is that invertibility (full rank) of the input-output expenditure share matrix does not have any relation with its diagonalizability. Though it is true that the Japan-1935 and Germany-1936 matrices are full-rank, that fact has no bearing on whether those matrices are diagonalizable.⁵ If a researcher is interested in applying the eigendecomposition analysis of [Liu and Tsyvinski \(2020\)](#) to study shock propagation, then she can very well ignore the in-

⁴We worked in R.

⁵Readers might note from Table 1 that rank of the matrix for Japan 1935 and Germany 1936 is one less than the dimension of the matrix. That is because in both these tables there is one sector which is disconnected from the network - it neither supplies nor receives any input from any other sector. Once we remove these sectors from the tables, the dimension of the square matrix shrinks to 22 for Japan in 1935 and to 39 in Germany 1936.

vertibility or noninvertibility of the input-output expenditure share matrix. This will have no bearing on the possibility or otherwise of carrying out the eigendecomposition analysis.

References

- Carvalho, V. M. and Tahbaz-Salehi, A. (2019). Production networks: A primer. *Annual Review of Economics*, 11(1):635–663.
- Chakraverti, A. K. (1968). *Structure of the Indian Economy*. Statistical Pub. Society.
- Fremdling, R. and Staeglin, R. (2014). An Input-Output Table for Germany in 1936: A Documentation of Results, Sources and Research Strategy. *Jahrbuch für Wirtschaftsgeschichte/Economic History Yearbook*, 55(2):187–298.
- IARNIW (1960). *Papers on national income and allied topics*, volume 1. Asia Publishing House.
- Kurz, H. D. (2011). Who is going to kiss sleeping beauty? On the ‘classical’ analytical origins and perspectives of input–output analysis. *Review of Political Economy*, 23(1):25–47.
- Kurz, H. D. and Salvadori, N. (2000). ‘Classical’ roots of input-output analysis: A short account of its long prehistory. *Economic Systems Research*, 12(2):153–179.
- Leontief, W. W. (1941). *Structure of American economy, 1919-1929*. Harvard University Press.
- Liu, E. and Tsyvinski, A. (2020). Dynamical structure and spectral properties of input-output networks. Technical report, National Bureau of Eco-

- conomic Research. NBER Working Paper 28178, URL: <http://www.nber.org/papers/w28178>.
- Meyer, C. D. (2000). *Matrix Analysis and Applied Linear Algebra*. Society for Industrial and Applied Mathematics.
- Pasinetti, L. L. (1977). *Lectures on the Theory of Production*. Columbia University Press.
- Saluja, M. R. (1980). *Input-Output Tables for India: Concepts, Construction and Applications*. Wiley Eastern.
- Strang, G. (2006). *Linear Algebra and its Applications*. Cengage.
- Thomas, A. M. (2021). *Macroeconomics: An Introduction*. Cambridge University Press.
- UN (1961). *Formulating Industrial Development Programmes: Development Programming Techniques, Series no. 2*. United Nations Economic Commission for Asia and the Far East, Bangkok.