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**Working Paper**

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Working Paper, No. 417

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*Suggested Citation:* Aspect, Leopold; Ewerhart, Christian (2022) : Colonel Blotto games with a head start, Working Paper, No. 417, University of Zurich, Department of Economics, Zurich, <https://doi.org/10.5167/uzh-220546>

This Version is available at:

<https://hdl.handle.net/10419/266241>

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University of Zurich  
Department of Economics

Working Paper Series  
ISSN 1664-7041 (print)  
ISSN 1664-705X (online)

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# **Colonel Blotto Games with a Head Start**

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August 2022

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# Colonel Blotto Games with a Head Start\*

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August 30, 2022

**Abstract.** This paper studies Colonel Blotto games with two battlefields where one player has a head start in the form of additional troops on one of the battlefields. Such games arise naturally in marketing, electoral competition, and military conflict. Sion and Wolfe (1957) have shown that, if the strategy space is continuous, a mixed-strategy Nash equilibrium need not exist. Therefore, we consider a finite approximation. Using the iterated elimination of (weakly) dominated strategies, we identify an equilibrium for all parameter constellations and discuss its uniqueness properties. In equilibrium, resource decisions are typically not uniform but tend to concern units that roughly correspond in size to multiples of the head start. Moreover, competition takes the form of a hide-and-seek game, where the favorite tries to outguess the number of units that the underdog commits to the balanced battlefield. Somewhat unexpectedly, equilibrium payoffs of finite approximations of the Sion-Wolfe game accumulate around precisely three values. We also discuss the relation to the model with heterogeneous budgets but no head start.

**Keywords.** Colonel Blotto games · Head start · Nash equilibrium · Finite approximation

**JEL-Codes.** C62 – Existence and Stability Conditions of Equilibrium; C72 – Noncooperative Games; D72 – Political Processes: Rent-Seeking, Lobbying, Elections, Legislatures, and Voting Behavior

\*) For useful discussions, we are grateful to Sergiu Hart, Dan Kovenock, and Bill Zame.

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## 1. Introduction

In the early years of game-theoretic research, fundamental contributions established the existence of mixed-strategy solutions for noncooperative games in increasing generality. Notable results have been achieved, in particular, for finite two-person zero-sum games (von Neumann, 1928; von Neumann and Morgenstern, 1945), finite  $n$ -player games (Nash, 1950, 1951), and games with infinite strategy spaces (Glicksberg, 1952; Fan, 1952; Debreu, 1952). This line of research came to a sudden halt, however, when Sion and Wolfe (1957) presented an example of a two-person zero-sum game that does not have a value. What that example means for the modern theory of games that are not restricted by the zero-sum condition is indeed, that a game with infinite strategy spaces need not possess a Nash equilibrium in mixed strategies.

Incidentally, the game used by Sion and Wolfe (1957) is strategically equivalent to a Colonel Blotto game with two battlefields in which one player has a head start in the form of additional troops in one battlefield. In a standard Colonel Blotto game *without* a head start (Borel, 1921), each player allocates a budget of one unit of a perfectly divisible resource across several equivalued battlefields where, on each battlefield separately, the highest bidder wins (and the winner is drawn randomly in the case of a tie). Moreover, the resource is either used or lost. In a Colonel Blotto game *with* a head start, however, the bid of the privileged player in one battlefield is raised by the head start. The analysis of Sion and Wolfe (1957) assumed that the head start corresponds to precisely one half of the homogeneous budget. Even though this is just a special case, the non-existence clearly is a severe obstacle for the game-theoretic analysis of Colonel Blotto games. On the other hand, Colonel Blotto games with a head start have numerous applications, e.g., in marketing, electoral competition, and military conflict.

This paper continues the exploration of the nature of strategic interaction in Colonel Blotto games with a head start. To this end, we consider a class of *discrete* Colonel Blotto games with two battlefields, referred to as battlefield A and battlefield B, in which one player has a head start in the form of additional troops on battlefield A. The continuous strategy space is replaced by a finite equidistant grid, following prior work by Hart (2008), Hortala-Vallve and Llorente-Saguer (2012), and Liang et al. (2019), in particular. Denoting by  $n$  the number of soldiers available for allocation (assumed to be the same for each player), and by  $k$  the head start

consisting of additional soldiers for player 1 on battlefield A, we study the set of mixed-strategy Nash equilibria in the corresponding Colonel Blotto game, which will be denoted by  $\mathfrak{B}(n, k)$ . The analysis naturally leads to the consideration of five cases:

- (i) There is no head start or the head start is at least as large as the budget, i.e.,  $k = 0$  or  $k \geq n$ ;
- (ii) the head start is strictly larger than half of the budget but strictly smaller than the budget, i.e.,  $k \in \{\lfloor n/2 \rfloor + 1, \dots, n - 1\}$ ;<sup>1</sup>
- (iii) the head start consists of precisely one soldier, i.e.,  $k = 1$ ;
- (iv) the head start consists of at least two soldiers and is weakly lower than half of the budget, i.e.,  $k \in \{2, \dots, \lfloor n/2 \rfloor\}$ , and the budget  $n$  is divisible by the head start  $k$ ;
- (v) the head start consists of at least two soldiers and is weakly lower than half of the budget, i.e.,  $k \in \{2, \dots, \lfloor n/2 \rfloor\}$ , and the budget  $n$  is *not* divisible by the head start  $k$ ;

Note that cases (i) through (v) are mutually exclusive yet collectively exhaustive. In each case, we will identify a Nash equilibrium in mixed strategies and characterize the unique pair of equilibrium payoffs.

It turns out that, even though  $\mathfrak{B}(n, k)$  is dominance solvable (in the sense of Moulin, 1979) only in case (i), the identification of equilibrium candidates in Colonel Blotto games with a head start is largely simplified by the *iterated elimination of (weakly) dominated strategies*.<sup>2</sup> In addition, while there are in general multiple equilibria, the procedure of iterated elimination of dominated strategies helps to narrow down the equilibrium set in many cases of interest. In particular, it is shown that, in cases (ii) and (iii),  $\mathfrak{B}(n, k)$  admits a unique Nash equilibrium in iteratively undominated strategies. In fact, in case (iv), the equilibrium is unique even without the prior elimination of dominated strategies. However, in case (v), the equilibrium set, while known to be a Cartesian product of closed convex simplices by general arguments, may be quite large even after applying the procedure of iterated elimination of dominated strategies.

Based on the analysis, we discuss the nature of strategic interaction in Colonel Blotto games

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<sup>1</sup>As usual,  $\lfloor x \rfloor$  denotes the largest integer smaller than or equal to  $x$ . Note that, with  $k$  being an integer,  $k \in \{\lfloor n/2 \rfloor + 1, \dots, n - 1\}$  is equivalent to  $n/2 < k < n$ .

<sup>2</sup>Throughout the paper, we will use term *dominance* for what is commonly understood as weak dominance between pure strategies. See Section 2 for details.

with a head start. The analysis shows that resource commitments are typically not uniform but instead occur in units that roughly correspond in size to multiples of the head start. Moreover, competition takes the form of a hide-and-seek game, where the favorite tries to outguess the number of units that the underdog commits to the balanced battlefield. Intuitively, the favorite benefits the most from the head start when she anticipates the underdog’s strategy, because that allows her to overpower the underdog in both battlefields. The underdog, conversely, tries to hide her intentions so that, with the highest probability possible, the favorite’s advantage plays out in one battlefield only. In other words, the underdog aims at a scenario where the favorite’s advantage creates little impact because units of the resource turn out to be wrongly allocated ex post, viz. to a battlefield where they are not needed.<sup>3</sup>

Given the tractability of the discrete Colonel Blotto game with a head start, it seems natural to relate the findings back to the motivating example by Sion and Wolfe (1957). To this end, we consider sequences of finite approximations of the Colonel Blotto game with continuous strategy spaces. Somewhat unexpectedly, we find that the equilibrium payoffs in the finite approximations accumulate around a finite set consisting of just three values. In addition, there is no simple relationship between these three values on the one hand and the maximin and minimax values of the continuous game on the other.

*Related literature.* More than a century ago, Borel (1921) proposed the study of Colonel Blotto games. Since then, a sizable literature on Colonel Blotto games has emerged. Notable contributions concerning continuous strategy spaces include Borel and Ville (1938), Gross and Wagner (1950), Friedman (1958), and Roberson (2006), in particular.<sup>4</sup> Related to the present study is work by Macdonell and Mastronardi (2015) who solved the two-battlefield case with heterogeneous budgets. Washburn (2013, Sec. 5.1.3) constructed equilibria for the Colonel Blotto game with arbitrary head starts subject to a playability constraint (which amounts to the existence of a suitable copula that satisfies the budget constraint ex post). Vu and Loiseau (2021, Sec. 5) derived exact equilibria in Colonel Blotto games with three or more *homogeneous* battlefields, allowing for pre-allocated resources and different effectiveness across players. For this, they extended Rober-

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<sup>3</sup>We hasten to add, however, that we do not characterize the entire equilibrium set in all cases, so that it is perceivable (but intuitively unlikely) that there are other types of equilibria.

<sup>4</sup>For a survey, see Kovenock and Roberson (2010).

son’s approach. Further, they obtained *approximate* equilibria in more general classes of Colonel Blotto games with favoritism. They achieved this by first drawing realizations of the unilaterally optimal strategies for a sufficiently large number of battlefields, and subsequently rescaling the realizations to ensure the budget constraint.

Colonel Blotto games are particularly appealing if units of the budget are indivisible. E.g., Borel considered an example with three battlefields and a budget of  $n = 7$  soldiers. The number seven was chosen because it is “the smallest integer for which the game does not have simple manners of playing superior to all others” (Borel, 1921, p. 100).<sup>5</sup> An informative review of the early literature, some of which apparently is still classified, may be found in Beale and Heselden (1962). More recently, Hart (2008) derived optimal strategies in discrete Colonel Blotto games from optimal strategies in so-called General Lotto games. In a General Lotto game, each player chooses a one-dimensional distribution that, if applied to all battlefields in an i.i.d. fashion, satisfies the budget constraint in expectation. Then, provided that an optimal strategy in a General Lotto game may be represented as a mixed strategy in a Colonel Blotto game that is symmetric across all battlefields, the mixed strategy in a Colonel Blotto game is optimal as well, i.e., an equilibrium is found in the Colonel Blotto game. That method delivers optimal strategies for Colonel Blotto games in the case of homogeneous budgets, and a variety of partial results in the case of heterogeneous budgets.<sup>6</sup> Note, however, that the introduction of a head start creates an asymmetry between battlefields. Therefore, the “Lotto approach” just described does not extend in an obvious way to the setup considered in the present paper.<sup>7</sup> Hortala-Vallve and Llorente-Saguer (2012) studied the properties of pure-strategy equilibria in Colonel Blotto games, allowing for battlefield valuations that are heterogeneous both across battlefields and across players. The paper most closely related to the present analysis is Liang et al. (2019), who used brute force to

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<sup>5</sup>Similarly, the analysis below considers the simplest non-trivial case of a Colonel Blotto game with a head start, viz. the case of two battlefields.

<sup>6</sup>Cf. the discussion in Dziubiński (2013, Sec. 5).

<sup>7</sup>When checking the early literature, we found that Beale and Heselden (1962), using an approach that might be considered a predecessor of Hart’s (2008), allowed for asymmetries between battlefields. A solution is constructed in three steps. First, given a deterministic allocation of the budget across battlefields, optimal Lotto strategies in each battlefield are determined. Then, exploiting useful convexity properties, the value in the General Lotto game is maximized by choosing the deterministic resource allocation. Finally, an algorithm is applied to construct a mixed strategy in the Colonel Blotto game that induces the mixed strategy in the General Lotto game. While interesting, that approach leads in general to approximations of optimal strategies only. Therefore, we do not expect this approach to be useful for the general identification of equilibria in the case of asymmetries across battlefields. Cf. Washburn (2014, Sec. 6.3) whose discussion suggests a similar conclusion.

characterize the equilibrium set of the Colonel Blotto game with two battlefields and heterogeneous budgets (yet no head start). In contrast to the present study, however, they did not apply dominance arguments to simplify the equilibrium analysis, neither did they exploit the useful properties of symmetric Toeplitz matrices.<sup>8</sup>

From the experimental literature on Blotto games, we mention three contributions. Arad and Rubinstein (2012) conducted a web-based experiment of the Colonel Blotto game. Two subjects each had to allocate a budget of  $n = 120$  troops across six battlefields. The results very nicely illustrate the variety of strategic considerations that arise even in simple Colonel Blotto games. Chowdhury et al. (2013) examined the main qualitative predictions of the (continuous-strategy) equilibrium theory of Colonel Blotto games in the case of heterogeneous budgets. Avrahami and Kareev (2009) conducted an analogous study for a class of General Blotto games. However, we are not aware of any experiment of Colonel Blotto games with a head start.<sup>9</sup>

The remainder of the paper is structured as follows. Section 2 contains preliminaries. Section 3 presents the equilibrium analysis in the cases (i) through (v) outlined above. Section 4 offers some discussion. In Section 5, we derive some implications for the Sion-Wolfe example. Section 6 clarifies the relationship between our analysis and the corresponding analysis of Colonel Blotto games with heterogeneous budgets but no head start. Section 7 concludes. Technical proofs have been relegated to an Appendix.

## 2. Preliminaries

### 2.1 Set-up and notation

Two players, called player 1 and player 2, are competing on two battlefields, called battlefield A and battlefield B.<sup>10</sup> Each player commands an integer number  $n$  of indivisible soldiers, where we assume  $n \geq 1$  throughout. Players compete by allocating soldiers across battlefields. We denote by  $x_i$  the number of soldiers allocated by player  $i \in \{1, 2\}$  to battlefield A. Then, without

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<sup>8</sup>We discuss the relationship between the two models in a separate section at the end of the paper.

<sup>9</sup>According to Beaglehole et al. (2022), the Colonel Blotto game belongs to the most well studied problems in algorithmic game theory. See, in particular, Ahmadinejad et al. (2019), Behnezhad et al. (2017), and Perchet et al. (2022). However, purely computational methods cannot be used to identify equilibria for larger classes of games.

<sup>10</sup>The case of two battlefields with homogeneous budgets is indeed the simplest case to consider *with* a head start. However, *without* a head start, this case is of little interest. See Lemma 2 below.

loss of generality, the remainder  $y_i = n - x_i$  is the number of soldiers allocated by player  $i$  to battlefield B. Thus, strategy spaces for player 1 and 2 may be specified (in reduced form) as  $X_1 = \{x_1 = 0, \dots, x_1 = n\}$  and  $X_2 = \{x_2 = 0, \dots, x_2 = n\}$ , respectively.

Player 1 is assumed to have a head start on battlefield A in the form of a nonnegative integer number  $k$  of additional soldiers. Player 2, in contrast, does not have any additional soldiers. We will therefore refer to battlefields A and B alternatively as the unbalanced and the balanced battlefield, respectively. If a player has, in total, more soldiers on a battlefield than her opponent, then she wins on that battlefield and receives a reward of 1, while the loser suffers a penalty of  $-1$  from that battlefield. In the case of a tie on a battlefield, both players get a payoff of 0 from that battlefield. Note that, after both players have deployed their troops, player 1 has a total of  $x_1 + k$  soldiers on battlefield A, while player 2 has a total of  $x_2$  soldiers there. Player 1's payoff from battlefield A is therefore given as

$$\Pi_1^A(x_1, x_2) = \text{sgn}((x_1 + k) - x_2), \quad (1)$$

where the *sign function*  $\text{sgn}(\cdot)$  is defined as usual by  $\text{sgn}(\xi) = 1$  if  $\xi > 0$ , by  $\text{sgn}(\xi) = 0$  if  $\xi = 0$ , and by  $\text{sgn}(\xi) = -1$  if  $\xi < 0$ . Moreover, due to the zero-sum condition,  $\Pi_2^A(x_1, x_2) = -\Pi_1^A(x_1, x_2)$ . On battlefield B, player 1 has a total of  $y_1 = n - x_1$  soldiers, while player 2 has a total of  $y_2 = n - x_2$  soldiers. From the accounting identity

$$y_1 - y_2 = (n - x_1) - (n - x_2) = x_2 - x_1, \quad (2)$$

player 1's payoff from battlefield B is therefore seen to be given as

$$\Pi_1^B(x_1, x_2) = \text{sgn}(x_2 - x_1). \quad (3)$$

Again, we have  $\Pi_2^B(x_1, x_2) = -\Pi_1^B(x_1, x_2)$ . Payoffs are assumed additively separable across battlefields. Consequently, player  $i$ 's total payoff is  $\Pi_i(x_1, x_2) = \Pi_i^A(x_1, x_2) + \Pi_i^B(x_1, x_2)$ , for  $i = 1, 2$ . The two-person zero-sum game just defined will be referred to as the *Colonel Blotto game with budget  $n$  and head start  $k$* , in short  $\mathfrak{B}(n, k)$ . For convenience, we will refer to the privileged player 1 alternatively as the *favorite*, and to player 2 as the *underdog*.

The general structure of the payoff matrix is illustrated in Figure 1. Only player 1's payoffs are shown. Below the main diagonal, all entries vanish (player 1 wins on battlefield A, but loses

on battlefield B). On the main diagonal, all entries equal 1 (player 1 wins on battlefield A, but ties with player 2 on battlefield B). The entries of the neighboring  $k - 1$  upper off-diagonals equal 2 (player 1 wins on both battlefields), while the entries of the  $k$ -th upper off-diagonal equal one (player 1 ties with player 2 on battlefield A, but wins on battlefield B). The entries in the remaining upper off-diagonals vanish (player 1 loses on battlefield A, but wins on battlefield B).<sup>11</sup>

	$x_2=0$	$x_2=1$	$x_2=2$	...	$x_2=k-1$	$x_2=k$	$x_2=k+1$	...	$x_2=n-k-1$	$x_2=n-k$	$x_2=n-k+1$	...	$x_2=n-2$	$x_2=n-1$	$x_2=n$
$x_1=0$	1	2	2	...	2	1	0	...	0	0	0	...	0	0	0
$x_1=1$	0	1	2	...	2	2	1	...	0	0	0	...	0	0	0
$x_1=2$	0	0	1	...	2	2	2	...	0	0	0	...	0	0	0
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
$x_1=k-1$	0	0	0	...	1	2	2	...	...	...	...	...	0	0	0
$x_1=k$	0	0	0	...	0	1	2	...	...	...	...	...	0	0	0
$x_1=k+1$	0	0	0	...	0	0	1	...	...	...	...	...	0	0	0
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
$x_1=n-k-1$	0	0	0	...	...	...	...	...	1	2	2	...	2	1	0
$x_1=n-k$	0	0	0	...	...	...	...	...	0	1	2	...	2	2	1
$x_1=n-k+1$	0	0	0	...	...	...	...	...	0	0	1	...	2	2	2
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
$x_1=n-2$	0	0	0	...	0	0	0	...	0	0	0	...	1	2	2
$x_1=n-1$	0	0	0	...	0	0	0	...	0	0	0	...	0	1	2
$x_1=n$	0	0	0	...	0	0	0	...	0	0	0	...	0	0	1

Figure 1. Player 1's payoff matrix in  $\mathfrak{B}(n, k)$ .

The set of probability distributions over the finite set  $X_i$  will be denoted by  $\Delta(X_i)$ , and any element  $\mu_i$  thereof will be referred to as a *mixed strategy*. Given any pure strategy realization  $\nu \in \{0, \dots, n\}$ , the *Dirac distribution*  $\delta_\nu$  is the probability distribution on  $\{0, \dots, n\}$  that gives all probability weight to the realization  $\nu$ , i.e.,  $\delta_\nu(\{\nu\}) = 1$ , and  $\delta_\nu(\{\hat{\nu}\}) = 0$  for any  $\hat{\nu} \neq \nu$ . Then, any probability distribution  $\mu_i \in \Delta(X_i)$  may be expressed in a unique way as a weighted sum  $\mu_i = \sum_{\nu=0}^n p_\nu \delta_\nu$ , where  $p_\nu \in [0, 1]$  is the probability that player  $i$ 's mixed strategy  $\mu_i$  realizes as

<sup>11</sup>While Figure 1 illustrates the payoff matrix in general, some caveats apply. Specifically, depending on parameter values, the payoff matrix may degenerate in several ways. First, if  $k = 0$ , then all entries vanish (by symmetry). Second, in the special case  $k = 1$ , the matrix does not contain any off-diagonals with entries equal to 2. Third, if  $k = n$ , then the payoff matrix does not contain any zero above the diagonal, and there is only one entry of 1 at  $(x_1, x_2) = (0, n)$ . Fourth and finally, if  $k \geq n+1$ , then all upper diagonal entries equal 2. One should also note that the relative ordering of the strategies  $x_i = k$  and  $x_i = n - k$  suggested by the illustration is not valid in general. As will become clear, these differences are part of the reason why we will need to differentiate between several cases in the equilibrium analysis.

$x_i = \nu$ , and where  $\sum_{\nu=0}^n p_\nu = 1$ . For any  $\mu_i \in \Delta(X_i)$ , we call the set  $\text{supp}\{\mu_i^*\} = \{\nu : p_\nu > 0\} \subseteq X_i$  the *support* of  $\mu_i$ . As usual, players' payoff functions are extended to mixed strategies by taking expectations. We will use notation such as  $E_{\mu_i}[\cdot]$  and  $E_{\mu_1, \mu_2}[\cdot]$  for the respective expected values. A mixed-strategy Nash equilibrium is a pair  $(\mu_1^*, \mu_2^*) \in \Delta(X_1) \times \Delta(X_2)$  such that  $E_{\mu_1^*, \mu_2^*}[\Pi_1(x_1, x_2)] \geq E_{\mu_1, \mu_2^*}[\Pi_1(x_1, x_2)]$  for any  $\mu_1 \in \Delta(X_1)$ , and  $E_{\mu_1^*, \mu_2^*}[\Pi_2(x_1, x_2)] \geq E_{\mu_1^*, \mu_2}[\Pi_2(x_1, x_2)]$  for any  $\mu_2 \in \Delta(X_2)$ .

Given that the game is finite, a mixed-strategy Nash equilibrium exists by Nash's theorem. Moreover,  $\mathfrak{B}(n, k)$  is zero-sum, so that all equilibria are payoff-equivalent. We will refer to player 1's equilibrium payoff  $v_1$  as the *value* of the game. Since player 1 has an advantage on one battlefield, but not in both, we (correctly) anticipate that  $v_1 \in [0, 1]$ .

## 2.2 Dominated strategies

The consideration of dominance relationships between strategies (Farquharson, 1969; Brams, 1975; Moulin, 1979) turns out to be a useful tool for identifying equilibria in Colonel Blotto games with a head start. This will be so especially in the cases (i) through (iii) outlined in the Introduction. We recall the definitions.

Let  $\hat{X}_1 \subseteq X_1$  and  $\hat{X}_2 \subseteq X_2$  be nonempty sets of strategies for both players. We will say that  $x_1 \in \hat{X}_1$  is *dominated by*  $\hat{x}_1 \in \hat{X}_1$  for player 1 in  $\hat{X}_1 \times \hat{X}_2$  if the following two conditions are simultaneously satisfied. First, for any pure strategy  $x_2 \in \hat{X}_2$ , we have  $\Pi_1(x_1, x_2) \leq \Pi_1(\hat{x}_1, x_2)$ . Second, there exists a strategy  $\hat{x}_2 \in \hat{X}_2$  such that  $\Pi_1(x_1, \hat{x}_2) < \Pi_1(\hat{x}_1, \hat{x}_2)$ . Similarly, we will say that a pure strategy  $x_2 \in \hat{X}_2$  is dominated by a strategy  $\hat{x}_2 \in \hat{X}_2$  for player 2 in  $\hat{X}_1 \times \hat{X}_2$  if analogous conditions hold with the roles of players 1 and 2 exchanged. We will say that  $x_i \in \hat{X}_i$  is *dominated in*  $\hat{X}_1 \times \hat{X}_2$  if there exists a strategy  $\hat{x}_i \in \hat{X}_i$  such that  $x_i$  is dominated by  $\hat{x}_i$  for player  $i$  in  $\hat{X}_1 \times \hat{X}_2$ .<sup>12</sup>

We are interested in undominated strategies. Let, therefore,  $X_1^{(1)}$  and  $X_2^{(1)}$  denote the sets of strategies for player 1 and 2, respectively, that are *not* dominated in  $X_1 \times X_2$ . The following result characterizes these sets in the Colonel Blotto game with budget  $n$  and head start  $k$ .

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<sup>12</sup> As mentioned in the Introduction, the term dominance is used in this paper to refer to what is commonly known as weak dominance between pure strategies. In general, a pure strategy may be dominated by a mixed strategy even though it is not dominated by any pure strategy (cf. Pearce, 1984). However, the additional mileage gained by using the more powerful definition is limited in our present application. Hence, we stick to the basic concept.

**Lemma 1.** *The respective sets of undominated strategies  $X_1^{(1)}$  and  $X_2^{(1)}$  in  $\mathfrak{B}(n, k)$  are as follows.*

- (i) *If  $k = 0$ , then  $X_1^{(1)} = X_2^{(1)} = \{0, \dots, n\}$ ;*
- (ii) *if  $k = 1$ , then  $X_1^{(1)} = \{0, \dots, n - 1\}$  and  $X_2^{(1)} = \{0\} \cup \{2, \dots, n\}$ ;*
- (iii) *if  $k \in \{2, \dots, n\}$ , then  $X_1^{(1)} = \{0, \dots, n - k + 1\}$  and  $X_2^{(1)} = \{0\} \cup \{k + 1, \dots, n\}$ ;*
- (iv) *if  $k \geq n + 1$ , then  $X_1^{(1)} = X_2^{(1)} = \{0\}$ .*

**Proof.** (i) For  $k = 0$ , payoff functions are constant, and all strategies are equivalent. Hence, the claim is immediate. (ii) For  $k = 1$ , player 1's payoff matrix has entries of 1 on the main diagonal and on the first upper off-diagonal, while all other entries vanish. Therefore, for player 1, strategy  $x_1 = n$  is dominated by  $\hat{x}_1 = n - 1$ , and no other strategy is dominated. Similarly, for player 2, exploiting the zero-sum property, strategy  $x_2 = 1$  is dominated by  $\hat{x}_2 = 0$ , and no other strategy is dominated. The claim follows. (iii) Suppose that  $k \in \{2, \dots, n\}$ . Then, as suggested by Figure 1, the pure strategy  $\hat{x}_1 = n - k + 1$  dominates all strategies  $x_1 \in \{n - k + 2, \dots, n\}$  for player 1. Moreover, there are no other dominated strategies for player 1. Similarly, the pure strategy  $\hat{x}_2 = 0$  dominates all strategies  $x_2 \in \{1, \dots, k\}$  for player 2, while no other strategy is dominated for player 2. (iv) If  $k \geq n + 1$ , player 1's payoff matrix has entries of 0 below the main diagonal, entries of 1 on the main diagonal, and entries of 2 above the main diagonal. Clearly, therefore, all strategies  $x_1 \in \{1, \dots, n\}$  for player 1 are dominated by  $\hat{x}_1 = 0$ , and similarly, all strategies  $x_2 \in \{1, \dots, n\}$  for player 2 are dominated by  $\hat{x}_2 = 0$ .  $\square$

We provide some intuition for the case  $k \in \{2, \dots, n\}$ . In this case, strategy  $\hat{x}_1 = n - k + 1$  dominates all higher strategies for player 1. Indeed, any strategy  $x_1 \geq n - k + 1$  secures a win on battlefield A, but  $\hat{x}_1 = n - k + 1$  is the most parsimonious in doing so, i.e., it leaves the largest number of soldiers for battlefield B. Similarly, for player 2, bidding any  $x_2 \in \{1, \dots, k - 1\}$  is dominated by  $\hat{x}_2 = 0$ , because out of all strategies that imply a certain loss on battlefield A,  $\hat{x}_2 = 0$  brings most soldiers to battlefield B. In fact, also  $x_2 = k$  is dominated by  $\hat{x}_2 = 0$  because the only scenario in which player 2 achieves a tie on battlefield A with  $x_2 = k$  entails that player 1 sends  $n$  soldiers to battlefield B, so that player 2 loses on battlefield B rather than achieving a tie there with  $\hat{x}_2 = 0$ .

Starting from players' unrestricted strategy spaces, we may recursively define

$$X_i^{(0)} = X_i \quad (i \in \{1, 2\}), \quad (4)$$

$$X_i^{(t)} = \left\{ x_i \in X_i^{(t-1)} \text{ s.t. } x_i \text{ is not dominated in } X_1^{(t-1)} \times X_2^{(t-1)} \right\} \quad (i \in \{1, 2\}; t \in \{1, 2, \dots\}). \quad (5)$$

Needless to say, this extends our earlier definition of  $X_1^{(1)}$  and  $X_2^{(1)}$ . A simple induction argument shows that  $X_i^{(t)} \neq \emptyset$  for any  $i$  and  $t$ . Since  $\mathfrak{B}(n, k)$  is a finite game, no further eliminations take place for sufficiently high  $t$ , so that  $X_1^{(t)} \times X_2^{(t)} = X_1^{(t-1)} \times X_2^{(t-1)}$ . In this case, we refer to  $X_i^{(\infty)} \equiv X_i^{(t)}$  as the set of strategies surviving the IEDS (iterated elimination of dominated strategies) procedure. If payoff functions are even constant on  $X_1^{(\infty)} \times X_2^{(\infty)}$ , then we say that  $\mathfrak{B}(n, k)$  is *dominance-solvable*.

Two well-known facts about dominated strategies should be recalled. First, the elimination of dominated strategies may eliminate Nash equilibria.<sup>13</sup> Second, and more importantly for the identification of equilibrium strategies, a Nash equilibrium found in a reduced game obtained by iteratively eliminating dominated strategies remains a Nash equilibrium in the original game.<sup>14</sup>

We will say that a mixed strategy Nash equilibrium  $(\mu_1^*, \mu_2^*)$  in  $\mathfrak{B}(n, k)$  is an *equilibrium in undominated strategies* (an *equilibrium in iteratively undominated strategies*) if  $\text{supp}\{\mu_i^*\} \subseteq X_i^{(1)}$  (if  $\text{supp}\{\mu_i^*\} \subseteq X_i^{(\infty)}$ ) for  $i \in \{1, 2\}$ . Since  $X_i^{(\infty)} \subseteq X_i^{(1)} \subseteq X_i$  for  $i = 1, 2$ , any equilibrium in iteratively undominated strategies is an equilibrium in undominated strategies, and any equilibrium in undominated strategies is an equilibrium. Moreover, using the second fact recalled above, the set of equilibria in undominated strategies (in iteratively undominated strategies) is isomorphic to the set of equilibria in reduced games obtained by eliminating dominated strategies (by iteratively eliminating dominated strategies) in  $\mathfrak{B}(n, k)$ .

### 3. Equilibrium analysis

In this section, we will identify Nash equilibria in the finite Blotto game with homogeneous budget  $n$  and head start  $k$ , i.e., in the game  $\mathfrak{B}(n, k)$ . As discussed in the Introduction, there are several cases. These will be dealt with below.

<sup>13</sup>The analogous statement for strictly dominated strategies is not true, of course.

<sup>14</sup>For a formal argument, see the proof of Proposition 1 in Moulin (1979).

### 3.1 The trivial cases where either $k = 0$ or $k \geq n$

This section presents the equilibrium analysis for two trivial cases. First, for  $k = 0$ , there is no head start. Payoff functions are constant and, therefore, any resource allocation is optimal. Second, for  $k \geq n$ , the underdog is unable to secure a win on battlefield A. Even if she concentrates all her forces on battlefield A, the favorite's head start will at least match them, and even overpower them if  $k \geq n+1$ . Thus, the conflict is essentially about battlefield B, and it is therefore intuitively optimal for both players to commit all their troops to that battlefield.

#### **Lemma 2.**

- (i) For  $k = 0$ , any combination of mixed strategies is an equilibrium in  $\mathfrak{B}(n, k)$ . The value is  $v_1 = 0$ .
- (ii) For  $k \geq n$ , there is a unique equilibrium in undominated strategies in  $\mathfrak{B}(n, k)$ , viz.  $(\mu_1^*, \mu_2^*) = (\delta_0, \delta_0)$ . The value is  $v_1 = 1$ .<sup>15</sup>
- (iii) In both cases,  $\mathfrak{B}(n, k)$  is dominance-solvable.

**Proof.** (i) If  $k = 0$ , then  $\Pi_1(x_1, x_2) = \text{sgn}(x_1 - x_2) + \text{sgn}(x_2 - x_1) = 0$ , for any  $x_1$  and  $x_2$ . Thus, in this case, payoff functions are constant and independent of strategic choices made by the players. The claim follows. (ii) Suppose first that  $k \geq n + 1$ . Then, player 1 wins on battlefield A regardless of the strategies chosen by the two players. Thus,  $\Pi_1(x_1, x_2) = 1 + \text{sgn}(x_2 - x_1)$ . The corresponding payoff matrix for this case is illustrated in panel (a) of Figure 2, where dominated strategies are marked in grey. As can be seen,  $(x_1, x_2) = (0, 0)$  is a strict pure-strategy Nash equilibrium. The corresponding value is  $v_1 = 1$ . However, as a consequence of exchangeability, any strict Nash equilibrium in a two-person zero-sum game is unique. Therefore, the claim follows in the case  $k \geq n + 1$ . Suppose finally that  $k = n$ . Then,  $\Pi_1(x_1, x_2) = \text{sgn}(x_1 + n - x_2) + \text{sgn}(x_2 - x_1)$ . The payoff matrix differs from the previous case at the strategy profile  $(x_1, x_2) = (0, n)$  only. See panel (b) of Figure 2 for illustration, where again, dominated strategies are marked in grey. As before,  $(x_1, x_2) = (0, 0)$  is a pure-strategy Nash equilibrium. In fact, player 1's best response to  $x_2 = 0$  is unique, so that by exchangeability, player 1 chooses  $x_1 = 0$  in any equilibrium. Hence,  $\mu_1^* = \delta_0$ . The set of player 2's best responses to  $x_1 = 0$

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<sup>15</sup>With dominated strategies admitted, the equilibrium remains unique if  $k \geq n + 1$ .

is  $\{x_2 = 0, x_2 = n\}$ . Therefore,  $\mu_2^* = (1 - \lambda)\delta_0 + \lambda\delta_n$  for some  $\lambda \in [0, 1]$ . Clearly, this is an equilibrium if and only if  $\lambda \in [0, \frac{1}{2}]$ . Moreover, strategy  $x_2 = n$  is dominated by  $\hat{x}_2 = 0$  for player 2, so that only the equilibrium where  $\lambda = 0$  is in undominated strategies. (iii) Immediate in view of Figure 2.  $\square$

(a)	$x_2 = 0$	$x_2 = 1$	...	$x_2 = n-1$	$x_2 = n$
$x_1 = 0$	1	2	...	2	2
$x_1 = 1$	0	1	...	2	2
...	...	...	...	2	2
$x_2 = n-1$	0	0	0	1	2
$x_1 = n$	0	0	0	0	1

(b)	$x_2 = 0$	$x_2 = 1$	...	$x_2 = n-1$	$x_2 = n$
$x_1 = 0$	1	2	...	2	1
$x_1 = 1$	0	1	...	2	2
...	...	...	...	2	2
$x_2 = n-1$	0	0	0	1	2
$x_1 = n$	0	0	0	0	1

Figure 2. Payoff matrices and dominated strategies in the cases (a)  $k \geq n + 1$ , and (b)  $k = n$ .

Lemma 2 characterizes the equilibrium in the Colonel Blotto game for the trivial cases where  $k = 0$  or  $k \geq n$ . These cases will, therefore, be excluded from the further analysis. In the sequel, it will be assumed that  $k \in \{1, \dots, n - 1\}$ .

### 3.2 The case $k \in \{\lfloor n/2 \rfloor + 1, \dots, n - 1\}$

Suppose that the head start is strictly larger than half of the budget and strictly smaller than the budget. In this case, dominance arguments turn out to have a lot of bite and guide quickly to an equilibrium. The following example illustrates this fact.

**Example 1.** Let  $n = 5$  and  $k = 3$ . The payoff matrix of  $\mathfrak{B}(n, k)$  is shown in panel (a) of Figure 3, where dominated strategies are marked in grey. Panels (b) and (c) exhibit the reduced game after one and two rounds of elimination, respectively. As can be seen, the reduced game obtained after application of IEDS is represented by a simple two-by-two matrix. Player 1 either sends  $x_1 = 3$  soldiers to battlefield A (the minimum necessary to win with certainty there), or she sends all her troops to battlefield B (if  $x_1 = 0$  so that  $y_1 = 5$ ). Similarly, Player 2 throws all her troops either on battlefield A (if  $x_2 = 5$ ) or on battlefield B (if  $x_2 = 0$ , so that  $y_2 = 5$ ). In terms of payoffs, it is a game of hide and seek. Indeed, given the head start, player 1 tries to track down player 2's forces, whereas player 2 tries to avoid this scenario. We conclude that the strategies

$\mu_1^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_3$  and  $\mu_2^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_5$  form the unique equilibrium in iteratively undominated strategies.<sup>16</sup> Moreover, player 1's expected payoff is  $v_1 = \frac{2}{3}$ .

(a)	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$
$x_1 = 0$	1	2	2	1	0	0
$x_1 = 1$	0	1	2	2	1	0
$x_1 = 2$	0	0	1	2	2	1
$x_1 = 3$	0	0	0	1	2	2
$x_1 = 4$	0	0	0	0	1	2
$x_1 = 5$	0	0	0	0	0	1

(b)	$x_2 = 0$	$x_2 = 4$	$x_2 = 5$
$x_1 = 0$	1	0	0
$x_1 = 1$	0	1	0
$x_1 = 2$	0	2	1
$x_1 = 3$	0	2	2

(c)	$x_2 = 0$	$x_2 = 5$
$x_1 = 0$	1	0
$x_1 = 3$	0	2

Figure 3. Iterated elimination of dominated strategies in the case  $(n, k) = (5, 3)$ .

The general analysis in the case  $k \in \{\lfloor n/2 \rfloor + 1, \dots, n-1\}$  is entirely analogous. We, therefore, arrive at our first result.

**Theorem 1.** *Suppose that  $k \in \{\lfloor n/2 \rfloor + 1, \dots, n-1\}$ . Then, IEDS identifies after two rounds the reduced strategy set  $X_1^{(\infty)} = X_1^{(2)} = \{0, n-k+1\}$  for player 1, and  $X_2^{(\infty)} = X_2^{(2)} = \{0, n\}$  for player 2, respectively. Moreover, there is a unique equilibrium in iteratively undominated strategies in  $\mathfrak{B}(n, k)$ , which is given by  $\mu_1^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_{n-k+1}$  and  $\mu_2^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_n$ . The value is  $v_1 = \frac{2}{3}$ .*

**Proof.** See the Appendix.  $\square$

The identified equilibrium keeps the simple form suggested by Example 1. As before, the conflict leads to a hide and seek game between an evasive underdog and a searching favorite. Indeed, the underdog randomizes between sending all her soldiers to battlefield A, which occurs with probability  $1/3$ , and sending all her troops to battlefield B, which occurs with probability  $2/3$ . The favorite, in turn, randomizes between sending just as many soldiers to battlefield A to guarantee a victory there, which occurs with probability  $1/3$ , and sending all her troops to battlefield B, which

<sup>16</sup>For this, we make use of the aforementioned fact that a Nash equilibrium in a given game remains a Nash equilibrium if a dominated strategy is added to the game. Starting from the mixed equilibrium in the 2x2 game, and iteratively adding strategies in reversed order of the IEDS procedure shows that the Nash equilibrium in the reduced form is a Nash equilibrium also in  $\mathfrak{B}(n, k)$ .

occurs with probability  $2/3$ . Thus, both players focus with higher probability on the balanced battlefield B (intuitively because the underdog's chances of winning or reaching a tie are higher there), but they also hide and seek in the unbalanced battlefield A.

Admitting strategies eliminated by the IEDS procedure, the equilibrium is no longer unique. In Example 1, e.g., any strategy for player 2 of the form  $\mu_2^\lambda = \frac{2}{3}\delta_0 + \frac{\lambda}{3}\delta_4 + \frac{1-\lambda}{3}\delta_5$ , with  $\lambda \in [0, 1]$  is optimal. However, only for  $\lambda = 0$ , the corresponding equilibrium  $(\mu_1^*, \mu_2^\lambda)$  is in iteratively undominated strategies.

### 3.3 The case $k = 1$

The case  $k = 1$  is special because the head start is too small to allow player 1 to win both battlefields with certainty in any outcome of the game. As in the previously considered case, the iterated elimination of dominated strategies is a route to identifying the support of an equilibrium candidate. In contrast, however, the IEDS procedure does not stop after two rounds for  $k = 1$  but goes on for quite a while. We illustrate this point with an example.

**Example 2.** Let  $(n, k) = (6, 1)$ . The corresponding payoff matrix is shown in panel (a) of Figure 4. The IEDS procedure yields a sequence of reduced games shown in panels (b) through (d). For player 1, the set of iteratively undominated strategies is  $X_1^{(\infty)} = \{0, 2, 3, 5\}$ . For player 2, we have  $X_2^{(\infty)} = \{0, 2, 4, 6\}$ . Given that the corresponding payoff matrix, shown in panel (d), is an identity matrix, it is easy to see that uniform randomizations over  $X_1^{(\infty)}$  and  $X_2^{(\infty)}$  constitute the unique equilibrium in iteratively undominated strategies.

(a)

	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$	$x_2 = 6$
$x_1 = 0$	1	1	0	0	0	0	0
$x_1 = 1$	0	1	1	0	0	0	0
$x_1 = 2$	0	0	1	1	0	0	0
$x_1 = 3$	0	0	0	1	1	0	0
$x_1 = 4$	0	0	0	0	1	1	0
$x_1 = 5$	0	0	0	0	0	1	1
$x_1 = 6$	0	0	0	0	0	0	1

(b)

	$x_2 = 0$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$	$x_2 = 6$
$x_1 = 0$	1	0	0	0	0	0
$x_1 = 1$	0	1	0	0	0	0
$x_1 = 2$	0	1	1	0	0	0
$x_1 = 3$	0	0	1	1	0	0
$x_1 = 4$	0	0	0	1	1	0
$x_1 = 5$	0	0	0	0	1	1

(c)

	$x_2 = 0$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 6$
$x_1 = 0$	1	0	0	0	0
$x_1 = 2$	0	1	1	0	0
$x_1 = 3$	0	0	1	1	0
$x_1 = 4$	0	0	0	1	0
$x_1 = 5$	0	0	0	0	1

(d)

	$x_2 = 0$	$x_2 = 2$	$x_2 = 4$	$x_2 = 6$
$x_1 = 0$	1	0	0	0
$x_1 = 2$	0	1	0	0
$x_1 = 3$	0	0	1	0
$x_1 = 5$	0	0	0	1

Figure 4. Iterated elimination of dominated strategies in  $\mathfrak{B}(6, 1)$ .

We now turn to the general analysis of the case  $k = 1$ . We start with identifying the set of strategies that survive the iterated elimination of dominated strategies.

**Lemma 3.** *Suppose that the head start is  $k = 1$ . Then, the set of strategies  $X_1^{(\infty)}$  and  $X_2^{(\infty)}$  surviving the iterated elimination of dominated strategies are the following:*

(i) *If  $n$  is even, then  $X_1^{(\infty)} = \{0, 2, \dots, 2q, 2q + 1, \dots, n - 1\}$  and  $X_2^{(\infty)} = \{0, 2, \dots, n\}$ , with  $q = \lfloor n/4 \rfloor$ ;*

(ii) *if  $n$  is odd, then  $X_1^{(\infty)} = \{0, 2, \dots, n - 1\}$  and  $X_2^{(\infty)} = \{0, 2, \dots, 2q, 2q + 3, \dots, n\}$ .*

**Proof.** See the Appendix.  $\square$

As the example suggests, the IEDS procedure eliminates precisely one strategy for each player in each round. It starts at extremal strategies, viz. at  $x_1 = n$ , which is dominated by  $\hat{x}_1 = n - 1$ , and at  $x_2 = 1$ , which is dominated by  $\hat{x}_2 = 0$ . In each round, the procedure then picks on the respective opposite side of the strategy space an extremal strategy, skipping one strategy each time. Specifically, in the second round, one deletes  $x_1 = 1$ , which is dominated by  $\hat{x}_1 = 2$ , and  $x_2 = n - 1$ , which is dominated by  $\hat{x}_2 = n$ . In the third round, *unless the procedure has already stopped*, one deletes  $x_1 = n - 2$ , which is dominated by  $\hat{x}_1 = n - 3$ , and  $x_2 = 3$ , which is dominated by  $\hat{x}_2 = 2$ . And so on. Jumping back and forth in this way, the procedure continues until that strategy  $\hat{x}_i$  that would dominate the next strategy  $x_i$  of player  $i$  has already been eliminated in prior rounds. One can convince oneself that, for  $n$  even (odd), the procedure stops within player 1's (player 2's) strategy set, which leads to the characterization of  $X_1^{(\infty)}$  and  $X_2^{(\infty)}$  given in the lemma.

After the reduction of the strategy spaces, the equilibrium has a simple structure.

**Theorem 2.** *Suppose that  $k = 1$ . Then, there exists a unique Nash equilibrium in iteratively undominated strategies in  $\mathfrak{B}(n, k)$ . In this equilibrium, each player  $i \in \{1, 2\}$  randomizes uniformly over the set  $X_i^{(\infty)}$  characterized in Lemma 3. The value of the game is*

$$v_1 = \frac{1}{1 + \lfloor n/2 \rfloor}. \quad (6)$$

**Proof.** See the Appendix.  $\square$

Theorem 2 identifies an equilibrium in the Colonel Blotto game for the case  $k = 1$ . The equilibrium strategies randomize uniformly over a subset of the strategy space that includes, roughly speaking, every second strategy.

As in the previous cases, there are additional equilibria involving dominated strategies. For example, for  $n$  even, there is a Nash equilibrium in which both players randomize uniformly over all even strategies. The equilibrium in undominated strategies characterized in Theorem 2 is a variant thereof, where dominated strategies are replaced by undominated strategies that are equivalent in terms of expected payoffs against the opponent's optimal strategy.

### 3.4 The case $k \in \{2, \dots, \lfloor n/2 \rfloor\}$ , with $n$ divisible by $k$

This section deals with the case  $k \in \{2, \dots, \lfloor n/2 \rfloor\}$ , where  $n$  is divisible by  $k$ . Recall our general notational convention that  $m = \lfloor n/k \rfloor$ . Since  $n$  is divisible by  $k$ , of course  $m = n/k$  holds in this subsection.

In the considered case, the iterated elimination of dominated strategies proves to be less effective in general. However, this fact ultimately does not cause any problems for the analysis because the equilibrium turns out to be unique.

**Theorem 3.** *Suppose that  $n$  is divisible by  $k$ , where  $k \in \{2, \dots, \lfloor n/2 \rfloor\}$ . Then, the following is the unique Nash equilibrium in  $\mathfrak{B}(n, k)$ : Player 1 chooses a mixed strategy  $\mu_1^*$  specified as follows:*

- $x_1 = 0$  with probability  $\mu_1^*(x_1) = \frac{2m-1}{m^2+m-1}$ ,
- $x_1 = s_1 k$  with probability  $\mu_1^*(x_1) = \frac{m-s_1}{m^2+m-1}$ , for  $s_1 \in \{1, \dots, m-1\}$ ,
- $x_1 = s_1 k + 1$  with probability  $\mu_1^*(x_1) = \frac{s_1}{m^2+m-1}$ , for  $s_1 \in \{1, \dots, m-1\}$ .

*Player 2 chooses the following strategy  $\mu_2^*$ :*

- $x_2 = 0$  with probability  $\mu_2^*(x_2) = \frac{2m-1}{m^2+m-1}$ ,
- $x_2 = s_2 k$  with probability  $\mu_2^*(x_2) = \frac{s_2-1}{m^2+m-1}$ , for  $s_2 \in \{2, \dots, m\}$ ,
- $x_2 = s_2 k + 1$  with probability  $\mu_2^*(x_2) = \frac{m-s_2}{m^2+m-1}$ , for  $s_2 \in \{1, \dots, m-1\}$ ,

All other pure strategies are strictly suboptimal and chosen with probability zero. The value of  $\mathfrak{B}(n, k)$  is

$$v_1 = \frac{2m - 1}{m^2 + m - 1}. \quad (7)$$

**Proof.** See the Appendix.  $\square$

Theorem 3 characterizes the unique equilibrium in the considered case. Thus, the uniqueness property is stronger than those obtainable in the cases dealt with before.

It should be noted that player 2 uses the bid  $x_2 = n$  with positive probability, because  $n = mk$ . Further, the support of the equilibrium strategy is “thin” in the sense that it contains, in addition to the boundary strategies  $x_i \in \{0, n\}$ , strategies of the form  $s_i k$  and  $s_i k + 1$  only, where  $s_i \in \{1, \dots, m - 1\}$ . Thus, resource decisions tend to concern multiple units of  $k$  soldiers (or “battalions”) rather than individual soldiers. Indeed, given that  $n$  is divisible by  $k$ , pure-strategy realizations of randomized equilibrium strategies may be thought of as sending a number of  $s_i \in \{0, \dots, m\}$  units to battlefield A and  $m - s_i$  units to battlefield B. Moreover, in cases where  $s_i \neq m$ , one additional soldier (or “messenger”) may be withdrawn from battlefield B and sent to battlefield A. As in the case where  $k \in \{\lfloor n/2 \rfloor, \dots, n - 1\}$ , competition takes the form of a hide-and-seek game, where the favorite (player 1) tries to anticipate the hiding underdog’s (player 2) strategy, so as to slightly overpower her on both battlefields.

The proof of the equilibrium property in Theorem 3 is not particularly original. To verify that  $(\mu_1^*, \mu_2^*)$  is a Nash equilibrium, we check the optimality conditions for each player. Careful computations show that

$$E_{\mu_2^*}[\Pi_1(x_1, x_2)] = v_1 \quad (x_1 \in \text{Supp}(\mu_1^*)) \quad (8)$$

and

$$E_{\mu_2^*}[\Pi_1(x_1, x_2)] < v_1 \quad (x_1 \notin \text{Supp}(\mu_1^*)). \quad (9)$$

Clearly, this shows that player 1’s optimality condition is satisfied. The analysis of player 2’s optimality condition is similar, which establishes existence.

Relationship (9) implies that pure strategies not contained in the support of  $\mu_1^*$  are even *strictly* suboptimal for player 1. Again, a similar relationship holds for player 2. Jointly, these

observations are used to prove uniqueness. Indeed, interchangeability of Nash equilibria in two-person constant-sum games implies that only pure best responses to  $\mu_1$  and  $\mu_2$  appear with positive probability in any Nash equilibrium. It therefore suffices to show that the payoff matrices, cut back to represent only the set of pure best responses, are invertible. These matrices turn out to be closely related to what is known as *finite symmetric Toeplitz matrices* (Gohberg and Semencul, 1972; Rodman and Shalom, 1992).<sup>17</sup> We show by induction that invertibility is given, and thereby establish uniqueness.

### 3.5 The case $k \in \{2, \dots, \lfloor n/2 \rfloor\}$ , with $n$ not divisible by $k$

If  $n$  is not divisible by  $k$ , the structure of equilibrium changes slightly compared to the previous case. We start again with an example.

**Example 3.** Let  $n = 5$  and  $k = 2$ . The IEDS procedure simplifies the game, as shown in panels (a) through (c) of Figure 5. The unique equilibrium in iteratively undominated strategies is given by  $\mu_1^* = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_2 + \frac{1}{4}\delta_4$  and  $\mu_2^* = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_3 + \frac{1}{4}\delta_5$ . The value is  $v_1 = \frac{1}{2}$ .

(a)	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$
$x_1 = 0$	1	2	1	0	0	0
$x_1 = 1$	0	1	2	1	0	0
$x_1 = 2$	0	0	0	2	1	0
$x_1 = 3$	0	0	0	1	2	1
$x_1 = 4$	0	0	0	0	1	2
$x_1 = 5$	0	0	0	0	0	1

(b)	$x_2 = 0$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$
$x_1 = 0$	1	0	0	0
$x_1 = 1$	0	1	0	0
$x_1 = 2$	0	2	1	0
$x_1 = 3$	0	1	2	1
$x_1 = 4$	0	0	1	2

(c)	$x_2 = 0$	$x_2 = 3$	$x_2 = 4$	$x_2 = 5$
$x_1 = 0$	1	0	0	0
$x_1 = 2$	0	2	1	0
$x_1 = 3$	0	1	2	1
$x_1 = 4$	0	0	1	2

Figure 5. Iterated elimination of weakly dominated strategies in the case  $(n, k) = (5, 2)$ .

The following result identifies an equilibrium in this case.

<sup>17</sup>A *symmetric Toeplitz matrix* is a square matrix  $\{x_{\nu_1\nu_2}\}$  such that (i)  $x_{\nu_1\nu_2}$  depends only on  $\nu_1 - \nu_2$  (Toeplitz property), and (ii)  $x_{\nu_1\nu_2} = x_{\nu_2\nu_1}$  (symmetry). E.g., the lower three-by-three principal submatrix of the payoff matrix shown in panel (c) of Figure 5 is a symmetric Toeplitz matrix.

**Theorem 4.** *Let  $n \geq 3$  and  $k \in \{2, \dots, \lfloor n/2 \rfloor\}$ , with  $n$  not divisible by  $k$ . Then,  $n = mk + r$  for  $m = \lfloor n/k \rfloor \geq 2$  and  $r \in \{1, \dots, k-1\}$ . Then, an equilibrium in  $\mathfrak{B}(n, k)$  is given by*

$$\mu_1^* = \frac{2}{m+2}\delta_0 + \left( \frac{1}{m+2} \sum_{\nu=1}^{m-1} \delta_{k\nu} \right) + \frac{1}{m+2}\delta_{n-k+1}, \quad (10)$$

$$\mu_2^* = \frac{2}{m+2}\delta_0 + \frac{1}{m+2} \sum_{\nu=1}^m \delta_{k\nu+r}. \quad (11)$$

The value is

$$v_1 = \frac{2}{m+2}. \quad (12)$$

**Proof.** See the Appendix.  $\square$

Like in the previous case, resource decisions concern units of  $k$  soldiers rather than individual soldiers. This becomes most evident by focusing on the underdog's resource commitments to battlefield B. Indeed,  $y_2 = n - x_2$  ranges over the set  $\{0, k, 2k, \dots, mk\} \cup \{n\}$ , i.e., the underdog essentially decides about the number of units she sends to battlefield B, where the option of “all in” on battlefield B is used with positive probability as well. The favorite in response tries to outguess the underdog's decision by sending between zero and  $(m-1)$  units of size  $k$  to battlefield A, while also sending with positive probability the minimum number of troops  $x_1 = n - k + 1$  sufficient to guarantee a victory there.

From the zero-sum property, the value of the discrete Colonel Blotto game with head start is always unique. It should be noted, however, that Theorem 4 does not contain any general assertion regarding equilibrium uniqueness. The following result, which holds under additional parameter restrictions, generalizes Example 3.

**Lemma 4.** *The equilibrium specified in Theorem 4 is unique in iteratively undominated strategies if one of the following two conditions holds:*

- (i)  $k = 2$ ;
- (ii)  $n = mk + r$ , for  $m = \lfloor n/k \rfloor = 2$  and some  $r \in \{2, \dots, k-1\}$ .

**Proof.** See the Appendix.  $\square$

It should be noted that Lemma 4 provides sufficient conditions only. As part (i) shows, uniqueness in iteratively undominated strategies holds for  $k = 2$  and  $n$  odd. In that case, IEDS ends after

two rounds, and the resulting matrix is essentially a symmetric Toeplitz matrix that is invertible, as in panel (c) of Figure 5. Part (ii) requires that  $m = 2$  and that the residue term  $r$  differs from one. In that case, the equilibrium is given by

$$\mu_1^* = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_k + \frac{1}{4}\delta_{n-k+1}, \quad (13)$$

$$\mu_2^* = \frac{1}{2}\delta_0 + \frac{1}{4}\delta_{k+r} + \frac{1}{4}\delta_n. \quad (14)$$

Moreover, the corresponding value is  $v_1 = \frac{1}{2}$ . This latter observation will be useful for our review of the Sion-Wolfe example.

#### 4. Discussion

Combining the results from the previous section, we obtain the following characterization of equilibrium values.

**Corollary 1.** *Let  $m = \lfloor n/k \rfloor$ . Then, player 1's value  $v_1 = V(n, k)$  in a discrete Colonel Blotto game with homogeneous budget of size  $n$  and head start  $k$  for player 1 is given by*

$$V(n, k) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{1}{\lfloor n/2 \rfloor + 1} & \text{if } k = 1 \\ \frac{2m-1}{m^2+m-1} & \text{if } n \text{ is divisible by } k, \text{ and } k \in \{2, \dots, n-1\} \\ \frac{2}{m+2} & \text{if } n \text{ is not divisible by } k, \text{ and } k \in \{2, \dots, n-1\} \\ 1 & \text{if } k \geq n \end{cases} \quad (15)$$

**Proof.** Immediate from Lemma 2 and Theorems 1 through 4.  $\square$

Table I exhibits the value  $v_1$  as a function of the budget  $n$ , represented as rows, and the head start  $k$ , represented as columns. The table suggests that, for  $n$  kept fixed,  $v_1$  is weakly increasing in the head start, which of course makes sense. Conversely, for  $k$  kept fixed, the value is weakly decreasing in  $n$ . Again, this is in line with intuition because a head start of given size becomes less valuable as the budget grows. Going over the possible cases suggested by Lemma 2 and Theorems 1 through 4, we obtain this as a formal result as well.

**Lemma 5.**  *$V(n, k)$  is (i) weakly increasing in  $k$ , and (ii) weakly decreasing in  $n$ .*

**Proof.** See the Appendix.  $\square$

n \ k	0	1	2	3	4	5	6	7	8	9	10
1	0.000	1.000									
2	0.000	0.500	1.000								
3	0.000	0.500	0.667	1.000							
4	0.000	0.333	0.600	0.667	1.000						
5	0.000	0.333	0.500	0.667	0.667	1.000					
6	0.000	0.250	0.455	0.600	0.667	0.667	1.000				
7	0.000	0.250	0.400	0.500	0.667	0.667	0.667	1.000			
8	0.000	0.200	0.368	0.500	0.600	0.667	0.667	0.667	1.000		
9	0.000	0.200	0.333	0.455	0.500	0.667	0.667	0.667	0.667	1.000	
10	0.000	0.167	0.310	0.400	0.500	0.600	0.667	0.667	0.667	0.667	1.000
11	0.000	0.167	0.286	0.400	0.500	0.500	0.667	0.667	0.667	0.667	0.667
12	0.000	0.143	0.268	0.368	0.455	0.500	0.600	0.667	0.667	0.667	0.667
13	0.000	0.143	0.250	0.333	0.400	0.500	0.500	0.667	0.667	0.667	0.667
14	0.000	0.125	0.236	0.333	0.400	0.500	0.500	0.600	0.667	0.667	0.667
15	0.000	0.125	0.222	0.310	0.400	0.455	0.500	0.500	0.667	0.667	0.667
16	0.000	0.111	0.211	0.286	0.368	0.400	0.500	0.500	0.600	0.667	0.667
17	0.000	0.111	0.200	0.286	0.333	0.400	0.500	0.500	0.500	0.667	0.667
18	0.000	0.100	0.191	0.268	0.333	0.400	0.455	0.500	0.500	0.600	0.667
19	0.000	0.100	0.182	0.250	0.333	0.400	0.400	0.500	0.500	0.500	0.667
20	0.000	0.091	0.174	0.250	0.310	0.368	0.400	0.500	0.500	0.500	0.600

Table I. Player 1's equilibrium payoff as a function of the budget  $n$  and the head start  $k$ .

We end this section with a brief discussion of the nature of competition in  $\mathfrak{B}(n, k)$ . In the equilibria identified in the previous section, the favorite with a head start of at least two soldiers and strictly less than  $n + 1$  soldiers wins on both battlefields only if she correctly anticipates how many units of soldiers the underdog sends to the balanced battlefield. While this intuitive description of the equilibrium as a game of hide and seek may be appealing, it should be noted that it is somewhat at odds with the *principle of concentration* that features quite prominently in the applied military literature (e.g., Mixon, 1988). We see the reason for this discrepancy in the assumed auction technology. I.e., the fact that one or two additional soldiers may entirely tip the balance in a battlefield is clearly a strong assumption. To bring the theoretical analysis closer to military application, it therefore seems desirable to study, in future work, also contest technologies that are less decisive than the auction.

## 5. Implications for the Sion-Wolfe example

Sion and Wolfe (1957, Sec. 2) considered a Colonel Blotto game with two battlefields and homogeneous budgets in which one player has a head start in one battlefield corresponding in size to half of the budget. In contrast to the model considered above, however, they worked with continuous strategy spaces. In this section, we will review their main result and relate it to our analysis of finite approximations.

The continuous model is defined in analogy to the finite case. Given  $\kappa \geq 0$ , let the two-person zero-sum game  $\mathfrak{B}_\kappa$  on the unit square be given by payoff functions

$$\pi_1(x_1, x_2) = \text{sgn}((x_1 + \kappa) - x_2) + \text{sgn}(x_2 - x_1), \quad (16)$$

and  $\pi_2(x_1, x_2) = -\pi_1(x_1, x_2)$ , respectively. In the mixed extension, players choose arbitrary probability distributions on the Borel sets of the interval  $[0, 1]$ . We denote by  $\tilde{\mu}_i$  the mixed strategy of player  $i \in \{1, 2\}$ . The main result in Sion and Wolfe (1957) may now be summarized as follows.

**Lemma 6. (Sion and Wolfe)** *Suppose that players choose probability distributions  $\tilde{\mu}_1$  and  $\tilde{\mu}_2$ , respectively, on the unit interval. Then,*

$$\underline{v}_1 \equiv \inf_{\tilde{\mu}_1} \sup_{\tilde{\mu}_2} \iint_{[0,1] \times [0,1]} \pi_1(x_1, x_2) d\tilde{\mu}_1(x_1) d\tilde{\mu}_2(x_2) = \frac{4}{7} \simeq 0.571, \quad (17)$$

$$\bar{v}_1 \equiv \inf_{\tilde{\mu}_2} \sup_{\tilde{\mu}_1} \iint_{[0,1] \times [0,1]} \pi_1(x_1, x_2) d\tilde{\mu}_1(x_1) d\tilde{\mu}_2(x_2) = \frac{2}{3} \simeq 0.667. \quad (18)$$

*In particular, the game  $\mathfrak{B}_{0.5}$  has no value.*

**Proof.** See the Appendix.  $\square$

The maximin value  $\underline{v}_1$  of  $\mathfrak{B}_{0.5}$  (from player 1's perspective) is defined through equation (17). Intuitively, this value is the expected payoff level that player 1 can guarantee herself in a sequential setting in which a first-moving player 1's mixed strategy (but not its pure-strategy realization) is observed by a second-moving player 2. The minimax value  $\bar{v}_1$  of  $\mathfrak{B}_{0.5}$  is, in turn, defined through equation (18). The easiest way to interpret this number in a game-theoretic setting is to consider it as the negative of the maximin value from player 2's perspective, i.e.,  $\bar{v}_1 = -\underline{v}_2$ , where  $\underline{v}_2$  is defined in analogy to (17) with the roles of players 1 and 2 exchanged. The remarkable point about Lemma 6 is that  $\underline{v}_1 < \bar{v}_1$ , i.e., the maximin value is strictly smaller than the corresponding minimax value. As a result, the game  $\mathfrak{B}_{0.5}$  has no value, and there is no Nash equilibrium in mixed strategies.<sup>18</sup>

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<sup>18</sup>It may be noted that we departed from Sion and Wolfe's (1957) original notation, which leads to the potentially unfamiliar values for the maximin and minimax values shown in equations (17) and (18). Details on the transformation applied may be found in the Appendix. In a nutshell, we swapped the roles of the two players, and likewise the roles of the two battlefields. Moreover, we normalized payoffs so that a tie on a battlefield implies a payoff of zero from the battlefield. In fact, that same payoff normalization is proposed in the original article as well when the abstract game is interpreted as a Colonel Blotto game with head start.

We briefly survey existing approaches to resolving the non-existence problem. First, as pointed out by Parthasarathy (1970), if one of the players is restricted to use an absolutely continuous strategy, then the resulting game has a value. Second, Kindler (1983) proposed the use of finitely additive measures. While that approach is intuitive, it is not straightforward to give the resulting solution an economic interpretation.<sup>19</sup> Finally, Boudreau and Schwartz (2019) proposed to modify the Sion-Wolfe game at points of discontinuity. Even though this approach is consistent with Simon and Zame's (1990) idea of endogenizing the payoff implications of ties, it does not resolve the nonexistence problem for the original game.

As discussed in the Introduction, we try to shed light on the result above by considering discrete-grid approximations of  $\mathfrak{B}_{0.5}$ . Thus, we consider sequences of finite Colonel Blotto games  $\mathfrak{B}(n, k)$ , where  $n, k \rightarrow \infty$  and  $n/k \simeq 2$ . By the results obtained so far, each such  $\mathfrak{B}(n, k)$ , for  $n$  sufficiently large and for  $n/k$  not too far away from 2, admits a unique Nash equilibria in iteratively undominated strategies. Taking the respective limits of these sequences, we arrive at the three mixed strategy profiles

$$\tilde{\mu}^{\text{Huey}} = \left( \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{1/2}, \frac{1}{2}\delta_0 + \frac{1}{4}\delta_{1/2} + \frac{1}{4}\delta_1 \right), \quad (19)$$

$$\tilde{\mu}^{\text{Dewey}} = \left( \frac{3}{5}\delta_0 + \frac{2}{5}\delta_{1/2}, \frac{3}{5}\delta_0 + \frac{1}{5}\delta_{1/2} + \frac{1}{5}\delta_1 \right), \quad (20)$$

$$\tilde{\mu}^{\text{Louie}} = \left( \frac{2}{3}\delta_0 + \frac{1}{3}\delta_{1/2}, \frac{2}{3}\delta_0 + \frac{1}{3}\delta_1 \right), \quad (21)$$

where the Dirac distributions are defined in straightforward extension of the finite case. For example, in the profile  $\tilde{\mu}^{\text{Dewey}}$ , derived using Theorem 3 in the case where  $n = 2k$ , the privileged player 1 sends all of her troops to battlefield B with probability  $3/5$ , and splits her troops equally across the two battlefields with probability  $2/5$ . Player 2, however, sends all of her troops to battlefield B with probability  $3/5$ , splits her troops equally with probability  $1/5$ , and send all of her troops to battlefield A with probability  $1/5$ . The other profiles admit analogous interpretations.

Using this notation, we can make the following observation.

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<sup>19</sup>Finitely additive measures are a more flexible notion of probability distribution, where  $\sigma$ -additivity is replaced by finite additivity (Yanovskaya, 1970). Intuitively, this means that it is feasible to define a bid that exceeds a given (deterministic) bid by an arbitrarily small amount. However, from an economic perspective, finitely additive measures suffer from the problem that Fubini's theorem does not in general hold. In plain English, this means that ex-ante expected payoffs need not be well-defined. The reason is that, should both players make use of an arbitrarily small increment over an existing bid, then there would be ambiguity regarding who wins with what probability.

**Corollary 2.** *Consider a sequence of discrete Colonel Blotto games  $\mathfrak{B}(n, k)$  with  $n, k \rightarrow \infty$  such that  $n/k \rightarrow 2$ . Then, the accumulation points of the corresponding sequence of unique Nash equilibria in iteratively undominated strategies are contained in the set*

$$\mathfrak{M}_3 = \{\tilde{\mu}^{\text{Huey}}, \tilde{\mu}^{\text{Dewey}}, \tilde{\mu}^{\text{Louie}}\}. \quad (22)$$

*Conversely, there exists a sequence of discrete Colonel Blotto games with  $n, k \rightarrow \infty$  such that  $n/k \rightarrow 2$  and such that the accumulation points of the corresponding sequence of unique Nash equilibria in iteratively undominated strategies are precisely the elements of  $\mathfrak{M}_3$ .*

**Proof.** Consider a sequence of discrete Colonel Blotto games  $\mathfrak{B}(n, k)$  with  $n, k \rightarrow \infty$ ,  $n/k < 2$ , and  $n - 2k = o(n)$ .<sup>20</sup> Then, the corresponding sequence of iteratively undominated Nash equilibria characterized in Theorem 1 converges weakly to the limit profile  $\tilde{\mu}^{\text{Louie}}$ . Next, by Theorem 3, any sequence of unique Nash equilibria in  $\mathfrak{B}(n, k)$  with  $n, k \rightarrow \infty$  such that  $n/k = 2$  converges weakly to the limit profile  $\tilde{\mu}^{\text{Dewey}}$ . Consider, finally, the sequence of discrete Colonel Blotto games  $\mathfrak{B}(n, k)$  with  $n, k \rightarrow \infty$ ,  $n/k > 2$ , and  $n - 2k = o(n)$ . Then, the corresponding sequence of Nash equilibria characterized in Theorem 4, which have been seen to be unique in iteratively undominated strategies for  $n$  large enough, converges weakly to the limit profile  $\tilde{\mu}^{\text{Huey}}$ . This proves the claim.  $\square$

The corresponding limit values are given by

$$v^{\text{Huey}} = \frac{2}{3} \simeq 0.667 \quad (23)$$

$$v^{\text{Dewey}} = \frac{3}{5} \simeq 0.600 \quad (24)$$

$$v^{\text{Louie}} = \frac{1}{2} \simeq 0.500. \quad (25)$$

These values can alternatively be retrieved from the numerical values shown in Table I. What is a bit strange about Corollary 2 is that the accumulation points of values of finite discretizations of the Sion-Wolfe game do not correspond in a simple way to the minimax and maximin values of the continuous game shown in Lemma 6. Specifically,  $\underline{v}_1$  lies strictly between  $v^{\text{Louie}}$  and  $v^{\text{Dewey}}$ , while  $\bar{v}_1$  equals  $v^{\text{Huey}}$ . Moreover,  $v^{\text{Dewey}} = \frac{3}{5}$  is the limit of  $V(n, k)$  as  $n = 2k \rightarrow \infty$ , but this value does not appear in the statement of Lemma 6.

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<sup>20</sup>As usual, the notation  $f(n) = o(n)$  means that  $\lim_{n \rightarrow \infty} f(n)/n = 0$ .

The situation does not become clearer when one compares the underlying strategy profiles. Using our notation, players' maximin strategies identified by Sion and Wolfe (1957) read

$$\mu_1^{\text{SW}57} = \frac{4}{7}\delta_0 + \frac{2}{7}\delta_{1/2} + \frac{1}{7}\delta_{3/4}, \quad (26)$$

$$\mu_2^{\text{SW}57} = \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{1/2} + \frac{1}{3}\delta_1. \quad (27)$$

These strategies are not components of any element of  $\mathfrak{M}_3$ , neither do they relate to the members  $\mathfrak{M}_3$  in any transparent way. In fact, player 1's respective component strategies  $\tilde{\mu}_1^{\text{Huey}}$ ,  $\tilde{\mu}_1^{\text{Dewey}}$ , and  $\tilde{\mu}_1^{\text{Louie}}$  fail to be maximin strategies in the continuous game.<sup>21</sup> For player 2, it turns out that  $\tilde{\mu}_2^{\text{Louie}}$  is actually a maximin strategy since

$$\begin{aligned} & \inf_{x_1 \in [0,1]} E_{\tilde{\mu}_2^{\text{Louie}}}[\pi_2(x_1, x_2)] \\ &= \inf_{x_1 \in [0,1]} \frac{2}{3}\pi_2(x_1, 0) + \frac{1}{3}\pi_2(x_1, 1) \end{aligned} \quad (28)$$

$$= \inf_{x_1 \in [0,1]} -\frac{2}{3} \left( \text{sgn}(x_1 + \frac{1}{2}) - \text{sgn}(x_1) \right) - \frac{1}{3} \left( \text{sgn}(x_1 - \frac{1}{2}) + \text{sgn}(1 - x_1) \right) \quad (29)$$

$$= -\frac{2}{3} = \underline{v}_2. \quad (30)$$

However,  $\tilde{\mu}_2^{\text{Dewey}}$  and  $\tilde{\mu}_2^{\text{Huey}}$  fail to be maximin strategies for player 2.<sup>22</sup> Further, none of the six strategies appearing as components of a profile in  $\mathfrak{M}_3$  plays any role in the analysis of Sion and Wolfe (1957). Thus, there does not seem to be any straightforward connection between the limits of maximin values and strategies in the finite approximations and the maximin values and strategies in the continuous game.

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<sup>21</sup>To see this for  $\tilde{\mu}_1^{\text{Louie}}$ , for example, it suffices to check that

$$\begin{aligned} E_{\tilde{\mu}_1^{\text{Louie}}}[\pi_1(x_1, 1)] &= \frac{2}{3}\pi_1(0, 1) + \frac{1}{3}\pi_1(\frac{1}{2}, 1) \\ &= \frac{2}{3} \left( \text{sgn}((0 + \frac{1}{2}) - 1) + \text{sgn}(1 - 0) \right) + \frac{1}{3} \left( \text{sgn}((\frac{1}{2} + \frac{1}{2}) - 1) + \text{sgn}(1 - \frac{1}{2}) \right) \\ &= \frac{1}{3} < \underline{v}_1. \end{aligned}$$

Analogous strict inequalities, likewise for  $x_2 = 1$ , hold for  $\tilde{\mu}_1^{\text{Dewey}}$  and  $\tilde{\mu}_1^{\text{Huey}}$ .

<sup>22</sup>Indeed, we have

$$\begin{aligned} E_{\tilde{\mu}_2^{\text{Dewey}}}[\pi_2(0, x_2)] &= \frac{3}{5}\pi_2(0, 0) + \frac{1}{5}\pi_2(0, \frac{1}{2}) + \frac{1}{5}\pi_2(0, 1) \\ &= -\frac{3}{5} \left( \text{sgn}((0 + \frac{1}{2}) - 0) + \text{sgn}(0 - 0) \right) - \frac{1}{5} \left( \text{sgn}((0 + \frac{1}{2}) - \frac{1}{2}) + \text{sgn}(\frac{1}{2} - 0) \right) \\ &\quad - \frac{1}{5} \left( \text{sgn}((0 + \frac{1}{2}) - 1) + \text{sgn}(1 - 0) \right) \\ &= -\frac{4}{5} < \underline{v}_2. \end{aligned}$$

A similar inequality holds for  $\tilde{\mu}_2^{\text{Huey}}$ , again with  $x_1 = 0$ .

## 6. Comparison with the case of heterogeneous budgets but no head start

A class of games related to those considered above are finite Colonel Blotto games with heterogeneous budgets yet no head start. As discussed in the Introduction, that type of model was first considered in some generality by Hart (2008) who established, in particular, general bounds on the values. Our analysis below draws from the precise characterization of values accomplished by Liang et al. (2019).

To compare the two models, note that our homogeneous-budgets model  $\mathfrak{B}(n, k)$  may be reinterpreted as a variant of the heterogeneous-budgets model in which the favorite's budget is  $n + k$  while the underdog's budget is  $n$ , and where the favorite's strategy is constrained by the additional requirements that at least  $k$  soldiers must be sent to battlefield A. For clarity, we will refer to this additional constraint in the heterogeneous-budgets model as the *head start constraint*. In the absence of this constraint, we have a heterogeneous-budgets model  $\widehat{\mathfrak{B}}(n, k)$  without head start, as considered in the literature. We will denote by  $\widehat{V}(n, k)$  player 1's value in  $\widehat{\mathfrak{B}}(n, k)$ . The following result quantifies the economic cost for the favorite of being restricted by this constraint.

**Corollary 3.** *Let  $k \geq 1$  and  $m = \lfloor n/k \rfloor$ . Then,*

$$\widehat{V}(n, k) - V(n, k) = \begin{cases} 0 & \text{if } k = 1 \\ \frac{2}{(m+2)(m+1)} & \text{if neither } n \text{ nor } n+1 \text{ is divisible by } k, \text{ and } k \in \{2, \dots, n-1\} \\ \frac{m-1}{(m+1)(m+m^2-1)} & \text{if } n \text{ is divisible by } k, \text{ but } n+1 \text{ is not, and } k \in \{2, \dots, n\} \\ \frac{1}{(m+2)(m+1)} & \text{if } n+1 \text{ is divisible by } k, \text{ but } n \text{ is not, and } k \in \{2, \dots, n-1\} \cup \{n+1\} \\ 1 & \text{if } k \geq n+2. \end{cases} \quad (31)$$

**Proof.** See the Appendix.  $\square$

Corollary 3 gives an explicit expression for the shadow cost of the head start constraint. As can be seen, the shadow cost vanishes for  $k = 1$ .<sup>23</sup> This may not be too surprising. In this case, the head start constraint implies that the favorite cannot send  $n + 1$  soldiers to battlefield B. Sending  $n + 1$  soldiers to battlefield B, however, is payoff-equivalent to sending  $n$  soldiers to battlefield B, because the only scenario where the difference could matter is that the underdog sends all

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<sup>23</sup>It obviously also vanishes for  $k = 0$ .

her soldiers to battlefield B. In that case, however, the two strategies each yield a draw in one battlefield and a win for the favorite in the other, so the payoff is indeed the same between the two models.

The shadow cost also vanishes for  $m = 1$  if  $n$  is divisible by  $k$  but  $n + 1$  is not. A moment's reflection shows that this is the case where  $k = n$ . In the model with heterogeneous budgets, this means that the favorite's budget is just twice the size of the underdog's budget. The head start constraint has no implications on the value in that case because it is an optimal strategy for the favorite in both models to bid equally on both battlefields. For  $k = n + 1$ , however, the shadow cost is  $\frac{1}{2}$ , which is due to the fact that the head start constraint prohibits a type of randomization that is optimal in the heterogeneous-budgets model.

The shadow cost is highest in the case where  $k \geq n + 2$ . This is intuitive because these are the cases where being able to move troops from one battlefield to the other is most valuable.

In all intermediate cases, the shadow cost lies strictly between 0 and 1. Moreover, it is strictly declining in  $m$  within each of the three “interior” cases distinguished in Corollary 3.

## 7. Conclusion

The research efforts documented in this paper have been motivated by the desire to better understand the nature of the Sion-Wolfe example. It is simply hard to accept that noncooperative game theory fails to deliver any prediction for the outcome of a simple game with straightforward economic interpretation. In an attempt to resolve the issue, we have fully characterized the values of discrete-grid approximations of homogeneous-budget Blotto games with a head start. In some but not all cases, the equilibrium was shown to be unique in iteratively undominated strategies, unique in undominated strategies, or even unique without qualification. Linking the findings back to the motivating example, two main observations have been made. First, the set of limit points of values in approximating finite game consists of three points. Second, one of the three limit points lies outside of the interval formed by the maximin and minimax values in the continuous Blotto game. We do not have an explanation for why this is happening. We conclude, however, that finite approximations are highly ineffective as a remedy against the equilibrium non-existence problem in Colonel Blotto Games with a head start and continuous strategy spaces.

## Appendix

This Appendix contains technical material omitted from the body of the paper. The following lemma is needed in the proof of Theorem 1.

**Lemma A.1.** *Suppose that  $n \geq 2$  and  $k \in \{\lfloor n/2 \rfloor + 1, \dots, n-1\}$ . Then, the set of iteratively undominated strategies is  $X_1^{(\infty)} = X_1^{(2)} = \{0, n-k+1\}$  and  $X_2^{(\infty)} = X_2^{(2)} = \{0, n\}$ .*

**Proof.** By Lemma 1, the respective sets of undominated strategies are given by  $X_1^{(1)} = \{0, \dots, n-k+1\}$  and  $X_2^{(1)} = \{0\} \cup \{k+1, \dots, n\}$ . See panel (a) of Figure A.1 for illustration.

(a)

	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	...	$x_2 = k-1$	$x_2 = k$	$x_2 = k+1$	...	$x_2 = n$
$x_1 = 0$	1	2	2	...	2	1	0	...	0
$x_1 = 1$	0	1	2	...	2	2	1	...	0
$x_1 = 2$	0	0	1	...	2	2	2	...	0
...	...	...	...	...	...	...	...	...	1
$x_1 = n-k+1$	0	0	0	...	1	2	2	...	2
$x_1 = n-k+2$	0	0	0	...	0	1	2	...	2
...	...	...	...	...	...	...	...	...	...
$x_1 = n-1$	0	0	0	...	0	0	0	...	2
$x_1 = n$	0	0	0	...	0	0	0	...	1

(b)

	$x_2 = 0$	$x_2 = k+1$	...	$x_2 = n$
$x_1 = 0$	1	0	...	0
$x_1 = 1$	0	1	...	0
$x_1 = 2$	0	2	...	0
...	...	...	...	1
$x_1 = n-k+1$	0	2	...	2

(c)

	$x_2 = 0$	$x_2 = n$
$x_1 = 0$	1	0
$x_1 = n-k+1$	0	2

Figure A.1 Iterated elimination of dominated strategies in the case  $k \in \{\lfloor n/2 \rfloor + 1, \dots, n-1\}$ .

In round 2,  $\hat{x}_1 = n-k+1$  dominates any  $x_1 \in \{1, \dots, n-k\}$  for player 1. See panel (b) for illustration. Indeed, we know from the first round of elimination that player 2 either ignores battlefield A or sends at least  $k+1$  soldiers to battlefield A. In the former case,  $\hat{x}_1 = n-k+1$  is payoff-equivalent to any  $x_1 \in \{1, \dots, n-k\}$  for player 1, because she wins on battlefield A but loses on battlefield B, i.e., player 1's payoff is zero in that case. In the latter case, however, player

2 sends at most  $n - (k + 1)$  soldiers to battlefield B, i.e.,  $y_2 \leq n - (k + 1)$ . Since  $n \leq 2 \lfloor n/2 \rfloor + 1$  and, by assumption,  $\lfloor n/2 \rfloor \leq k - 1$ , we know that

$$y_2 \leq n - (k + 1) \leq 2 \lfloor n/2 \rfloor - k \leq k - 2. \quad (32)$$

For player 1, this means that sending  $\hat{y}_1 = k - 1$  soldiers to battlefield B, as the strategy  $\hat{x}_1 = n - k + 1$  does, wins on both battlefields for sure if player 2 sends any number  $x_2 \in \{k + 1, \dots, n\}$  of soldiers to battlefield A. Moreover, in the case where  $\hat{x}_2 = n$ , any strategy  $x_1 \in \{1, \dots, n - k\}$  is strictly inferior to  $\hat{x}_1 = n - k + 1$ , because player 1 either loses or ties on battlefield A. Thus,  $\hat{x}_1 = n - k + 1$  indeed dominates any  $x_1 \in \{1, \dots, n - k\}$  for player 1. For player 2, a similar consideration shows that any  $x_2 \in \{k + 1, \dots, n - 1\}$  is dominated by the strategy  $\hat{x}_2 = n$ . Indeed, if  $x_1 = 0$ , then player 2's payoff is zero for any  $x_2 \geq k + 1$  (player 2 wins on battlefield A, but loses on battlefield B). Further, if  $x_1 \in \{1, \dots, n - k - 1\}$ , then player 2's payoff from  $\hat{x}_2 = n$  is zero (she wins on battlefield A, but loses on battlefield B), whereas her payoff from any  $x_2 \in \{k + 1, \dots, n - 1\}$  is weakly lower simply because a positive payoff is not feasible for player 2. If  $x_1 = n - k$ , then  $\hat{x}_2 = n$  is even strictly better than any  $x_2 \in \{k + 1, \dots, n - 1\}$ , because  $\hat{x}_2 = n$  yields a tie on battlefield A rather than a loss. Finally, if  $x_1 = n - k + 1$ , then player 2 loses on battlefield A for sure, but player 2 loses also on battlefield B for any  $x_2 \geq k + 1$ , because in that case,  $y_1 = k - 1$  and  $y_2 \leq n - (k + 1) \leq k - 2$  by relationship (32). The reduced payoff matrix obtained after two rounds of elimination is, consequently, the two-by-two diagonal matrix shown in panel (c). Clearly, further eliminations are not feasible.  $\square$

**Proof of Theorem 1.** The game shown in panel (c) of Figure A.1 admits a unique equilibrium  $(\mu_1^*, \mu_2^*)$ , given by strategies  $\mu_1^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_{n-k+1}$  and  $\mu_2^* = \frac{2}{3}\delta_0 + \frac{1}{3}\delta_n$ . As explained in the body of the paper,  $(\mu_1^*, \mu_2^*)$  is an equilibrium also in the game shown in panel (a), i.e., in  $\mathfrak{B}(n, k)$ . It is, therefore, the unique equilibrium in strictly undominated strategies. Obviously, player 1's equilibrium payoff is  $v_1 = \frac{2}{3}$ .  $\square$

**Proof of Lemma 3.** The payoff matrix in the case  $k = 1$ , with  $n$  general, is outlined in Figure A.2.

	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$x_2 = 3$	...	$x_2 = n-4$	$x_2 = n-3$	$x_2 = n-2$	$x_2 = n-1$	$x_2 = n$
$x_1 = 0$	1	1	0	0	...	0	0	0	0	0
$x_1 = 1$	0	1	1	0	...	0	0	0	0	0
$x_1 = 2$	0	0	1	1	...	0	0	0	0	0
$x_1 = 3$	0	0	0	1	...	1	0	0	0	0
$x_1 = 4$	0	0	0	0	...	1	1	0	0	0
...	...	...	...	...	...	...	...	...	...	...
$x_1 = n-3$	0	0	0	0	...	0	1	1	0	0
$x_1 = n-2$	0	0	0	0	...	0	0	1	1	0
$x_1 = n-1$	0	0	0	0	...	0	0	0	1	1
$x_1 = n$	0	0	0	0	...	0	0	0	0	1

Figure A.2 Iterated elimination of dominated strategies in the case  $k = 1$ .

In round 1,  $x_1 = n$  is dominated by  $\hat{x}_1 = n-1$ . Moreover,  $x_2 = 1$  is dominated by  $\hat{x}_2 = 0$ . In round 2, strategy  $x_1 = 1$  is dominated by  $\hat{x}_1 = 2$ , provided that the latter exists (the strategy either may have never existed or may have been deleted in an earlier round). Moreover, strategy  $x_2 = n-1$  is dominated by strategy  $\hat{x}_2 = n$ , provided the latter exists. In round 3,  $x_1 = n-2$  is dominated by  $\hat{x}_1 = n-3$ , provided the latter strategy exists. Moreover,  $x_2 = 3$  is dominated by  $\hat{x}_2 = 2$ , provided that the latter exists. In round 4,  $x_1 = 3$  is dominated by  $\hat{x}_1 = 4$ , provided the latter exists. A pattern emerges. As illustrated in Figure A.2, for player 1, odd strategies  $x_1 = 1, 3, \dots$  get sequentially eliminated starting from the bottom, and “co-even” strategies  $x_1 = n, n-2, \dots$  get sequentially eliminated starting from the top. Similarly, for player 1, odd strategies  $x_1 = 1, 3, \dots$  get eliminated sequentially starting from the bottom, and “co-odd” strategies  $x_1 = n-1, n-3, \dots$  get sequentially eliminated starting from the top. The procedure stops once one of the dominating strategies fails to exist. Consider first player 1. If  $n$  is odd, then only odd strategies are eliminated. Therefore, in this case, all even strategies survive the iterated elimination of dominated strategies, i.e.,  $X_1^{(\infty)} = \{0, 2, \dots, n-1\}$ . If  $n$  is even, there are two cases. Suppose first that  $n = 4q$ . Then, the set of iteratively undominated strategies is  $X_1^{(\infty)} = \{0, 2, \dots, 2q, 2q+1, \dots, n-1\}$ . Suppose, next that  $n = 4q+2$ . Then, again,  $X_1^{(\infty)} = \{0, 2, \dots, 2q, 2q+1, \dots, n-1\}$ . Consider next player 2. If  $n$  is even, then all odd strategies are sequentially eliminated, so that  $X_2^{(\infty)} = \{0, 2, \dots, n\}$ . If  $n$  is odd, however, there are again two cases. If  $n = 4q+1$ , then  $X_2^{(\infty)} = \{0, 2, \dots, 2q, 2q+3, \dots, n\}$ .

But also if  $n = 4q + 3$ , then  $X_2^{(\infty)} = \{0, 2, \dots, 2q, 2q + 3, \dots, n\}$ . This proves the lemma.  $\square$

**Proof of Theorem 2.** We have to check that no pure strategy for player 1 can get a payoff exceeding  $v_1$ . But this is immediate from the fact that player 2 does not use any two neighboring pure strategies in the candidate equilibrium. We have to check that, likewise, no pure strategy for player 2 can get a payoff exceeding  $-v_1$ . Indeed, as player 1 randomizes uniformly over  $X_1 = \{0, 2, \dots, 2q, 2q + 1, \dots, n - 1\}$ , and player 2 loses one util against  $x_1$  and  $x_1 + 1$  alike (unless  $x_1 = n$ ), it is not feasible for player 2 to get more than  $-v_1$ . If  $n = 4q$  or  $n = 4q + 1$ , then both  $X_1$  and  $X_2$  have a cardinality of  $2q + 1$ , and the reduced payoff matrix is an identity matrix. Since this matrix is invertible, the equilibrium is unique. If  $n = 4q + 2$  or  $n = 4q + 3$ , then both  $X_1$  and  $X_2$  have a cardinality of  $2q + 2$ , but the reduced payoff matrix is again an identity matrix. This proves the claim.  $\square$

**Proof of Theorem 3.** It can be readily verified that the respective probabilities sum up to one for each player. For player 1, for example, we have

$$\sum_{x_1 \in X_1} \mu_1^*(x_1) = \frac{2\rho - 1}{\rho^2 + \rho - 1} + \sum_{r_1=1}^{\rho-1} \left( \frac{\rho - r_1}{\rho^2 + \rho - 1} + \frac{r_1}{\rho^2 + \rho - 1} \right) \quad (33)$$

$$= \frac{2\rho - 1}{\rho^2 + \rho - 1} + \frac{\rho(\rho - 1)}{\rho^2 + \rho - 1} = 1. \quad (34)$$

For  $\mu_2^*$ , the calculation is similar. Thus,  $\mu_1^*$  and  $\mu_2^*$  indeed define mixed strategies.

Next, we check the optimality of player 1's strategy  $\mu_1^*$  against  $\mu_2^*$ . This is done by testing all pure strategies  $x_1 \in X_1$ . Player 1's expected payoff against  $\mu_2^*$  is given by

$$E_{\mu_2^*}[\Pi_1(x_1, x_2)] = \sum_{\tilde{x}_2=0}^n \mu_2^*(\{\tilde{x}_2\}) \Pi_1(x_1, \tilde{x}_2) \quad (35)$$

$$= \frac{2m - 1}{m^2 + m - 1} \Pi_1(x_1, 0) + \left( \sum_{s_2=2}^m \frac{s_2 - 1}{m^2 + m - 1} \Pi_1(x_1, s_2 k) \right) + \left( \sum_{s_2=1}^{m-1} \frac{m - s_2}{m^2 + m - 1} \Pi_1(x_1, s_2 k + 1) \right). \quad (36)$$

To evaluate the right-hand side of (36) for a given pure strategy  $x_1 \in X_1$ , we compute the terms  $\Pi_1(x_1, 0)$ ,  $\Pi_1(x_1, s_2 k)$ ,  $\Pi_1(x_1, s_2 k + 1)$ , and subsequently simplify the resulting expression. For a start, suppose that player 1 chooses  $x_1 = 0$ . In this case, we know that  $\Pi_1(0, 0) = 1$  (player 1

wins on battlefield A and ties on battlefield B), and  $\Pi_1(0, x_2) = 0$  for any  $x_2 > k$  (player 1 loses on battlefield A but wins on battlefield B). Plugging this into (36) for  $x_1 = 0$ , one finds

$$E_{\mu_2^*}[\Pi_1(0, x_2)] = \frac{2m - 1}{m^2 + m - 1} = \Pi_1^*, \quad (37)$$

which is just the equilibrium payoff for player 1. Suppose next that player 1 chooses  $x_1 = s_1 k$ , for some  $s_1 \in \{1, \dots, m - 1\}$ . Note that  $\Pi_1(x_1, 0) = 0$  if  $x_1 > 0$  (player 1 wins on battlefield A, but loses on battlefield B). Moreover,

$$\Pi_1(s_1 k, s_2 k) = \begin{cases} 1 & \text{if } s_2 = s_1 \text{ or } s_2 = s_1 + 1 \\ 0 & \text{otherwise,} \end{cases} \quad (38)$$

$$\Pi_1(s_1 k, s_2 k + 1) = \begin{cases} 2 & \text{if } s_2 = s_1 \\ 0 & \text{otherwise.} \end{cases} \quad (39)$$

Hence,

$$E_{\mu_2^*}[\Pi_1(s_1 k, x_2)] = \frac{s_1 - 1}{m^2 + m - 1} \cdot 1 + \frac{(s_1 + 1) - 1}{m^2 + m - 1} \cdot 1 + \frac{m - s_1}{m^2 + m - 1} \cdot 2 = \Pi_1^*, \quad (40)$$

as claimed. Suppose next that player 1 chooses  $x_1 = s_1 k + 1$  for some  $s_1 \in \{1, \dots, m - 1\}$ . Note that

$$\Pi_1(s_1 k + 1, s_2 k) = \begin{cases} 2 & \text{if } s_2 = s_1 + 1 \\ 0 & \text{otherwise,} \end{cases} \quad (41)$$

$$\Pi_1(s_1 k + 1, s_2 k + 1) = \begin{cases} 1 & \text{if } s_2 = s_1 \text{ or } s_2 = s_1 + 1 \\ 0 & \text{otherwise.} \end{cases} \quad (42)$$

Hence,

$$E_{\mu_2^*}[\Pi_1(s_1 k + 1, x_2)] = \frac{(s_1 + 1) - 1}{m^2 + m - 1} \cdot 2 + \frac{m - s_1}{m^2 + m - 1} \cdot 1 + \frac{m - (s_1 + 1)}{m^2 + m - 1} \cdot 1 = \Pi_1^*. \quad (43)$$

We have shown that each pure strategy  $x_1 \in \text{supp}(\mu_1^*)$  yields the same payoff against  $\mu_2^*$ . To establish that  $\mu_1^*$  is a best response to  $\mu_2^*$ , we still have to check that any  $x_1 \in X_1 \setminus \text{supp}(\mu_1^*)$  gives player 1 a weakly lower expected payoff than  $\Pi_1^*$ . In fact, we will show that the expected payoff is strictly lower than  $\Pi_1^*$  in these cases. Suppose that player 1 chooses  $x_1 = s_1 k + r$ , for some  $s_1 \in \{1, \dots, m - 1\}$  and some  $r \in \{2, \dots, k - 1\}$ . We note that

$$\Pi_1(s_1 k + r, s_2 k) = \Pi_1(s_1 k + r, s_2 k + 1) = \begin{cases} 2 & \text{if } s_2 = s_1 + 1 \\ 0 & \text{otherwise.} \end{cases} \quad (44)$$

Therefore,

$$E_{\mu_2^*}[\Pi_1(s_1k + r, x_2)] = \frac{(s_1 + 1) - 1}{m^2 + m - 1} \cdot 2 + \frac{m - (s_1 + 1)}{m^2 + m - 1} \cdot 2 = \frac{2m - 2}{m^2 + m - 1} < \Pi_1^*, \quad (45)$$

which proves the claim for any strategy  $x_1 = s_1k + r$ , where  $s_1 \in \{1, \dots, m - 1\}$  and  $r \in \{2, \dots, k - 1\}$  yields. Suppose next that player 1 chooses  $x_1 \in \{1, \dots, k - 1\}$ . In this case,  $\Pi_1(x_1, s_2k) = 0$  for any  $s_2 \in \{2, \dots, n\}$  (player 1 loses on battlefield A but wins on battlefield B).

If  $x_1 = 1$ , then,

$$\Pi_1(x_1, s_2k + 1) = \begin{cases} 1 & \text{if } s_2 = 1 \\ 0 & \text{if } s_2 \geq 2. \end{cases} \quad (46)$$

Thus,

$$E_{\mu_2^*}[\Pi_1(x_1, x_2)] = \frac{m - 1}{m^2 + m - 1} \cdot 1 < \Pi_1^*. \quad (47)$$

If, however,  $x_1 \in \{2, \dots, k - 1\}$ , then

$$\Pi_1(x_1, s_2k + 1) = \begin{cases} 2 & \text{if } s_2 = 1 \\ 0 & \text{if } s_2 \geq 2. \end{cases} \quad (48)$$

Therefore,

$$E_{\mu_2^*}[\Pi_1(x_1, x_2)] = \frac{m - 1}{m^2 + m - 1} \cdot 2 < \Pi_1^*. \quad (49)$$

Finally, suppose that player 1 chooses  $x_1 = n$ . One notes that

$$\Pi_1(n, s_2k) = \Pi_1(n, s_2k + 1) = 0 \quad (s_2 \in \{2, \dots, m - 1\}), \quad (50)$$

$$\Pi_1(n, n) = 1. \quad (51)$$

Therefore,

$$E_{\mu_2^*}[\Pi_1(n, x_2)] = \frac{m - 1}{m^2 + m - 1} < \Pi_1^*. \quad (52)$$

We have shown that no pure strategy in  $X_1$  yields a higher expected payoff against  $\mu_2^*$  than  $\mu_1^*$ .

Thus, we have shown that  $\mu_1^*$  is a best response to  $\mu_2^*$ .

We now show that  $\mu_2^*$  in turn is a best response to  $\mu_1^*$ . Player 2's expected payoff from playing  $x_2$  against  $\mu_1^*$  is

$$\begin{aligned} E_{\mu_1^*}[\Pi_2(x_1, x_2)] &= \frac{2m - 1}{m^2 + m - 1} \Pi_2(0, x_2) + \left( \sum_{s_1=1}^{m-1} \frac{m - s_1}{m^2 + m - 1} \Pi_2(s_1k, x_2) \right) \\ &\quad + \left( \sum_{s_1=1}^{m-1} \frac{s_1}{m^2 + m - 1} \Pi_2(s_1k + 1, x_2) \right). \end{aligned} \quad (53)$$

Suppose first that  $x_2 = 0$ . We have  $\Pi_2(0, 0) = -\Pi_1(0, 0) = -1$ . Moreover,  $\Pi_2(x_1, 0) = 0$  for any  $x_1 > 0$  (because player 2 loses on battlefield A yet wins on battlefield B). Hence,

$$E_{\mu_1^*}[\Pi_2(x_1, 0)] = -\frac{2m-1}{m^2+m-1} = \Pi_2^*, \quad (54)$$

where  $\Pi_2^*$  is the equilibrium payoff for player 2. Suppose next that player 2 chooses  $x_2 = s_2 k$ , for some  $s_2 \in \{2, \dots, m-1\}$ . We know that

$$\Pi_2(s_1 k, s_2 k) = \begin{cases} -1 & \text{if } s_1 = s_2 \text{ or } s_1 = s_2 - 1 \\ 0 & \text{otherwise,} \end{cases} \quad (55)$$

$$\Pi_2(s_1 k + 1, s_2 k) = \begin{cases} -2 & \text{if } s_1 = s_2 - 1 \\ 0 & \text{otherwise.} \end{cases} \quad (56)$$

Therefore,

$$E_{\mu_1^*}[\Pi_2(x_1, s_2 k)] = \frac{m-s_2}{m^2+m-1} \cdot (-1) + \frac{m-(s_2-1)}{m^2+m-1} \cdot (-1) + \frac{s_2-1}{m^2+m-1} \cdot (-2) = \Pi_2^*, \quad (57)$$

as claimed. Suppose now that player 2 chooses  $x_2 = s_2 k + 1$ , for some  $s_2 \in \{1, \dots, m-1\}$ . Note that, since  $k \geq 2$ ,

$$\Pi_2(s_1 k, s_2 k + 1) = \begin{cases} -2 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise,} \end{cases} \quad (58)$$

$$\Pi_2(s_1 k + 1, s_2 k + 1) = \begin{cases} -1 & \text{if } s_1 = s_2 \text{ or } s_1 = s_2 - 1 \\ 0 & \text{otherwise.} \end{cases} \quad (59)$$

Therefore,

$$E_{\mu_1^*}[\Pi_2(x_1, s_2 k + 1)] = \frac{m-s_2}{m^2+m-1} \cdot (-2) + \frac{s_2}{m^2+m-1} \cdot (-1) + \frac{s_2-1}{m^2+m-1} \cdot (-1) = \Pi_2^*, \quad (60)$$

as claimed. Finally, suppose that player 2 chooses  $x_2 = n$ . Since  $k \geq 2$ , we have

$$\Pi_2(s_1 k, n) = \begin{cases} -1 & \text{if } s_1 = m-1 \\ 0 & \text{otherwise,} \end{cases} \quad (61)$$

$$\Pi_2(s_1 k + 1, n) = \begin{cases} -2 & \text{if } s_1 = m-1 \\ 0 & \text{otherwise.} \end{cases} \quad (62)$$

Hence,

$$E_{\mu_1^*}[\Pi_2(x_1, n)] = \frac{1}{m^2+m-1} \cdot (-1) + \frac{m-1}{m^2+m-1} \cdot (-2) = -\frac{2m-1}{m+m^2-1} = \Pi_2^*. \quad (63)$$

We have shown that each strategy  $x_2$  that is chosen with positive probability against the proposed strategy  $\mu_1^*$  yields the same payoff. We now have to show that no other strategy  $x_2$  gives player

2 a payoff strictly higher than  $\Pi_2^*$ . Suppose that player 2 chooses  $x_2 = s_2k + r$ , for some  $s_2 \in \{1, \dots, m-1\}$  and some  $r \in \{2, \dots, k-1\}$ . Note that, since  $k \geq 2$ ,

$$\Pi_2(0, s_2k + r) = 0, \quad (64)$$

$$\Pi_2(s_1k, s_2k + r) = \Pi_2(s_1k + 1, s_2k + r) = \begin{cases} -2 & \text{if } s_2 = s_1 \\ 0 & \text{if } s_2 \neq s_1. \end{cases} \quad (65)$$

Therefore

$$E_{\mu_1^*}[\Pi_2(x_1, s_2k + r)] = \frac{m - s_2}{m^2 + m - 1} \cdot (-2) + \frac{s_2}{m^2 + m - 1} \cdot (-2) = -\frac{2m}{m + m^2 - 1} < \Pi_2^*. \quad (66)$$

Suppose next that player 2 chooses  $x_2 \in \{1, \dots, k-1\}$ . Then,  $\Pi_2(0, x_2) = -2$  (player 2 loses on both battlefields). Moreover,  $\Pi_2(s_1k, x_2) = \Pi_2(s_1k + 1, x_2) = 0$  (player 2 loses on battlefield A, but wins on battlefield B). Therefore,

$$E_{\mu_1^*}[\Pi_2(x_1, x_2)] = \frac{2m - 1}{m^2 + m - 1} \cdot (-2) < \Pi_2^*. \quad (67)$$

Finally, we have to check the payoff if player 2 plays  $x_2 = k$  against  $\mu_1^*$ . In this case,  $\Pi_2(0, k) = -1$  (player 2 ties on battlefield A but loses on battlefield B). Moreover,

$$\Pi_2(s_1k, k) = \begin{cases} -1 & \text{if } s_1 = 1 \\ 0 & \text{if } s_1 \geq 2, \end{cases} \quad (68)$$

and  $\Pi_2(s_1k + 1, k) = 0$  (player 2 loses on battlefield A but wins on battlefield B). Hence,

$$E_{\mu_1^*}[\Pi_2(x_1, k)] = \frac{2m - 1}{m^2 + m - 1} \cdot (-1) + \frac{m - 1}{m^2 + m - 1} \cdot (-1) = -\frac{3m - 2}{m^2 + m - 1} < \Pi_2^*. \quad (69)$$

Every strategy  $x_2$  that is played with probability  $\mu_2^*(x_2) = 0$  yields a strictly smaller payoff than  $\mu_2^*$  against  $\mu_1^*$ . Thus, we have shown that  $\mu_2^*$  is a best response to  $\mu_1^*$ . Hence, both players play a best response and we have identified a Nash equilibrium.

We claim that, under the assumptions of Theorem 3,  $(\mu_1^*, \mu_2^*)$  is the unique mixed-strategy Nash equilibrium. To provoke a contradiction, suppose that  $(\mu_1^{**}, \mu_2^{**}) \neq (\mu_1^*, \mu_2^*)$  is another equilibrium. Since  $\mathfrak{B}(n, k)$  is a zero-sum game, Nash equilibria in  $\mathfrak{B}(n, k)$  are interchangeable (see, e.g., Osborne and Rubinstein, 1994, Section 2.5). Therefore, the profile  $(\mu_1^*, \mu_2^{**})$  is an equilibrium. Now, as shown above, the set of pure best responses to  $\mu_1^*$  is

$$B_2 = \{x_2^1, \dots, x_2^L\} \quad (70)$$

$$\equiv \{0, k+1, 2k, 2k+1, \dots, n-k, n-k+1, n\}, \quad (71)$$

where  $L = 2m - 1$ . Since  $\mu_2^{**}$  is a best response to  $\mu_1^*$ , the support of  $\mu_2^{**}$  is necessarily a subset of  $B_2$ , i.e.,  $\text{supp}(\mu_2^{**}) \subseteq B_2$ . Hence,

$$\mu_2^{**} = \sum_{l=1}^L p_2^l \delta_{x_2^l}, \quad (72)$$

for probabilities  $p_2^1, \dots, p_2^L \in [0, 1]$  such that  $p_2^1 + \dots + p_2^L = 1$ . But also  $\mu_1^*$  is a best response to  $\mu_2^{**}$ . Therefore, any pure strategy in the support of  $\mu_1^*$ ,

$$B_1 = \text{supp}(\mu_1^*) \quad (73)$$

$$= \{x_1^1, \dots, x_1^L\} \quad (74)$$

$$\equiv \{0, k, k+1, 2k, 2k+1, \dots, n-k, n-k+1\}, \quad (75)$$

yields the same expected payoff  $\Pi_1^*$  against  $\mu_2^{**}$ . Thus,

$$\underbrace{\begin{pmatrix} \Pi_1(x_1^1, x_2^1) & \cdots & \Pi_1(x_1^1, x_2^L) \\ \vdots & \ddots & \vdots \\ \Pi_1(x_1^L, x_2^1) & \cdots & \Pi_1(x_1^L, x_2^L) \end{pmatrix}}_{\equiv \mathbf{M}_1} \begin{pmatrix} p_2^1 \\ \vdots \\ p_2^L \end{pmatrix} = \begin{pmatrix} \Pi_1^* \\ \vdots \\ \Pi_1^* \end{pmatrix}. \quad (76)$$

From Lemma A.2 below, the matrix  $\mathbf{M}_1$  is invertible. Therefore, there is at most one solution for the probabilities vector  $(p_2^1, \dots, p_2^L)$ . In particular,  $p_2^l = \mu_2^*(\{x_2^l\})$  for all  $l \in \{1, \dots, L\}$ , so that  $\mu_2^{**} = \mu_2^*$ . An analogous argument, using that  $(\mu_1^{**}, \mu_2^*)$  is an equilibrium and that the matrix  $\mathbf{M}_2 = -\mathbf{M}_1^T$  is invertible, shows that  $\mu_1^{**} = \mu_1^*$ . However, we assumed that  $(\mu_1^*, \mu_2^*) \neq (\mu_1^{**}, \mu_2^{**})$ . The contradiction shows that the equilibrium is indeed unique.  $\square$

**Lemma A.2** *Player 1's restricted payoff matrix,*

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 0 & & \vdots \\ 0 & 1 & 2 & 1 & \ddots & \vdots \\ 0 & 0 & 1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{pmatrix}, \quad (77)$$

*is invertible, and so is player 2's restricted payoff matrix,  $\mathbf{M}_2 = -\mathbf{M}_1^T$ .*

**Proof.** For  $\nu \in \{1, 2, \dots\}$ , consider the symmetric Toeplitz matrix

$$\mathbf{S}(\nu) = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & 1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 2 \end{pmatrix} \in \mathbb{R}^{\nu \times \nu}. \quad (78)$$

Recalling that  $L = 2m - 1$  is odd, it can be readily verified that  $\det \mathbf{M}_1 = \det \mathbf{M}_2 = \det \mathbf{S}(L - 1)$ . Moreover, since  $n$  is divisible by  $k$ , but  $k \leq \lfloor n/2 \rfloor \leq n/2$ , we have  $n/k \geq 2$ , so that  $m = \lfloor n/k \rfloor \geq 2$ . Hence,  $L = 2m - 1 \geq 3$ . To prove the lemma, it therefore suffices to show that  $\mathbf{S}(\nu)$  is invertible for all  $\nu \geq 2$ . But, clearly,

$$\det \mathbf{S}(1) = 2, \quad (79)$$

$$\det \mathbf{S}(2) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3. \quad (80)$$

Moreover, by Laplacian expansion with respect to the first row, it is easy to see that

$$\det \mathbf{S}(\nu) = 2 \det \mathbf{S}(\nu - 1) - \det \underbrace{\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & \ddots & \vdots \\ \vdots & 1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 1 & 2 \end{pmatrix}}_{\in \mathbb{R}^{(\nu-1) \times (\nu-1)}} \quad (\nu \geq 3). \quad (81)$$

Expanding the second determinant on the right-hand side another time with respect to the first row yields

$$\det \mathbf{S}(\nu) = 2 \det \mathbf{S}(\nu - 1) - \det \mathbf{S}(\nu - 2) \quad (\nu \geq 3). \quad (82)$$

Based on equations (79) and (80), an induction argument using relationship (82) shows that  $\det \mathbf{S}(\nu) = \nu + 1$  holds for any  $\nu \geq 1$ . Thus,  $\det \mathbf{M}_1 \neq 0$ , so that  $\mathbf{M}_1$  and  $\mathbf{M}_2 = -\mathbf{M}_1^T$  are indeed invertible. This proves the claim.  $\square$

**Proof of Theorem 4.** The idea of the proof is to compute expected payoffs against the candidate equilibrium strategies in an efficient way. We start with player 1's expected payoff against  $\mu_2^*$ . We decompose  $\mu_2^*$  as follows:

$$\mu_2^* = \underbrace{\left( \frac{2}{m+2} \delta_0 - \frac{1}{m+2} \delta_r \right)}_{=\mu_2'} + \underbrace{\left( \frac{1}{m+2} \sum_{\nu=0}^m \delta_{k\nu+r} \right)}_{=\mu_2''}. \quad (83)$$

It should be noted that  $\mu'_2$  and  $\mu''_2$  are signed measures and therefore expected payoffs against those measures are well-defined (even they lack a direct intuitive interpretation). It is then not hard to check that

$$\Pi_1(x_1, \mu'_2) = \begin{cases} -\frac{2}{m+2} & \text{if } x_1 \in \{1, \dots, r-1\} \\ -\frac{1}{m+2} & \text{if } x_1 = r \\ 0 & \text{otherwise,} \end{cases} \quad (84)$$

$$\Pi_1(x_1, \mu''_2) = \begin{cases} \frac{1}{m+2} & \text{if } x_1 = n \\ \frac{2}{m+2} & \text{otherwise.} \end{cases} \quad (85)$$

It follows that

$$\Pi_1(x_1, \mu_2^*) = \Pi_1(x_1, \mu'_2) + \Pi_1(x_1, \mu''_2) \quad (86)$$

$$= \begin{cases} 0 & \text{if } x_1 \in \{1, \dots, r-1\} \\ \frac{1}{m+2} & \text{if } x_1 \in \{r, n\} \\ \frac{2}{m+2} & \text{otherwise,} \end{cases} \quad (87)$$

i.e., any pure strategy  $x_1 \in \{0\} \cup \{r+1, \dots, n-1\}$  is a best response to  $\mu_2^*$ . This shows that  $\mu_1^*$  is a best response against  $\mu_2^*$ . Turning to player 2's expected payoff against  $\mu_1^*$ , we decompose  $\mu_1^*$  as follows:

$$\mu_1^* = \frac{2}{m+2}\delta_0 + \left(\frac{1}{m+2}\sum_{\nu=1}^{m-1}\delta_{k\nu}\right) + \frac{1}{m+2}\delta_{n-k+1} \quad (88)$$

$$= \underbrace{\frac{1}{m+2}\delta_0}_{=\mu'_1} + \underbrace{\left(\frac{1}{m+2}\sum_{\nu=0}^m\delta_{k\nu}\right)}_{=\mu''_1} + \underbrace{\frac{1}{m+2}\delta_{n-k+1}}_{=\mu'''_1} + \underbrace{\left(-\frac{1}{m+2}\delta_{km}\right)}_{=\mu''''_1}. \quad (89)$$

We find

$$\Pi_2(\mu'_1, x_2) = \begin{cases} -\frac{1}{m+2} & \text{if } x_2 \in \{0, k\} \\ -\frac{2}{m+2} & \text{if } x_2 \in \{1, \dots, k-1\} \\ 0 & \text{otherwise,} \end{cases} \quad (90)$$

$$\Pi_2(\mu''_1, x_2) = \begin{cases} -\frac{1}{m+2} & \text{if } x_2 = 0 \\ -\frac{2}{m+2} & \text{otherwise.} \end{cases} \quad (91)$$

Since  $n - r = km$ , there are now two cases. Suppose first that  $r = k - 1$ . Then,  $\mu'''_1 + \mu''''_1 = 0$ , and we may ignore the last two signed measures. This yields

$$\Pi_2(\mu_1^*, x_2) = \Pi_2(\mu'_1, x_2) + \Pi_2(\mu''_1, x_2) \quad (92)$$

$$= \begin{cases} -\frac{4}{m+2} & \text{if } x_2 \in \{1, \dots, k-1\} \\ -\frac{3}{m+2} & \text{if } x_2 = k \\ -\frac{2}{m+2} & \text{otherwise.} \end{cases} \quad (93)$$

Thus, in the case  $r = k - 1$ , all pure strategies  $x_2 \in \{0\} \cup \{k + 1, \dots, n\}$  are best responses to  $\mu_1^*$ . This proves the equilibrium property in this case. Suppose next that  $r \neq k - 1$ , which is equivalent to  $r \in \{1, \dots, k - 2\}$ . Then,

$$\Pi_2(\mu_1''', x_2) = \frac{1}{m+2} \Pi_2(n - k + 1, x_2) \quad (94)$$

$$= \begin{cases} -\frac{2}{m+2} & \text{if } x_2 \in \{n - k + 2, \dots, n\} \\ -\frac{1}{m+2} & \text{if } x_2 = n - k + 1 \\ 0 & \text{otherwise,} \end{cases} \quad (95)$$

and

$$\Pi_2(\mu_1'''' , x_2) = -\frac{1}{m+2} \Pi_2(n - r, x_2) \quad (96)$$

$$= \begin{cases} \frac{2}{m+2} & \text{if } x_2 \in \{n - r + 1, \dots, n\} \\ \frac{1}{m+2} & \text{if } x_2 = n - r \\ 0 & \text{otherwise.} \end{cases} \quad (97)$$

Recall that  $m \geq 2$  by assumption. Therefore,  $n - k + 1 = k(m - 1) + r - 1 \geq k$ , so that the cases do not overlap. Summing up yields

$$\Pi_2(\mu_1^*, x_2) = \Pi_2(\mu_1', x_2) + \Pi_2(\mu_1'', x_2) + \Pi_2(\mu_1''', x_2) + \Pi_2(\mu_1'''' , x_2) \quad (98)$$

$$= \begin{cases} -\frac{4}{m+2} & \text{if } x_2 \in \{1, \dots, k - 1\} \cup \{n - k + 2, \dots, n - r - 1\} \\ -\frac{3}{m+2} & \text{if } x_2 \in \{k, n - k + 1, n - r\} \\ -\frac{2}{m+2} & \text{otherwise.} \end{cases} \quad (99)$$

Thus, in the case  $r \in \{1, \dots, k - 2\}$ , all strategies  $x_2 \in \{0\} \cup \{k + 1, \dots, n - k\} \cup \{n - r + 1, \dots, n\}$  are best responses to  $\mu_1^*$ . But

$$\mu_2^* = \frac{2}{m+2} \delta_0 + \left( \frac{1}{m+2} \sum_{\nu=1}^{m-1} \delta_{k\nu+r} \right) + \frac{1}{m+2} \delta_n. \quad (100)$$

It follows that, likewise,  $\mu_2^*$  is a best response to  $\mu_1^*$ . This proves the theorem.  $\square$

(a)	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	...	$x_2 = n-2$	$x_2 = n-1$	$x_2 = n$
$x_1 = 0$	1	2	1	...	...	...	...
$x_1 = 1$	0	1	2	...	...	...	...
$x_1 = 2$	...	0	1	...	...	...	...
...	...	...	...	...	...	...	...
$x_1 = n-2$	...	...	...	...	1	2	1
$x_1 = n-1$	...	...	...	...	0	1	2
$x_1 = n$	...	...	...	...	...	0	1

(b)	$x_2 = 0$	$x_2 = 3$	$x_2 = 4$	...	$x_2 = n-1$	$x_2 = n$
$x_1 = 0$	1	0	...	...	...	...
$x_1 = 1$	0	1	0	...	...	...
$x_1 = 2$	...	2	1	...	...	...
...	...	...	...	...	...	...
$x_1 = n-2$	...	...	...	...	2	1
$x_1 = n-1$	...	...	...	...	1	2

Figure A.3 IEDS in the case  $k = 2$ .

**Proof of Lemma 4.** (i) By assumption,  $k = 2$ . For  $n = 1$ , the claim follows from Lemma 1(iv). Suppose, therefore, that  $n \geq 2$ . Then, in round 1, as evident from panel (a) of Figure A.3,  $x_1 = n$  is dominated by  $\hat{x}_1 = n - 1$ , and  $x_2 \in \{1, 2\}$  are dominated by  $\hat{x}_2 = 0$ . Hence,  $X_1^{(1)} = \{0, \dots, n - 1\}$  and  $X_2^{(1)} = \{0\} \cup \{3, \dots, n\}$ .<sup>24</sup> In round two,  $x_1 = 1$  is dominated by  $\hat{x}_1 = 2$  for player 1. No other strategies are dominated, however. See panel (b) of Figure A.3 for illustration. The reduced-form payoff matrix is, therefore, essentially a symmetric Toeplitz matrix, which has been seen to be invertible above. Hence,  $X_1^{(\infty)} = X_1^{(2)} = \{0\} \cup \{2, \dots, n - 1\}$  and  $X_2^{(\infty)} = X_2^{(2)} = \{0\} \cup \{3, \dots, n\}$ , and the equilibrium in iteratively undominated strategies is unique. (ii) By assumption,  $n = 2k + r$ , with  $r \in \{2, \dots, k - 1\}$ . We start by determining  $X_1^{(\infty)}$  and  $X_2^{(\infty)}$  in  $\mathfrak{B}(n, k)$ . Clearly,  $k \in \{3, \dots, n - 1\}$ . Therefore, invoking Lemma 1(iii), we see that  $X_1^{(1)} = \{0, \dots, n - k + 1\}$  and  $X_2^{(1)} = \{0\} \cup \{k + 1, \dots, n\}$ . The reduced game obtained after the elimination of dominated strategies therefore looks as shown in Figure A.4. In round 2 of the IEDS procedure, we see that strategies  $x_1 \in \{1, \dots, k - 1\}$ , marked in grey, are dominated by  $\hat{x}_1 = k$  for player 1. Moreover, strategies  $x_2 \in \{k + r + 2, \dots, 2k + r - 1\}$  are dominated by  $\hat{x}_2 = 2k + r = n$  for player 2. Therefore,  $X_1^{(2)} = \{0\} \cup \{k, \dots, n - r + 1\}$  and  $X_2^{(2)} = \{0\} \cup \{k + 1, \dots, k + r + 1\} \cup \{n\}$ .

		$x_2 = 0$	$x_2 = k + 1$	$\dots$	$x_2 = k + r$	$x_2 = k + r + 1$	$\dots$	$x_2 = 2k + r - 1$	$x_2 = 2k + r$
$x_1 = 0$	1	0	...	...	...	...	...	...	0
$x_1 = 1$	0	1	0	...	...	...	...	...	...
$x_1 = 2$	...	2	1	0	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
$x_1 = k$	...	2	...	2	1	...	...	...	...
$x_1 = k + 1$	...	1	...	...	2	...	0	...	...
...	...	0	...	...	...	...	1	0	...
...	...	...	...	1	2	...	2	1	...
$x_1 = k + r + 1$	0	0	...	0	1	...	...	...	2

Figure A.4 2nd round eliminations in the case  $m = 2$ .

In round 3 of IEDS, there are two cases. Suppose first that  $r = k - 1$ . Then, clearly,  $k \neq 2$ . Panels (a) through (d) of Figure A.5 illustrate further eliminations in this case. It can be readily verified that  $X_1^{(\infty)} = X_1^{(5)} = \{0\} \cup \{k\} \cup \{n - k + 1\}$  and  $X_2^{(\infty)} = X_2^{(5)} = \{0\} \cup \{k + 1\} \cup \{n\}$ .

<sup>24</sup>This, of course, is simply Lemma 1(iii) in the case  $k = 2$ .

Moreover, the resulting reduced payoff matrix is a diagonal matrix, which implies uniqueness of the equilibrium.

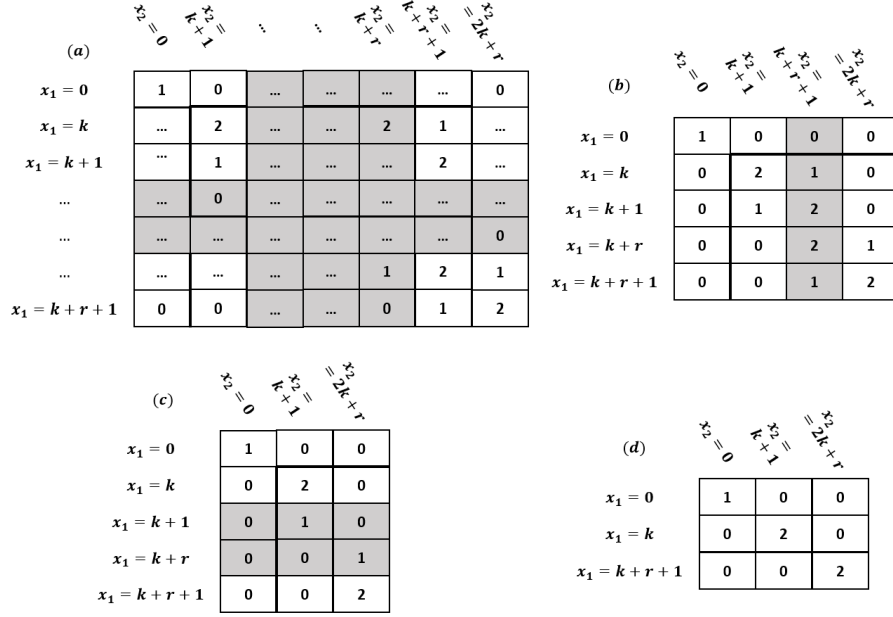


Figure A.5 Higher round eliminations in the case  $m = 2$  and  $r = k - 1$ .

Suppose next that  $r \in \{2, \dots, k - 2\}$ . In this case, going through a sequence of eliminations outlined in panels (a) through (c) of Figure A.6, we get  $X_1^{(\infty)} = X_1^{(4)} = \{0\} \cup \{k\} \cup \{n - k + 1\}$  and  $X_2^{(\infty)} = X_2^{(4)} = \{0\} \cup \{k + 1\} \cup \{n\}$ . Again, the resulting payoff matrix is a diagonal matrix, which implies uniqueness.

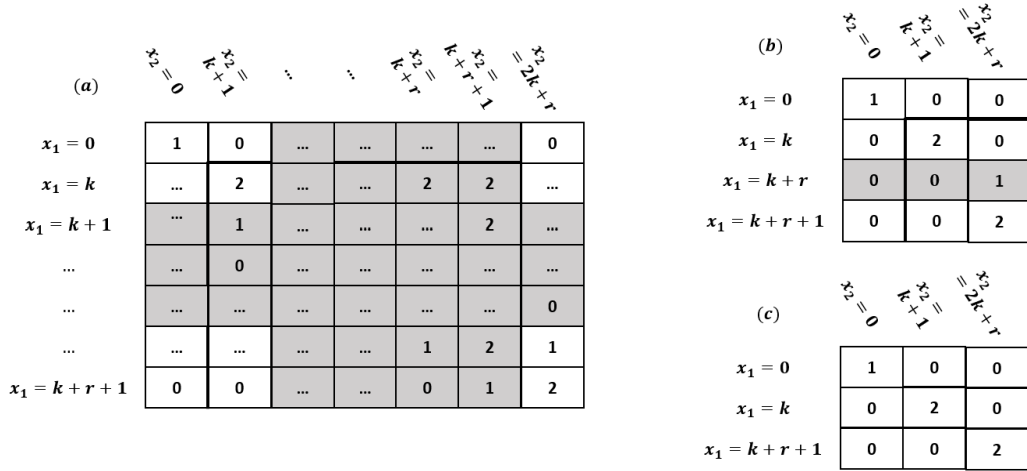


Figure A.6 Higher round eliminations in the case  $m = 2$  and  $r \in \{2, \dots, k - 2\}$ .

The rest of the proof is now immediate.  $\square$

**Proof of Lemma 5.** (i) Fix  $n \geq 1$ . For  $k \in \{0, \dots, n\}$ , we denote the corresponding value as  $v_1 = V(n, k)$ . Because of Lemma 2(ii), it suffices to show that  $V(n, k) \leq V(n, k+1)$  for  $k \in \{0, \dots, n-1\}$ . Suppose first that  $k = 0$ . Then,  $V(n, k) = V(n, 0) = 0$  by Lemma 2(i). Moreover, by Theorem 2,  $V(n, k+1)$  is given

$$V(n, 1) = \frac{1}{1 + \lfloor n/2 \rfloor} > 0. \quad (101)$$

This proves the claim for  $k = 0$ . Suppose next that  $k = 1$ . Note that this implies  $n \geq 2$ . Then,  $V(n, k)$  is given by (101). Moreover, using Theorems 3 and 4,  $V(n, k+1)$  is given by

$$V(n, 2) = \begin{cases} \frac{2}{m+2} & \text{if } n \text{ is odd} \\ \frac{2m-1}{m^2+m-1} & \text{if } n \text{ is even,} \end{cases} \quad (102)$$

where  $m = \lfloor n/2 \rfloor \geq 1$ . Hence, if  $n$  is odd, then  $V(n, 1) = \frac{1}{1+m} < \frac{2}{2+m} = V(n, 2)$ . Similarly, if  $n$  is even, then  $V(n, 1) = \frac{1}{1+m} < \frac{2m-1}{m^2+m-1} = V(n, 2)$ . Thus, the claim holds for  $k = 1$  as well. Next, suppose that  $k \in \{2, \dots, n-1\}$ . Then, proceeding as before, and invoking additionally Theorem 1 if  $k > n/2$ , we have

$$V(n, k) = \begin{cases} \frac{2}{m+2} & \text{if } n \text{ is not divisible by } k \\ \frac{2m-1}{m^2+m-1} & \text{if } n \text{ is divisible by } k, \end{cases} \quad (103)$$

where  $m = \lfloor n/k \rfloor$ . To compare this with  $V(n, k)$ , note first that  $m \geq \lfloor n/(k+1) \rfloor = m' \geq 1$ . There are now four subcases. Suppose first that  $n$  is not divisible by  $k$  and neither by  $k+1$ . Then,

$$V(n, k) = \frac{2}{m+2} \leq \frac{2}{m'+2} = V(n, k+1), \quad (104)$$

as claimed. Suppose next that  $n$  is not divisible by  $k$ , but by  $k+1$ .<sup>25</sup> Then, using  $m \geq m' \geq 1$ ,

$$V(n, k) = \frac{2}{m+2} \leq \frac{2}{m'+2} < \frac{2m'-1}{m'^2+m'-1} = V(n, k+1). \quad (105)$$

Suppose now that  $n$  is divisible by  $k$ , but not by  $k+1$ . Then, we have both  $m \geq 1$  and  $m \geq m'+1$ , so that

$$V(n, k) = \frac{2m-1}{m^2+m-1} \leq \frac{2}{m+1} \leq \frac{2}{m'+2} = V(n, k+1). \quad (106)$$

Suppose, finally, that  $n$  is divisible by both  $k$  and  $k+1$ . Then, as before,

$$V(n, k) = \frac{2m-1}{m^2+m-1} \leq \frac{2}{m+1} \leq \frac{2}{m'+2} \leq \frac{2m'-1}{m'^2+m'-1} = V(n, k+1). \quad (107)$$

---

<sup>25</sup>For any example, think of  $n = 12$  and  $k = 5$ .

This proves the claim. (ii) Fix  $n \geq 1$  and  $k \geq 0$ . We wish to show that  $V(n, k) \geq V(n+1, k)$ . The claim is immediate for  $k = 0$ , since  $V(n, 0) = V(n+1, 0) = 0$  by Lemma 2(i). Similarly, for  $k = 1$ , we see from Theorem 2 that  $V(n, 1) = \frac{1}{1+\lfloor n/2 \rfloor} \geq \frac{1}{1+\lfloor (n+1)/2 \rfloor} = V(n+1, 1)$ . Let now  $k \in \{2, \dots, n-1\}$ . Then, as noted above,  $V(n, k)$  is given by (103), where  $m = \lfloor n/k \rfloor$ . To compare this with  $V(n+1, k)$ , note first that  $m \leq \lfloor (n+1)/k \rfloor = \widehat{m}$ . Again, we have four subcases. Suppose first that neither  $n$  nor  $n+1$  are divisible by  $k$ . Then,

$$V(n, k) = \frac{2}{m+2} \geq \frac{2}{\widehat{m}+2} = V(n+1, k). \quad (108)$$

Suppose instead that  $n$  is divisible by  $k$ , but that  $n+1$  is not. Then, it is not hard to see that  $m = \widehat{m}$ , and therefore

$$V(n, k) = \frac{2m-1}{m^2+m-1} = \frac{2\widehat{m}-1}{\widehat{m}^2+\widehat{m}-1} > \frac{2}{m'+2} = V(n+1, k). \quad (109)$$

Next, suppose that  $n$  is not divisible by  $k$ , while  $n+1$  is divisible by  $k$ . Then,  $m+1 \leq \widehat{m}$ , and therefore

$$V(n, k) = \frac{2}{m+2} \geq \frac{2}{\widehat{m}+1} = \frac{2\widehat{m}-1}{\widehat{m}^2+\widehat{m}-1} = V(n+1, k). \quad (110)$$

Finally, suppose that both  $n$  and  $n+1$  are divisible by  $k$ . Clearly this is in conflict with  $k \geq 2$ , so this case cannot occur. The case  $k \geq n$  is again trivial. As all cases have been covered, the claim has been proved.  $\square$

**Proof of Lemma 6.** Sion and Wolfe (1957) consider a two-person zero-sum game on the square, with player 1 choosing  $\mathbf{x}$  and player 2 choosing  $\mathbf{y}$ , where player 1's payoff function is given as

$$\mathbf{K}(\mathbf{x}, \mathbf{y}) = \begin{cases} -1 & \text{if } \mathbf{x} < \mathbf{y} < \mathbf{x} + \frac{1}{2} \\ 0 & \text{if } \mathbf{x} = \mathbf{y} \text{ or } \mathbf{y} = \mathbf{x} + \frac{1}{2} \\ 1 & \text{otherwise.} \end{cases} \quad (111)$$

In comparison, one notes that in  $\mathfrak{B}_{0.5}$ ,

$$\pi_1(x_1, x_2) = \begin{cases} 2 & \text{if } x_1 < x_2 < x_1 + \frac{1}{2} \\ 1 & \text{if } x_2 = x_1 \text{ or } x_2 = x_1 + \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \quad (112)$$

It is now easy to check that, by letting  $\mathbf{x} = 1 - x_2$  and  $\mathbf{y} = 1 - x_1$ , we obtain

$$\pi_1(x_1, x_2) = 1 - \mathbf{K}(\mathbf{x}, \mathbf{y}). \quad (113)$$

Thus, exchanging the roles of players 1 and 2, and applying the linear transformations of strategy spaces and payoff functions, the Sion-Wolfe game is indeed strategically equivalent to  $\mathfrak{B}_{0.5}$ . Moreover, as the maximin value in the Sion-Wolfe game is  $\sup \inf \mathbf{K} = \frac{1}{3}$  and the minimax value is  $\inf \sup \mathbf{K} = \frac{3}{7}$ , we obtain

$$\sup_{\tilde{\mu}_1} \inf_{\tilde{\mu}_2} \iint_{[0,1] \times [0,1]} \pi_1(x_1, x_2) d\tilde{\mu}_1(x_1) d\tilde{\mu}_2(x_2) = 1 - \inf \sup \mathbf{K} = \frac{4}{7}, \quad (114)$$

$$\inf_{\tilde{\mu}_2} \sup_{\tilde{\mu}_1} \iint_{[0,1] \times [0,1]} \pi_1(x_1, x_2) d\tilde{\mu}_1(x_1) d\tilde{\mu}_2(x_2) = 1 - \sup \inf \mathbf{K} = \frac{2}{3}, \quad (115)$$

as claimed.  $\square$

We prepare the proof of Corollary 2. The following result is essentially due to Liang et al. (2019). It has, however, been reformulated and also been enriched by the inclusion of trivial cases for better comparability with our Corollary 1.

**Lemma A.3 (Liang, Wang, Cao, and Yang)** *Let  $\hat{V}(n, k)$  denote player 1's value in a heterogeneous-budgets discrete Colonel Blotto game with budget  $n + k$  for player 1 and budget  $n$  for player 2 (but no head start). Then,*

$$\hat{V}(n, k) = \begin{cases} 0 & \text{if } k = 0 \\ \frac{1}{\lfloor n/2 \rfloor + 1} & \text{if } k = 1 \\ \frac{2}{m+1} & \text{if } n+1 \text{ is not divisible by } k, \text{ and } k \in \{2, \dots, n+1\} \\ \frac{1}{m+1} + \frac{1}{m+2} & \text{if } n+1 \text{ is divisible by } k, \text{ and } k \in \{2, \dots, n+1\} \\ 2 & \text{if } k \geq n+2. \end{cases} \quad (116)$$

**Proof.** We go over the cases in equation (116). The case  $k = 0$  follows from a straightforward symmetry consideration in zero-sum games. For the remaining cases, let  $k \geq 1$  and denote by  $r \in \{0, \dots, k-1\}$  the remainder of an integer division of  $n$  by  $k$ , as before. Table 2 presents in bold face the symbols used by Liang et al. (2019).<sup>26</sup> Considering the case  $k = 1$  first, we have  $\Delta = 1$  (the “small-gap case”). For that case, Liang et al. (2019, Thm. 3(i)) says that

$$v(\mathbf{A}, \mathbf{B}) = \begin{cases} \frac{1}{k} & \text{if } \mathbf{A} \text{ is even} \\ \frac{1}{k+1} & \text{if } \mathbf{A} \text{ is odd.} \end{cases} \quad (117)$$

<sup>26</sup>Thus, in  $\hat{\mathfrak{B}}(n, k)$ , the budget of the favorite is  $\mathbf{A} = n + k$ , while the budget of the underdog is  $\mathbf{B} = n$ . Moreover, the gap between the budgets is  $\Delta = \mathbf{A} - \mathbf{B} = k$ . It is easy to see that with  $\mathbf{r} \in \{0, \dots, \Delta - 1\}$  being the remainder of an integer division of  $\mathbf{A} = n + k$  by  $\Delta = k$ , we have  $r = \mathbf{r}$ . Moreover, from  $\mathbf{A} = k\Delta + \mathbf{r}$ , we see that  $k = m + 1 = \lfloor n/k \rfloor + 1$ . Note, finally, that the value  $v(\mathbf{A}, \mathbf{B})$  in Liang et al.'s (2019) corresponds to precisely half of our value  $\hat{V}(n, k)$ .

Using that  $m = n$  holds for  $k = 1$ , this translates into the desired

$$\widehat{V}(n, k) = 2v(\mathbf{A}, \mathbf{B}) = \left\{ \begin{array}{ll} \frac{2}{m+1} & \text{if } n \text{ is odd} \\ \frac{2}{m+2} & \text{if } n \text{ is even} \end{array} \right\} = \frac{1}{\lfloor n/2 \rfloor + 1}. \quad (118)$$

Let now  $k \in \{2, \dots, n+1\}$ . For that case, Liang et al. (2019, Thm. 3, ii-iii) show that

$$v(\mathbf{A}, \mathbf{B}) = \left\{ \begin{array}{ll} \frac{1}{k} & \text{if } r \in \{0, \dots, \Delta - 2\} \\ \frac{1}{2} \left( \frac{1}{k} + \frac{1}{k+1} \right) & \text{if } r = \Delta - 1. \end{array} \right. \quad (119)$$

Translating into our notation, this reads

$$\widehat{V}(n, k) = 2v(\mathbf{A}, \mathbf{B}) = \left\{ \begin{array}{ll} \frac{2}{m+1} & \text{if } r \in \{0, \dots, k-2\} \\ \frac{1}{m+1} + \frac{1}{m+2} & \text{if } r = k-1, \end{array} \right. \quad (120)$$

as claimed. Finally, consider the case where  $k \geq n+2$ . Then, clearly, the favorite can easily win both battlefields by sending at least  $n+1$  soldiers to each of them. Thus,  $\widehat{V}(n, k) = 2$  in that case. As all cases have been covered, the proof is complete.  $\square$

Symbol used by Liang et al. (2019)	Interpretation in Liang et al. (2019)	Equivalent (this paper)
$\mathbf{A}$	Player 1's total amount of the resource	$n + k$
$\mathbf{B}$	Player 2's total amount of the resource	$n$
$\Delta$	Gap between the resource amounts of the two players	$k$
$r$	Remainder in $\mathbf{A} = k\Delta + r$ (*) $r \in \{0, \dots, \Delta - 1\}$	$r$
$k$	Nonnegative integer defined by relationship (*)	$m + 1$
$v(\mathbf{A}, \mathbf{B})$	Value of the two-battlefield Colonel Blotto game	$\frac{1}{2}\widehat{V}(n, k)$

Table A.1 Notation used by Liang et al. (2019).

**Proof of Corollary 3.** We will make use of Corollary 1 and Lemma A.3 without explicit reference. Suppose first that  $k \in \{0, 1\}$ . Then,  $\widehat{V}(n, k) = V(n, k)$ , so that  $\widehat{V}(n, k) - V(n, k) = 0$ , as claimed. Suppose next that neither  $n$  nor  $n+1$  is divisible by  $k$ , and that  $k \in \{2, \dots, n-1\}$ . Then,  $\widehat{V}(n, k) = \frac{2}{m+1}$ , while  $V(n, k) = \frac{2}{m+2}$ . Hence,  $\widehat{V}(n, k) - V(n, k) = \frac{2}{(m+2)(m+1)}$ , as claimed.

Suppose now that  $n$  is divisible by  $k$ , but that  $n + 1$  is not, and that  $k \in \{2, \dots, n\}$ . Then,  $\widehat{V}(n, k) = \frac{2}{m+1}$ , while  $V(n, k) = \frac{2m-1}{m^2+m-1}$  (resorting to Lemma 2(ii) in the case  $k = n$ , where  $m = 1$  and  $V(n, k) = 1$ ). Therefore,  $\widehat{V}(n, k) - V(n, k) = \frac{m-1}{(m+1)(m+m^2-1)}$  in this case. Assume further that  $n + 1$  is divisible by  $k$ , but that  $n$  is not, and that  $k \in \{2, \dots, n-1\} \cup \{n+1\}$ . Then,  $\widehat{V}(n, k) = \frac{1}{m+1} + \frac{1}{m+2}$ , while  $V(n, k) = \frac{2}{m+2}$ , where we used the fact that  $m = 0$  if  $k = n + 1$ . Hence, in the considered case,  $\widehat{V}(n, k) - V(n, k) = \frac{1}{(m+2)(m+1)}$ . Suppose, finally, that  $k \geq n + 2$ . Then,  $\widehat{V}(n, k) - V(n, k) = 2 - 1 = 1$ . This concludes the proof.  $\square$

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