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# **AGGREGATION IN NETWORKS**

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# AGGREGATION IN NETWORKS

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**ABSTRACT.** In this paper, we show that a concept of aggregation can hold in large network games with linear best replies. Breaking up large networks into smaller sub-networks, which can be replaced by representative players, leads to a coarse-grained description of strategic interactions. This method of summarizing complex strategic interactions by simple ones can be applied to compute all Nash equilibria for the special network structure of cograph. A key finding is that a stable Nash equilibrium of the large network game can be decomposed into a collection of Nash equilibria of subnetwork games. Thereby, we establish a systematic relationship between player's position in a subnetwork and his equilibrium action in the large network game.

*JEL classification:* C72, D31, D85, H41.

*Keywords:* aggregation, modular decomposition, network games, public goods, stability.

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## 1. INTRODUCTION

The economics of networks, which focuses on modeling and understanding varied economic interactions, has recently become one of the most active and dynamic fields in economics. It has the potential for important and lasting policy implications—see, for example, Jackson (2016) and Goyal (2017). It is notable that most economic interactions take place in large networks, whose sheer sizes and complex structures make economic analysis quite a challenging task. Nevertheless, it is well known that economic networks have a rich degree of symmetry due to similar linkage patterns for individuals having similar economic characteristics such as income, education, and preferences, for firms facing the same competitors, and for countries having similar bilateral agreement policies.

In this paper, we show that advantage can be taken of the symmetric features of large economic networks. More specifically, we show that a concept of aggregation that ensures a group of players behaves like a single player holds for network games with linear best replies, a subject of ongoing research as in Ballester, Calvó-Armengol, and Zenou (2006) for externalities, Bramoullé and Kranton (2007) for public good provision, and Bramoullé, Kranton, and D’Amours (2014) for various economic interactions. For recent related contributions, see also, Zhou and Chen (2015), Acemoglu, Malekian, and Ozdaglar (2016), Günther and Hellmann (2017), Elliott and Golub (2019), and Parise and Ozdaglar (2020). It is worth noting that our concept of aggregation holds for players, as in the aggregative game for pure public goods introduced by Cornes and Hartley (2007), rather than individual preferences.

A key ingredient of our analysis is a group of players, called a module, such that players in the group have exactly the same neighbors outside the group.<sup>1</sup> In interpretation, since players in a module are indistinguishable by players outside the module in terms of their network position, outside players are affected either by their aggregate action or by nothing and hence one can substitute players in the group with a single representative player. Modules feature frequently in the economics of networks literature. For instance, a nested split graph, which has been extensively studied in König, Tessone, and Zenou (2014), Hiller (2017), Kinatered and Merlino (2017), and Olaizola and Valenciano (2020) as the outcome of network formation with a strong empirical support, has many modules formed by the various nested neighborhoods. A concrete example, which arises naturally,

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<sup>1</sup>The notion of a modular set has been rediscovered several times in many fields including cooperative game theory by Shapley (1967) under the name of committee.

is a set of firms competing domestically while facing the same overseas competition, which can then be replaced by just one big firm. Another concrete example is several countries privately providing a public good, such as cybersecurity, which could be made accessible via bilateral agreements. Assume that there is a group of countries with identical outside bilateral agreements. Therefore the group of countries can be represented by a single country regardless of the group's architecture of bilateral agreements.

We first consider a partition of players of a network game with linear best replies into groups that gives rise to two-level nested games—a group game played within each group and an intergroup game played between representative players of the groups. Then, we show that each Nash equilibrium of the network game corresponds to a combination of Nash equilibria of the nested games if and only if the groups are modules. Furthermore, we show that a network game can be decomposed into a unique hierarchy of nested games. Key to this are the modules that overlap with no other modules, called strong modules, which, when ordered by inclusion, define a unique tree, called the modular decomposition tree, whose root is the set of players and whose leaves are the single players.<sup>2</sup> By fitting nested games into each other along the nodes of the modular decomposition tree, we obtain a unique hierarchical decomposition of the network game with linear best replies, which is useful for the analysis of strategic interactions. First, it can be used to carry out a recursive computation of Nash equilibria, which could be of great algorithmic interest. In addition, we show that the computation can be especially useful to characterize all stable Nash equilibria for a special class of networks, called cograph. More specifically, we show that a stable Nash equilibrium can be decomposed into a collection of Nash equilibria of module games. Thereby, we establish a systematic relationship between player's position in a module, and his Nash equilibrium actions in the large network game.

We also provide an application of our results to the model of public goods in networks with large direct spillover effects to further illustrate the usefulness of our results. The key question addressed in Bramoullé and Kranton (2007) is how the network architecture of spillovers influences public goods provision, in the absence of coordination. It is shown in their analysis that despite the attractive normative feature of sharing the burden of public goods among all players, an equilibrium where all players are active is not always guaranteed to exist. Our aggregation approach complements the above analysis, as it provides a necessary condition on the network architecture in order to have a Nash

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<sup>2</sup>The concept of the modular decomposition tree was introduced in Gallai (1967) and a similar decomposition also appeared in Shapley (1967).

equilibrium with only active players—see also Sun (2012) for a different approach. This necessary condition for the existence of an equilibrium where all players are active, also becomes sufficient for the special network class of cograph.

The paper is organized as follows. In Section 2, we present the basic model of network games. In Section 3, we introduce the concept of aggregation and nested games. In Section 4, we investigate and relate the stability of the various Nash equilibria. In Section 5, we show that aggregation yields a unique hierarchy of nested games and characterize Nash equilibria for cograph. In Section 6, we provide an application of our results to public goods in networks. Section 7 concludes the paper.

## 2. THE MODEL

We consider a strategic form game  $\Gamma(\mathbf{g}, \delta)$  with  $N = \{1, \dots, n\}$  players embedded on an undirected and unweighted connected network  $\mathbf{g}$  of interactions, and where  $\delta \in [0, 1]$  measures how much player  $i$ 's action is affected by his neighbors' actions. Each player  $i$  chooses an action  $x_i \in \mathbb{R}_+$ . Given a subset of players  $S \subset N$  and a profile of actions  $\mathbf{x} = (x_1, \dots, x_n)$ , let  $\mathbf{x}_S = (x_i)_{i \in S}$  denote the actions of the players in  $S$  and  $x_S = \sum_{i \in S} x_i$  denote their sum. As usual, let  $\mathbf{x}_{-i} = \mathbf{x}_{N \setminus \{i\}}$  denote the actions of all other players than  $i$ . The payoffs of player  $i$  for the profile of actions  $\mathbf{x} = (x_1, \dots, x_n)$  are  $U_i(\mathbf{x}) = U_i(x_i, \mathbf{x}_{-i})$ . Player  $i$  seeks to maximize his payoffs and has a best-reply function

$$x_i = f_i(\mathbf{x}_{-i}) \stackrel{\text{def}}{=} \max\{1 - \delta x_{\mathcal{N}_i(\mathbf{g})}, 0\},$$

where  $\mathcal{N}_i(\mathbf{g})$  denotes  $i$ 's neighbors in  $\mathbf{g}$  and 1 is the action player  $i$  chooses in isolation.

As shown in Bramoullé, Kranton, and D'Amours (2014), this type of game,  $\Gamma(\mathbf{g}, \delta)$ , can be used to represent various types of economic interactions, including the model of public goods in networks, introduced in Bramoullé and Kranton (2007), and the model of negative externalities with linear-quadratic payoffs, introduced in Ballester, Calvó-Armengol, and Zenou (2006).

At a Nash equilibrium  $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$  of the game  $\Gamma(\mathbf{g}, \delta)$ , each player's action is a best-reply to his neighbors' actions, that is,  $x_i^* = f_i(\mathbf{x}_{-i}^*)$  for each player  $i \in N$ . The existence of a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  is guaranteed by Brouwer's fixed point theorem by restricting strategies of players to  $[0, 1]^n$ .

Let  $\mathbf{p} = \{M_1, \dots, M_K\}$  denote a partition of the set of players  $N$  into groups. Given two disjoint groups  $M_k$  and  $M_h$  either there is a player in  $M_k$  adjacent to a player in  $M_h$  or no player in  $M_k$  is adjacent to a player in  $M_h$ . Thus, the relationship between two disjoint groups is either adjacent (if they have two adjacent players) or nonadjacent. Hence the partition  $\mathbf{p}$  gives rise to a new network,  $\mathcal{G}_{\mathbf{p}}$ , called the quotient network, whose vertices are the groups of the partition  $\mathbf{p}$  and links are the adjacencies of these groups.

Now we define an *intergroup game* played on the quotient network  $\mathcal{G}_{\mathbf{p}}$ , denoted by  $\Gamma(\mathcal{G}_{\mathbf{p}}, \delta; \mathbf{z})$ , where  $\mathbf{z} = (z_1, \dots, z_K) \in \mathbb{R}_+^K$  is a vector of weights determined exogenously. This set-up means that in the quotient network, players positions are filled by the groups. For each group  $M_k$ , there is a representative player  $k$ , who chooses an action  $r_k \in [0, 1]$ . Player  $k$ 's payoffs depend on his own action  $r_k$  and the actions of the other players  $\mathbf{r}_{-k}$ . We denote the payoffs of player  $k$  by  $V_k$ , which are assumed to yield the best-reply function:

$$r_k = F_k(\mathbf{r}_{-k}) \stackrel{\text{def}}{=} \max\{1 - \delta \sum_{h \in \mathcal{N}_k(\mathcal{G}_{\mathbf{p}})} z_h r_h, 0\}.$$

Finally, an *independent set* is a set of players in the network, no two of which are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set.

### 3. MODULAR AGGREGATION

We now introduce a network position similarity of a group of players, which ensures that it can behave like a single player. A group of players  $M$  is called a module if they have exactly the same neighbors outside the module, that is, for any player  $i \in N \setminus M$ , either  $i$  is adjacent to every player in  $M$  or  $i$  is adjacent to no player in  $M$ . It is easy to notice that each single player  $\{1\}, \dots, \{n\}$  and the entire set of players  $N = \{1, \dots, n\}$  are always modules, called trivial modules.

While not every network has a non-trivial module, it is well known that economic networks have a rich degree of symmetry due to similar linkage patterns for individuals having similar economic characteristics. For instance, a *nested split graph*, which has been extensively studied as the outcome of network formation in König, Tessone, and Zenou (2014), Hiller (2017), Kinatader and Merlino (2017), and Olaizola and Valenciano (2020) as the outcome of network formation with a strong empirical support, has many modules formed by the various nested neighborhoods. A partition  $\mathbf{p} = \{M_1, \dots, M_K\}$  of the set of players  $N$  is called modular if each group  $M_k$  of the partition is a module.

The following result shows that a Nash equilibrium of the network game corresponds to a combination of Nash equilibria of the nested games provided that the partition is modular.

**Theorem 1.** *Given a modular partition  $\mathbf{p} = \{M_1, \dots, M_K\}$ . Then, the following are equivalent:*

- (1)  $\mathbf{x}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$
- (2)  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, \dots, r_K^* \mathbf{y}_{M_K}^*)$  such that
  - (a)  $\mathbf{y}_{M_k}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$ , for each  $k = 1, \dots, K$ , and
  - (b)  $\mathbf{r}^*$  is a Nash equilibrium of  $\Gamma(\mathcal{G}_{\mathbf{p}}, \delta; y_{M_1}^*, \dots, y_{M_K}^*)$ .

Conversely, given a partition  $\mathbf{p} = \{M_1, \dots, M_K\}$ . If (1) and (2) are equivalent for any  $\delta > 0$ , then  $\mathbf{p} = \{M_1, \dots, M_K\}$  is modular.

Hence, it follows from Theorem 1 that if the partition is modular, then finding the Nash equilibria of the nested games could provide significant insights into the Nash equilibria of the network game. In particular, note that players' actions in a Nash equilibrium of the network game are proportional to their actions in a Nash equilibrium of the module game.

The following example given in Ballester, Calvó-Armengol, and Zenou (2006) illustrates a network game and the corresponding nested games.

**Example 1.** Consider the network with eleven players depicted in Figure 1. Clearly, the partition  $\mathbf{p} = \{M_1, M_2, M_3, M_4, M_5\}$ , where  $M_1 = \{1\}$ ,  $M_2 = \{2, 6\}$ ,  $M_3 = \{7, 11\}$ ,  $M_4 = \{3, 4, 5\}$ , and  $M_5 = \{8, 9, 10\}$ , is modular. Below, we depict the underlying networks of the various games.

- (1) The network game  $\Gamma(\mathbf{g}, \delta)$ .

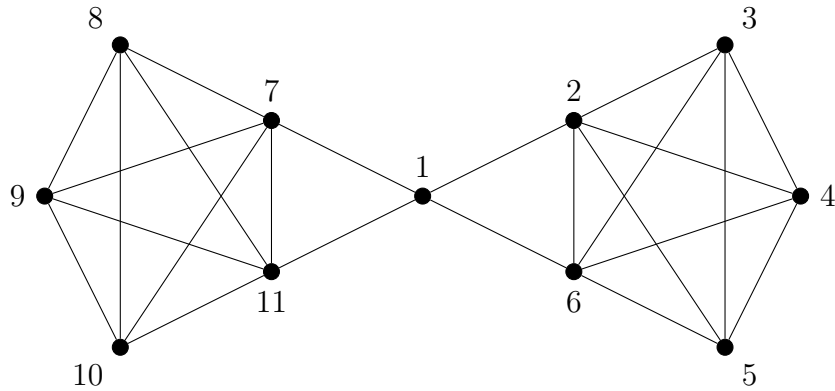


Figure 1: The network  $\mathbf{g}$ .



(2) The nested games.

(a) The module game  $\Gamma(\mathbf{g}_{M_k}, \delta)$ , for each  $k = 1, \dots, 5$ .

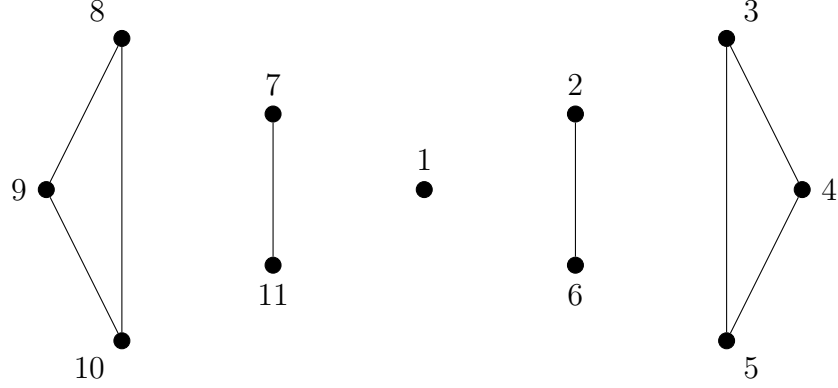


Figure 2: The module network  $\mathbf{g}_{M_k}$ , for each  $k = 1, \dots, 5$ .

(b) The intergroup game  $\Gamma(\mathcal{G}_{\mathbf{p}}, \delta; y_{M_1}^*, \dots, y_{M_5}^*)$ .

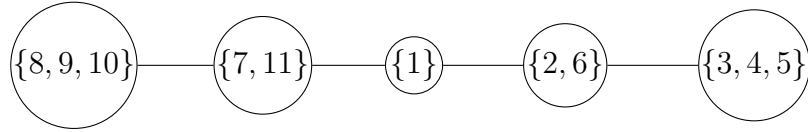


Figure 3: The quotient network  $\mathcal{G}_{\mathbf{p}}$ .

**3.1. Stable equilibria.** We now consider the issue of stability, which is often invoked to refine the set of Nash equilibria. In this respect, we consider a myopic adjustment process defined, for each consumer  $i = 1, \dots, n$ , by

$$\dot{x}_i = f_i(\mathbf{x}_{-i}) - x_i,$$

where  $f_i(\mathbf{x}_{-i})$  is player  $i$ 's best-reply function. The Nash equilibrium  $\mathbf{x}^*$  is “locally asymptotically stable” if there exists a neighborhood of  $\mathbf{x}^*$  such that if the above system starts at any point inside this neighborhood, it converges back to  $\mathbf{x}^*$ . In interpretation, stable equilibria are robust to small perturbations in players' actions.<sup>3</sup>

The following result relates the stability of the Nash equilibrium to the stability of the Nash equilibria of the nested games. Since active players induce a modular partition, for simplicity of notations, we will assume that all players are active.

<sup>3</sup>See Bervoets and Faure (2019) for a more recent contribution on stability in network games.

**Theorem 2.** *Given a modular partition  $\mathbf{p} = \{M_1, \dots, M_K\}$  and a Nash equilibrium  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, \dots, r_K^* \mathbf{y}_{M_K}^*)$  such that for each  $k = 1, \dots, K$ ,  $\mathbf{y}_{M_k}^*$  is a Nash equilibrium of the module game  $\Gamma(\mathbf{g}_{M_k}, \delta)$  and  $\mathbf{r}^*$  is a Nash equilibrium of the intergroup game  $\Gamma(\mathcal{G}_{\mathbf{p}}, \delta; y_{M_1}^*, \dots, y_{M_K}^*)$ . If  $\mathbf{x}^*$  is stable, then  $\mathbf{y}_{M_k}^*$ , for each  $k = 1, \dots, K$ , and  $\mathbf{r}^*$  are stable.*

Theorem 2 shows that, given a modular partition, the stability of a Nash equilibrium of the network game implies the stability of the Nash equilibria of the nested games.<sup>4</sup> Hence, as we will show later on, a convenient way to check whether a Nash equilibrium of the network game is unstable is to first check if any Nash equilibrium of the nested games is unstable.

#### 4. STRONG MODULES

Now, we will further exploit the decomposition of the network game into nested games by focusing on distinguished modules, called strong modules. A module  $M$  is called a strong module if, for any module  $M' \neq M$ , it holds that either  $M' \cap M = \emptyset$  or one module is included in the other. We say that a strong module  $M$  is a descendant of another strong module  $M'$  if  $M \subset M'$  and there is no other strong module  $M^*$  such that  $M \subset M^* \subset M'$ .

Given a strong module  $M$ , let  $\mathbf{p}_M = (D_1, \dots, D_T)$  denote the descendants' partition. Understandably, as it will become clear later on, the descendants's partition will provide a convenient way to organize the various modular partitions. A strong module  $M$  is labeled in three ways: *parallel* when the descendants are all non-neighbors of each other (that is,  $\mathcal{G}_{\mathbf{p}_M}$  is empty), *series* when the descendants are all neighbors of each other (that is,  $\mathcal{G}_{\mathbf{p}_M}$  is complete), and *prime* otherwise (that is,  $\mathcal{G}_{\mathbf{p}_M}$  is neither empty nor complete).

The following two propositions exploit the particular network architectures of parallel and series modules to compute the Nash equilibria of the intergroup game.

**Proposition 1.** *Given a parallel module  $M$ . Then  $\mathbf{r}_M^* = (1, \dots, 1)$  is the unique Nash equilibrium of the  $\Gamma(\mathcal{G}_{\mathbf{p}_M}, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$ .*

Let  $M$  be a series module, we say that a descendant  $D_t$  is *over-threshold* if  $y_{D_t}^* > \frac{1}{\delta}$ . We say that  $M$  is *separating* if the set of over-threshold descendants  $O_M = \{D_t \mid y_{D_t}^* > \frac{1}{\delta}\}$

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<sup>4</sup>Although we don't have a counterexample for the opposite, we conjecture that it doesn't generally hold: the stability of the Nash equilibrium cannot be inferred from the stability of the Nash equilibria of the nested games since, intuitively, there are fewer possible perturbations in the Nash equilibria of the nested games.

is non-empty.<sup>5</sup> Let  $A_M$  denote the set of active descendants at the Nash equilibrium of the intergroup game played by the descendants.

**Proposition 2.** *Given a series module  $M$ . Then,  $\mathbf{r}_M^*$  is a Nash equilibrium of the intergroup game  $\Gamma(\mathcal{G}_{\mathbf{p}_M}, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$  if and only if, for almost every  $\delta$ ,<sup>6</sup> it holds that,*

$$r_t^* = \frac{\frac{1}{1-\delta y_{D_t}^*}}{1 + \sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1-\delta y_{D_s}^*}},$$

where  $A_M$  is a subset of  $O_M$ , if  $M$  is separating and  $A_M = \{1, \dots, T\}$ , otherwise.

Proposition 2 provides several sharp predictions on equilibrium actions in a Nash equilibrium, especially, for separating series modules.

**Corollary 1.** *Given a separating series module  $M$ . Then it holds that*

- (i) *there exists always a Nash equilibrium with free-riders.*
- (ii) *all players not in an over-threshold descendant are free-riders in any Nash equilibrium.*
- (iii) *if  $\mathbf{x}_M$  and  $\mathbf{x}'_M$  are Nash equilibria such that  $A_M \subset A'_M$ , then the equilibrium level of aggregate play is higher for  $\mathbf{x}_M$  than  $\mathbf{x}'_M$ .*

Corollary 1 shows that given a separating series module there always exists an equilibrium with free-riders (even when all descendants are over-threshold). In particular, players not in an over-threshold descendant are free-riders in any Nash equilibrium. Moreover, the equilibrium level of aggregate play is decreasing in terms of inclusion of active descendants. As a consequence, equilibria with just one active over-threshold descendant yield the highest equilibrium level of aggregate play in terms of inclusion of active players.

In the following, we will show that Nash equilibria with just one active over-threshold descendant have another remarkable property. More specifically, equilibria with just one active over-threshold descendant for every separating series module are the only stable equilibria.

**Corollary 2.** *Given a stable Nash equilibrium. Then for every separating series module there is just one active over-threshold descendant.*

<sup>5</sup>As it will become clear later on, we call it separating in the sense, it can always separate players into active players and free-riders.

<sup>6</sup>We say that a property holds for almost every  $\delta$  if it holds for every  $\delta$  except a finite number of values.

## 5. HIERARCHICAL DECOMPOSITION

We now take advantage of the descendant relation and decompose the network game with linear best replies into a unique hierarchy of nested games. More specifically, the descendant relation yields a tree on the set of strong modules, called the modular decomposition tree of the network, where the set of players  $\{1, \dots, n\}$  is the root, the single players  $\{1\}, \dots, \{n\}$  are the leaves, and any other strong module is an internal node. The modular decomposition tree of a network is unique (see Gallai (1967) and Shapley (1967)), as illustrated in Figure 4.

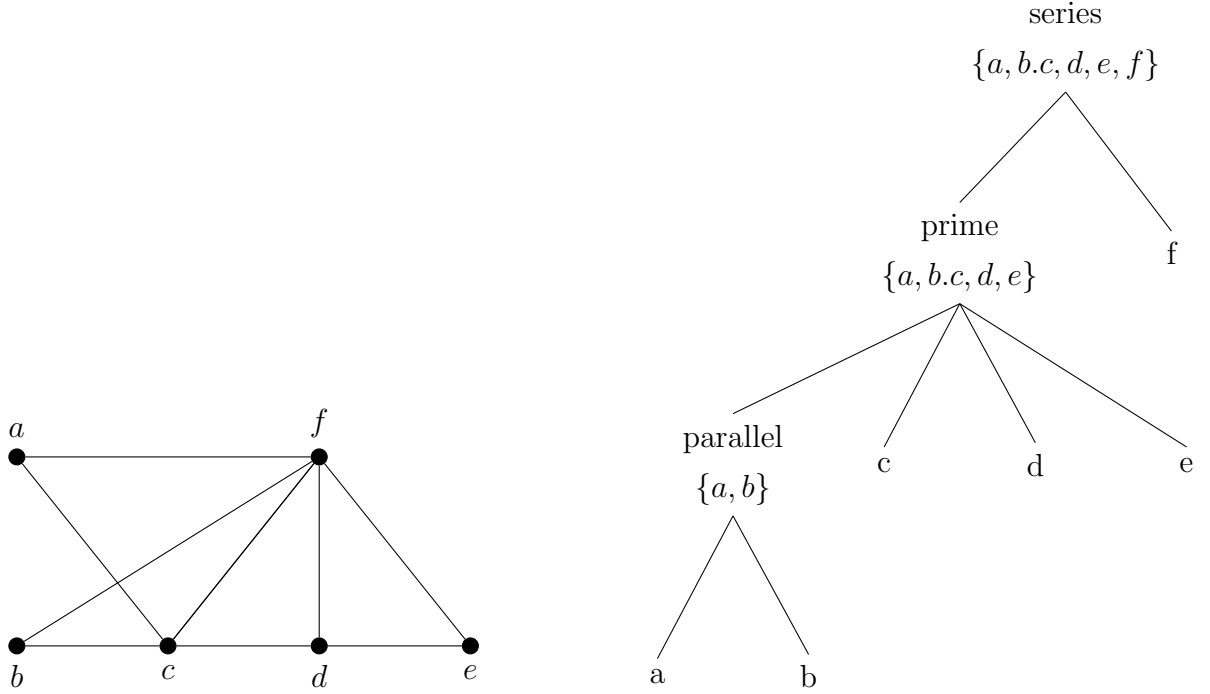


Figure 4: Modular decomposition tree of a network.

Note that the modular decomposition tree constitutes an exact alternative representation of the network whenever the structure of each prime module is depicted. As a consequence, nested games can be used to compute Nash equilibria using the bottom-up technique along the modular decomposition tree.

**5.1. Cograph.** Now we consider the special class of networks known as *cograph*, which consist of networks with only parallel and series modules in their modular decomposition tree.<sup>7</sup>

<sup>7</sup>The class of cographs has been intensively studied since it was discovered independently by several authors in the 1970s.

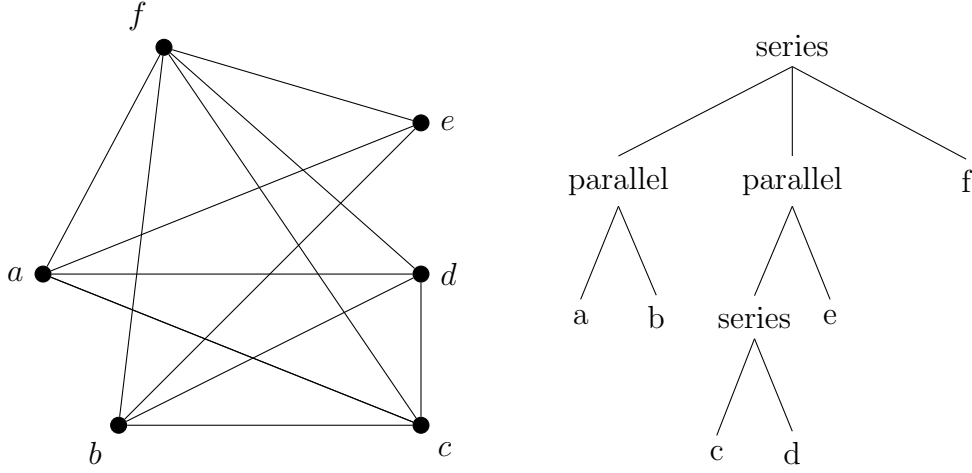


Figure 5: Cograph network with six players.

For cographs, it follows from Propositions 1 and 2 that, for almost every  $\delta$ , Nash equilibria can be computed immediately using the bottom-up technique. In the following, based on the above computation, we will characterize all stable Nash equilibria for a cograph.

To do so, we say a partition of descendants  $(M_k)_{k \in \mathcal{I}}$  is *independent* if every over-threshold  $M_k$  is minimal and not adjacent to any  $M_h$ , for each  $h \neq k$ . We say an independent partition is *maximal* if it is the coarsest and maximal in terms of set inclusion among independent partitions.<sup>8</sup> A maximal independent partition can be easily constructed from the modular decomposition tree by following a bottom-top approach if each minimal over-threshold descendant is replaced by  $M_k$ , descendants of all separating series modules containing  $M_k$  are deleted and each descendant of a parallel module containing  $M_k$  is replaced by  $M_h$ .<sup>9</sup>

The following example illustrates all maximal independent partitions in a cograph.

**Example 2.** Consider the cograph network with six players depicted in Figure 5. Then using the computations in Propositions 1 and 2 bottom-up along the modular decomposition tree, it follows that the maximal independent partition is the entire set of players

<sup>8</sup>That is, there is no independent partition  $(M_{k'})_{k' \in \mathcal{I}'}$  such that for some  $k_0$  and  $k'_0$ , it holds that  $M_{k_0} \subsetneq M_{k'_0}$ , or  $\cup_{k \in \mathcal{I}} M_k \subsetneq \cup_{k' \in \mathcal{I}'} M_{k'}$ .

<sup>9</sup>In interpretation, a maximal independent partition corresponds to the maximal independent set of the quotient network where each descendant up to a minimal over-threshold descendant is replaced by a representative player, not over-threshold descendants of all separating series modules are deleted, and each descendant of parallel modules that contains a minimal over-threshold descendant is replaced by a representative player.

$(\{a, b, c, d, e, f\})$  for low  $\delta \in [0, 0.22[$  and  $(\{c, d, e\})$  for intermediate  $\delta \in [0.23, 0.5[$ . For high  $\delta \in [0.5, 1[$  there are two maximal independent partitions  $(\{a, b\})$  and  $(\{c, d, e\})$ .

The next theorem provides a full characterization of all stable Nash equilibria for a cograph.

**Theorem 3.** *Consider a cograph  $\mathbf{g}$ . A profile  $\mathbf{x}^*$  is a stable Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  if and only if there exists a maximal independent partition  $(M_k)_{k \in \mathcal{I}}$  such that  $\mathbf{x}^* = ((\mathbf{x}_{M_k}^*)_{k \in \mathcal{I}}, \mathbf{0})$ , where  $\mathbf{x}_{M_k}^*$  is the unique Nash equilibrium  $\Gamma(\mathbf{g}_{M_k}, \delta)$  and all players in  $M_k$  are active.*

Note that the Nash equilibria action of each player in  $M_k$  in the network game  $\Gamma(\mathbf{g}, \delta)$  is identical to his action in the unique Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$ . Thereby, we establish a systematic relationship between each player's network position in a module and his Nash equilibrium action in the network game. Note also that the multiplicity of Nash equilibria in the network game arises from the multiplicity of the maximal independent partitions.

A special case of cographs is a nested split graph, where the neighborhood of every player is contained in the neighborhoods of the players with higher degrees. The modular decomposition tree of a nested split graph is a caterpillar, that is, a tree in which the removal of all terminal nodes yields a path.

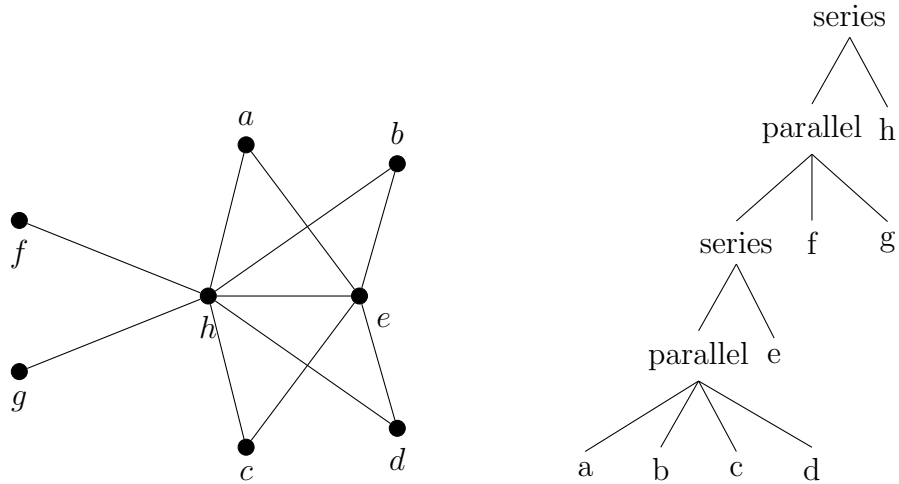


Figure 6: Modular decomposition tree of a nested split graph.

Observe that for a nested split graph there is always a unique maximal independent partition since the modular decomposition tree is a caterpillar. Observe also that players

who are descendants of parallel modules are always active since they belong to the unique maximal independent partition. The following example illustrates these points.

**Example 3.** Consider the nested split graph with eight players depicted in Figure 6. Then using the computations in Propositions 1 and 2 bottom-up along the caterpillar, it follows that the maximal independent partition is the entire set of players  $(\{a, b, c, d, e, f, g, h\})$  for low  $\delta \in [0, 0.12[$ ,  $(\{a, b, c, d, e, f, g\})$  for intermediate  $\delta \in ]0.12, 0.25[$ , and  $(\{a, b, c, d\}, \{f\}, \{g\})$  for high  $\delta \in ]0.25, 1[$ .

In the following, we further link the Nash equilibrium actions to degrees in a nested split graph.

**Proposition 3.** *Given a nested split graph. If  $\delta \in [0, 1[$  then in the unique stable Nash equilibrium the action of the players is decreasing with degree. In particular, there is a threshold degree (the highest degree of players in the maximal independent partition) such that players whose degree is below the threshold are active and players above the threshold are free-riders.*

## 6. AN APPLICATION: PUBLIC GOODS IN NETWORKS

Now, we provide a further application of our results to the public goods in a network model, introduced in Bramoullé and Kranton (2007), which can be investigated as a  $\Gamma(\mathbf{g}, 1)$  game. Recall that for a profile of contributions to be a Nash equilibrium, it has to be the case that every player contributes nothing to the public good if the sum of his neighbors' contributions exceeds 1 or contributes exactly the difference between 1 and the sum of his neighbors' contributions. Therefore, at a Nash equilibrium, we may distinguish three types of players: *free-riders*, who contribute nothing; *experts*, who make full contributions; and the *others*. Bramoullé and Kranton (2007) insightfully show that specialized equilibria—that is, equilibria with only experts and free-riders—correspond to maximal independent sets of the network and therefore are always guaranteed to exist.

Specialized equilibria are of interest as they illustrate in an acute form how the network can lead to specialization. However, beyond specialized equilibria, very little is known about other equilibria such as distributed equilibria, where all players make positive contributions, and hybrid equilibria, which are neither specialized nor distributed. Distributed equilibria can be especially of interest given their normative importance, because all players share the burden of contributing to the public good, but they are not always guaranteed to exist. For instance, distributed equilibria are not possible in star

networks. Moreover, even when distributed equilibria exist, very little is known about their properties beyond the symmetric contribution equilibrium in regular networks.

In the following, we will provide a condition on the modular decomposition of the network that is necessary for the existence of a distributed equilibrium. We say that a series module is *uncentered* if all (or none) of its descendants are single players. More specifically, an uncentered series module rules out the possibility of having both a single player and a non-single player as descendants, which, as shown below, rules out the distributed equilibrium.

**Proposition 4.** *If a distributed equilibrium exists, then all series modules are uncentered.*

The intuition for the necessary condition of Proposition 4 can be explained as follows. In a distributed equilibrium, it must be the case that every player makes a strictly positive contribution. However, the (simultaneous) presence of a single player and a non-single player as descendants of a series module brings about a mismatch between what these players contribute and consume of the public goods, leading one of them to become a free-rider. The next result shows that the necessary condition becomes also sufficient for the special class of cographs.

**Proposition 5.** *If the network is a cograph, then a distributed equilibrium exists if and only if all series modules are uncentered.*

An example of a cograph where all series modules are uncentered is a nested split graph such as a star network.

**Corollary 3.** *There is no distributed equilibrium in a nested split graph.*

It is worth noting that Sun (2012) also provides a sufficient and necessary condition, called the Monotonicity Condition, for the existence of a distributed equilibrium for the games with heterogeneous targets on a general class of networks. While, by following the proof of Proposition 4, the Monotonicity Condition implies all series modules are uncentered, unfortunately, we were not able to establish that the converse holds for cographs. Our approach differs from Sun (2012) in at least two key aspects. First, it illustrates the role particular network architectures play in determining public goods provision. Second, it provides an algorithm to compute the distributed equilibria for the special class of cograph. The following example illustrates these points.



**Example 4.** Consider again the cograph network with six players depicted in Figure 5. Then using the computations in Propositions 1 and 2 bottom-up along the modular decomposition tree, we can compute all Nash equilibria as shown in Figure 8. Observe that there is no distributed equilibrium, which can be explained by the fact that the series module consisting of the entire set of players has both a single player and a non-single player as descendants.

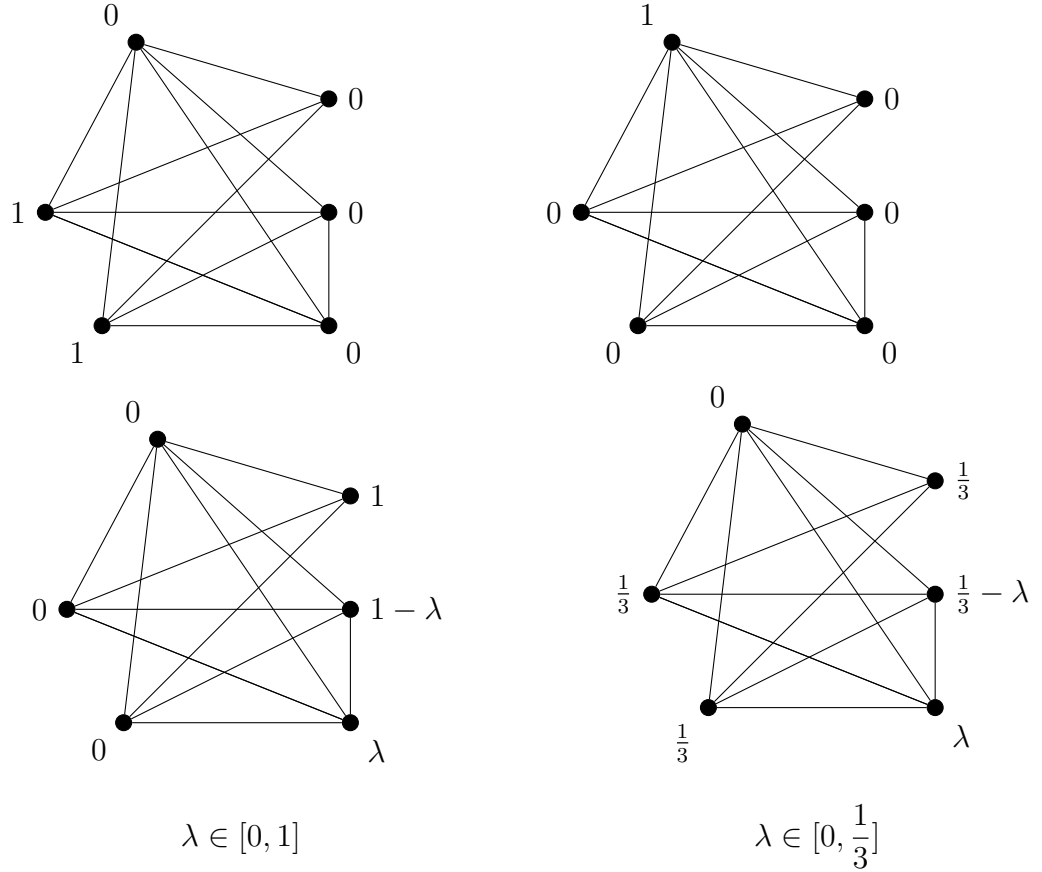


Figure 7: Nash equilibria of a cograph network.

## 7. CONCLUSION

Understanding, and making sense of large economic networks is an increasingly important problem from an economic perspective due to the ever-widening gap between technological advances in constructing such networks, and our ability to predict and estimate their properties. Our finding shows that, by exploiting network position similarity, it can be possible to decompose a large network into subnetworks that can be analyzed independently. This could potentially have empirical applications to many large network

models in economics, including public goods and targeting/finding the key players policies as in Galeotti, Golub, and Goyal (2020).

Throughout history, various concepts have been developed to reduce the inherent complexity found in large economic systems, thereby rendering them more amenable to economic analysis. One prominent example is aggregation, which aims to devise representative concepts that can be analyzed in a more tractable manner. For instance, a key question, which appeared in the seminal contributions of von Neumann and Morgenstern (1944), Chapter IX, Gorman (1953), and Shapley (1967), is: when does a group of individuals behave as if it were a single individual? Our investigation of aggregation in network games is quite similar in motivation. Often, the reason such an argument holds in the above literature appears to hinge on having identical preferences or accessibility as in Cornes and Hartley (2007). Our approach suggests that aggregation holds for a similar reason in network games; however, the homogeneity is brought about by the network architecture rather than economic characteristics.

Finally, it remains to be seen whether other approaches from the vast and important literature on network position similarity, across myriad disciplines, ranging from biology and sociology to computer science—see, for example, Gagneur et al. (2004) and Newman (2006)—could be useful to further analyze complex economic interactions.

## 8. APPENDIX

**Proof of Theorem 1.** First, we will prove given a modular partition  $\mathbf{p} = \{M_1, \dots, M_K\}$ , then (1) and (2) are equivalent. Observe that a profile of actions  $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  if and only if for each player  $i \in N$

$$x_i^* = \begin{cases} 1 - \delta x_{\mathcal{N}_i(\mathbf{g})}^* & \text{if } \delta x_{\mathcal{N}_i(\mathbf{g})}^* \leq 1 \\ 0 & \text{if } \delta x_{\mathcal{N}_i(\mathbf{g})}^* > 1. \end{cases} \quad (8.1)$$

Since  $M_k$  is a module, for each  $i \in M_k$  and for each  $h \neq k$ , it holds that the set of neighbors of  $i$  in  $M_h$ , that is,  $\mathcal{N}_i(\mathbf{g}_{M_h})$ , is independent of the choice of  $i \in M_k$ . Let us posit

$$r_k^* \stackrel{\text{def}}{=} \max\{1 - \delta \sum_{h \in \mathcal{N}_k(\mathcal{G}_{\mathbf{p}})} x_{\mathcal{N}_i(\mathbf{g}_{M_h})}^*, 0\}.$$

Then, since for each  $i \in M_k$

$$\mathcal{N}_i(\mathbf{g}) = \bigcup_{h \in k \cup \mathcal{N}_k(\mathcal{G}_{\mathbf{p}})} \mathcal{N}_i(\mathbf{g}_{M_h}),$$

it holds that

$$\delta x_{\mathcal{N}_i(\mathbf{g})}^* = \begin{cases} \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* - r_k^* + 1 & \text{if } r_k^* > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (8.2)$$

Also let

$$\mathbf{y}_{M_k}^* \stackrel{\text{def}}{=} \begin{cases} \frac{\mathbf{x}_{M_k}^*}{r_k^*} & \text{if } r_k^* > 0 \\ \text{a Nash equilibrium of } \Gamma(\mathbf{g}_{M_k}, \delta) & \text{otherwise.} \end{cases}$$

Hence, in view of (8.1) and (8.2),  $\mathbf{x}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  if and only if for each module  $k = 1, \dots, K$

$$r_k^* = \max\{1 - \delta \sum_{h \in \mathcal{N}_h(\mathcal{G}_{\mathbf{p}})} y_{\mathcal{N}_i(\mathbf{g}_{M_h})}^* r_h^*, 0\}$$

and for each player  $i \in M_k$  it holds that

$$x_i^* = \begin{cases} r_k^* - \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* & \text{if } \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* \leq r_k^* \\ 0 & \text{if } \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* > r_k^* \end{cases}$$

or, equivalently,

$$y_i^* = \begin{cases} 1 - \delta y_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* & \text{if } \delta y_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* \leq 1 \\ 0 & \text{if } \delta y_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* > 1. \end{cases}$$

Therefore,  $\mathbf{x}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}, \delta)$  if and only if  $\mathbf{x}^* = (r_1^* \mathbf{y}_{M_1}^*, \dots, r_K^* \mathbf{y}_{M_K}^*)$  such that  $\mathbf{r}^*$  is a Nash equilibrium of  $\Gamma(\mathcal{G}_{\mathbf{p}}, \delta; y_{M_1}^*, \dots, y_{M_K}^*)$  and  $\mathbf{y}_{M_k}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$ , for each  $k = 1, \dots, K$ .

Conversely, we will prove that if (1) and (2) are equivalent, then the partition  $\mathbf{p} = \{M_1, \dots, M_K\}$  is modular. Let's consider  $\delta$  small enough so that all players are active. Then, from (8.1) it holds that

$$x_i^* = 1 - \delta x_{\mathcal{N}_i(\mathbf{g})}^* = 1 - \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* - \delta \sum_{h \in \mathcal{N}_k(\mathcal{G}_{\mathbf{p}})} x_{\mathcal{N}_i(\mathbf{g}_{M_h})}^*.$$

Suppose that  $i$  is not connected in all players in  $M_{h_0}$  for some  $h_0 \in \mathcal{N}_k(\mathcal{G}_{\mathbf{p}})$ . Then, it holds that  $y_{\mathcal{N}_i(\mathbf{g}_{M_{h_0}})}^* < y_{M_{h_0}}^*$ . Therefore,

$$x_i^* = 1 - \delta x_{\mathcal{N}_i(\mathbf{g})}^* = 1 - \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^* - \delta \sum_{h \in \mathcal{N}_k(\mathcal{G}_{\mathbf{p}})} x_{\mathcal{N}_i(\mathbf{g}_{M_h})}^* < r_k^* - \delta x_{\mathcal{N}_i(\mathbf{g}_{M_k})}^*,$$

which implies  $y_i^* < 1 - \delta y_{\mathcal{N}_i(\mathbf{g}_{M_k})}^*$ . This is a contradiction to  $\mathbf{y}_{M_k}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$ .  $\square$

**Proof of Theorem 2.** The proof relies on the Brouwer and Haemers (2011) version of the generalized interlacing eigenvalue theorem, as stated below.

**Theorem.** (*Brouwer and Haemers*). Let  $\mathbf{S}$  be a real  $n \times m$  matrix such that  $\mathbf{S}^T \mathbf{S} = \mathbf{I}$ . Let  $\mathbf{B}$  be a real symmetric matrix of order  $n$ . Define  $\mathbf{J} = \mathbf{S}^T \mathbf{B} \mathbf{S}$ . Then the eigenvalues of  $\mathbf{J}$  interlace those of  $\mathbf{B}$ .

Note that the classical interlacing eigenvalue theorem holds as a special case of the generalized interlacing eigenvalue theorem if one takes  $\mathbf{S} = [\mathbf{I}, \mathbf{0}]$ . The Nash equilibrium  $\mathbf{x}^*$  is stable if and only if all eigenvalues of the matrix  $\mathbf{I} + \delta \mathbf{G}$  have positive real parts, which is equivalent to  $\lambda_{\min}(\mathbf{G}) > -\frac{1}{\delta}$  since all the eigenvalues of  $\mathbf{G}$  are real.

First observe that from the classical interlacing eigenvalue theorem it follows that  $\lambda_{\min}(\mathbf{G}_{M_k}) \geq \lambda_{\min}(\mathbf{G}) > -\frac{1}{\delta}$ , for each for each module  $k = 1, \dots, K$ . Therefore  $\mathbf{y}_{M_k}^*$  is stable, for each module  $k = 1, \dots, K$ .

Second observe that rows and columns of  $\mathbf{G}$  can be partitioned as

$$\mathbf{G} = (\mathbf{G}_{kh})_{1 \leq k, h \leq K}$$

where  $\mathbf{G}_{kh}$  lists the links connecting players in  $M_k$  to players in  $M_h$ .<sup>10</sup> Let  $\mathbf{S}$  be the  $n \times K$  matrix defined as follows:

$$s_{ik} \stackrel{\text{def}}{=} \begin{cases} \frac{y_i^*}{\|\mathbf{y}_{M_k}^*\|} & \text{if } i \in M_k \\ 0 & \text{otherwise} \end{cases},$$

$$\mathbf{U} = \text{diag}\left(\frac{\|\mathbf{y}_{M_k}^*\|}{\sqrt{y_{M_k}^*}}\right)_{k \in K},$$

and

$$\mathbf{V} = \text{diag}(\sqrt{y_{M_k}^*})_{k \in K}.$$

Since  $(\mathbf{I} + \delta \mathbf{G}_{kk}) \mathbf{y}_{M_k}^* = \mathbf{1}$  and

$$\mathbf{G}_{kh} \mathbf{y}_{M_k}^* = \begin{cases} y_{M_k}^* \mathbf{1} & \text{if } \mathbf{G}_{kh} \neq \mathbf{0}, \\ \mathbf{0} & \text{otherwise} \end{cases},$$

it follows that

$$\mathbf{V} \mathbf{U}^T (\mathbf{I} + \delta \mathbf{G}) \mathbf{S} \mathbf{U} \mathbf{V}^{-1} = \mathbf{I} + \delta \mathbf{G}_{\mathbf{p}},$$

where  $\mathbf{G}_{\mathbf{p}}$  denotes the adjacencies of modules in the Nash equilibrium  $\mathbf{r}^*$ .

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<sup>10</sup>Observe that  $\mathbf{G}_{kk} = \mathbf{G}_{M_k}$ .

Observe that  $\mathbf{S}^T \mathbf{S} = \mathbf{I}$ . Hence, it follows from the generalized interlacing eigenvalue theorem that the eigenvalues of  $\mathbf{S}^T(\mathbf{I} + \delta \mathbf{G})\mathbf{S}$  are positive since they interlace those of  $\mathbf{I} + \delta \mathbf{G}$ . From the sharp bounds provided by Ostrowski (1959), it holds that the eigenvalues of the symmetric matrix  $\mathbf{U} \mathbf{S}^T(\mathbf{I} + \delta \mathbf{G})\mathbf{S} \mathbf{U}$  are also positive since they are given by  $\psi_i \lambda_i$ , where  $\lambda_i$  is an eigenvalue of  $\mathbf{S}^T(\mathbf{I} + \delta \mathbf{G})\mathbf{S}$  and  $\psi_i$  lies between the smallest and the largest eigenvalues of  $\mathbf{U}$ . Hence, the matrix  $\mathbf{I} + \delta \mathbf{G}_{\mathbf{p}}$  also has positive eigenvalues, being similar to  $\mathbf{U} \mathbf{S}^T(\mathbf{I} + \delta \mathbf{G})\mathbf{S} \mathbf{U}$ . Therefore,  $\mathbf{r}^*$  is a stable Nash equilibrium.  $\square$

**Proof of Proposition 1.** If  $M$  is parallel, then  $\mathbf{r}_M^*$  is a Nash equilibrium of the intergroup game  $\Gamma(\mathcal{G}_{p_M}, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$  is equivalent to  $r_t^* = 1$  since  $\mathcal{N}_t(\mathcal{G}_{p_M}) = \emptyset$ .  $\square$

**Proof of Proposition 2.** If  $M$  is a series module, then  $\mathbf{r}_M^*$  is a Nash equilibrium of the intergroup game  $\Gamma(\mathcal{G}_{p_M}, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$  is equivalent to

$$r_t^* = 1 - \delta \sum_{s \in A \setminus \{t\}} y_{D_s}^* r_s^* \text{ for each } t \in A_M \quad (8.3)$$

and

$$\delta \sum_{s \in A} y_{D_s}^* r_s^* \geq 1 \text{ if } A_M \neq \{1, \dots, T\}. \quad (8.4)$$

Let

$$\mathbf{v} \stackrel{\text{def}}{=} \left( \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*} \right)_{s \in A_M} \text{ and } \mathbf{U} \stackrel{\text{def}}{=} \text{diag}(1 - \delta y_{D_s}^*)_{s \in A_M}.$$

Then (8.3) is equivalent to

$$(\mathbf{I} + \mathbf{1} \mathbf{v}^T) \mathbf{U} \mathbf{r}_{A_M}^* = \mathbf{1}.$$

From the Sherman–Morrison formula, provided that  $1 + \mathbf{v}^T \mathbf{1} \neq 0$ , it holds that

$$\mathbf{r}_{A_M}^* = \mathbf{U}^{-1}(\mathbf{I} + \mathbf{1} \mathbf{v}^T)^{-1} \mathbf{1} = \mathbf{U}^{-1} \left( \mathbf{I} - \frac{\mathbf{1} \mathbf{v}^T}{1 + \mathbf{v}^T \mathbf{1}} \right) \mathbf{1} = \mathbf{U}^{-1} \left( \mathbf{1} - \frac{\mathbf{v}^T \mathbf{1}}{1 + \mathbf{v}^T \mathbf{1}} \mathbf{1} \right) = \frac{1}{1 + \mathbf{v}^T \mathbf{1}} \mathbf{U}^{-1} \mathbf{1}.$$

Hence, for each  $t \in A_M$ , it holds that

$$r_t^* = \frac{\frac{1}{1 - \delta y_{D_t}^*}}{1 + \sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}}.$$

Note that since  $r_t^* > 0$  for each  $t \in A_M$ , it follows from above that either  $y_{D_t}^* > \frac{1}{\delta}$  for each  $t \in A_M$  or  $y_{D_t}^* < \frac{1}{\delta}$  for each  $t \in A_M$ . Moreover, in view of (8.4), if  $A_M \neq \{1, \dots, T\}$  then

$$\frac{\sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}}{1 + \sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}} = 1 - \frac{1}{1 + \sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}} \geq 1,$$

which implies that

$$\sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*} < -1.$$

Hence if  $A_M \neq \{1, \dots, T\}$ , then it holds that  $y_{D_t}^* > \frac{1}{\delta}$  for each  $t \in A_M$ .

Conversely, it is easy to check that if either for each  $t \in A_M$  it holds that  $y_{D_t}^* > \frac{1}{\delta}$ , or for each  $t \in A_M = \{1, \dots, T\}$  it holds that  $y_{D_t}^* < \frac{1}{\delta}$ , and

$$r_t^* = \frac{\frac{1}{1 - \delta y_{D_t}^*}}{1 + \sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*}},$$

then  $\mathbf{r}_M^*$  is a Nash equilibrium of  $\Gamma(\mathcal{G}_{p_M}, \delta; y_{D_1}^*, \dots, y_{D_T}^*)$ .  $\square$

**Proof of Corollary 1.** (i) and (ii) follow immediately from Proposition 2.

(ii) Note that

$$x_M^* = \sum_{t \in A_M} r_t^* y_{D_T}^* = \frac{\sum_{t \in A_M} \frac{\delta y_{D_t}^*}{1 - \delta y_{D_t}^*}}{\delta(1 + \sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*})} = \frac{1}{\delta} - \frac{1}{\delta(1 + \sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*})}.$$

Hence if  $A_M \subset A'_M$ , then it holds that

$$\sum_{s \in A'_M} \frac{\delta y'_{D_s}}{1 - \delta y'_{D_s}} < \sum_{s \in A_M} \frac{\delta y_{D_s}^*}{1 - \delta y_{D_s}^*} < 0,$$

which implies  $x'_M \leq x_M^*$ .  $\square$

**Proof of Corollary 2.** Suppose not. Then, there is a stable Nash equilibrium with at least two over-threshold descendants:  $D_t$  and  $D_s$ . From Theorem 2, it follows that the intergroup game is stable, which implies that the eigenvalues of the quotient network induced by active modules are positive. By the classical interlacing eigenvalues theorem, the eigenvalues of

$$\mathbf{C} = \begin{pmatrix} 1 & \delta y_{D_t}^* \\ \delta y_{D_s}^* & 1 \end{pmatrix}.$$

interlace the eigenvalues of the quotient network induced by the active modules. Note that the lowest eigenvalue of  $\mathbf{C}$  is negative, which is a contradiction.  $\square$

**Proof of Theorem 3.** Consider a maximal independent partition  $(M_k)_{k \in \mathcal{I}}$  of the cograph and a profile  $\mathbf{x}^* = ((\mathbf{x}_{M_k}^*)_{k \in \mathcal{I}}, \mathbf{0})$ , where  $\mathbf{x}_{M_k}^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_{M_k}, \delta)$ . Since every over-threshold descendant is minimal it follows that for each  $M_k$  using Propositions 1 and 2 recursively along the nodes of the modular decomposition tree that the Nash

equilibrium  $\mathbf{x}_{M_k}^*$  of  $\Gamma(\mathbf{g}_{M_k}, \delta)$  is unique and all players are active. Clearly, each player  $i \in \cup_{k \in \mathcal{I}} M_k$  is optimally responding to his neighbors' actions. Moreover, each player  $i \notin \cup_{k \in \mathcal{I}} M_k$  is connected to all players in at least one active minimal over-threshold descendant otherwise  $(M_k)_{k \in \mathcal{I}}$  is not a maximal independent partition. Hence he has access to a provision of public goods that exceeds  $\frac{1}{\delta}$ . Therefore, his optimal best-response is zero. Finally, from Corollary 2 in Bramoullé, Kranton, and D'Amours (2014) it follows that the unique Nash equilibrium  $\mathbf{x}_{M_k}^*$  of  $\Gamma(\mathbf{g}_{M_k}, \delta)$  is stable. Hence the eigenvalues of the Jacobian matrix associated with  $\mathbf{x}_{M_k}^*$  are all negative. The stability of  $\mathbf{x}^* = ((\mathbf{x}_{M_k}^*)_{k \in \mathcal{I}}, \mathbf{0})$  follows from the fact that the eigenvalues of the Jacobian matrix associated with  $\mathbf{x}^*$  are negative since they are the union of the eigenvalues of the Jacobian matrices associated with  $(\mathbf{x}_{M_k}^*)_{k \in \mathcal{I}}$ .  $\square$

**Proof of Proposition 3.** Note that a caterpillar may be formed by repeatedly adding one player, either connected to all previous players, which makes him a descendant of a series module, or to none of them, which makes him a descendant of a parallel module. Hence, traversing the caterpillar bottom-up, the degree sequence is decreasing for single players descendants of parallel modules and is increasing for single players descendants of series modules. Note also that the the action of each player is decreasing if descendant of series modules and increasing. if descendant of parallel modules. Finally, observe that if there exists a minimal over-threshold descendant, then all active descendants will have a degree below the maximal degree in the minimal over-threshold descendant.  $\square$

**Proof of Proposition 4.** Let  $\mathbf{x}^*$  be a Nash equilibrium of  $\Gamma(\mathbf{g}, 1)$  such that  $x_i^* > 0$ , for each  $i \in N$ . Let  $M$  be a series module. From Theorem 1, there exists a real number  $r_M > 0$  such that  $\mathbf{x}^* = r_M \mathbf{y}_M^*$ , where  $\mathbf{y}_M^*$  is a Nash equilibrium of  $\Gamma(\mathbf{g}_M, 1)$ . Suppose that  $M$  is not uncentered. Let  $\mathbf{p}_M = (D_1, \dots, D_T)$  denote the descendants' partition of  $M$ . Then, there exists  $1 \leq t_1 \neq t_2 \leq T$  such that  $D_{t_1} = \{i_1\}$  is a single player and  $D_{t_2}$  is not a single player. Note that each player in  $D_{t_2}$  is not connected to all other players in  $D_{t_2}$ . Otherwise,  $D_{t_2}$  is not a direct descendant of  $M$ .

At the Nash equilibrium  $\mathbf{y}_M^*$ , each player's action is a best reply to his neighbors' actions. In particular, it holds for player  $i_1$  that

$$y_{i_1}^* + \sum_{i \in D_{t_2}} y_i^* + \sum_{t \neq t_1, t_2} y_{D_t}^* = 1$$

and for a player  $i_2 \in D_{t_2}$  that

$$y_{i_2}^* + \sum_{i \in \mathcal{N}_{i_2}(\mathbf{g}_M) \cap D_{t_2}} y_i^* + y_{i_1}^* + \sum_{t \neq t_1, t_2} y_{D_t}^* = 1,$$

which together imply

$$\sum_{i \in \{i_2 \cup \mathcal{N}_{i_2}(\mathbf{g}_M)\}^c \cap D_{t_2}} y_i^* = 0.$$

This is a contradiction since  $\{i_2 \cup \mathcal{N}_{i_2}(\mathbf{g}_M)\}^c \cap D_{t_2} \neq \emptyset$  and  $y_i^* > 0$ , for each  $i \in M$ .  $\square$

**Proof of Proposition 5.** Suppose the network  $\mathbf{g}$  is a cograph. Therefore, the network  $\mathbf{g}$  has only parallel and series modules in its modular decomposition tree. If all series module are uncentered, then, given a series module  $M$ , with direct descendants' partition  $\mathbf{p}_M = (D_1, \dots, D_T)$ , either all or none of the direct descendants are single players. If all direct descendants are single players, then the symmetric contribution  $\frac{1}{T+1}$  is a Nash equilibrium of  $\mathbf{\Gamma}(\mathbf{g}_M, 1)$ . If none of  $M$ 's direct descendants is a single player, then for each  $t = 1, \dots, T$  and for any Nash equilibrium  $\mathbf{y}_t^*$  of  $\mathbf{\Gamma}(\mathbf{g}_{D_t}, \delta)$ , it holds that  $y_{D_t}^* \geq 2$  since  $D_t$  is a parallel module with at least two direct descendants. From (2)(iii) in Theorem 2, it follows that there exists a Nash equilibrium of the quotient game such that  $r_t^* > 0$  for each  $t = 1, \dots, T$ . Therefore one can use in Propositions 1 and 2 recursively along the nodes of the modular decomposition tree in order to construct a distributed equilibrium.  $\square$

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