

Sproule, Robert

Working Paper

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Suggested Citation: Sproule, Robert (2022) : On The History and The Mathematics of The Wold-Juréen (1953) Utility Function, And Its Basis For The Modeling of Giffen Behavior, ZBW - Leibniz Information Centre for Economics, Kiel, Hamburg

This Version is available at:

<https://hdl.handle.net/10419/263912>

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On The History and The Mathematics of The Wold-Juréen (1953) Utility Function, And Its Basis For The Modeling of Giffen Behavior

Robert Sproule¹

Department of Economics, Bishop's University, Sherbrooke, Québec, J1M 1Z7, Canada

Monday, September 5, 2022

Abstract: Twenty years ago, Peter Moffatt (2002) posed this general question: “Is Giffen behavior compatible with the axioms of consumer theory?” The present paper addresses this very same question, but only as it applies to the Wold-Juréen (1953) utility function. In this paper, we demonstrate that the Wold-Juréen (1953) utility function exhibits both Giffenity and compatibility with the axioms of consumer theory. Our singular interest in the Wold-Juréen (1953) utility function stems from the fact that this utility function continues to be the preeminent theoretical benchmark in the study of Giffenity.

Keywords: Axioms of Consumer Theory, Wold-Juréen (1953) utility function, Slutsky decomposition, Giffen Behavior

JEL Classification: A22, A23, D11

Word Count: 2,821

¹ The author can be reached at ra_sproule@shaw.ca.

1. Introduction

Twenty years ago, Peter Moffatt (2002) posed this general question: “Is Giffen behaviour compatible with the axioms of consumer theory?” The goal of the present paper is revisit Moffatt’s question, but only as it applies to the Wold-Juréen (1953) utility function. In particular, the goal of the present paper is to demonstrate that the Wold-Juréen (1953) utility function can exhibit Giffenity and that it is compatible with the axioms of consumer theory. Our singular interest in the Wold-Juréen (1953) utility function stems from the fact that this utility function continues to be the preeminent theoretical benchmark in the study of Giffenity.

This paper is organized as follows. In Section 2, we offer an overview to six historical facts about the study of Giffenity and the role played by the Wold-Juréen (1953) utility function. All six of these facts occurred in the period 1997-2020, and two of these six are related to comments made by Moffatt (2002 and 2011). In Section 3, we define the Wold-Juréen (1953) utility function, and then we enumerate some of its essential properties. In Section 4, we offer an overview to two papers which define Giffen behavior for the Wold-Juréen (1953) utility function. These two are the papers by Christian Weber (1997) and Robert Sproule (2020). In Section 5, we demonstrate that the Wold-Juréen (1953) utility function is compatible with the axioms of consumer theory. We offer our concluding remarks in Section 6, and these are: (a) that the Wold-Juréen (1953) utility function is capable of exhibiting Giffen behavior, and (b) that the Wold-Juréen (1953) utility function is compatible with the axioms of consumer theory.

2. The History: The Wold-Juréen (1953) Utility Function As A Basis For Modeling Giffen Behavior

In this section, we offer an overview to six historical facts about the role played by the Wold-Juréen (1953) utility function in the study of Giffenity. All six of these facts occurred during the period 1997 -2020.

- **Fact 1:** In 2002, Peter Moffatt published a paper entitled, “Is Giffen behaviour compatible with the axioms of consumer theory?” In it, he wrote: “One of the most familiar pathological situations described in the microeconomics literature is the ‘Giffen’ paradox, in which the quantity demanded of a good rises in response to an increase in its own price. ... Over the years, several authors have illustrated the Giffen paradox by specifying particular direct utility functions, e.g. Wold and Jureen (1953), Vandermeulen (1972), Spiegel (1994), Weber (1997, 2001) and Butler and Moffatt (2000). A common feature of all of these proposed utility functions is that they are not globally quasi-concave on the positive quadrant: each one exhibits a region in which the indifference curves are concave to the origin.” Moffatt then went on to define a new class of utility functions which is globally quasi-concave on the positive quadrant and which exhibits Giffenity. The reader should note that in Moffatt’s list of those authors who “have illustrated the Giffen paradox by specifying particular direct utility functions” there are two documents of special interest to us, Wold and Jureen (1953) and Weber (1997).
- **Fact 2:** In 2011, Wim Heijman and Pierre van Mouche (2011a) released an edited collection of contributions with the title, “New Insights into the Theory of Giffen Behaviour”. The sole motivation for this tome was the further exploration and study of Giffen behavior. In the introductory chapter, Heijman and van Mouche (2011b) reported that the genesis for their book sprang from a solitary question -- one which was posed by one of the volume’s contributors, Peter Moffatt. That one question is this: “(T)o what extent are Giffen goods, for a single consumer, theoretically possible in the neoclassical framework of utility maximization under a budget restriction?” [Heijman and van Mouche (2011b, pages 1-2)]. In some sense, this question resonates with the question posed by Moffatt in 2002, namely: “Is Giffen behaviour compatible with the axioms of consumer theory?”

- **Fact 3:** One major object-of-interest within the volume edited by Heijman and van Mouche (2011a) is the particular utility function due to Herman Wold and Lars Juréen -- a utility function which we refer to as the Wold-Juréen (1953) utility function.² The reason for this interest is that (over the course of nearly seventy years) the Wold-Juréen (1953) utility function has become the preeminent theoretical benchmark in the study of Giffen behavior – so much so, that Sproule (2020, page 2) proclaimed the Wold-Juréen (1953) utility function as the progenitor of all theoretical research on Giffenity, (a) which use a two-good utility functions, and (b) which have the potential to exhibit Giffen behavior.
- **Fact 4:** Sproule’s proclamation represents no more than a crystallization of a related statement by Peter Moffatt (2011): Moffatt wrote: “Ever since Wold and Jureen’s attempt to illustrate the Giffen paradox by specifying a particular direct utility function, there has been a stream of contributions from authors pursuing similar objectives, for example Spiegel (1994), Weber (1997 and 2001), Moffatt (2002), and Sørensen (2007). One of the lessons learned from this strand of literature is that it is not easy to specify a direct utility function that predicts ‘Giffen behaviour’ and simultaneously satisfies the basic axioms of consumer theory.” [Moffatt (2011, page 127)]. Once again, we have a statement by Moffatt which resonates with the question posed by Moffatt in 2002, namely: “Is Giffen behaviour compatible with the axioms of consumer theory?”
- **Fact 5:** While the Wold-Juréen (1953) utility function has become pivotal to the study of Giffen behavior, so too has a related paper by Weber (1997), in that Weber’s paper provides the first thorough enumeration of the core properties of the Wold-Juréen (1953) utility function.
- **Fact 6:** Since the appearance of the tome by Heijman and van Mouche (2011a), and the prior appearance of the paper by Weber (1997), no other significant

² For an auto-biographical sketch of the academic career of the lead author, Herman Ole Andreas Wold (1908-1992), see Gani (1982, pages 189-212).

contribution to this literature has been made; that is, until the appearance of the paper by Sproule (2020). In this paper, Sproule (2020) presented two arguments:

- **Argument 1:** Using the Wold-Juréen (1953) utility function, Weber (1997) demonstrated that the Giffenness of the inferior good (Good 1) depends upon the relative magnitudes of the decision maker's (hereafter DM) income vis-à-vis the price of the normal good (Good 2). In particular, Weber stated that: "Giffen behavior is more likely for higher ... incomes" and that the Giffenness of Good 1 "is more likely at lower values of the price for Good 2" [Weber (1997, page 40)] In response to these two claims, Sproule (2020, page 2) stated: "Weber's precondition is so vague that it lacks broad appeal" and that Weber's precondition does not accord "with a core tenet of microeconomics, which is that economic decision-making is predicated on (changes in) relative prices."
- **Argument 2:** Sproule (2020) then went on to offer a new precondition for Giffen behavior under the Wold-Juréen (1953) utility function – one which accords with the core tenet of microeconomics regarding the import of relative prices in the decision-making process. In particular, Sproule's (2020) precondition is this: if the DM has the Wold-Juréen (1953) utility function, and if the price of the inferior good (Good 1) is greater than or equal to the price of the normal good (Good 2), then the inferior good (Good 1) is a Giffen good.

3. The Mathematics: The Definition of The Wold-Juréen (1953) Utility Function and Its Key Properties

In this section, we define the Wold-Juréen (1953) utility function, and then we enumerate some of its key properties.

Remark 1 -- The Wold-Juréen (1953) Utility Function Defined: Following Wold and Juréen (1953), Weber (1997), and Sproule (2020), the Wold-Juréen (1953) utility function is defined as

$$U = (x_1 - 1)(x_2 - 2)^{-2} \quad (1)$$

where U denotes the DM's utility level, where x_1 denotes the quantity of Good 1, where x_2 denotes the quantity of Good 2, and where (by assumption) $x_1 > 1$ and $0 < x_2 < 2$

Remark 2 – An Enumeration of Key Properties of The Wold-Juréen (1953) Utility Function: Several properties about the Wold-Juréen (1953) utility function are pertinent to the present discussion. These properties are contained below in Propositions 1 and 2, and both Propositions 1 and 2 are due to Weber (1997):

Proposition 1 [Weber (1997, page 39)]: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, then

$$\frac{\partial U}{\partial x_1} = U_1 = (x_2 - 2)^{-2} > 0 \quad (2)$$

$$\frac{\partial U}{\partial x_2} = U_2 = -2(x_1 - 1)(x_2 - 2)^{-3} > 0 \quad (3)$$

$$\frac{\partial^2 U}{\partial x_1^2} = U_{11} = 0 \quad (4)$$

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} = U_{12} = -2(x_2 - 2)^{-3} > 0 \quad (5)$$

$$\frac{\partial^2 U}{\partial x_2^2} = U_{22} = 6(x_1 - 1)(x_2 - 2)^{-4} > 0 \quad (6)$$

Proposition 2 [Weber (1997, page 39)]: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, then the sign of the bordered Hessian determinant,

$$|\bar{B}| = \begin{vmatrix} 0 & U_1 & U_2 \\ U_1 & U_{11} & U_{12} \\ U_2 & U_{12} & U_{22} \end{vmatrix}$$

is positive.

Proof: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, then

$$\begin{aligned} |\bar{B}| &= \begin{vmatrix} 0 & U_1 & U_2 \\ U_1 & U_{11} & U_{12} \\ U_2 & U_{12} & U_{22} \end{vmatrix} \\ &= \begin{vmatrix} 0 & (x_2 - 2)^{-2} & -2(x_1 - 1)(x_2 - 2)^{-3} \\ (x_2 - 2)^{-2} & 0 & -2(x_2 - 2)^{-3} \\ -2(x_1 - 1)(x_2 - 2)^{-3} & -2(x_2 - 2)^{-3} & 6(x_1 - 1)(x_2 - 2)^{-4} \end{vmatrix} \\ &= 2(x_1 - 1)(x_2 - 2)^{-8} > 0 \quad \bullet \end{aligned}$$

4. The Mathematics: The Wold-Juréen (1953) Utility Function In The Modeling of Giffen Behavior

In this section, we offer an overview to two papers which define Giffen behavior for the Wold-Juréen (1953) utility function. These papers are: Weber (1997) and Sproule (2020). But before we discuss these in detail, we must first offer the following conceptual preliminaries:

4.1. A Few Conceptual Preliminaries: Following the lead of Weber (1997) and Sproule (2020), both of whom use the Wold-Juréen (1953) utility function, the choice-theoretical framework for the present discussion begins with the following Lagrange function:

$$\text{maximize}_{x_1, x_2, \lambda} L = U(x_1, x_2) + \lambda \left(p_1 x_1 + p_2 x_2 - m \right) \quad (7)$$

where $U(x_1, x_2) = (x_1 - 1)(x_2 - 2)^2$ such that $x_1 > 1$ and $0 < x_2 < 2$, where p_1 denotes the price of the inferior good (Good 1), p_2 denotes the price of the normal good (Good 2), and m denotes the DM's income such that $m = p_1 x_1 + p_2 x_2$.

The solution to Equation (7) requires that the first-order conditions (hereafter FOCs) for an interior maximum hold, and that the second-order condition (hereafter SOC) for an interior maximum also holds. If both sets of conditions hold, then it follows that there exists a Marshallian demand function for both Goods 1 and 2. Since Proposition 2 above is tantamount to the SOC for an interior maximum, then it is clear that the Marshallian demand functions for both Goods 1 and 2 exist and that these demand functions take the form of:

$$x_1^* = x_1^*(p_1, p_2, m) = 2 + \frac{2p_2 - m}{p_1} \quad (8)$$

$$x_2^* = x_2^*(p_1, p_2, m) = 2 \left(\frac{m - p_1}{p_2} - 1 \right) \quad (9)$$

Remark 3: Equation (8) represents the Marshallian demand function for the inferior good (Good 1), when the DM's utility function is the Wold-Juréen (1953) utility function.

Remark 4: Equation (9) represents the Marshallian demand function for the normal good (Good 2), when the DM's utility function is the Wold-Juréen (1953) utility function.

4.2. The Wold-Juréen (1953) Utility Function Can Exhibit Giffenity: We turn next to Weber's (1997) precondition for Giffenity, which is then followed by Sproule's (2020) precondition. Weber's (1997) precondition for Giffenity is defined next Proposition 3.

Proposition 3 [Weber (1997, page 40)]: If the DM has the Wold-Juréen (1953) utility function, then Good 1 is a Giffen good if and only if $m - 2p_2 > 0$; that is, if m is “relatively large” and if p_2 is “relatively small”.

Proof: (a) $\text{Sign}\left(\frac{\partial x_1^*}{\partial p_1}\right) = \text{sign}(m - 2p_2)$ [Weber (1997, page 40)].

(b) If m is “relatively large” and if p_2 is “relatively small”, then $\frac{\partial x_1^*}{\partial p_1} > 0$ and Good 1 is a Giffen good [Weber (1997, page 40)]. •

As stated in Proposition 3, the essence of Weber’s (1997) precondition for Giffenity is that the DM’s income be “relatively large” and the price of Good 2 be “relatively small”. In contrast, Sproule’s (2020) precondition for Giffenity is defined below in Proposition 4. Sproule’s precondition is that the price of the inferior good (Good 1) be greater than or equal to the price of the normal good (Good 2). In order to define Proposition 4, we must first entertain the following three lemmas, all of which are due to Sproule:

Lemma 1 [Sproule (2020)]: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, then $m < p_1 + 2p_2$.

Lemma 2 [Sproule (2020)]: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, then $p_1 + p_2 < m$

Lemma 3 [Sproule (2020)]: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, then (by Lemmas 1 and 2) $p_1 - p_2 < m - 2p_2 < p_1$.

Proposition 4 [Sproule (2020)]: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, and if $p_1 \geq p_2$, then Good 1 is a Giffen good.

Proof: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, and if $p_1 \geq p_2$, then

$$m - 2p_2 > 0 \text{ [Lemma 3] and } \frac{\partial x_1^*}{\partial p_1} = \frac{m - 2p_2}{(p_1)^2} > 0 \bullet$$

In words, Proposition 4 states this: if the DM has the Wold-Juréen (1953) utility function, and if the price of the inferior good (Good 1) is greater than or equal to the price of the normal good (Good 2), then the inferior good (Good 1) is a Giffen good.

5. The Mathematics: The Wold-Juréen (1953) Utility Function and The Axioms of Consumer Theory

Having just shown that the Wold-Juréen (1953) utility function can exhibit Giffen behavior, the objective of this section is to complete our two-part argument, and that is by showing that the Wold-Juréen (1953) utility function is also compatible with the axioms of consumer theory. But before we do this, we begin by offering Moffatt's (2002) definition of the axioms of consumer theory for the arbitrary utility function.

5.1. Moffatt's (2002) Definition of the Axioms of Consumer Theory for the Arbitrary Utility Function:

Consider a DM, whose utility function is the two-good arbitrary utility function, $U = U(x_1, x_2)$, where U denotes the DM's utility level, x_1 denotes the quantity of Good 1, and x_2 denotes the quantity of Good 2. This then leads to Moffatt's definition of the axioms of consumer theory:

Proposition 5 [Moffatt (2002, page 260)]: The axioms of consumer theory and $U = U(x_1, x_2)$ are compatible: (a) if $U = U(x_1, x_2)$ is non-decreasing in Good 1 and non-decreasing in Good 2 [that is, if $\partial U / \partial x_1 = U_1 \geq 0$ and $\partial U / \partial x_2 = U_2 \geq 0$] and (b) if $U = U(x_1, x_2)$ is globally quasi-concave in the first quadrant [that is, if $dx_2/dx_1 \geq 0$ for all $x_1, x_2 \geq 0$].

5.2. On The Compatibility of the Axioms of Consumer Theory with The Wold-Juréen (1953) Utility Function:

Here we show that the particular two-good (the Wold-Juréen (1953) utility function) is compatible with the axioms of consumer theory over a subset of the first quadrant as defined by $x_1 > 1$ and $0 < x_2 < 2$. Consider

Proposition 6: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, then $U_1 > 0$,

$U_2 > 0$, and $dx_2/dx_1 > 0$ for all $x_1 > 1$ and $0 < x_2 < 2$.

Proof: If $U = (x_1 - 1)(x_2 - 2)^{-2}$, where $x_1 > 1$ and $0 < x_2 < 2$, then: (a) $U_1 = (x_2 - 2)^{-2} > 0$ [by Proposition 1]; (b) $U_2 = -2(x_1 - 1)(x_2 - 2)^{-3} > 0$ [by Proposition 1]; and (c) $dx_2/dx_1 = |\bar{B}|/(U_2)(p_2)^2 > 0$ [by virtue of Chiang and Wainwright (2005, page 377), and Propositions 1 and 2] •

6. Concluding Remarks

The intent of this paper was to provide a terse overview to the history and the mathematics of the Wold-Juréen (1953) utility function, as a basis for the modeling of Giffen behavior.

The original motivation of this paper was rooted in two objects-of-interest: (a) Moffatt's (2002) question, "Is Giffen behaviour compatible with the axioms of consumer theory?", and (b) the Wold-Juréen (1953) utility function. These two interests led us to the preparation of:

- Section 2, which offered our review of the history of the Wold-Juréen (1953) utility function and Giffen behavior from 1997 to 2020;
- Section 3, which offered our formal definition of the Wold-Juréen (1953) utility function, and its key properties;
- Section 4, which offered our review of two papers, which use the Wold-Juréen (1953) utility function as a basis for modeling Giffen behavior [viz., Weber (1997) and Sproule (2020)]; and

- Section 5, which offered our analysis of the compatibility of the Wold-Juréen (1953) utility function with the axioms of consumer theory.

It is the combination of all of these findings which leads to our summary conclusion; and that is this: the Wold-Juréen (1953) utility function is capable of exhibiting Giffen behavior, and it is compatible with the axioms of consumer theory.

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