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General Equilibrium and Dynamic Inconsistency

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General Equilibrium and Dynamic Inconsistency

Abstract

We study the role of expectations of naive agents in a general equilibrium version of the Ramsey model with quasi-hyperbolic discounting. When agents recognize others' naivete, as strongly suggested by empirical evidence, they revise consumption paths, correctly anticipating prices in a resulting sliding equilibrium (perfect foresight). When agents are unaware of others' naivete, as is typically assumed in the literature, they revise both consumption paths and price expectations (quasi-perfect foresight). We prove the existence of sliding equilibrium under perfect foresight for the class of isoelastic utility functions. We show that generically quasi-hyperbolic discounting matters for saving behavior: sliding equilibrium under perfect foresight is observationally equivalent to some optimal path in the standard Ramsey model if and only if utility is logarithmic. We compare sliding equilibria under different types of foresight and show that perfect foresight implies a higher saving rate, long-run capital stock, and consumption level than quasi-perfect foresight.

JEL-Codes: D150, D840, D910, E210, O400.

Keywords: quasi-hyperbolic discounting, time inconsistency, naivete, sliding equilibrium, perfect foresight, observational equivalence.

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1 Introduction

Most of us tend to revise and change our previously chosen course of action. Economists refer to this time-inconsistent behavior as *naivete*. Many examples of naive behavior such as procrastination are ubiquitous and empirically well-documented. Moreover, there is experimental evidence that while people are unaware of their own time inconsistency, they hold remarkably correct beliefs about others' time inconsistency.¹ This evidence suggests that naivete originates not so much from *unawareness* but rather from *self-deception*: people are naive not because they fail to realize that their preferences will change but due to overconfidence about their self-control. This distinction between the sources of naivete does not matter for the analysis of individual decision making but appears crucial for the analysis of general equilibrium. In a standard dynamic model without time inconsistency, equilibrium is based on perfect foresight: agents correctly anticipate the sequences of prices in the economy, that is, they form correct expectations about others' behavior. However, when decisions are time-inconsistent, the notion of perfect foresight in equilibrium is unclear. In this paper, we study and clarify the role of expectations of naive agents in a general equilibrium framework.

Specifically, this paper studies the behavior of naive agents in a general equilibrium version of the Ramsey model with quasi-hyperbolic discounting. We argue that self-deceptive naivete leads to perfect foresight about prices, while unawareness-related naivete leads to quasi-perfect foresight, as discussed in detail below. We define a sliding equilibrium path under perfect foresight (perfect sliding equilibrium, PSE), prove that within the class of isoelastic utility functions there exists a PSE starting from any initial state, and show that a non-stationary PSE is observationally equivalent to an optimal path in the standard Ramsey model *if and only if* utility is logarithmic. We also define a sliding equilibrium path under quasi-perfect foresight (quasi-perfect sliding equilibrium, QSE), and compare PSE with QSE for both the case with log-utility and Cobb–Douglas production technology and the case of stationary equilibria. Our results suggest that perfect foresight implies a higher saving rate and a higher long-run consumption level than quasi-perfect foresight.

Time-inconsistent behavior, as first recognized by Strotz (1955), is induced by non-constant (time-declining) discounting. Strotz shows that if an agent cannot commit to their actions, then the preferred path as viewed from any future date coincides with the initially preferred path only for an agent with constant (exponential) discounting. An agent with another type of discounting is time-

¹ See, e.g., the discussion and references in Ericson and Laibson, 2019.

inconsistent and has a new preferred path at each date. It is common to model time-inconsistent behavior employing the assumption of quasi-hyperbolic discounting (see, e.g., Phelps and Pollak, 1968; Laibson, 1996), which captures the empirical observation that discount rates are higher in the short run than in the long run (see, e.g., Ainslie, 1992; DellaVigna, 2009).

Pollak (1968) clarifies the ideas of Strotz and distinguishes between two types of agents. A *naive* agent revises their optimal path at each date, which results in a naive path.² A *sophisticated* agent recognizes their time inconsistency and acts strategically, which results in a time-consistent sophisticated path.³ There are supporting arguments for both approaches: naivete is closer to real-world decision making, while sophistication is more consistent with the standard notion of rationality (see, e.g., the discussion and references in Findley and Caliendo, 2014).

In the prior literature, the effects of quasi-hyperbolic discounting on consumption and savings decisions within a neoclassical growth model are studied for both naive and sophisticated agents. Phelps and Pollak (1968) characterize naive and sophisticated paths in a model with general isoelastic utility and linear production technology, and show that under log-utility both paths coincide. In another seminal contribution, Barro (1999) studies the Ramsey model with general time-declining discounting and log-utility, and provides a solution for an agent who revises their consumption path at each instant of time, that is, he implicitly assumes a naive agent and obtains a naive path.⁴

Two important features of the literature on time-inconsistent decision making are worth noting. First, most of the models focus on the agent's beliefs about their own future behavior. The formation of expectations about *others' behavior*, which is crucial for the analysis of general equilibrium, is typically overlooked in the characterization of agents' decisions.⁵ Second, almost all results about both naive and sophisticated paths are obtained only for those cases in which expectations

² The concept of a naive path resembles the notion of a sliding path or a rolling plan (see, e.g., Goldman, 1968; Kaganovich, 1985).

³ A sophisticated agent is typically modelled as a sequence of autonomous temporal selves with conflicting preferences who play a dynamic game among themselves. Sophisticated paths correspond to the symmetric Markov perfect Nash equilibria of such a game.

⁴ Barro implicitly considers a naive agent, but since under log-utility the propensity to consume is constant, his solution looks like a time-consistent one. Perhaps, this is the reason why many subsequent authors mistook Barro's naive agent for a sophisticated agent.

⁵ Notable exceptions are game-theoretic models with different types of time-inconsistent agents. Such models (see, e.g., Akin, 2007; Schweighofer-Kodritsch, 2018) explicitly acknowledge that equilibrium outcome depends not only on agents' beliefs about their future selves, but also on agents' beliefs about opponents.

play no role: (i) for log-utility; (ii) for an exogenous and constant interest rate.⁶ Our paper contributes to the existing literature by studying those cases in which expectations about others' behavior matter. We fully characterize the behavior of naive agents in a general equilibrium framework with an endogenously changing interest rate and isoelastic utility, and clarify the notion of perfect foresight under time inconsistency.

In the Ramsey model with quasi-hyperbolic discounting, a naive agent at each date chooses a consumption path and believes that in the future they will commit to this path. However, due to dynamic inconsistency, the agent revises their decision at any future date, and a different consumption path is obtained. A natural concept to describe the behavior of a naive agent is a *sliding equilibrium path* (cf. Borissov, 2013). The idea is to consider a step-by-step procedure where an agent at each date revises their consumption path and implements only the current step.

The consumption path chosen by an agent at each date depends on their expectations about future interest and wage rates. Thereby, every sliding equilibrium path critically depends on the formation of expectations about prices. The question is: would a naive agent who revises their consumption path also revise their expectations about interest and wage rates? Clearly, the answer to this question depends on the beliefs about others' behavior that are affected by the source of naivete. If a decision maker forms expectations about their own future behavior, then it can be shown that “it does not matter whether a self holds an explicit belief on the lack of change in her preferences, or whether she believes she will simply fail to act on such changes, or that she has strong beliefs in her own will- or pre-commitment power” (Meder et al., 2017, p. 47). However, if a decision maker forms expectations about others' behavior, then whether the agent is naive about their own naivete becomes crucial for the type of their foresight. Depending on the source of naivete, we distinguish between two types of expectations in equilibrium.

Consider a continuum of identical naive agents in the economy. Suppose that their naivete originates from *unawareness*: any agent believes they will commit to the originally chosen path because the agent fails to take into account their time-inconsistent preferences. Then it is clear that the agent is also unaware of time inconsistency of other agents and (incorrectly) believes that all the others would also commit to the original path. In this case, at each date τ , the agent expects

⁶ Krusell et al. (2002) provide a closed-form solution for the sophisticated path in the Ramsey model with quasi-hyperbolic discounting, log-utility and Cobb–Douglas production technology. Sophisticated paths under general isoelastic utility and a constant interest rate are also studied in Bernheim et al. (2015); Cao and Werning (2018). Ahn et al. (2020) develop an axiomatic theory of naivete and apply it to a consumption-savings problem with constant interest rate.

those interest and wage rates that occur on a “temporary” date- τ equilibrium path, as if this path would be followed in the future. We refer to this type of foresight as *quasi-perfect foresight*: at each date the agent revises both their consumption path and expectations about interest and wage rates.

Suppose now that an agent’s naivete originates from *self-deception*. It is known that people tend to have unrealistically positive self-images which results in over-estimating the extent to which they have control over their future actions (see, e.g., discussion in Benabou and Tirole, 2002). Moreover, it is not the absolute level of self-image that matters but the *relative* one (see, e.g., Johansson-Stenman and Martinsson, 2006).⁷ People are over-confident about themselves, while they are aware of the self-control problems of others. There is recent experimental evidence that clearly confirms this observation: Ludwig and Nafziger (2011) report that most subjects overestimate their abilities while at the same time believing that others are unbiased; and Fedyk (2021) shows that subjects are unaware of their own time inconsistency but correctly expect time inconsistency in others.

The implications of this relative self-deception in a general equilibrium framework are as follows. The agent incorrectly believes that they will commit to the originally chosen path, but at the same time correctly believes that all the others are time-inconsistent, cannot commit to their actions and would revise their decisions at each date. We refer to this type of foresight as *perfect foresight*: at each date the agent revises their consumption path but correctly anticipates interest and wage rates that occur on the resulting sliding equilibrium path.⁸

Thus, by distinguishing between two sources of naivete, we can distinguish between two types of sliding equilibria which differ in the type of foresight. First, a *sliding equilibrium path under perfect foresight* (PSE) is an outcome of interaction of agents who are self-deceptive about their own decisions but can perfectly foresee the decisions of the others. This is a novel equilibrium concept, so we prove that it is well-defined and study its properties. Second, a *sliding equilibrium path under quasi-perfect foresight* (QSE) is an outcome of interaction of agents who are unaware both of their own time inconsistency and of the others’ time inconsistency. A QSE picks only the date- τ elements of each “temporary” date- τ equilibrium

⁷There is a large theoretical and empirical literature emphasizing the importance of social comparison in terms of status (see, e.g., Wendner and Goulder, 2008; Dioikitopoulos et al., 2020).

⁸ It can be argued that perfect foresight captures the case of a partially naive agent. Following O’Donoghue and Rabin (2001), the literature on partial naivete assumes that an agent recognizes their naivete but underestimates its impact. If a decision maker acts in isolation, then the behavior of a partially naive agent who is completely unaware of the degree of their naivete is the same as that of a naive agent who is unaware of their naivete. Our paper contributes to the discussion of partial naivete by noting that in a general equilibrium framework an agent may not recognize their own time inconsistency but can have perfect foresight about future prices.

path. Therefore, its existence is guaranteed by the existence of a “temporary” date- τ equilibrium path.

The difference between a PSE and a QSE is illustrated in Fig. 1. In both cases, agents believe that they will commit to the currently chosen path but revise their path, which is an indication of naivete. If this naivete originates from unawareness, then agents’ beliefs about others behavior coincide with their beliefs about themselves, and they form expectations which are correct only from the current perspective. However, if this naivete originates from self-deception, then agents’ beliefs about the others are model-consistent and are correct from the sliding perspective.

<i>Source of naivete</i>	<i>Belief about theirsself</i>	<i>Belief about others</i>	<i>Action</i>	
Self-deception	Commit	Revise	Revise	PSE
Unawareness	Commit	Commit	Revise	QSE

Figure 1: Different sources of naivete lead to different types of foresight

For both PSE and QSE, we investigate the impact of quasi-hyperbolic discounting on macroeconomic variables. Specifically, we study the question of *observational equivalence*, that is, whether the observable outcome of consumption and savings decisions made by agents with quasi-hyperbolic discounting differ from that of agents with exponential discounting. A number of models (e.g., Laibson, 1996; Krusell et al., 2002) show that observational equivalence holds for sophisticated paths under log-utility. Barro (1999) shows that a naive path under log-utility and a concave production function is observationally equivalent to an optimal path in the standard Ramsey model. These results are extended by Findley and Caliendo (2014), who prove that for any fixed and constant finite planning horizon, there is observational equivalence of naive paths under quasi-hyperbolic discounting either for log-utility or for general isoelastic utility and an exogenous and constant interest rate.

In contrast to the existing literature, which emphasizes the prevalence of observational equivalence, we prove that within the class of isoelastic utility functions, there is *no observational equivalence* for PSE beyond the standard case of log-utility. We also note that, to the best of our knowledge, observational equivalence of QSE holds only in the case with log-utility and Cobb–Douglas production technology.⁹ Our results imply that under naivete, quasi-hyperbolic discounting

⁹ Observational equivalence also holds for stationary PSE and stationary QSE, where the interest rate is constant.

almost always matters for saving behavior.

The rest of the paper is organized as follows. Section 2 introduces the central object of our study, a PSE, and Section 3 proves its existence. In Section 4 we report the general results concerning observational equivalence of PSE. In Section 5 we consider quasi-perfect foresight and compare PSE with QSE. Section 6 concludes. The Appendix contains proofs and mathematical details supporting the analysis in the main text.

2 The model

We consider a general equilibrium version of the Ramsey model with quasi-hyperbolic discounting and perfect foresight.¹⁰ Time is discrete. The consumption side is characterized by a continuum of agents of measure one. Each agent at each date τ has (intertemporal) utility of the form

$$u(c_\tau) + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} u(c_t) = u(c_\tau) + \beta \delta u(c_{\tau+1}) + \beta \delta^2 u(c_{\tau+2}) + \dots,$$

where c denotes consumption, $u(c)$ is the instantaneous utility function, β is the present bias parameter, and δ is the long-run discount factor. We assume that $0 < \beta < 1$, $0 < \delta < 1$, and $u(c)$ is isoelastic: $u(c) = c^{1-\rho}/(1-\rho)$ for $\rho > 0$, with the convention that $\rho = 1$ refers to the logarithmic case $u(c) = \ln c$. We assume that each agent is naive, that is, at each date τ the agent believes they would commit to the path chosen at date τ but revises this path at date $\tau + 1$.

At date τ , given initial savings $s_{\tau-1}$, each agent faces the following problem:

$$\begin{aligned} \max_{c_t \geq 0} \quad & u(c_\tau) + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} u(c_t), \quad \text{s. t.} \quad c_\tau + \sum_{t=\tau+1}^{\infty} \frac{c_t}{(1+r_{\tau+1}) \cdots (1+r_t)} \\ & \leq (1+r_\tau)s_{\tau-1} + w_\tau + \sum_{t=\tau+1}^{\infty} \frac{w_t}{(1+r_{\tau+1}) \cdots (1+r_t)}, \end{aligned} \quad (1)$$

where interest rates $\{r_t\}_{t=\tau}^{\infty}$ and wage rates $\{w_t\}_{t=\tau}^{\infty}$ are taken as given by the agent.

We call a sequence $\{c_t^\tau, s_t^\tau\}_{t=\tau}^{\infty}$, a *date- τ consumer optimum starting from $s_{\tau-1}$ at given $\{r_t\}_{t=\tau}^{\infty}$ and $\{w_t\}_{t=\tau}^{\infty}$* , if $\{c_t^\tau\}_{t=\tau}^{\infty}$ is a solution to problem (1) and $\{s_t^\tau\}_{t=\tau}^{\infty}$ is determined recursively by $s_t^\tau = (1+r_t)s_{t-1}^\tau + w_t - c_t^\tau$.

¹⁰ This model resembles the general equilibrium version of the standard Ramsey model (Cass, 1965; Koopmans, 1965).

Due to dynamic inconsistency, each agent at each date τ revises their consumer optimum and implements only the current step. If a solution to problem (1) exists, it satisfies the budget constraint as equality and the following first-order conditions: $c_{\tau+1}^\tau = (\beta\delta(1+r_{\tau+1}))^{\frac{1}{\rho}} c_\tau^\tau$, and $c_t^\tau = (\delta(1+r_t))^{\frac{1}{\rho}} c_{t-1}^\tau$ for $t \geq \tau + 2$. Substituting the first-order conditions into the budget constraint, we obtain the date- τ consumption in a date- τ consumer optimum:

$$c_\tau^\tau = \frac{(1+r_\tau)s_{\tau-1} + w_\tau + \sum_{t=\tau+1}^{\infty} \frac{w_t}{(1+r_{\tau+1}) \cdots (1+r_t)}}{1 + (\beta\delta)^{\frac{1}{\rho}} (1+r_{\tau+1})^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta)^{\frac{1}{\rho}} ((1+r_{\tau+1}) \cdots (1+r_{\tau+t}))^{\frac{1-\rho}{\rho}} + \dots}. \quad (2)$$

The production side is characterized by a neoclassical production function $F(k, l)$, where l is labor input and function F is homogeneous of degree one. The amount of labor is fixed and normalized to 1. The production function in intensive form, f , is given by $f(k) = F(k, 1)$. Capital is the only variable factor, and it is assumed to depreciate completely within one period. The production function $f(k)$ satisfies the standard assumptions: $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$, $\lim_{k \rightarrow 0} f'(k) = +\infty$, and $\lim_{k \rightarrow \infty} f'(k) = 0$. At each date t , producers take as given the interest rate r_t and solve the following profit maximization problem:

$$\max_{k_t \geq 0} f(k_t) - (1+r_t)k_t. \quad (3)$$

Under perfect foresight, each agent correctly anticipates the sequences of interest and wage rates that would prevail in the economy. At each date τ , the agent maximizes their utility and determines the current capital supply which coincides with the date- τ savings in a date- τ consumer optimum at equilibrium interest and wage rates. Producers maximize profit and determine the current capital demand as the required level of investment. Interest and wage rates are given by the respective marginal products. An equilibrium occurs when the output and capital markets clear at each date, that is, savings are equal to investment. We define a PSE as follows.

Definition 1. A sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^{\infty}$ is a sliding equilibrium path under perfect foresight (PSE) starting from s_{-1}^* , if

1. Consumption and savings at each date τ are obtained from the date- τ consumer optimum starting from $s_{\tau-1}^*$ at given $\{r_t^*\}_{t=\tau}^{\infty}$ and $\{w_t^*\}_{t=\tau}^{\infty}$;
2. Prices at each date τ are equal to marginal products: $1+r_\tau^* = f'(k_\tau^*)$ and $w_\tau^* = f(k_\tau^*) - f'(k_\tau^*)k_\tau^*$;
3. Savings at each date τ are equal to investment: $s_\tau^* = k_{\tau+1}^*$.

the same for different dates τ (that is, in each row in Fig. 2). While the agent does not recognize their time inconsistency (hence revising their consumer optimum at each date), the agent has perfect foresight about sliding equilibrium prices (hence never recalculating the expected prices). This is a natural consequence of self-deceptive naivete. If agents' naivete originates from perceiving that they have more willpower than others, then each agent overlooks their own time inconsistency but recognizes time inconsistency of the others. At each date, agents (incorrectly) believe that in the future they will behave in a manner consistent with their long-run preferences but (correctly) expect that all other agents will revise their plans to satisfy their short-run impatience. In this case the resulting interest and wage rates are those that prevail in a sliding equilibrium, which is precisely the sense in which "the market knows best" in a model with quasi-hyperbolic discounting.

Third, our definition of a PSE essentially clarifies some ideas of Barro (1999). His argument implies that the agent revises their consumption path at each instant of time but does not revise their expectations. Hence Barro implicitly assumes a naive agent who correctly anticipates prices taking into account that the equilibrium itself will change. Thus, Barro (1999) considers a PSE in terms of our Definition 1.

3 Equilibrium existence theorem

The following theorem proves the existence of a PSE.

Theorem 1. *There exists a sliding equilibrium path under perfect foresight starting from any $s_{-1}^* > 0$.*

Proof. We prove the existence of a PSE in two steps. First, we consider a PSE in the finite horizon model and show that for any $T \in \mathbb{N}$ there exists a finite T -horizon PSE. Second, we construct a candidate for a PSE in the infinite horizon model by applying a diagonalization procedure to the sequence of finite T -horizon PSE, and show that this candidate is indeed a PSE.

Fix a finite horizon $T \geq 1$. For any date $0 \leq \tau \leq T$, consider the following T -horizon date- τ problem:

$$\begin{aligned} \max_{c_t \geq 0} \quad & u(c_\tau) + \beta \sum_{t=\tau+1}^{T+1} \delta^{t-\tau} u(c_t), & \text{s. t. } & c_\tau + \sum_{t=\tau+1}^{T+1} \frac{c_t}{(1+r_{\tau+1}) \cdots (1+r_t)} \\ & & & \leq f(k_\tau) + \sum_{t=\tau+1}^{T+1} \frac{w_t}{(1+r_{\tau+1}) \cdots (1+r_t)}, \end{aligned} \quad (4)$$

where the initial capital stock $k_\tau > 0$ and sequences of interest rates $\{r_{t+1}\}_{t=\tau}^T$ and wage rates $\{w_{t+1}\}_{t=\tau}^T$ are taken as given by the agent.

We call the sequence $\{c_t^\tau, s_t^\tau\}_{t=\tau}^{T+1}$ a T -horizon date- τ consumer optimum starting from k_τ at given $\{r_{t+1}\}_{t=\tau}^T$ and $\{w_{t+1}\}_{t=\tau}^T$ if $\{c_t^\tau\}_{t=\tau}^{T+1}$ is the solution to problem (4), and $\{s_t^\tau\}_{t=\tau}^{T+1}$ is given recursively by $s_\tau^\tau = f(k_\tau) - c_\tau^\tau$ and $s_t^\tau = (1 + r_t)s_{t-1}^\tau + w_t - c_t^\tau$.

A finite T -horizon PSE is formally defined as follows.

Definition 2. A sequence $\{c_t^*(T), s_t^*(T), k_{t+1}^*(T), r_{t+1}^*(T), w_{t+1}^*(T)\}_{t=0}^T$ is a T -horizon sliding equilibrium path under perfect foresight starting from $s_{-1}^* = k_0^*$ if

1. Consumption and savings at date τ are the elements of the T -horizon date- τ consumer optimum starting from $k_\tau^*(T)$ at given $\{r_{t+1}^*(T)\}_{t=\tau}^T$ and $\{w_{t+1}^*(T)\}_{t=\tau}^T$;
2. Prices at each date are equal to marginal products: for $1 \leq t \leq T + 1$, $1 + r_t^*(T) = f'(k_t^*(T))$ and $w_t^*(T) = f(k_t^*(T)) - f'(k_t^*(T))k_t^*(T)$;
3. Savings at each date are equal to investment: $s_t^*(T) = k_{t+1}^*(T)$ for $0 \leq t \leq T$.

Lemma 1.1. There exists a T -horizon sliding equilibrium path under perfect foresight starting from any $s_{-1}^* = k_0^* > 0$.

Proof. See Appendix A. □

Importantly enough, the sequence of capital stocks on a T -horizon PSE is bounded from both below and above. To define the bounds, let $\{\underline{c}_t(T), \underline{k}_{t+1}(T)\}_{t=0}^{T+1}$ be a solution to the following problem given $\underline{k}_0 = k_0^*$:

$$\max_{c_t \geq 0, k_{t+1} \geq 0} \sum_{t=0}^{T+1} (\beta\delta)^t u(c_t), \quad \text{s. t. } c_t + k_{t+1} = f(k_t), \quad 0 \leq t \leq T + 1, \quad (5)$$

and let $\{\bar{c}_t(T), \bar{k}_{t+1}(T)\}_{t=0}^{T+1}$ be a solution to the following problem given $\bar{k}_0 = k_0^*$:

$$\max_{c_t \geq 0, k_{t+1} \geq 0} \sum_{t=0}^{T+1} \delta^t u(c_t), \quad \text{s. t. } c_t + k_{t+1} = f(k_t), \quad 0 \leq t \leq T + 1. \quad (6)$$

Lemma 1.2. For all $T \geq 1$, and for all $0 \leq t \leq T$,

$$\underline{k}_{t+1}(T) < k_{t+1}^*(T) < \bar{k}_{t+1}(T). \quad (7)$$

Proof. See Appendix A. □

Now consider the sequence $[\{k_{t+1}^*(T)\}_{t=0}^T]_{T \geq 1}$, whose elements are the sequences of capital stocks on the T -horizon PSE starting from the same $s_{-1}^* = k_0^*$, for increasing horizons $T = 1, 2, \dots$

Let us apply the following procedure to the sequence $[\{k_{t+1}^*(T)\}_{t=0}^T]_{T \geq 1}$. At the first step of the procedure, consider the sequence $\{k_1^*(T)\}_{T \geq 1}$, take a cluster point k_1^* of this sequence and extract a subsequence $\{T_{1n}\}_{n=1}^\infty$ from $\{T\}_{T \geq 1}$ such that $\{k_1^*(T_{1n})\}_{n=1}^\infty$ converges to k_1^* . At the second step, consider the sequence $\{k_2^*(T_{1n})\}_{n=1}^\infty$, take a cluster point k_2^* of this sequence and extract a subsequence $\{T_{2n}\}_{n=1}^\infty$ from the sequence $\{T_{1n}\}_{n=1}^\infty$ such that $T_{21} > 1$ and $\{k_2^*(T_{2n})\}_{n=1}^\infty$ converges to k_2^* . This procedure continues *ad infinitum*.

Finally, consider the sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_{t+1}^*, w_{t+1}^*\}_{t=0}^\infty$, where $\{k_{t+1}^*\}_{t=0}^\infty$ is obtained by the diagonal procedure described above and for all $\tau \geq 0$,

$$c_\tau^* = f(k_\tau^*) - k_{\tau+1}^*, \quad s_\tau^* = k_{\tau+1}^*, \quad 1 + r_\tau^* = f'(k_\tau^*), \quad w_\tau^* = f(k_\tau^*) - f'(k_\tau^*)k_\tau^*. \quad (8)$$

The following lemma shows that this sequence is a PSE.

Lemma 1.3. *The sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_{t+1}^*, w_{t+1}^*\}_{t=0}^\infty$ is a sliding equilibrium path under perfect foresight starting from $s_{-1}^* = k_0^*$.*

Proof. See Appendix A. □

Thus, we obtain a PSE and prove the theorem. ■

4 Observational equivalence

After defining a general equilibrium in the model with quasi-hyperbolic discounting and proving its existence, we investigate the effects of discounting. It is frequently argued that the observable outcome of consumption and savings decisions made by agents with quasi-hyperbolic discounting does not necessarily differ from that of agents with exponential discounting, which is known as *observational equivalence*.

Consider the *standard Ramsey model*, that is, the optimal growth model with exponential discounting:

$$\max_{c_t \geq 0, k_{t+1} \geq 0} \sum_{t=0}^{\infty} \gamma^t u(c_t), \quad \text{s. t. } c_t + k_{t+1} = f(k_t), \quad t \geq 0, \quad (9)$$

where $0 < \gamma < 1$ is a constant discount factor. We call a solution to problem (9) the γ -*optimal path* starting from k_0 .

The reasonable question to ask is whether the sequence of consumption and

capital in a PSE coincides with some optimal path in the standard Ramsey model. We formally define observational equivalence of PSE as follows.

Definition 3. *A sliding equilibrium path under perfect foresight in the Ramsey model with quasi-hyperbolic discounting starting from s_{-1}^* , $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^\infty$, is observationally equivalent to a γ -optimal path, if there exists γ for which the sequence $\{c_t^*, k_{t+1}^*\}_{t=0}^\infty$ is a solution to problem (9) starting from $k_0^* = s_{-1}^*$.*

In what follows we show that in the cases with log-utility and with stationary sliding equilibria observational equivalence holds. We then prove that there is no observational equivalence in the general case: within the class of isoelastic utility functions, observational equivalence of PSE holds *if and only if* either utility is logarithmic or the interest rate is constant.

4.1 Logarithmic utility

Consider the log-utility case. Then the following result holds.

Proposition 1. *Suppose that $u(c) = \ln c$. Then, irrespective of the production technology, a sliding equilibrium path under perfect foresight exists, is unique, and is observationally equivalent to the γ^* -optimal path, where $\gamma^* = \frac{\beta\delta}{1-\delta+\beta\delta}$.*

Proof. It follows from (2) that when $\rho = 1$, the date- τ consumption in the date- τ consumer optimum starting from $s_{\tau-1}^*$ at given $\{r_t^*\}_{t=\tau}^\infty$ and $\{w_t^*\}_{t=\tau}^\infty$, satisfies $c_\tau^{**\tau} = \frac{1-\delta}{1-\delta+\beta\delta} M_\tau^*$, where $M_\tau^* = (1+r_\tau^*)s_{\tau-1}^* + w_\tau^* + \sum_{t=\tau+1}^\infty \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}$ is the present (date- τ) value of the expected date- τ lifetime income. For the formal argument that $M_\tau^* < \infty$, see the proof of Lemma 1.3 in Appendix A.

By definition, a PSE starting from s_{-1}^* is a sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^\infty$ such that for all $\tau \geq 0$,

$$\begin{aligned} c_\tau^* &= \frac{1-\delta}{1-\delta+\beta\delta} M_\tau^*, & k_{\tau+1}^* &= s_\tau^* = (1+r_\tau^*)s_{\tau-1}^* + w_\tau^* - c_\tau^*, \\ 1+r_\tau^* &= f'(k_\tau^*), & w_\tau^* &= f(k_\tau^*) - f'(k_\tau^*)k_\tau^*. \end{aligned}$$

It follows that lifetime incomes expected at dates τ and $\tau+1$ are linked: $M_\tau^* - (1+r_\tau^*)s_{\tau-1}^* - w_\tau^* = \frac{M_{\tau+1}^*}{1+r_{\tau+1}^*} - s_\tau^*$, and hence $M_{\tau+1}^* = (1+r_{\tau+1}^*)(M_\tau^* - c_\tau^*)$. Therefore,

$$c_{\tau+1}^* = \frac{1-\delta}{1-\delta+\beta\delta} M_{\tau+1}^* = (1+r_{\tau+1}^*) \left(c_\tau^* - \frac{1-\delta}{1-\delta+\beta\delta} c_\tau^* \right) = \frac{\beta\delta}{1-\delta+\beta\delta} (1+r_{\tau+1}^*) c_\tau^*.$$

Thus, in a PSE, the consumption levels at two adjacent dates are linked via the following “first-order conditions”: $c_{t+1}^* = \frac{\beta\delta}{1-\delta+\beta\delta}(1+r_{t+1}^*)c_t^*$. Now it is clear that the sequence $\{c_t^*, k_{t+1}^*\}_{t=0}^\infty$ extracted from a PSE is the γ^* -optimal path, where $\gamma^* = \frac{\beta\delta}{1-\delta+\beta\delta}$, and hence observational equivalence holds. ■

Three comments about Proposition 1 are in order. First, Proposition 1 implies that in the case with log-utility, observing only the path $\{c_t^*, k_{t+1}^*\}_{t=0}^\infty$, one cannot determine whether the agent has $\beta\text{--}\delta$ discounting and is time-inconsistent or the agent has γ^* discounting and is time-consistent. The equivalent discount factor γ^* does not depend on technology and lies in between the short-run discount factor $\beta\delta$ and the long-run discount factor δ , $\beta\delta < \gamma^* < \delta$.

Second, Krusell et al. (2002) prove observational equivalence of a sophisticated equilibrium path in the Ramsey model with quasi-hyperbolic discounting, log-utility and Cobb–Douglas production technology, and obtain the same equivalent discount factor γ^* . However, Krusell et al. (2002) consider sophisticated agents, while we consider naive agents; and they consider only a Cobb–Douglas technology, while our result holds irrespective of the technology. Thus, Proposition 1 implies that self-deceptive naivete might be an alternative to sophistication for explaining high saving rates of agents with quasi-hyperbolic discounting.

Third, Proposition 1 establishes the link between observational equivalence and a controlled comparison of discount functions (see, e.g., Myerson et al., 2001; Caliendo and Findley, 2014). Note that the equivalent discount factor is such that γ^* discounting provides the same degree of overall impatience as $\beta\text{--}\delta$ discounting: $\sum_{t=0}^\infty (\gamma^*)^t = 1 + \beta \sum_{t=1}^\infty \delta^t$. Under log-utility, controlling for overall impatience implies that the paths of consumption and capital under exponential and quasi-hyperbolic discounting are observationally equivalent.

4.2 Stationary sliding equilibrium

Before turning to the general results, consider a stationary sliding equilibrium under perfect foresight (stationary perfect sliding equilibrium, SPSE).

Definition 4. *A tuple $\{c^*, s^*, k^*, r^*, w^*\}$ is a stationary sliding equilibrium under perfect foresight if the sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^\infty$, satisfying for each $t \geq 0$, $c_t^* = c^*$, $s_t^* = s^*$, $k_{t+1}^* = k^*$, $r_t^* = r^*$, and $w_t^* = w^*$, is a sliding equilibrium path under perfect foresight starting from s^* .*

By the definition of PSE, $\{c^*, s^*\}$ are obtained from the associated date- τ consumer optimum starting from s^* at given constant interest rate r^* and wage rate w^* . It is clear that this underlying date- τ consumer optimum, which determines

a SPSE, does not depend on τ . The following theorem states that a SPSE exists, is unique and there is always observational equivalence.

Theorem 2. *There is a unique stationary sliding equilibrium under perfect foresight. It is observationally equivalent to a stationary γ^* -optimum, where*

$$\gamma^* = \frac{(\gamma^*)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}. \quad (10)$$

Proof. Take some $r^* > 0$ and $w^* > 0$. Let $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ be consumption in a date- τ consumer optimum starting from s^* at given constant interest rate r^* and wage rate w^* , that is, the solution to problem (1) for $\{r_t\}_{t=\tau}^{\infty} = \{r^*, r^*, \dots\}$ and $\{w_t\}_{t=\tau}^{\infty} = \{w^*, w^*, \dots\}$. The following lemma characterizes this consumer optimum.

Lemma 2.1. *Suppose that $\delta(1+r^*)^{1-\rho} < 1$. A date- τ consumer optimum starting from s^* at given r^* and w^* exists, is unique, and the date- τ consumption satisfies*

$$c_{\tau}^{**\tau} = \frac{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}} \cdot \frac{1+r^*}{r^*} \cdot (r^*s^* + w^*). \quad (11)$$

Proof. See Appendix B. □

By the definition of SPSE, $c_{\tau}^{**\tau} = c^* = (1+r^*)s^* + w^* - s^* = r^*s^* + w^*$. It now follows from (11) that the interest rate in a SPSE is such that

$$1+r^* = \frac{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}},$$

that is, r^* satisfies

$$\frac{1}{1+r^*} = \frac{\left(\frac{1}{1+r^*}\right)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}. \quad (12)$$

The following lemma shows that there exists a unique solution to equation (12), and it is compatible with the existence of a date- τ consumer optimum.

Lemma 2.2. *There is a unique solution r^* to equation (12), and $\delta(1+r^*)^{1-\rho} < 1$.*

Proof. See Appendix B. □

It follows that the tuple $\{c^*, s^*, k^*, r^*, w^*\}$, where r^* is the solution to equation (12), and $s^* = k^* = (f')^{-1}(1+r^*)$, $w^* = f(k^*) - f'(k^*)k^*$, and $c^* = r^*s^* + w^*$, is the unique SPSE.

Furthermore, denote $\gamma^* = \frac{1}{1+r^*}$. It is clear that since r^* satisfies (12), γ^* satisfies (10). Since $f'(k^*) = 1/\gamma^*$, it follows that $\{c^*, k^*\}$ is a stationary optimum

in the standard Ramsey model with the discount factor γ^* . Therefore, a SPSE is observationally equivalent to a stationary γ^* -optimum. \blacksquare

It is easily checked that the stationary equivalent discount factor γ^* lies in between the short-run discount factor $\beta\delta$ and the long-run discount factor δ , $\beta\delta < \gamma^* < \delta$, and is increasing both in β and in δ . Note also that in the case of stationary equilibria with log-utility, Theorem 2 and Proposition 1 yield the same result.

4.3 The general case

Consider now the question of observational equivalence of a PSE for general isoelastic utility. The following theorem proves that there is no observational equivalence in the general case: a PSE is observationally equivalent to some γ -optimal path if and only if either utility is logarithmic or the interest rate is constant.

Theorem 3. *A sliding equilibrium path under perfect foresight starting from $s_{-1}^* \neq s^*$ is observationally equivalent to a γ -optimal path if and only if $\rho = 1$.*

Proof. Suppose that a PSE, $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^\infty$, is observationally equivalent to some γ -optimal path. It follows from Theorem 2 that the equivalent discount factor must be γ^* , and $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^\infty$ converges to the SPSE $\{c^*, s^*, k^*, r^*, w^*\}$ characterized in Theorem 2.

Let Δ_{t+1} be given by

$$\Delta_{t+1} = \delta^{\frac{1}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}} + \delta^{\frac{2}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}}(1 + r_{t+3}^*)^{\frac{1-\rho}{\rho}} + \dots \quad (13)$$

It is easily seen that

$$\Delta_{t+1} = \delta^{\frac{1}{\rho}}(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}}(1 + \Delta_{t+2}). \quad (14)$$

Lemma 3.1. *Let $\{c_t^*, s_t^*, k_{t+1}^*, r_t^*, w_t^*\}_{t=0}^\infty$ be a PSE. Then*

$$c_{t+1}^* = c_t^* (\beta\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}} \frac{1 + \Delta_{t+1}}{1 + \beta^{\frac{1}{\rho}}\Delta_{t+1}}. \quad (15)$$

Proof. See Appendix C. \square

Due to observational equivalence, $c_{t+1}^* = c_t^* (\gamma^*(1 + r_{t+1}^*))^{\frac{1}{\rho}}$. Taking account of (15), we obtain that for all $t \geq 0$,

$$\left(\frac{\gamma^*}{\beta\delta}\right)^{\frac{1}{\rho}} = \frac{1 + \Delta_{t+1}}{1 + \beta^{\frac{1}{\rho}}\Delta_{t+1}}, \quad \text{and hence} \quad \Delta_{t+1} = \frac{(\gamma^*)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{(\beta\delta)^{\frac{1}{\rho}} - (\beta\gamma^*)^{\frac{1}{\rho}}}.$$

Therefore, the value of Δ_{t+1} is constant over time. Using (14) and (10), we can restate this condition in terms of interest rates as follows:

$$(1 + r_{t+2}^*)^{\frac{1-\rho}{\rho}} = \frac{1}{(\gamma^*)^{\frac{1}{\rho}}} \cdot \frac{(\gamma^*)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}} = (\gamma^*)^{1-\frac{1}{\rho}},$$

or

$$(\gamma^*(1 + r_{t+2}^*))^{\frac{1-\rho}{\rho}} = 1, \quad t \geq 0. \quad (16)$$

Clearly, (16) holds only in the following two cases. First, $\rho = 1$, which is the log-utility case. Second, $r_t^* = r^*$, which is the case of stationary equilibrium with $s_{-1}^* = s^*$. Thus, in the Ramsey model with quasi-hyperbolic discounting and isoelastic utility, a PSE starting from $s_{-1}^* \neq s^*$ is not observationally equivalent to an optimal path in the Ramsey model with exponential discounting unless $\rho = 1$. ■

The intuition behind Theorem 3 is as follows. Under perfect foresight the expected prices are not revised and remain the same at each date. Therefore, in a PSE the consumption levels at two adjacent dates are linked and satisfy the “first-order conditions” (15). These conditions are compatible with the first-order conditions in the standard Ramsey model if and only if either utility is logarithmic or the interest rate is constant. Only then a PSE is observationally equivalent to some γ -optimal path. In the case with log-utility considered in Proposition 1 the equivalent discount factor is given by $\gamma^* = \frac{\beta\delta}{1-\delta+\beta\delta}$. In the case of a SPSE considered in Theorem 2 the equivalent discount factor γ^* satisfies Eq. (10). In all other cases, a PSE is not observationally equivalent to any γ -optimal path.

5 Quasi-perfect foresight

As we have seen, a self-deceptive naive agent has perfect foresight, being overconfident about their time inconsistency but correctly anticipating the behavior of all other agents in the economy. Here we consider how the outcome of the model changes if agent’s naivete is unawareness-related, that is, an agent overlooks not only their own time inconsistency but also time inconsistency of all other agents. In this case, the agent at each date revises both their consumer optimum and their expectations about prices, and hence has quasi-perfect foresight. In what follows we formally define a QSE, introduce a corresponding sliding optimal path (SOP) and characterize their properties.

5.1 Sliding equilibrium path under quasi-perfect foresight

Consider the general equilibrium version of the Ramsey model with quasi-hyperbolic discounting described in Section 2. At date τ , agents solve problem (1). Producers at each date $t \geq \tau$ solve problem (3). Suppose that at date τ each agent believes that their date- τ consumer optimum would be followed in the future and expects those interest and wage rates that are consistent with this belief. This yields the “temporary” date- τ equilibrium path.

Formally, a *date- τ equilibrium path* (under quasi-hyperbolic discounting) starting from $s_{\tau-1}$ is a sequence $\{c_t^{*\tau}, s_t^{*\tau}, k_{t+1}^{*\tau}, r_t^{*\tau}, w_t^{*\tau}\}_{t=\tau}^{\infty}$, such that (i) agents maximize utility correctly anticipating the prices from the date- τ perspective: $\{c_t^{*\tau}, s_t^{*\tau}\}_{t=\tau}^{\infty}$ is a date- τ consumer optimum starting from $s_{\tau-1}$ at given $\{r_t^{*\tau}\}_{t=\tau}^{\infty}$ and $\{w_t^{*\tau}\}_{t=\tau}^{\infty}$; (ii) at each date prices are equal to marginal products from the date τ perspective: $1 + r_t^{*\tau} = f'(k_t^{*\tau})$ and $w_t^{*\tau} = f(k_t^{*\tau}) - f'(k_t^{*\tau})k_t^{*\tau}$; (iii) at each date savings are equal to investment: $s_{t-1}^{*\tau} = k_t^{*\tau}$.

Clearly, the date- τ equilibrium path differs from the truncation of any previous-date equilibrium path, and hence at each date τ there arises a new equilibrium.¹¹ Therefore, each agent revises their expectations at each date: on the date- τ equilibrium path, the date- τ consumer optimum is obtained under expectations $\{r_t^{*\tau}\}_{t=\tau}^{\infty}$ and $\{w_t^{*\tau}\}_{t=\tau}^{\infty}$, while on the date- τ' equilibrium path, the date- τ' consumer optimum is obtained under different expectations $\{r_t^{*\tau'}\}_{t=\tau'}^{\infty}$ and $\{w_t^{*\tau'}\}_{t=\tau'}^{\infty}$. It turns out that each agent correctly anticipates prices on the date- τ equilibrium path but cannot take into account that the equilibrium itself will change. This motivates the definition of a QSE which is constructed by applying at each date τ the corresponding date- τ equilibrium path.

Definition 5. A sequence $\{c_t^{\circ}, s_t^{\circ}, k_{t+1}^{\circ}, r_t^{\circ}, w_t^{\circ}\}_{t=0}^{\infty}$ is a sliding equilibrium path under quasi-perfect foresight (QSE) starting from s_{-1}° , if

1. Consumption and savings at each date τ are obtained from the date- τ equilibrium path, that is, from the date- τ consumer optimum starting from $s_{\tau-1}^{\circ}$ at given $\{r_t^{*\tau}\}_{t=\tau}^{\infty}$ and $\{w_t^{*\tau}\}_{t=\tau}^{\infty}$;
2. Prices at each date τ are equal to marginal products: $1 + r_{\tau}^{\circ} = f'(k_{\tau}^{\circ})$ and $w_{\tau}^{\circ} = f(k_{\tau}^{\circ}) - f'(k_{\tau}^{\circ})k_{\tau}^{\circ}$;
3. Savings at each date τ are equal to investment: $s_{\tau}^{\circ} = k_{\tau+1}^{\circ}$.

¹¹ Indeed, agent’s discount factor between periods $\tau + 1$ and τ equals $\beta\delta$ from the date τ perspective, while it is equal to δ from any earlier perspective. Therefore, the value $c_t^{*\tau}$ planned at date τ for the date- t consumption will never be optimal when date t comes.

Definition 5 also deserves several comments. First, note that a QSE depends on all date- τ equilibrium paths for $\tau \geq 0$. Under our assumptions, a date- τ equilibrium path exists and is unique, and therefore *there exists a unique QSE*. The structure of a QSE is illustrated in Fig. 3.

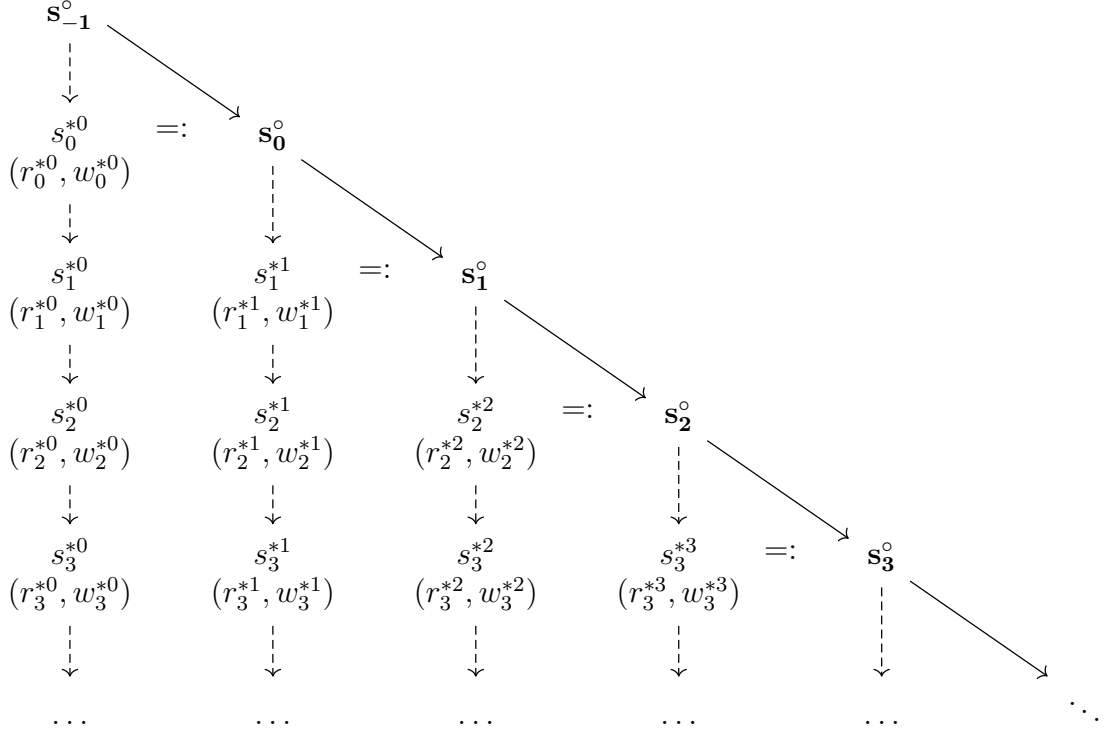


Figure 3: Savings on date- τ equilibria and QSE

Second, the label *quasi-perfect foresight* highlights unawareness-related naivete of the agent. In a QSE, an agent at each date is unaware of both their own time inconsistency (hence revising their consumer optimum) and time inconsistency of all other agents (hence revising the expected prices). The important difference between a PSE and a QSE is precisely the formation of price expectations in the consumer optimum. Under quasi-perfect foresight, the sequences of interest and wage rates expected at each date τ coincide with those realized on the date- τ equilibrium path (in each column in Fig. 3) but not on the sliding path. In contrast, under perfect foresight, the sequences of interest and wage rates expected at date τ coincide with those realized in PSE.

5.2 Sliding optimal paths

In order to study the properties of a QSE, it is convenient to introduce the corresponding sliding optimal path (see Definition 6 below). Given an initial capital

stock $k_\tau > 0$, consider the following date- τ utility maximization problem:

$$\max_{c_t \geq 0, k_{t+1} \geq 0} u(c_\tau) + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} u(c_t), \quad \text{s. t. } c_t + k_{t+1} = f(k_t), \quad t \geq \tau. \quad (17)$$

A solution to problem (17), $\{c_t^{*\tau}, k_{t+1}^{*\tau}\}_{t=\tau}^{\infty}$, is the *date- τ optimal path* (under quasi-hyperbolic discounting) starting from k_τ .

Note that the *date- τ equilibrium and the date- τ optimal paths are essentially the same* — a sequence $\{c_t^{*\tau}, k_{t+1}^{*\tau}\}_{t=\tau}^{\infty}$ extracted from the date- τ equilibrium path is the date- τ optimal path (starting from $k_\tau = s_{\tau-1}$).¹² Similarly, a QSE is essentially the same as a SOP which picks only the date- τ elements of each date- τ optimal path.

Definition 6. A sequence $\{c_t^\circ, k_{t+1}^\circ\}_{t=0}^{\infty}$ is a sliding optimal path starting from k_0 in the Ramsey model with quasi-hyperbolic discounting, if for each $\tau \geq 0$, consumption and capital stock at date τ are obtained from the date- τ optimal path under quasi-hyperbolic discounting starting from k_τ° : $c_\tau^\circ = c_\tau^{*\tau}$ and $k_{\tau+1}^\circ = k_{\tau+1}^{*\tau}$.

Clearly, a SOP exists and is unique. Note that a SOP is essentially characterized by the first step in problem (17), that is, by the first elements of the date- τ optimal path $\{c_t^{*\tau}, k_{t+1}^{*\tau}\}_{t=\tau}^{\infty}$. This fact allows us to describe a SOP in terms of dynamic programming (see Appendix D).

By construction, a sequence $\{c_t^\circ, k_{t+1}^\circ\}_{t=0}^{\infty}$ extracted from a QSE is a SOP starting from $k_0 = s_{-1}$. Thus, instead of studying the properties of a QSE, we can study the properties of the corresponding SOP, which turns out to be much easier.

In particular, we are again interested in whether a SOP under quasi-hyperbolic discounting coincides with some γ -optimal path. We formally define observational equivalence of SOP as follows.

Definition 7. A sliding optimal path in the Ramsey model with quasi-hyperbolic discounting, $\{c_t^\circ, k_{t+1}^\circ\}_{t=0}^{\infty}$, is observationally equivalent to the γ -optimal path if there exists γ for which $\{c_t^\circ, k_{t+1}^\circ\}_{t=0}^{\infty}$ is a solution to problem (9).

To the best of our knowledge, a SOP is observationally equivalent to a γ -optimal path only in the following two cases. First, in the case of a stationary sliding optimum.

Definition 8. A pair $\{c^\circ, k^\circ\}$ is a stationary sliding optimum if the sequence $\{c_t^\circ, k_{t+1}^\circ\}_{t=0}^{\infty}$, where for each $t \geq 0$, $c_t^\circ = c^\circ$ and $k_{t+1}^\circ = k^\circ$, is a sliding optimal path starting from k° .

¹² This should be taken as a loose way of saying that in the Ramsey model the first and the second fundamental theorems of welfare economics hold.

Clearly, a stationary sliding optimum is observationally equivalent to a stationary γ° -optimum for $\gamma^\circ = 1/f'(k^\circ)$. It can be checked that $\gamma^\circ < \delta$ (see Appendix D).

Second, in the case with log-utility and Cobb–Douglas production technology.

Proposition 2. *Suppose that $u(c) = \ln c$ and $f(k) = k^\alpha$. Then a sliding optimal path in the Ramsey model with quasi-hyperbolic discounting is observationally equivalent to the γ° -optimal path, where $\gamma^\circ = \frac{\beta\delta}{1-\alpha\delta+\alpha\beta\delta}$.*

Proof. See Appendix D. ■

For a QSE, observational equivalence holds only in two cases:¹³ the case with log-utility and Cobb–Douglas production technology and the case of stationary equilibrium.

5.3 Comparison

Finally, we compare the properties of a PSE and a QSE in terms of saving rates, long-run capital stocks and consumption.

As we have seen, already in the simplest case where $u(c) = \ln c$ and $f(k) = k^\alpha$, PSE and QSE differ. By Proposition 1, the PSE is observationally equivalent to the γ^* -optimal path, where $\gamma^* = \frac{\beta\delta}{1-\delta+\beta\delta}$ (which does not depend on the production technology). By Proposition 2, the QSE is observationally equivalent to the γ° -optimal path, where $\gamma^\circ = \frac{\beta\delta}{1-\alpha\delta+\alpha\beta\delta}$ (which depends on the technology parameter α). Comparing the equivalent discount factors, it is easily seen that $\beta\delta < \gamma^\circ < \gamma^* < \delta$. Thus, the equivalent exponential discount factor is higher under perfect foresight than under quasi-perfect foresight.

It is well known that for $u(c) = \ln c$ and $f(k) = k^\alpha$, the saving rate on the γ -optimal path is constant and given by $\alpha\gamma$. Therefore, the saving rate is always higher under perfect foresight than under quasi-perfect foresight. It follows that if the two economies start from the same initial condition, then the capital stock at each date on the PSE is higher than the capital stock on the QSE. Consequently, while initially consumption is higher on the QSE, starting from some date, the PSE provides higher consumption. Therefore, the stationary capital stock and consumption level are higher under perfect foresight than under quasi-perfect foresight. Loosely speaking, the source of naivete is crucial for savings decisions. In the considered case, when agents are unaware of their naivete, they save less than sophisticated agents. However, when agents' naivete originates from self-deception, they save precisely as much as sophisticated agents.

¹³ We say that a QSE is observationally equivalent to a γ -optimal path if the corresponding SOP is observationally equivalent to a γ -optimal path.

The result about stationary states can be generalized — it also holds for stationary sliding equilibria when intertemporal elasticity of substitution in consumption is sufficiently high. Recall that a SPSE was defined in Section 4.2. A stationary sliding equilibrium under quasi-perfect foresight (SQSE) can be defined similarly.

Definition 9. *A tuple $\{c^\circ, s^\circ, k^\circ, r^\circ, w^\circ\}$ is a stationary sliding equilibrium under quasi-perfect foresight if the sequence $\{c_t^\circ, s_t^\circ, k_{t+1}^\circ, r_t^\circ, w_t^\circ\}_{t=0}^\infty$, satisfying for each $t \geq 0$, $c_t^\circ = c^\circ$, $s_t^\circ = s^\circ$, $k_{t+1}^\circ = k^\circ$, $r_t^\circ = r^\circ$, and $w_t^\circ = w^\circ$, is a sliding equilibrium path under quasi-perfect foresight starting from $s^\circ = k^\circ$.*

By the definition of QSE, $\{c^\circ, s^\circ, k^\circ, r^\circ, w^\circ\}$ are the elements of the associated date- τ equilibrium path starting from s° . While this underlying date- τ equilibrium path, $\{c_t^{*\tau}, s_t^{*\tau}, k_{t+1}^{*\tau}, r_t^{*\tau}, w_t^{*\tau}\}_{t=\tau}^\infty$, does not depend on τ (all columns in Fig. 3 are identical), the sequences of prices on this date- τ equilibrium path are not constant.

An important difference between SPSE and SQSE again lies in the formation of price expectations in the consumer optimum. Consumption and savings on a SQSE, $\{c^\circ, s^\circ\}$, are the first elements of the consumer optimum starting from s° at given non-constant sequences $\{r^\circ, r^\circ, \{r_t^{*\tau}\}_{t=\tau+2}^\infty\}$ and $\{w^\circ, w^\circ, \{w_t^{*\tau}\}_{t=\tau+2}^\infty\}$. At the same time, consumption and savings on a SPSE, $\{c^*, s^*\}$, are the first elements of the consumer optimum starting from s^* at given constant sequences $\{r^*, r^*, r^*, \dots\}$ and $\{w^*, w^*, w^*, \dots\}$.

The comparison of consumer optima at given constant and non-constant sequences of interest and wage rates, allows us to compare SPSE with SQSE. The following theorem shows that when $0 < \rho \leq 1$, a stationary capital stock is higher under perfect foresight than under quasi-perfect foresight, irrespective of a production technology.

Theorem 4. *Let $\{c^*, s^*, k^*, r^*, w^*\}$ be a stationary sliding equilibrium under perfect foresight, and $\{c^\circ, s^\circ, k^\circ, r^\circ, w^\circ\}$ be a stationary sliding equilibrium under quasi-perfect foresight. If $0 < \rho \leq 1$, then*

$$c^* > c^\circ, \quad s^* > s^\circ, \quad k^* > k^\circ, \quad r^* < r^\circ, \quad w^* > w^\circ.$$

Proof. For the formal proof, see Appendix E. The idea is to assume the opposite and obtain a contradiction. Suppose that $k^* \leq k^\circ$. It then follows that $r^\circ \leq r^*$ and $w^\circ \geq w^*$. Let $c(r^\circ, w^\circ)$ be the date- τ consumption in the date- τ consumer optimum starting from s° at given constant interest rate r° and wage rate w° .

Since c° is consumption in a SQSE, it can be checked that $k^\circ \geq k^*$ implies $c(r^\circ, w^\circ) \geq c^\circ$. However, when $0 < \rho \leq 1$, it is easily seen from (2) that the date- τ consumption in the date- τ consumer optimum starting from $s_{\tau-1}$ at given

$\{r_t\}_{t=\tau}^\infty$ and $\{w_t\}_{t=\tau}^\infty$ is monotonically increasing in w_t and decreasing in r_t for all $t \geq \tau + 1$. Therefore, $c(r^\circ, w^\circ)$ is lower than the date- τ consumption on the date- τ equilibrium path starting from s° , that is, $c(r^\circ, w^\circ) < c^\circ$. This contradiction shows that for $0 < \rho \leq 1$ we must have $k^* > k^\circ$. It then follows that $s^* > s^\circ$, $r^* < r^\circ$, $w^* > w^\circ$, and $c^* = f(k^*) - k^* > f(k^\circ) - k^\circ = c^\circ$. ■

We conjecture that the above result also holds for $\rho > 1$. A large number of simulations with different values of β and δ supports this conjecture.

6 Conclusion

We study a general equilibrium version of the Ramsey model with quasi-hyperbolic discounting. In order to capture the prevalent time-inconsistent behavior, naivete of agents is taken as a point of departure. Naive agents revise their consumer optima at each date, which results in a sliding equilibrium path. In general equilibrium, a naive agent's behavior depends on their expectations about other naive agents' behavior which determine future prices. We distinguish between two sources of naivete (self-deception and unawareness), and argue that different sources of naivete lead to different expectations about others' future behavior which in turn lead to different types of foresight about prices in equilibrium.

An agent whose naivete originates from self-deception has perfect foresight. Such an agent is over-confident about their own time inconsistency but correctly anticipates that all others will revise their paths, in accordance with significant recent empirical evidence. Under perfect foresight, the expected sequences of interest and wage rates at each date coincide with those realized in a sliding equilibrium. The resulting sliding equilibrium path under perfect foresight (PSE) is a novel equilibrium concept.

In contrast, an agent whose naivete originates from unawareness has quasi-perfect foresight. Such an agent is unaware both of their own time inconsistency and of the others' time inconsistency, as is typically assumed in the literature. Under quasi-perfect foresight, an agent at each date revises both their consumer optimum and expectations about interest and wage rates. The resulting sliding equilibrium path under quasi-perfect foresight (QSE) is constructed by applying at each date τ the corresponding "temporary" date- τ equilibrium path.

We prove the existence of a PSE for the class of isoelastic utility functions and note that the implications of perfect foresight and quasi-perfect foresight differ. We compare PSE with QSE in terms of saving rates for the case with log-utility and Cobb–Douglas production technology and for the case of station-

ary equilibria with any utility function and production technology. We show that self-deceptive naivete leads to higher capital accumulation in the long run than unawareness-related naivete: perfect foresight implies a higher capital stock and a higher consumption level than quasi-perfect foresight.

We show that neither PSE nor QSE is generically observationally equivalent to some optimal path in the standard Ramsey model. We prove that observational equivalence of non-stationary PSE holds if and only if utility is logarithmic. We also prove that observational equivalence of QSE holds in the case with log-utility and Cobb–Douglas production technology.

There are several open questions to be addressed by future studies. Of special interest is the dynamics of PSE, that is, whether or not a PSE converges to a stationary PSE. Also, the conditions under which observational equivalence holds for QSE are yet to be fully characterized.

Our research can be extended in a number of ways. It is worthwhile to compare PSE with QSE in terms of welfare, which is not an obvious task, as welfare criteria under time inconsistency are not clearly defined. Another possible direction of future research is to consider sliding equilibrium paths in a model where agents are heterogeneous in their time preferences or differ in their source of naivete (that is, some agents have perfect foresight, and some have quasi-perfect foresight).

This paper clarifies the concept of naivete and its relation to expectations in the neoclassical growth model with quasi-hyperbolic discounting. We believe that our approach can be used in a much wider range of applications and will be useful for studying many other problems related to time-inconsistent decision making.

Appendix

A Proof of Theorem 1

A.1 A T -horizon PSE

Fix a finite horizon $T \geq 1$ and consider a T -horizon date- τ consumer optimum. Similarly to the infinite horizon case (cf. Eq. (2)), it is easily checked that for all $0 \leq \tau \leq T$, date- τ consumption level on a T -horizon date- τ consumer optimum starting from k_τ at given $\{r_{t+1}\}_{t=\tau}^T$ and $\{w_{t+1}\}_{t=\tau}^T$ satisfies

$$c_\tau^\tau = \frac{f(k_\tau) + \sum_{t=\tau+1}^{T+1} \frac{w_t}{(1+r_{\tau+1}) \cdots (1+r_t)}}{1 + (\beta\delta)^{\frac{1}{\rho}} (1+r_{\tau+1})^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta^{T+1-\tau})^{\frac{1}{\rho}} ((1+r_{\tau+1}) \cdots (1+r_{T+1}))^{\frac{1-\rho}{\rho}}}.$$

Also, similarly to the infinite horizon case, a T -horizon PSE is associated with $T + 1$ optimization problems of the form (4), i.e., with the sequence of the corresponding T -horizon consumer optima. More precisely, a T -horizon PSE starting from $s_{-1}^* = k_0^*$ is a sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_{t+1}^*, w_{t+1}^*\}_{t=0}^T$ characterized as follows:

- there exists a T -horizon date-0 consumer optimum starting from k_0^* at given $\{r_{t+1}^*\}_{t=0}^T$ and $\{w_{t+1}^*\}_{t=0}^T$, which we denote by $\{c_t^{**0}, s_t^{**0}\}_{t=0}^{T+1}$, and its first elements are precisely the date-0 consumption and savings on a T -horizon PSE: $c_0^{**0} = c_0^*$ and $s_0^{**0} = s_0^*$;
- there exists a T -horizon date-1 consumer optimum starting from k_1^* at given $\{r_{t+1}^*\}_{t=1}^T$ and $\{w_{t+1}^*\}_{t=1}^T$, denoted by $\{c_t^{**1}, s_t^{**1}\}_{t=1}^{T+1}$, and its first elements are the date-1 consumption and savings on a T -horizon PSE: $c_1^{**1} = c_1^*$ and $s_1^{**1} = s_1^*$;
- and so on, till the T -horizon date- T consumer optimum starting from k_T^* given r_{T+1}^* and w_{T+1}^* , denoted by $\{c_t^{**T}, s_t^{**T}\}_{t=T}^{T+1}$, whose first elements are the date- T consumption and savings on a T -horizon PSE: $c_T^{**T} = c_T^*$ and $s_T^{**T} = s_T^*$;
- so that the resulting sliding equilibrium capital stock sequence $\{k_{t+1}^*\}_{t=0}^T = \{s_t^{**t}\}_{t=0}^T$ determines precisely those interest and wage rates which were correctly expected by the agent when solving for consumer optima at each date.

A.2 Proof of Lemma 1.1

Step 1. Lower bounds for capital.

Let $\{c_t(T), \underline{k}_{t+1}(T)\}_{t=0}^{T+1}$ be a solution to problem (5). Let for $0 \leq t \leq T + 1$, $\underline{s}_t(T) = \underline{k}_{t+1}(T)$, $1 + \underline{r}_t(T) = f'(\underline{k}_t(T))$, and $\underline{w}_t(T) = f(\underline{k}_t(T)) - f'(\underline{k}_t(T))\underline{k}_t(T)$. It is well known that the sequence $\{\underline{c}_t(T), \underline{s}_t(T), \underline{k}_{t+1}(T), \underline{r}_t(T), \underline{w}_t(T)\}_{t=0}^{T+1}$ is a T -horizon equilibrium path starting from $\underline{s}_{-1} = \underline{k}_0$ in the standard Ramsey model with the discount factor $\beta\delta$. In what follows, a finite horizon T is fixed, so we shall omit the notation “ (T) ” as long as it does not lead to confusion.

In particular, $\{\underline{c}_t\}_{t=0}^{T+1}$ is a solution to the problem

$$\begin{aligned} \max_{c_t \geq 0} \quad & \sum_{t=0}^{T+1} (\beta\delta)^t u(c_t), \quad \text{s. t.} \quad c_0 + \sum_{t=1}^{T+1} \frac{c_t}{(1 + \underline{r}_1) \cdots (1 + \underline{r}_t)} \\ & \leq f(\underline{k}_0) + \sum_{t=1}^{T+1} \frac{\underline{w}_t}{(1 + \underline{r}_1) \cdots (1 + \underline{r}_t)}, \end{aligned}$$

and the following observation holds.

Claim A.1. For all $0 \leq \tau \leq T$,

$$\underline{c}_\tau = \frac{f(\underline{k}_\tau) + \sum_{t=1}^{T+1} \frac{w_t}{(1+r_1)\cdots(1+r_t)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1})^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta)^{\frac{T+1-\tau}{\rho}} ((1+r_{\tau+1})\cdots(1+r_{T+1}))^{\frac{1-\rho}{\rho}}}.$$

Proof. The solution $\{\underline{c}_t\}_{t=0}^{T+1}$ satisfies the following first-order conditions:

$$\underline{c}_{t+1} = (\beta\delta(1+r_{t+1}))^{\frac{1}{\rho}} \underline{c}_t, \quad 0 \leq t \leq T. \quad (\text{A.1})$$

Substituting (A.1) into the budget constraint which holds as equality, we get

$$\underline{c}_0 \left(1 + \sum_{t=1}^{T+1} (\beta\delta)^{\frac{t}{\rho}} ((1+r_1)\cdots(1+r_t))^{\frac{1-\rho}{\rho}} \right) = f(\underline{k}_0) + \sum_{t=1}^{T+1} \frac{w_t}{(1+r_1)\cdots(1+r_t)},$$

which proves the claim for $\tau = 0$. To prove it for $1 \leq \tau \leq T$, it is sufficient to note that the corresponding budget constraint holds for any τ :

$$\underline{c}_\tau + \sum_{t=\tau+1}^{T+1} \frac{\underline{c}_t}{(1+r_{\tau+1})\cdots(1+r_t)} = f(\underline{k}_\tau) + \sum_{t=\tau+1}^{T+1} \frac{w_t}{(1+r_{\tau+1})\cdots(1+r_t)}, \quad (\text{A.2})$$

and repeat the argument. ■

Step 2. Constructing a fixed point.

Consider the following set consisting of bounded sequences of length $T+1$:

$$\mathcal{S}_T = \{ \{k_{t+1}\}_{t=0}^T \mid \underline{k}_1 \leq k_1 \leq f(k_0^*), \text{ and } \underline{k}_{t+1} \leq k_{t+1} \leq f(k_t), \ 1 \leq t \leq T \}.$$

Clearly, \mathcal{S}_T is a non-empty compact convex set. Now, given $\{k_{t+1}\}_{t=0}^T \in \mathcal{S}_T$, consider the sequence $\{\tilde{k}_{t+1}\}_{t=0}^T$ constructed recursively as follows:

$$\tilde{k}_1 = \max\{\underline{k}_1, f(k_0^*) - c_0^0\}, \quad \tilde{k}_{\tau+1} = \max\{\underline{k}_{\tau+1}, f(\tilde{k}_\tau) - c_\tau^\tau\}, \quad 1 \leq \tau \leq T,$$

where c_τ^τ is the date- τ consumption on a T -horizon date- τ consumer optimum starting from k_τ at given interest and wage rates determined by the sequence $\{k_{t+1}\}_{t=\tau}^T$ as $1+r_t = f'(k_t)$, and $w_t = f(k_t) - f'(k_t)k_t$.

By construction, we have $\tilde{k}_{t+1} \geq \underline{k}_{t+1}$ for all $0 \leq t \leq T$. Moreover, since $c_\tau^\tau \geq 0$ and $\underline{k}_{t+1} \leq f(\underline{k}_t)$, it also follows that $\tilde{k}_1 \leq f(k_0^*)$ and $\tilde{k}_{t+1} \leq f(\tilde{k}_t)$ for $1 \leq t \leq T$. Therefore, $\{\tilde{k}_{t+1}\}_{t=0}^T \in \mathcal{S}_T$.

Thus we have a continuous mapping from a compact convex set \mathcal{S}_T to itself such that $\{k_{t+1}\}_{t=0}^T$ maps into $\{\tilde{k}_{t+1}\}_{t=0}^T$. By the Brouwer's fixed point theorem,

there is a sequence $\{k_{t+1}^*\}_{t=0}^T \in \mathcal{S}_T$, which is a fixed point of this mapping.

Step 3. Useful claims.

For the obtained fixed point $\{k_{t+1}^*\}_{t=0}^T$, let $\{r_{t+1}^*\}_{t=0}^T$ and $\{w_{t+1}^*\}_{t=0}^T$ be given by $1 + r_t^* = f'(k_t^*)$ and $w_t^* = f(k_t^*) - f'(k_t^*)k_t^*$ for all $0 \leq t \leq T$. Let $\{c_t^{**\tau}, s_t^{**\tau}\}_{t=\tau}^{T+1}$ be a T -horizon date- τ consumer optimum starting from k_τ^* at given $\{r_{t+1}^*\}_{t=\tau}^T$ and $\{w_{t+1}^*\}_{t=\tau}^T$. Denote the date- τ consumption on this T -horizon date- τ consumer optimum by $c_\tau^* = c_\tau^{**\tau}$. For all $0 \leq \tau \leq T$ (cf. Section A.1),

$$c_\tau^* = \frac{f(k_\tau^*) + \sum_{t=\tau+1}^{T+1} \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta^{T+1-\tau})^{\frac{1}{\rho}} \left((1+r_{\tau+1}^*) \cdots (1+r_{T+1}^*) \right)^{\frac{1-\rho}{\rho}}}. \quad (\text{A.3})$$

The following claim establishes a useful property of the sequence $\{c_t^*\}_{t=0}^T$.

Claim A.2. For $0 \leq t \leq T-1$,

$$c_{t+1}^* \geq c_t^* (\beta\delta(1+r_{t+1}^*))^{\frac{1}{\rho}} \frac{1 + \Delta_t^T}{1 + \beta^{\frac{1}{\rho}} \Delta_t^T} \quad (= \text{if } k_{t+1}^* > \underline{k}_{t+1}), \quad (\text{A.4})$$

where $\Delta_t^T = \delta^{\frac{1}{\rho}}(1+r_{t+2}^*)^{\frac{1-\rho}{\rho}} + \dots + \delta^{\frac{T+1-t}{\rho}}(1+r_{t+2}^*)^{\frac{1-\rho}{\rho}} \cdots (1+r_{T+1}^*)^{\frac{1-\rho}{\rho}}$.

Proof. Note that for $0 \leq t \leq T-2$, we have $\Delta_t^T = \delta^{\frac{1}{\rho}}(1+r_{t+2}^*)^{\frac{1-\rho}{\rho}} (1 + \Delta_{t+1}^T)$ (cf. Eq. (14)). Then (A.3) can be rewritten as

$$c_t^* \left(1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{t+1}^*)^{\frac{1-\rho}{\rho}} (1 + \Delta_t^T) \right) = f(k_t^*) + \sum_{s=t+1}^{T+1} \frac{w_s^*}{(1+r_{t+1}^*) \cdots (1+r_s^*)}. \quad (\text{A.5})$$

By construction of $k_{\tau+1}^*$, we have $c_\tau^* + k_{\tau+1}^* \geq f(k_\tau^*)$ ($=$ if $k_{\tau+1}^* > \underline{k}_{\tau+1}$), and hence it follows from (A.5) for $t = \tau$ that

$$c_\tau^* (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} (1 + \Delta_\tau^T) \leq k_{\tau+1}^* + \sum_{t=\tau+1}^{T+1} \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}, \quad (\text{A.6})$$

(with equality when $k_{\tau+1}^* > \underline{k}_{\tau+1}$).

Consider the value $c_{\tau+1}^* \left(1 + \beta^{\frac{1}{\rho}} \Delta_\tau^T\right)$. By (A.5) for $t = \tau + 1$ and (A.6), we get

$$\begin{aligned} c_{\tau+1}^* \left(1 + \beta^{\frac{1}{\rho}} \Delta_\tau^T\right) &= c_{\tau+1}^* \left(1 + (\beta\delta)^{\frac{1}{\rho}} (1 + r_{\tau+2}^*)^{\frac{1-\rho}{\rho}} (1 + \Delta_{\tau+1}^T)\right) \\ &= (1 + r_{\tau+1}^*) k_{\tau+1}^* + w_{\tau+1}^* + \sum_{t=\tau+2}^{T+1} \frac{w_t^*}{(1 + r_{\tau+2}^*) \cdots (1 + r_t^*)} \\ &= (1 + r_{\tau+1}^*) \left(k_{\tau+1}^* + \sum_{t=\tau+1}^{T+1} \frac{w_t^*}{(1 + r_{\tau+1}^*) \cdots (1 + r_t^*)} \right) \geq c_\tau^* (\beta\delta)^{\frac{1}{\rho}} (1 + r_{\tau+1}^*)^{\frac{1}{\rho}} (1 + \Delta_\tau^T) \end{aligned}$$

(with equality when $k_{\tau+1}^* > \underline{k}_{\tau+1}$).

Therefore, the consumption levels $c_{\tau+1}^*$ and c_τ^* are linked as follows:

$$c_{\tau+1}^* \geq c_\tau^* (\beta\delta(1 + r_{\tau+1}^*))^{\frac{1}{\rho}} \frac{1 + \Delta_\tau^T}{1 + \beta^{\frac{1}{\rho}} \Delta_\tau^T}, \quad (= \text{ if } k_{\tau+1}^* > \underline{k}_{\tau+1}),$$

and this holds true for all $0 \leq \tau \leq T - 1$. ■

Claim A.2 allows us to prove that the lower bound in \mathcal{S}_T is never binding for our fixed point.

Claim A.3. *For all $0 \leq t \leq T$, we have $k_{t+1}^* > \underline{k}_{t+1}$.*

Proof. First, let us prove that $k_{t+1}^* > \underline{k}_{t+1}$ for all $t < T$. Suppose the opposite, i.e., that $k_{\tau+1}^* = \underline{k}_{\tau+1}$ for some $\tau \leq T - 1$. Then, by construction of $k_{\tau+1}^*$,

$$c_\tau^* \geq f(k_\tau^*) - k_{\tau+1}^* \geq f(\underline{k}_\tau) - k_{\tau+1}^* = f(\underline{k}_\tau) - \underline{k}_{\tau+1} = \underline{c}_\tau. \quad (\text{A.7})$$

Let us show that in this case $k_{\tau+2}^* = \underline{k}_{\tau+2}$. Indeed, suppose the opposite, i.e., $k_{\tau+2}^* > \underline{k}_{\tau+2}$. Then, by construction of $k_{\tau+2}^*$,

$$c_{\tau+1}^* = f(k_{\tau+1}^*) - k_{\tau+2}^* = f(\underline{k}_{\tau+1}) - k_{\tau+2}^* < f(\underline{k}_{\tau+1}) - \underline{k}_{\tau+2} = \underline{c}_{\tau+1}.$$

However, since $1 + \Delta_\tau^T > 1 + \beta^{\frac{1}{\rho}} \Delta_\tau^T$, it follows from (A.4), (A.7) and (A.1) that

$$c_{\tau+1}^* \geq c_\tau^* (\beta\delta(1 + r_{\tau+1}^*))^{\frac{1}{\rho}} \frac{1 + \Delta_\tau^T}{1 + \beta^{\frac{1}{\rho}} \Delta_\tau^T} > (\beta\delta(1 + r_{\tau+1}^*))^{\frac{1}{\rho}} c_\tau^* \geq (\beta\delta(r_{\tau+1}))^{\frac{1}{\rho}} \underline{c}_\tau = \underline{c}_{\tau+1}.$$

This contradiction shows that $k_{\tau+2}^* = \underline{k}_{\tau+2}$. Similarly to (A.7), it follows that

$$c_{\tau+1}^* \geq f(k_{\tau+1}^*) - k_{\tau+2}^* = f(\underline{k}_{\tau+1}) - \underline{k}_{\tau+2} = \underline{c}_{\tau+1}. \quad (\text{A.8})$$

Repeating the argument, we obtain that if $k_{\tau+1}^* = \underline{k}_{\tau+1}$, then $k_{t+1}^* = \underline{k}_{t+1}$ for

all $\tau + 1 \leq t \leq T$, and hence the sequence $\{k_{t+1}^*\}_{t=\tau}^T$ coincides with the sequence $\{\underline{k}_{t+1}\}_{t=\tau}^T$. However, in this case for all $\tau \leq t \leq T$, we have $1 + r_{t+1}^* = 1 + \underline{r}_{t+1}$ and $w_{t+1}^* = \underline{w}_{t+1}$, and it follows from Claim A.1 and (A.3) that

$$\begin{aligned} \underline{c}_{\tau+1} &= \frac{f(k_{\tau+1}^*) + \sum_{t=\tau+2}^{T+1} \frac{w_t^*}{(1+r_{\tau+2}^*) \cdots (1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}} (1+r_{\tau+2}^*)^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta)^{\frac{T+1-\tau}{\rho}} \left((1+r_{\tau+2}^*) \cdots (1+r_{T+1}^*) \right)^{\frac{1-\rho}{\rho}}} > \\ &= \frac{f(k_{\tau+1}^*) + \sum_{t=\tau+2}^{T+1} \frac{w_t^*}{(1+r_{\tau+2}^*) \cdots (1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}} (1+r_{\tau+2}^*)^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta^{T+1-\tau})^{\frac{1}{\rho}} \left((1+r_{\tau+2}^*) \cdots (1+r_{T+1}^*) \right)^{\frac{1-\rho}{\rho}}} = c_{\tau+1}^*, \end{aligned}$$

because in the denominator $\beta < 1$. This contradiction to (A.8) shows that for all $0 \leq t \leq T-1$, we have $k_{t+1}^* > \underline{k}_{t+1}$.

Second, let us prove that $k_{T+1}^* > \underline{k}_{T+1}$. Suppose the opposite, i.e., $k_{T+1}^* = \underline{k}_{T+1}$, and hence $1 + r_{T+1}^* = 1 + \underline{r}_{T+1}$ and $w_{T+1}^* = \underline{w}_{T+1}$. By construction of k_{T+1}^* ,

$$c_T^* \geq f(k_T^*) - k_{T+1}^* \geq f(\underline{k}_T) - k_{T+1}^* = f(\underline{k}_T) - \underline{k}_{T+1} = \underline{c}_T. \quad (\text{A.9})$$

Consider now the T -horizon date- T consumer optimum starting from k_T^* at given $1 + r_{T+1}^*$ and w_{T+1}^* . By the first-order condition, (A.9) and (A.1), we have $c_{T+1}^{**T} = (\beta\delta(1+r_{T+1}^*))^{\frac{1}{\rho}} c_T^* = (\beta\delta(1+\underline{r}_{T+1}))^{\frac{1}{\rho}} c_T^* > (\beta\delta(1+\underline{r}_{T+1}))^{\frac{1}{\rho}} \underline{c}_T = \underline{c}_{T+1}$. At the same time, it follows from the budget constraint, (A.9) and (A.2) that

$$\begin{aligned} c_{T+1}^{**T} &= (1+r_{T+1}^*) (f(k_T^*) - c_T^*) + w_{T+1}^* \leq (1+r_{T+1}^*) k_{T+1}^* + w_{T+1}^* \\ &= f(k_{T+1}^*) = f(\underline{k}_{T+1}) = (1+\underline{r}_{T+1}) (f(\underline{k}_T) - \underline{c}_T) + \underline{w}_{T+1} = \underline{c}_{T+1}. \end{aligned}$$

This contradiction shows that $k_{T+1}^* > \underline{k}_{T+1}$, which proves the claim. \blacksquare

Step 4. Ensuring existence.

Claim A.3 shows that for all $0 \leq \tau \leq T$, our obtained fixed point satisfies $k_{\tau+1}^* > \underline{k}_{\tau+1}$. Let for $0 \leq \tau \leq T$, $s_\tau^* = k_{\tau+1}^*$. Then $s_\tau^* = f(k_\tau^*) - c_\tau^*$, and hence by construction c_τ^* and s_τ^* are the first elements of the T -horizon date- τ consumer optimum starting from k_τ^* at given $\{r_{t+1}^*\}_{t=\tau}^T$ and $\{w_{t+1}^*\}_{t=\tau}^T$.

It follows that the obtained fixed point determines the T -horizon PSE starting from $s_{-1}^* = k_0^*$. Formally, the sequence $\{c_t^*, s_t^*, k_{t+1}^*, r_{t+1}^*, w_{t+1}^*\}_{t=0}^T$, where $\{k_{t+1}^*\}_{t=0}^T$ is the fixed point of the described mapping $\mathcal{S}_T \rightarrow \mathcal{S}_T$, and for all $0 \leq \tau \leq T$, $c_\tau^* = f(k_\tau^*) - k_{\tau+1}^*$, $s_\tau^* = k_{\tau+1}^*$, $1 + r_\tau^* = f'(k_\tau^*)$, and $w_\tau^* = f(k_\tau^*) - f'(k_\tau^*) k_\tau^*$, satisfies Definition 2 and hence is a T -horizon PSE.

A.3 Proof of Lemma 1.2

It follows from Claim A.3 that the sequence of capital stocks on a T -horizon PSE is bounded from below by the sequence of capital stocks $\{\bar{k}_{t+1}(T)\}_{t=0}^T$. It remains to show that it is also bounded from above.

Step 1. Upper bounds for capital.

Let $\{\bar{c}_t(T), \bar{k}_{t+1}(T)\}_{t=0}^{T+1}$ be a solution to problem (6). Let for $0 \leq t \leq T+1$, $\bar{s}_t(T) = \bar{k}_{t+1}(T)$, $1 + \bar{r}_t(T) = f'(\bar{k}_t(T))$, and $\bar{w}_t(T) = f(\bar{k}_t(T)) - f'(\bar{k}_t(T))\bar{k}_t(T)$. Then the sequence $\{\bar{c}_t(T), \bar{s}_t(T), \bar{k}_{t+1}(T), \bar{r}_t(T), \bar{w}_t(T)\}_{t=0}^{T+1}$ is a T -horizon equilibrium path starting from $\bar{s}_{-1} = \bar{k}_0$ in the standard Ramsey model with the discount factor δ . In what follows we omit the notation “ (T) ”, as there will be no confusion.

Similarly to (A.1) and (A.2), it is easily checked that $\{\bar{c}_\tau\}_{\tau=0}^{T+1}$ satisfies

$$\bar{c}_{\tau+1} = (\delta(1 + \bar{r}_{\tau+1}))^{\frac{1}{\rho}} \bar{c}_\tau, \quad (\text{A.10})$$

$$\bar{c}_\tau + \sum_{t=\tau+1}^{T+1} \frac{\bar{c}_t}{(1 + \bar{r}_{\tau+1}) \cdots (1 + \bar{r}_t)} = f(\bar{k}_\tau) + \sum_{t=\tau+1}^{T+1} \frac{\bar{w}_t}{(1 + \bar{r}_{\tau+1}) \cdots (1 + \bar{r}_t)}. \quad (\text{A.11})$$

Step 2. Upper and lower bounds for the consumption growth rate.

Let $\{c_t^*, s_t^*, k_{t+1}^*, r_{t+1}^*, w_{t+1}^*\}_{t=0}^T$ be a T -horizon PSE starting from $s_{-1}^* = k_0^*$. We already know that date- τ consumption on this path is given by (A.3). The following claim provides upper and lower bounds for the consumption growth rate on a T -horizon PSE.

Claim A.4. *On a T -horizon PSE we have*

$$(\beta\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}} < \frac{c_{t+1}^*}{c_t^*} < (\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}}. \quad (\text{A.12})$$

Proof. By Claims A.2 and A.3, on a T -horizon PSE the consumption levels at any two adjacent dates are linked via the following “first-order conditions” (cf. Eq. (15)): $c_{t+1}^* = c_t^* (\beta\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}} \frac{1 + \Delta_t^T}{1 + \beta^{\frac{1}{\rho}} \Delta_t^T}$. Since $1 + \Delta_t^T > 1 + \beta^{\frac{1}{\rho}} \Delta_t^T > \beta^{\frac{1}{\rho}} + \beta^{\frac{1}{\rho}} \Delta_t^T$, we have

$$\frac{c_{t+1}^*}{c_t^*} = (\beta\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}} \frac{1 + \Delta_t^T}{1 + \beta^{\frac{1}{\rho}} \Delta_t^T} > (\beta\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}},$$

and

$$\frac{c_{t+1}^*}{c_t^*} = (\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}} \frac{\beta^{\frac{1}{\rho}} + \beta^{\frac{1}{\rho}} \Delta_t^T}{1 + \beta^{\frac{1}{\rho}} \Delta_t^T} < (\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}},$$

and hence (A.12) holds. ■

Step 3. Proof of the lemma.

Now we can show that the sequence of capital stocks on a T -horizon PSE is bounded from above by the sequence $\{\bar{k}_{t+1}(T)\}_{t=0}^T$.

Claim A.5. *For all $0 \leq t \leq T$, we have $k_{t+1}^* < \bar{k}_{t+1}$.*

Proof. Suppose the opposite, i.e., that while $k_0^* = \bar{k}_0$ and $k_\tau^* < \bar{k}_\tau$ (if $\tau > 0$), we have $k_{\tau+1}^* \geq \bar{k}_{\tau+1}$ for some $\tau \leq T - 1$. Then

$$c_\tau^* = f(k_\tau^*) - k_{\tau+1}^* \leq f(\bar{k}_\tau) - k_{\tau+1}^* \leq f(\bar{k}_\tau) - \bar{k}_{\tau+1} = \bar{c}_\tau. \quad (\text{A.13})$$

Let us show that also $k_{\tau+2}^* \geq \bar{k}_{\tau+2}$. Suppose the opposite, i.e., $k_{\tau+2}^* < \bar{k}_{\tau+2}$. Then $c_{\tau+1}^* = f(k_{\tau+1}^*) - k_{\tau+2}^* \geq f(\bar{k}_{\tau+1}) - k_{\tau+2}^* > f(\bar{k}_{\tau+1}) - \bar{k}_{\tau+2} = \bar{c}_{\tau+1}$. At the same time, it follows from (A.12), (A.13) and (A.10) that

$$c_{\tau+1}^* < (\delta(1 + r_{t+1}^*))^{\frac{1}{\rho}} c_\tau^* \leq (\delta(1 + \bar{r}_{t+1}))^{\frac{1}{\rho}} c_\tau^* \leq (\delta(1 + \bar{r}_{t+1}))^{\frac{1}{\rho}} \bar{c}_\tau = \bar{c}_{\tau+1}.$$

This contradiction shows that $k_{\tau+2}^* \geq \bar{k}_{\tau+2}$.

Repeating the argument, we obtain that if $k_{\tau+1}^* \geq \bar{k}_{\tau+1}$, then $k_{t+1}^* \geq \bar{k}_{t+1}$ for all $\tau + 1 \leq t \leq T$. It then follows from (A.12) that $c_t^* \leq \bar{c}_t$ for all $\tau \leq t \leq T$.

Consider now the T -horizon date- T consumer optimum starting from k_T^* at given $1 + r_{T+1}^*$ and w_{T+1}^* . Since $\beta < 1$, by the first-order condition and (A.10), $c_{T+1}^{**T} < (\delta(1 + r_{T+1}^*))^{\frac{1}{\rho}} c_T^* \leq (\delta(1 + \bar{r}_{t+1}))^{\frac{1}{\rho}} c_T^* \leq (\delta(1 + \bar{r}_{t+1}))^{\frac{1}{\rho}} \bar{c}_T = \bar{c}_{T+1}$. However, it follows from the budget constraint and (A.11) that

$$\begin{aligned} c_{T+1}^{**T} &= (1 + r_{T+1}^*) (f(k_T^*) - c_T^*) + w_{T+1}^* = (1 + r_{T+1}^*) k_{T+1}^* + w_{T+1}^* \\ &= f(k_{T+1}^*) \geq f(\bar{k}_{T+1}) = (1 + \bar{r}_{T+1}) (f(\bar{k}_T) - \bar{c}_T) + \bar{w}_{T+1} = \bar{c}_{T+1}, \end{aligned}$$

and this contradiction shows that indeed $k_{t+1}^* < \bar{k}_{t+1}$ for all $0 \leq t \leq T$. ■

A.4 Proof of Lemma 1.3

Step 1. Upper and lower bounds for capital.

Let $\{\underline{c}_t, \underline{k}_{t+1}\}_{t=0}^\infty$ be the $\beta\delta$ -optimal path, i.e., the optimal path starting from $\underline{k}_0 = k_0^*$ in the standard Ramsey model with a discount factor $\beta\delta$. It is clear that $\underline{c}_t = \lim_{T \rightarrow \infty} \underline{c}_t(T)$ and $\underline{k}_{t+1} = \lim_{T \rightarrow \infty} \underline{k}_{t+1}(T)$.

Similarly, let $\{\bar{c}_t, \bar{k}_{t+1}\}_{t=0}^\infty$ be the δ -optimal path, i.e., the optimal path starting from $\bar{k}_0 = k_0^*$ in the standard Ramsey model with a discount factor δ . As above, it is clear that $\bar{c}_t = \lim_{T \rightarrow \infty} \bar{c}_t(T)$ and $\bar{k}_{t+1} = \lim_{T \rightarrow \infty} \bar{k}_{t+1}(T)$.

Step 2. A closed-form expression for consumption.

Now note that by construction of the sequence (8), $\{c_t^*, s_t^*, k_{t+1}^*, r_{t+1}^*, w_{t+1}^*\}_{t=0}^\infty$, we have $c_t^* = \lim_{T \rightarrow \infty} c_t^*(T)$, $s_t^* = \lim_{T \rightarrow \infty} s_t^*(T)$, $k_{t+1}^* = \lim_{T \rightarrow \infty} k_{t+1}^*(T)$, $r_{t+1}^* = \lim_{T \rightarrow \infty} r_{t+1}^*(T)$, and $w_{t+1}^* = \lim_{T \rightarrow \infty} w_{t+1}^*(T)$.

Taking the limit $T \rightarrow \infty$ in (7), we obtain that $\underline{k}_{t+1} \leq k_{t+1}^* \leq \bar{k}_{t+1}$ for all $t \geq 0$. It immediately follows that $\underline{s}_t \leq s_t^* \leq \bar{s}_t$, $\underline{w}_t \leq w_t^* \leq \bar{w}_t$, and $\bar{r}_t \leq r_t^* \leq \underline{r}_t$.

Note also that $c_0^* = f(k_0^*) - k_1^* = f(\bar{k}_0) - k_1^* \geq f(\bar{k}_0) - \bar{k}_1 = \bar{c}_0$, and taking the limit $T \rightarrow \infty$ in (A.12), we get

$$\frac{c_{t+1}^*}{c_t^*} = \lim_{T \rightarrow \infty} \frac{c_{t+1}^*(T)}{c_t^*(T)} \geq (\beta\delta(1+r_{t+1}^*))^{\frac{1}{\rho}} \geq (\beta\delta(1+\bar{r}_{t+1}))^{\frac{1}{\rho}},$$

so that $c_t^* > 0$ for all $t \geq 0$.

Let us show that c_τ^* is given by a limit of (A.3) as $T \rightarrow \infty$.

Claim A.6. *For all $\tau \geq 0$, we have*

$$c_\tau^* = \frac{f(k_\tau^*) + \sum_{t=\tau+1}^\infty \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta)^{\frac{1}{\rho}}((1+r_{\tau+1}^*) \cdots (1+r_{\tau+t}^*))^{\frac{1-\rho}{\rho}} + \dots}. \quad (\text{A.14})$$

Proof. Note that (A.3) can be written as

$$\begin{aligned} c_\tau^*(T) & \left(1 + \sum_{t=\tau+1}^{T+1} (\beta\delta^{t-\tau})^{\frac{1}{\rho}} ((1+r_{\tau+1}^*(T)) \cdots (1+r_t^*(T)))^{\frac{1-\rho}{\rho}} \right) \\ & = f(k_\tau^*(T)) + \sum_{t=\tau+1}^{T+1} \frac{w_t^*(T)}{(1+r_{\tau+1}^*(T)) \cdots (1+r_t^*(T))}. \end{aligned} \quad (\text{A.15})$$

By (7), the right hand side in (A.15) is bounded from above for any T , and hence there exists a finite limit as $T \rightarrow \infty$:

$$\begin{aligned} & \lim_{T \rightarrow \infty} \left\{ f(k_\tau^*(T)) + \sum_{t=\tau+1}^{T+1} \frac{w_t^*(T)}{(1+r_{\tau+1}^*(T)) \cdots (1+r_t^*(T))} \right\} \\ & = f(k_\tau^*) + \sum_{t=\tau+1}^\infty \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)} \leq f(\bar{k}_\tau) + \sum_{t=\tau+1}^\infty \frac{\bar{w}_t}{(1+\bar{r}_{\tau+1}) \cdots (1+\bar{r}_t)}. \end{aligned}$$

It then follows from (A.15) that there exists a finite limit of its left hand side

as $T \rightarrow \infty$, and hence the following series converges:

$$\begin{aligned} & \lim_{T \rightarrow \infty} \left(1 + \sum_{t=\tau+1}^{T+1} (\beta\delta^{t-\tau})^{\frac{1}{\rho}} \left((1+r_{\tau+1}^*(T)) \cdots (1+r_t^*(T)) \right)^{\frac{1-\rho}{\rho}} \right) \\ &= 1 + (\beta\delta)^{\frac{1}{\rho}} (1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta^t)^{\frac{1}{\rho}} \left((1+r_{\tau+1}^*) \cdots (1+r_{\tau+t}^*) \right)^{\frac{1-\rho}{\rho}} + \dots \end{aligned}$$

Therefore, taking the limit $T \rightarrow \infty$ in (A.15), we obtain (A.14). \blacksquare

Step 3. Infinite horizon date- τ consumer optimum.

For each $\tau \geq 0$, consider the sequence $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$, which is defined as follows:

$$\begin{aligned} c_{\tau}^{**\tau} &= c_{\tau}^*, & c_{\tau+1}^{**\tau} &= (\beta\delta(1+r_{\tau+1}^*))^{\frac{1}{\rho}} c_{\tau}^{**\tau}, \\ c_{t+1}^{**\tau} &= (\delta(1+r_{t+1}^*))^{\frac{1}{\rho}} c_t^{**\tau}, & t &\geq \tau+1. \end{aligned} \tag{A.16}$$

Claim A.7. *The sequence $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ is the sequence of consumptions on the date- τ consumer optimum starting from k_{τ}^* at given $\{r_{t+1}^*\}_{t=\tau}^{\infty}$ and $\{w_{t+1}^*\}_{t=\tau}^{\infty}$.*

Proof. Let us show that $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ is the solution to the following problem:

$$\begin{aligned} \max_{c_t \geq 0} \quad & u(c_{\tau}) + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} u(c_t), & \text{s. t.} \quad & c_{\tau} + \sum_{t=\tau+1}^{\infty} \frac{c_t}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)} \\ & & & \leq f(k_{\tau}^*) + \sum_{t=\tau+1}^{\infty} \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}. \end{aligned}$$

It follows from (A.16) that $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ satisfies the first-order conditions for the solution to this problem. By combining (A.16) and (A.14), it is easily seen that

$$\begin{aligned} c_{\tau}^{**\tau} + \sum_{t=\tau+1}^{\infty} \frac{c_t^{**\tau}}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)} &= c_{\tau}^* \left(1 + (\beta\delta)^{\frac{1}{\rho}} (1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} + \dots \right. \\ & \left. + (\beta\delta^t)^{\frac{1}{\rho}} \left((1+r_{\tau+1}^*) \cdots (1+r_{\tau+t}^*) \right)^{\frac{1-\rho}{\rho}} + \dots \right) = f(k_{\tau}^*) + \sum_{t=\tau+1}^{\infty} \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}, \end{aligned}$$

so that $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$ satisfies the budget constraint in this problem.

Consider the utility on the path $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$, $U_{\tau} = u(c_{\tau}^{**\tau}) + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} u(c_t^{**\tau})$.

Using (A.16) and (A.14), we obtain that

$$\begin{aligned}
U_\tau &= \frac{(c_\tau^{**\tau})^{1-\rho}}{1-\rho} + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} \frac{(c_t^{**\tau})^{1-\rho}}{1-\rho} = \frac{(c_\tau^{**\tau})^{1-\rho}}{1-\rho} \left(1 + \beta \sum_{t=\tau+1}^{\infty} \delta^{t-\tau} \left(\frac{c_t^{**\tau}}{c_\tau^{**\tau}} \right)^{1-\rho} \right) \\
&= \frac{1}{1-\rho} \frac{c_\tau^{**\tau}}{(c_\tau^{**\tau})^\rho} \left(1 + \sum_{t=\tau+1}^{\infty} (\beta \delta^{t-\tau})^{\frac{1}{\rho}} \left((1+r_{\tau+1}^*) \cdots (1+r_t^*) \right)^{\frac{1-\rho}{\rho}} \right) \\
&= \frac{1}{1-\rho} \frac{1}{(c_\tau^*)^\rho} \left(f(k_\tau^*) + \sum_{t=\tau+1}^{\infty} \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)} \right).
\end{aligned}$$

It is now clear that $-\infty < U_\tau < \infty$, which completes the proof of the claim. \blacksquare

Step 4. Ensuring existence.

Claim A.7 shows that the sequence $\{c_t^{**\tau}\}_{t=\tau}^{\infty}$, determined by (A.16), is the sequence of consumptions on the date- τ consumer optimum starting from k_τ^* at given $\{r_{t+1}^*\}_{t=\tau}^{\infty}$ and $\{w_{t+1}^*\}_{t=\tau}^{\infty}$. Therefore, for each τ , c_τ^* and $s_\tau^* = f(k_\tau^*) - c_\tau^*$ are the first elements of the date- τ consumer optimum starting from k_τ^* at given $\{r_{t+1}^*\}_{t=\tau}^{\infty}$ and $\{w_{t+1}^*\}_{t=\tau}^{\infty}$. Thus the sequence $\{c_t^*, s_t^*, k_t^*, r_t^*, w_t^*\}_{t=0}^{\infty}$, given by (8), is a sliding equilibrium path under perfect foresight starting from $s_{-1}^* = k_0^*$, which completes the proof of Lemma 1.3 and the proof of existence theorem as well.

B Proof of Theorem 2

B.1 Proof of Lemma 2.1

Since $\frac{1}{1+r^*} < 1$, the right-hand side of the budget constraint in problem (1) under constant interest rate r^* and wage rate w^* is finite:

$$(1+r^*)s^* + \sum_{t=0}^{\infty} \frac{w^*}{(1+r^*)^t} = (1+r^*)s^* + \frac{w^*}{1-\frac{1}{1+r^*}} = \frac{1+r^*}{r^*} (r^*s^* + w^*) < +\infty.$$

It now follows from (2) that

$$c_\tau^{**\tau} = \frac{\frac{1+r^*}{r^*} (r^*s^* + w^*)}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta^t)^{\frac{1}{\rho}}(1+r^*)^{\frac{t(1-\rho)}{\rho}} + \dots}.$$

Since $\delta(1+r^*)^{1-\rho} < 1$, the sum in the denominator is finite:

$$\begin{aligned} & 1 + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} \left(1 + (\delta(1+r^*)^{1-\rho})^{\frac{1}{\rho}} + (\delta(1+r^*)^{1-\rho})^{\frac{2}{\rho}} + \dots \right) \\ &= 1 + \frac{(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}} = \frac{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}. \end{aligned}$$

Thus a date- τ consumer optimum at given r^* and w^* exists and is unique. In this optimum, $c_\tau^{**\tau}$ is given by (11).

B.2 Proof of Lemma 2.2

The interest rate on a SPSE, r^* , satisfies

$$1 + r^* = \frac{1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}{(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}}}.$$

Rearranging the above equation, we get

$$(\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1}{\rho}} = 1 - \delta^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}} + (\beta\delta)^{\frac{1}{\rho}}(1+r^*)^{\frac{1-\rho}{\rho}},$$

and hence $\frac{1}{1+r^*} = \frac{\left(\frac{1}{1+r^*}\right)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}$. Thus r^* is the solution to equation (12).

Denote $\gamma^* = \frac{1}{1+r^*}$. It is clear that equations (10) and (12) are essentially the same. Let us show that there exists a unique solution to equation (10), and this solution satisfies $\delta < (\gamma^*)^{1-\rho}$, i.e., $\delta(1+r^*)^{1-\rho} < 1$.

Consider the functions $L(\gamma) = \gamma$, and $R(\gamma) = \frac{\gamma^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}$. Both $L(\gamma)$ and $R(\gamma)$ are monotonically increasing in γ . Moreover, $R(\beta\delta) = 0 < \beta\delta = L(\beta\delta)$, and $R(\delta) = 1 > \delta = L(\delta)$. Since $R(\gamma)$ is strictly convex for $\rho < 1$, linear for $\rho = 1$, and strictly concave for $\rho > 1$, for any ρ there is a unique γ^* such that $L(\gamma^*) = R(\gamma^*)$. This γ^* is the solution to equation (10), and it is clear that $\beta\delta < \gamma^* < \delta$.

Since $\rho > 0$, we also have $(\beta\delta)^{\frac{1}{\rho}} < (\gamma^*)^{\frac{1}{\rho}} < \delta^{\frac{1}{\rho}}$, and hence

$$\gamma^* = \frac{(\gamma^*)^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}} < \frac{(\gamma^*)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}}}.$$

Therefore, $\delta < (\gamma^*)^{1-\rho}$, which completes the proof of the lemma.

C Proof of Lemma 3.1

Let $c_\tau^{**\tau}$ be the date- τ consumption in the date- τ consumer optimum starting from $s_{\tau-1}^*$ at given $\{r_t^*\}_{t=\tau}^\infty$ and $\{w_t^*\}_{t=\tau}^\infty$. It follows from (2) that

$$c_\tau^{**\tau} = \frac{(1+r_\tau^*)s_{\tau-1}^* + w_\tau^* + \sum_{t=\tau+1}^\infty \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta)^{\frac{1}{\rho}} \left((1+r_{\tau+1}^*) \cdots (1+r_{\tau+t}^*) \right)^{\frac{1-\rho}{\rho}} + \dots}.$$

Due to the properties of r^* (cf. Lemma 2.2), $c_\tau^{**\tau}$ is well-defined for all τ :

$$\begin{aligned} \frac{w_{t+1}^*}{(1+r_{\tau+1}^*) \cdots (1+r_{t+1}^*)} \frac{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}{w_t^*} &= \frac{w_{t+1}^*}{w_t^*} \frac{1}{1+r_{t+1}^*} \xrightarrow{t \rightarrow \infty} \frac{1}{1+r^*} < 1, \\ \frac{(\beta\delta^{t+1})^{\frac{1}{\rho}} \left((1+r_{\tau+1}^*) \cdots (1+r_{t+1}^*) \right)^{\frac{1-\rho}{\rho}}}{(\beta\delta^t)^{\frac{1}{\rho}} \left((1+r_{\tau+1}^*) \cdots (1+r_t^*) \right)^{\frac{1-\rho}{\rho}}} &= \delta^{\frac{1}{\rho}} (1+r_{t+1}^*)^{\frac{1-\rho}{\rho}} \xrightarrow{t \rightarrow \infty} \delta^{\frac{1}{\rho}} (1+r^*)^{\frac{1-\rho}{\rho}} < 1. \end{aligned}$$

By the d'Alembert's ratio test both infinite series in the equation for $c_\tau^{**\tau}$ converge.

Let Δ_{t+1} be given by (13). By the same argument as above, $0 < \Delta_t < \infty$ for all t . Now, in a PSE, we have for all $\tau \geq 0$,

$$c_\tau^* = c_\tau^{**\tau} = \frac{(1+r_\tau^*)s_{\tau-1}^* + w_\tau^* + \sum_{t=\tau+1}^\infty \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}} (1+\Delta_{\tau+1})}, \quad (\text{C.1})$$

$$k_{\tau+1}^* = s_\tau^* = (1+r_\tau^*)s_{\tau-1}^* + w_\tau^* - c_\tau^*, \quad 1+r_\tau^* = f'(k_\tau^*), \quad w_\tau^* = f(k_\tau^*) - f'(k_\tau^*)k_\tau^*.$$

Consider the value $c_{\tau+1}^*(1 + \beta^{\frac{1}{\rho}}\Delta_{\tau+1})$. By (14) and (C.1), we have

$$\begin{aligned} c_{\tau+1}^* \left(1 + \beta^{\frac{1}{\rho}}\Delta_{\tau+1} \right) &= c_{\tau+1}^* \left(1 + (\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+2}^*)^{\frac{1-\rho}{\rho}} (1+\Delta_{\tau+2}) \right) \\ &= (1+r_{\tau+1}^*) \left(s_\tau^* + \sum_{t=\tau+1}^\infty \frac{w_t^*}{(1+r_{\tau+1}^*) \cdots (1+r_t^*)} \right) = (1+r_{\tau+1}^*) (s_\tau^* - (1+r_\tau^*)s_{\tau-1}^* \\ &\quad - w_\tau^* + c_\tau^* + c_\tau^*(\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}}(1+\Delta_{\tau+1})) = c_\tau^*(\beta\delta)^{\frac{1}{\rho}}(1+r_{\tau+1}^*)^{\frac{1-\rho}{\rho}}(1+\Delta_{\tau+1}). \end{aligned}$$

Thus, in a PSE the consumption levels at two adjacent dates are linked via the “first-order conditions” (15), which proves the lemma.

D Characterization of sliding optimal paths

A SOP is essentially characterized by the first step in problem (17), i.e., by the first elements from the date- τ optimal path $\{c_t^{*\tau}, k_{t+1}^{*\tau}\}_{t=\tau}^\infty$. After the first step

is implemented at date τ ($c_{\tau}^{*\tau}$ is consumed and $k_{\tau+1}^{*\tau}$ remains as the new capital stock), the remaining problem is problem (9) with the constant discount factor δ .

Let $V_{\gamma}(k)$ be the value function of problem (9):

$$V_{\gamma}(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \gamma V_{\gamma}(k'). \quad (\text{D.1})$$

The associated policy function $g_{\gamma}(k)$ is implicitly given by the functional equation:

$$u'(f(k) - g_{\gamma}(k)) = \gamma V'_{\gamma}(g_{\gamma}(k)). \quad (\text{D.2})$$

Thus the γ -optimal path in the standard Ramsey model is fully determined by value function $V_{\gamma}(k)$ or policy function $g_{\gamma}(k)$.

Since the continuation of the date- τ optimal path is the δ -optimal path, the first step in problem (17) can be described as follows:

$$\max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta \delta V_{\delta}(k').$$

Therefore, the policy function $h(k)$ which solves the functional equation

$$u'(f(k) - h(k)) = \beta \delta V'_{\delta}(h(k)), \quad (\text{D.3})$$

determines a SOP which is a sequence $\{c_t^{\circ}, k_{t+1}^{\circ}\}_{t=0}^{\infty}$ such that $c_t^{\circ} = f(k_t^{\circ}) - h(k_t^{\circ})$ and $k_{t+1}^{\circ} = h(k_t^{\circ})$.

Clearly, a SOP under β - δ discounting is observationally equivalent to some γ -optimal path if and only if $h(k) = g_{\gamma}(k)$ for all k , which, using (D.2) and (D.3), can be written as

$$\gamma V'_{\gamma}(k) = \beta \delta V'_{\delta}(k), \quad \forall k. \quad (\text{D.4})$$

Therefore, observational equivalence of SOP depends on the properties of a value function of the standard Ramsey model.

It is very hard to expect that the derivative of a value function, $\gamma V'_{\gamma}(k)$, simply scales when a discount factor changes, i.e., that equation (D.4) holds. To the best of our knowledge, a SOP under quasi-hyperbolic discounting is observationally equivalent to a γ -optimal path either for log-utility and Cobb–Douglas production technology or for a stationary sliding optimum.

Proof of Proposition 2. When $u(c) = \ln c$ and $f(k) = k^{\alpha}$, there is a closed-form solution for (D.1): $V_{\gamma}(k) = \frac{1}{1-\gamma} \left(\ln(1-\alpha\gamma) + \frac{\alpha\gamma}{1-\alpha\gamma} \ln(\alpha\gamma) \right) + \frac{\alpha}{1-\alpha\gamma} \ln k$, and the associated policy function is given by $g_{\gamma}(k) = \alpha\gamma k^{\alpha}$. Then equation (D.3) which determines the policy function $h(k)$ associated with a SOP takes the form

$\frac{1}{k^\alpha - h(k)} = \beta \delta V'_\delta(h(k))$, which can be rewritten as $\frac{1}{k^\alpha - h(k)} = \frac{\alpha \beta \delta}{1 - \alpha \delta} \frac{1}{h(k)}$, so that $h(k) = \frac{\alpha \beta \delta}{1 - \alpha \delta + \alpha \beta \delta} k^\alpha$. Hence $h(k)$ coincides with the policy function $g_{\gamma^\circ}(k)$ associated with the γ° -optimal path for $\gamma^\circ = \frac{\beta \delta}{1 - \alpha \delta + \alpha \beta \delta}$. Therefore, a SOP is observationally equivalent to γ° -optimal path. \blacksquare

A stationary sliding optimum is observationally equivalent to a stationary γ° -optimum for $\gamma^\circ = 1/f'(k^\circ)$. Let us show that $\gamma^\circ < \delta$, which is equivalent to $k^\circ < k^\delta$, where k^δ is the modified golden rule capital stock for the discount factor δ . Indeed, k° is the solution to equation (D.3) for $h(k) = k$:

$$f(k) - k = (\beta \delta V'_\delta(k))^{-\frac{1}{\rho}}. \quad (\text{D.5})$$

Consider the functions $L(k) = f(k) - k$ and $R(k) = \frac{1}{(\beta \delta V'_\delta(k))^{\frac{1}{\rho}}}$. Clearly, $L(k)$ monotonically increases for $k < k^\delta$, is concave, and $L(0) = 0$. At the same time, since $V_\delta(k)$ is concave, we have $R'(k) = \left(1/\rho(\beta \delta)^{\frac{1}{\rho}}\right) |V''_\delta(k)| / (V'_\delta(k))^{1+\frac{1}{\rho}} > 0$. Thus, $R(k)$ also monotonically increases for $k < k^\delta$, and $R(0) = 0$. When $\beta = 1$, the solution to (D.5) is k^δ . As β decreases, $R(k)$ shifts upward, and hence the capital stock which solves equation (D.5) decreases. This means that for $\beta < 1$, we have $k^\circ < k^\delta$, i.e., $\gamma^\circ < \delta$.

E Proof of Theorem 4

The proof is by contradiction. Suppose that $k^* \leq k^\circ$, so $r^\circ \leq r^*$ and $w^\circ \geq w^*$. Let $c(r^\circ, w^\circ)$ be the date- τ consumption in the date- τ consumer optimum starting from s° at given constant r° and w° . The following lemma shows that $c(r^\circ, w^\circ) \geq c^\circ$.

Lemma E.1. *Suppose that $0 < \rho \leq 1$, and $k^* \leq k^\circ$. Then $c(r^\circ, w^\circ) \geq c^\circ$.*

Proof. Since $r^\circ \leq r^*$ and $0 < \rho \leq 1$, we have $\delta(1 + r^\circ)^{1-\rho} \leq \delta(1 + r^*)^{1-\rho} < 1$. It then follows from Lemma 2.1 that $c(r^\circ, w^\circ)$ exists and is given by

$$c(r^\circ, w^\circ) = \frac{1 - \delta^{\frac{1}{\rho}}(1 + r^\circ)^{\frac{1-\rho}{\rho}}}{1 - \delta^{\frac{1}{\rho}}(1 + r^\circ)^{\frac{1-\rho}{\rho}} + (\beta \delta)^{\frac{1}{\rho}}(1 + r^\circ)^{\frac{1-\rho}{\rho}}} \cdot \frac{1 + r^\circ}{r^\circ} \cdot (r^\circ s^\circ + w^\circ).$$

Taking into account that $c^\circ = (1 + r^\circ)s^\circ + w^\circ - s^\circ = r^\circ s^\circ + w^\circ$, we get $c(r^\circ, w^\circ) = Q(r^\circ)c^\circ$, where the function $Q(r)$ is defined as

$$Q(r) = \frac{1 - \delta^{\frac{1}{\rho}}(1 + r)^{\frac{1-\rho}{\rho}}}{1 - \delta^{\frac{1}{\rho}}(1 + r)^{\frac{1-\rho}{\rho}} + (\beta \delta)^{\frac{1}{\rho}}(1 + r)^{\frac{1-\rho}{\rho}}} \cdot \frac{1 + r}{r}.$$

It follows from the proof of Lemma 2.1 that $Q(r^*) = 1$, where r^* is the interest rate on a SPSE.

Let us show that for all $r \leq r^*$, $Q(r) \geq Q(r^*) = 1$. Repeating the argument used in the proof of Theorem 2, we find that $Q(r) \geq 1$ for $\frac{1}{1+r} \leq \frac{(\frac{1}{1+r})^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}$.

Let $\gamma = \frac{1}{1+r}$, and consider the functions $L(\gamma) = \gamma$ and $R(\gamma) = \frac{\gamma^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}{\delta^{\frac{1}{\rho}} - (\beta\delta)^{\frac{1}{\rho}}}$ (cf. proof of Lemma 2.2). Since they are monotone and $R(\gamma^*) = L(\gamma^*)$, while $R(\delta) = 1 > \delta = L(\delta)$, it follows that $R(\gamma) \geq L(\gamma)$ for all γ such that $\gamma^* \leq \gamma < \delta$. In terms of interest rates, $Q(r) \geq 1$ for $r \leq r^*$. Since by assumption $r^\circ \leq r^*$, $Q(r^\circ) \geq Q(r^*) = 1$. Thus, $c(r^\circ, w^\circ) = Q(r^\circ)c^\circ \geq c^\circ$, which proves the lemma. ■

However, the following lemma shows that the opposite inequality holds.

Lemma E.2. *Suppose that $0 < \rho \leq 1$, and $k^* \leq k^\circ$. Then $c^\circ > c(r^\circ, w^\circ)$.*

Proof. Recall that c° is the date- τ consumption in the date- τ consumer optimum starting from s° at given $\{r_t^{*\tau}\}_{t=\tau}^\infty$ and $\{w_t^{*\tau}\}_{t=\tau}^\infty$. It follows from (2) that

$$c^\circ = \frac{(1+r^\circ)s^\circ + w^\circ + \frac{w^\circ}{1+r^\circ} + \sum_{t=\tau+2}^\infty \frac{w_t^{*\tau}}{(1+r^\circ)\cdots(1+r_t^{*\tau})}}{1 + (\beta\delta)^{\frac{1}{\rho}}(1+r^\circ)^{\frac{1-\rho}{\rho}} + \dots + (\beta\delta^t)^{\frac{1}{\rho}}((1+r^\circ)\cdots(1+r_{\tau+t}^{*\tau}))^{\frac{1-\rho}{\rho}} + \dots}. \quad (\text{E.1})$$

Let us check that c° is well-defined. Since the truncation of the date- τ optimal path which starts at date $\tau + 1$ is the δ -optimal path, $r_t^{*\tau}$ and $w_t^{*\tau}$ converge to the corresponding modified golden rule levels for the discount factor δ . Thus,

$$\begin{aligned} \frac{w_{t+1}^{*\tau}}{(1+r^\circ)\cdots(1+r_{t+1}^{*\tau})} \frac{(1+r^\circ)\cdots(1+r_t^{*\tau})}{w_t^{*\tau}} &= \frac{w_{t+1}^{*\tau}}{w_t^{*\tau}} \frac{1}{1+r_{t+1}^{*\tau}} \xrightarrow{t \rightarrow \infty} \delta < 1, \\ \frac{(\beta\delta^{t+1})^{\frac{1}{\rho}}((1+r^\circ)\cdots(1+r_{t+1}^{*\tau}))^{\frac{1-\rho}{\rho}}}{(\beta\delta^t)^{\frac{1}{\rho}}((1+r^\circ)\cdots(1+r_t^{*\tau}))^{\frac{1-\rho}{\rho}}} &= \delta^{\frac{1}{\rho}}(1+r_{t+1}^{*\tau})^{\frac{1-\rho}{\rho}} \xrightarrow{t \rightarrow \infty} \delta^{\frac{1}{\rho}}\delta^{\frac{\rho-1}{\rho}} = \delta < 1. \end{aligned}$$

By the d'Alembert's ratio test both infinite series in (E.1) converge.

Let us compare c° with $c(r^\circ, w^\circ)$. Since $\{r_t^{*\tau}\}_{t=\tau+2}^\infty$ is the sequence of interest rates on the δ -equilibrium path starting from $k^\circ < k^\delta$, it is decreasing, and $r_{t+2}^{*\tau} < r^\circ$. Similarly, the sequence $\{w_t^{*\tau}\}_{t=\tau+2}^\infty$ is increasing, and $w_{t+2}^{*\tau} > w^\circ$.

It is evident from (E.1) that c° is increasing in $w_t^{*\tau}$ for all $t \geq \tau + 2$. The numerator in (E.1) is decreasing in $r_t^{*\tau}$, and for $0 < \rho \leq 1$, the denominator in (E.1) is non-decreasing in $r_t^{*\tau}$. Hence when $0 < \rho \leq 1$, c° is decreasing in $r_t^{*\tau}$ for any $t \geq \tau + 2$. Therefore, $c^\circ > c(r^\circ, w^\circ)$, which proves the lemma. ■

Since Lemmas E.1 and E.2 contradict each other, $k^* > k^\circ$ for $0 < \rho \leq 1$, and hence $s^* > s^\circ$, $r^* < r^\circ$, $w^* > w^\circ$, and $c^* = f(k^*) - k^* > f(k^\circ) - k^\circ = c^\circ$.

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References

- [1] Ahn D. S., Iijima R., Sarver T. (2020). Naivete about Temptation and Self-control: Foundations for Recursive Naive Quasi-hyperbolic Discounting. *Journal of Economic Theory*, **189**, 105087.
- [2] Ainslie G. (1992). *Picoeconomics*. Cambridge: Cambridge University Press.
- [3] Akin Z. (2007). Time Inconsistency and Learning in Bargaining Games. *International Journal of Game Theory*, **36** (2), pp. 275–299.
- [4] Barro R. J. (1999). Ramsey Meets Laibson in the Neoclassical Growth Model. *Quarterly Journal of Economics*, **114** (4), pp. 1125–1152.
- [5] Benabou R., Tirole J. (2002). Self-confidence and Personal Motivation. *Quarterly Journal of Economics*, **117** (3), pp. 871–915.
- [6] Bernheim B. D., Ray D., Yeltekin Ş. (2015). Poverty and Self-control. *Econometrica*, **83** (5), pp. 1877–1911.
- [7] Borissov K. (2013). Growth and Distribution in a Model with Endogenous Time Preferences and Borrowing Constraints. *Mathematical Social Sciences*, **66**, pp. 117–128.
- [8] Caliendo F. N., Findley T. S. (2014). Discount Functions and Self-control Problems. *Economics Letters*, **122** (3), pp. 416–419.
- [9] Cao D., Werning I. (2018). Saving and Dissaving with Hyperbolic Discounting. *Econometrica*, **86** (3), pp. 805–857.

- [10] Cass D. (1965). Optimum Growth in an Aggregative Model of Capital Accumulation. *Review of Economic Studies*, **32** (3), pp. 233–240.
- [11] DellaVigna S. (2009). Psychology and Economics: Evidence from the Field. *Journal of Economic Literature*, **47** (2), pp. 315–372.
- [12] Dioikitopoulos E., Turnovsky S., Wendner R. (2020). Dynamic Status Effects, Savings, and Income Inequality. *International Economic Review*, **61** (1), pp. 351–382.
- [13] Ericson K. M., Laibson D. (2019). Intertemporal Choice. In *Handbook of Behavioral Economics: Applications and Foundations*, Vol. 2, pp. 1–67. North-Holland.
- [14] Fedyk A. (2021). Asymmetric Naivete: Beliefs about Self-control. SSRN Working Paper 2727499.
- [15] Findley T. S., Caliendo F. N. (2014). Interacting Mechanisms of Time Inconsistency. *Journal of Economic Psychology*, **41**, pp. 68–76.
- [16] Goldman S. M. (1968). Optimal Growth and Continual Planning Revision. *Review of Economic Studies*, **35** (2), pp. 145–154.
- [17] Johansson-Stenman O., Martinsson P. (2006). Honestly, Why Are You Driving a BMW? *Journal of Economic Behavior and Organization*, **60** (2), pp. 129–146.
- [18] Kaganovich M. (1985). Efficiency of Sliding Plans in a Linear Model with Time-dependent Technology. *Review of Economic Studies*, **52** (4), pp. 691–702.
- [19] Koopmans T. (1965). On the Concept of Optimal Economic Growth. In *The Econometric Approach to Development Planning*. Amsterdam: North Holland.
- [20] Krusell P., Kuruşçu B., Smith A. A. (2002). Equilibrium Welfare and Government Policy with Quasi-geometric Discounting. *Journal of Economic Theory*, **105** (1), pp. 42–72.
- [21] Laibson D. I. (1996). Hyperbolic Discount Functions, Undersaving, and Savings Policy. NBER Working Paper No. 5635.
- [22] Ludwig S., Nafziger J. (2011). Beliefs about Overconfidence. *Theory and Decision*, **70** (4), pp. 475–500.

- [23] Meder Z. Z., Flesch J., Peeters R. (2017). Naivete and Sophistication in Dynamic Inconsistency. *Mathematical Social Sciences*, **87**, pp. 40–54.
- [24] Myerson J., Green L., Warusawitharana M. (2001). Area under the Curve as a Measure of Discounting. *Journal of the Experimental Analysis of Behavior*, **76**, pp. 235–243.
- [25] O’Donoghue T., Rabin M. (2001). Choice and Procrastination. *Quarterly Journal of Economics*, **116** (1), pp. 121–160.
- [26] Phelps E. S., Pollak R. A. (1968). On Second-best National Saving and Game-equilibrium Growth. *Review of Economic Studies*, **35** (2), pp. 185–199.
- [27] Pollak R. A. (1968). Consistent Planning. *Review of Economic Studies*, **35** (2), pp. 201–208.
- [28] Schweighofer-Kodritsch S. (2018). Time Preferences and Bargaining. *Econometrica*, **86** (1), pp. 173–217.
- [29] Strotz R. H. (1955). Myopia and Inconsistency in Dynamic Utility Maximization. *Review of Economic Studies*, **23** (3), pp. 165–180.
- [30] Wendner R., Goulder L. H. (2008). Status Effects, Public Goods Provision, and Excess Burden. *Journal of Public Economics*, **92**, pp. 1968–1985.