

Barros, Maria Filomena; Ortega, Fernando

Article

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Journal of Economic Structures

Provided in Cooperation with:

Pan-Pacific Association of Input-Output Studies (PAPAIOS)

Suggested Citation: Barros, Maria Filomena; Ortega, Fernando (2019) : An optimal equilibrium for a reformulated Samuelson economic discrete time system, Journal of Economic Structures, ISSN 2193-2409, Springer, Heidelberg, Vol. 8, Iss. 29, pp. 1-10,
<https://doi.org/10.1186/s40008-019-0162-2>

This Version is available at:

<https://hdl.handle.net/10419/261532>

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RESEARCH

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An optimal equilibrium for a reformulated Samuelson economic discrete time system

Maria Filomena Barros^{1,2*} and Fernando Ortega^{1,2}

*Correspondence:
barrosmariafilomena@gmail.com

¹ Universitat Autònoma de Barcelona, Barcelona, Spain
Full list of author information is available at the end of the article

Abstract

This paper studies the equilibrium of an extended case of the classical Samuelson's multiplier–accelerator model for national economy. This case has incorporated some kind of memory into the system. We assume that total consumption and private investment depend upon the national income values. Then, delayed difference equations of third order are employed to describe the model, while the respective solutions of third-order polynomial correspond to the typical observed business cycles of real economy. We focus on the case that the equilibrium is not unique and provide a method to obtain the optimal equilibrium.

Keywords: Economic modelling, Samuelson model, Difference equations, Equilibrium, Optimal

1 Introduction

Keynesian macroeconomics inspired the seminal work of Samuelson, who actually introduced the business cycle theory. Although primitive and using only the demand point of view, the Samuelson's prospect still provides an excellent insight into the problem and justification of business cycles appearing in national economies. In the past decades, other models have been proposed and studied by other researchers for several applications, see Chari (1994), Chow (1985), Dassios et al. (2014a), Dassios and Zimbidis (2014), Dassios and Kalogeropoulos (2014), Dassios and Baleanu (2018), Dassios (2018b), Dassios and Devine (2016), Dorf (1983), Kuo (1996), Milano and Dassios (2016), Liu et al. (2017, 2019a, b), Puu et al. (2004), Rosser (2000), Samuelson (1939), Schinnar (1978), Westerhoff (2006), and Wincoop (1996). All these models use mechanisms involving monetary aspects, inventory issues, business expectation, borrowing constraints, welfare gains, and multi-country consumption correlations. Some of the previous articles also contribute to the discussion for the inadequacies of Samuelson's model. The basic shortcoming of the original model is: the incapability to produce a stable path for the national income when realistic values for the different parameters (multiplier and accelerator parameters) are entered into the system of equations. Of course, this statement contradicts with the empirical evidence which supports temporary or long-lasting business cycles. In this article, we propose a special case, i.e., a modification of the typical model incorporating delayed variables into the system of equations and focusing on consumption and investments.

Actually, the proposed modification succeeds to provide a more comprehensive explanation for the emergence of business cycles while also producing a stable trajectory for national income. The final model is a discrete time system of first order and its equilibrium, i.e., equilibrium of the proposed reformulated Samuelson economical model, is not always unique. For the case that we have infinite equilibriums, we provide an optimal equilibrium for the model.

2 The model

The original version of Samuelson's model is based on the following assumptions:

Assumption 1 National income T_k at time k equals to the summation of three elements: consumption, C_k , private investment, I_k , and governmental expenditure G_k :

$$T_k = C_k + I_k + G_k.$$

Assumption 2 Consumption C_k at time k depends on past income (only on last year's value) and on marginal tendency to consume, modelled with a , the multiplier parameter, where $0 < a < 1$:

$$C_k = aT_{k-1}.$$

Assumption 3 Private investment at time k depends on consumption changes and on the accelerator factor b , where $b > 0$. Consequently, I_k depends on national income changes:

$$I_k = b(C_k - C_{k-1}) = ab(T_{k-1} - T_{k-2}).$$

Assumption 4 Governmental expenditure G_k at time k remains constant:

$$G_k = \bar{G}.$$

Hence, the national income is determined via the following second-order linear difference equation:

$$T_{k+2} - a(1+b)T_{k+1} + abT_k = \bar{G}$$

Our reformulated (delayed) version of Samuelson's model is based on the following assumptions:

Assumption 5 National income T_k at time k equals to the summation of two elements: consumption, C_k and private investment, I_k :

$$T_k = C_k + I_k. \quad (1)$$

Assumption 6 Consumption C_k at time k is a linear function of the incomes of the two preceding periods. The governmental expenditures in our model are included in the consumption C_k :

$$C_k = c_1 T_{k-1} + c_2 T_{k-2} + P$$

or, equivalently,

$$C_{k+3} = c_1 T_{k+2} + c_2 T_{k+1} + P. \quad (2)$$

Here, P , c_1 , and c_2 are constant, and $c_1 > 0$, $c_2 > 0$, and $0 < c_1 + c_2 < 1$.

Assumption 7 Private investment I_k at time k , depends on consumption changes and on the positive accelerator factors b . Consequently, I_k depends on the respective national income changes:

$$I_k = b(C_k - C_{k-1})$$

or, using (2), we get the following:

$$I_k = bc_1 T_{k-1} + b(c_2 - c_1) T_{k-2} - bc_2 T_{k-3}$$

or, equivalently,

$$I_{k+3} = bc_1 T_{k+2} + b(c_2 - c_1) T_{k+1} - bc_2 T_k. \quad (3)$$

Hence, using (2) and (3) into (1), the national income is determined via the following high-order linear difference equation:

$$T_{k+3} - c_1(1 + b) T_{k+2} - [c_2 + b(c_2 - c_1)] T_{k+1} + bc_2 T_k = P \quad (4)$$

3 The equilibrium

Consumption, C_k , depends only on past year's income value, while private investment I_k , depends on consumption changes within the last 2 years and governmental expenditure, G_k , depends on past year's income value. From (4), the national income is then determined via the following third-order linear difference equation:

$$T_{k+3} - c_1(1 + b) T_{k+2} - [c_2 + b(c_2 - c_1)] T_{k+1} + bc_2 T_k = P.$$

Lemma 1 The difference equation (4) is equivalent to the following matrix difference equation

$$Y_{k+1} = FY_k + V. \quad (5)$$

Here

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -bc_2 & c_2 + b(c_2 - c_1) & c_1(1 + b) \end{bmatrix}, \quad V = \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix}, \quad (6)$$

and

$$Y_k = \begin{bmatrix} Y_{k,1} \\ Y_{k,2} \\ Y_{k,3} \end{bmatrix}, \quad Y_{k,1} = T_k.$$

Proof We consider (4) and adopt the following notations:

$$\begin{aligned} Y_{k,1} &= T_k, \\ Y_{k,2} &= T_{k+1}, \\ Y_{k,3} &= T_{k+3}. \end{aligned}$$

and

$$\begin{aligned} Y_{k+1,1} &= T_{k+1} = Y_{k,2}, \\ Y_{k+1,2} &= T_{k+2} = Y_{k,3}, \\ Y_{k+1,3} &= T_{k+3} = c_1(1 + b) T_{k+2} + [c_2 + b(c_2 - c_1)] T_{k+1} - bc_2 T_k + P. \end{aligned}$$

Then

$$\begin{bmatrix} Y_{k+1,1} \\ Y_{k+1,2} \\ Y_{k+1,3} \end{bmatrix} = \begin{bmatrix} Y_{k,2} \\ Y_{k,3} \\ c_1(1+b)T_{k+2} + [c_2 + b(c_2 - c_1)]T_{k+1} - bc_2T_k + P \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Y_{k+1,1} \\ Y_{k+1,2} \\ Y_{k+1,3} \end{bmatrix} = \begin{bmatrix} Y_{k,2} \\ Y_{k,3} \\ c_1(1+b)T_{k+2} + [c_2 + b(c_2 - c_1)]T_{k+1} - bc_2T_k + P \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} Y_{k+1,1} \\ Y_{k+1,2} \\ Y_{k+1,3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -bc_2 & c_2 + b(c_2 - c_1) & c_1(1+b) \end{bmatrix} \begin{bmatrix} Y_{k,1} \\ Y_{k,2} \\ Y_{k,3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix},$$

or, equivalently,

$$Y_{k+1} = FY_k + V.$$

The proof is completed.

The discrete time system of first order can be studied in terms of solutions, stability, and control, see Apostolopoulos and Ortega (2018), Dai (1988), Dassios (2012, 2015a, 2018a), Dassios and Szajowski (2016), Dassios and Kalogeropoulos (2013), Dassios et al. (2017), Leonard (1996), Ogata (1987), Ortega and Apostolopoulos (2018), Rugh (1996), Sandefur (1990), Steward and Sun (1990), and Verde-Star (1994). Next, we provide a Lemma for the equilibrium of this system.

Lemma 2 *The equilibrium(s) s_e of the reformulated Samuelson economical model (4) is given by the solution of the following algebraic system:*

$$(I_3 - F)Y^* = V,$$

where

$$Y^* = \begin{bmatrix} s_e \\ s_2 \\ s_3 \end{bmatrix}.$$

Proof From Lemma 1, the reformulated Samuelson economical model (4) is equivalent to (5). Then, to find the equilibrium state of this matrix difference equation, we have the following:

$$\lim_{k \rightarrow +\infty} Y_k = Y^*,$$

that is

$$\lim_{k \rightarrow +\infty} \begin{bmatrix} Y_{k,1} \\ Y_{k,2} \\ Y_{k,3} \end{bmatrix} = \begin{bmatrix} s_e \\ s_2 \\ s_3 \end{bmatrix},$$

and hence,

$$Y^* = FY^* + V,$$

or, equivalently,

$$(I_3 - F)Y^* = V.$$

The proof is completed.

If the equilibrium is unique, we can study its stability based on the eigenvalues of matrix F , see Boutarfa and Dassios (2017), Cheng and Yau (1997), Datta (1995), Dassios (2015b), Milano and Dassios (2017), and Lewis (1986, 1987, 1992). Next, we provide a Lemma which determines when the equilibrium of (5) and consequently of (4) is unique.

Lemma 3 *Consider the system (5) and let $G = I_3 - F$. Then, G is a regular matrix if and only if*

$$1 - c_1 - c_2 \neq 0.$$

Proof We consider (5), and then

$$G = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ bc_2 & -c_2 - b(c_2 - c_1) & 1 - c_1(1 + b) \end{bmatrix}. \quad (7)$$

The determinant of G is equal to

$$\det(G) = bc_2 - c_2 - b(c_2 - c_1) + 1 - c_1(1 + b),$$

or, equivalently,

$$\det(G) = -c_2 + 1 - c_1.$$

Hence, the matrix G is regular if and only if

$$\det(G) \neq 0,$$

or, equivalently,

$$1 - c_2 - c_1 \neq 0.$$

The proof is completed.

We are now ready to state our main Theorem:

Theorem 1 *Consider the system (5) and the matrices F , V , and G as defined in (6) and (7) respectively, i.e., let $G = I_3 - F$. Then*

(a) *If G is full rank, the solution Y^* of (5) is given by the following:*

$$Y^* = (I_3 - F)^{-1}V,$$

and consequently, the unique equilibrium of the reformulated Samuelson economical model (4) is given by the following:

$$s_e = (1 - c_2 - c_1)^{-1}P.$$

(b) *If G is rank deficient, then an optimal solution \hat{Y}^* of (5) is given by the following:*

$$\hat{Y}^* = (G^T G + E^T E)^{-1} G^T V. \quad (8)$$

Here, E is a matrix, such that $G^T G + E^T E$ is invertible and $\|E\|_2 = \theta$, $0 < \theta \ll 1$. Where $\|\cdot\|_2$ is the Euclidean norm.

Proof Let $G = I_3 - F$. For the proof of (a), since G is full rank, from Lemma 3, we have $1 - c_2 - c_1 \neq 0$. Then, where G is equal to the following:

$$G = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ bc_2 & -c_2 - b(c_2 - c_1) & 1 - c_1(1 + b) \end{bmatrix}.$$

Hence, the equilibrium Y^* is given by the unique solution of system (5), that is

$$Y^* = G^{-1} V,$$

or, equivalently, since

$$G^{-1} = \frac{1}{1 - c_1 - c_2} \begin{bmatrix} -c_2 + bc_1 & 1 - c_1(1 + b) & 1 \\ -bc_2 & 1 - c_1(1 + b) & 1 \\ -bc_2 & c_2 - bc_1 & 1 \end{bmatrix};$$

we have the following:

$$Y^* = \frac{1}{1 - c_1 - c_2} \begin{bmatrix} -c_2 + bc_1 & 1 - c_1(1 + b) & 1 \\ -bc_2 & 1 - c_1(1 + b) & 1 \\ -bc_2 & c_2 - bc_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix},$$

or, equivalently,

$$Y^* = \frac{1}{1 - c_1 - c_2} \begin{bmatrix} P \\ P \\ P \end{bmatrix},$$

or, equivalently,

$$Y^* = \frac{P}{1 - c_1 - c_2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For the proof of (b), since G is rank deficient, if $V \notin \text{colspan} G$ system (5) has no solutions and if $V \in \text{colspan} G$ system (5) has infinite solutions. Let

$$\hat{V}(\hat{Y}_n^*) = \hat{V} + E\hat{Y}_n^*,$$

such that the linear system

$$G\hat{Y}_n^* = \hat{V}(\hat{Y}_n^*),$$

or, equivalently the system

$$(G - E)\hat{Y}_n^* = \hat{V},$$

has a unique solution. Where E is a matrix, such that $G^T G + E^T E$ is invertible, $\|E\|_2 = \theta$, $0 < \theta \ll 1$, and $E\hat{Y}_n^*$ is orthogonal to $\hat{V} - G\hat{Y}_n^*$. We use E , because G is rank deficient,

i.e., the matrix $G^T G$ is singular and not invertible. We want to solve the following optimization problem:

$$\begin{aligned} \min & \|V - \hat{V}\|_2^2, \\ \text{s.t. } & (G - E)\hat{Y}_n^* = \hat{V}, \end{aligned}$$

or, equivalently,

$$\min \|V - (G - E)\hat{Y}_n^*\|_2^2,$$

or, equivalently,

$$\min \|V - G\hat{Y}_n^*\|_2^2 + \|E\hat{Y}_n^*\|_2^2.$$

To sum up, we seek a solution \hat{Y}_n^* minimising the functional

$$D_1(\hat{Y}_n^*) = \|V - G\hat{Y}_n^*\|_2^2 + \|E\hat{Y}_n^*\|_2^2.$$

Expanding $D_1(\hat{Y}_n^*)$ gives the following:

$$D_1(\hat{Y}_n^*) = (V - G\hat{Y}_n^*)^T (V - G\hat{Y}_n^*) + (E\hat{Y}_n^*)^T E\hat{Y}_n^*,$$

or, equivalently,

$$D_1(\hat{Y}_n^*) = V^T V - 2V^T G\hat{Y}_n^* + (\hat{Y}_n^*)^T G^T G\hat{Y}_n^* + (\hat{Y}_n^*)^T E^T E\hat{Y}_n^*,$$

because $V^T G\hat{Y}_n^* = (\hat{Y}_n^*)^T G^T V$. Furthermore

$$\frac{\partial}{\partial \hat{Y}_n^*} D_1(\hat{Y}_n^*) = -2G^T V + 2G^T G\hat{Y}_n^* + 2E^T E\hat{Y}_n^*.$$

Setting the derivative to zero, $\frac{\partial}{\partial \hat{Y}_n^*} D_1(\hat{Y}_n^*) = 0$, we get the following:

$$(G^T G + E^T E)\hat{Y}_n^* = G^T V.$$

The solution is then given by the following:

$$\hat{Y}_n^* = (G^T G + E^T E)^{-1} G^T V.$$

Hence, the optimal equilibrium is given by (8). Note that similar optimization techniques have been applied to several problems of this type of algebraic systems, see Cuffe et al. (2016), Dassios et al. (2015), Dassios (2015c, 2019) and Dassios and Baleanu (2019). The proof is completed.

Further research is carried out for even higher order equations investigating qualitative results. For this purpose, we may use an interesting tool applied to difference equations with many delays, the fractional nabla operator, see Atici and Eloe (2011), Dassios and Baleanu (2013, 2015), Dassios et al. (2014b), Dassios (2017, 2015d), Klamka and Wyrwał (2008), Klamka (2010) and Podlubny (1999). The fractional nabla operator is a very interesting tool for this, since it succeeds to provide information from a specific year in the

past until the current year. The nabla operator of n th order, n *Natural*, applied to a vector of sequences $Y_k : \mathbb{N}_\alpha \rightarrow \mathbb{C}^m$ is defined by the following:

$$\nabla^n Y_k = \sum_{j=0}^n a_j Y_{k-j} = a_0 Y_k + a_1 Y_{k-1} + \cdots + a_n Y_{k-n},$$

where $a_j = (-1)^j \frac{1}{\Gamma(n+1)\Gamma(j+1)\Gamma(n-j+1)}$. The nabla fractional operator of n th order, n *Fractional*, applied to a vector of sequences $Y_k : \mathbb{N}_\alpha \rightarrow \mathbb{C}^m$ is defined by the following:

$$\nabla_\alpha^n Y_k = \sum_{j=\alpha}^k b_j Y_j = b_k Y_k + b_{k-1} Y_{k-1} + \cdots + b_\alpha Y_\alpha,$$

where $b_j = \frac{1}{\Gamma(-n)}(k-j+1)^{-n-1}$ and $(k-j+1)^{-n-1} = \frac{\Gamma(k-j-n)}{\Gamma(k-j+1)}$. For all this, there is already some ongoing research.

4 Conclusions

Closing this paper, we may argue that it is not only a theoretical extension of the basic version of Samuelson's model, but also a practical guide for obtaining the optimal equilibrium of this model; in the case, we have infinite many equilibriums. We studied the equilibrium of an extended case of the classical Samuelson's multiplier–accelerator model for national economy. We also focused on the case that the equilibrium is not unique and provided a method to obtain the optimal equilibrium.

Authors' contributions

Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

Funding

Not applicable.

Availability of data and materials

Data sharing was not applicable to this article as no data sets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

Author details

¹ Universitat Autònoma de Barcelona, Barcelona, Spain. ² Advanced Research Centre for Applied Mathematics (ARCAM), Athens, Greece.

Received: 21 October 2018 Revised: 28 April 2019 Accepted: 28 August 2019

Published online: 09 September 2019

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