

Ewerhart, Christian

Working Paper

Diagonal payoff security and equilibrium existence in quasi-symmetric discontinuous games

Working Paper, No. 414

Provided in Cooperation with:

Department of Economics, University of Zurich

Suggested Citation: Ewerhart, Christian (2022) : Diagonal payoff security and equilibrium existence in quasi-symmetric discontinuous games, Working Paper, No. 414, University of Zurich, Department of Economics, Zurich, <https://doi.org/10.5167/uzh-219039>

This Version is available at:

<https://hdl.handle.net/10419/260598>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



**University of
Zurich** ^{UZH}

University of Zurich
Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 414

Diagonal Payoff Security and Equilibrium Existence in Quasi-Symmetric Discontinuous Games

Christian Ewerhart

June 2022

Diagonal Payoff Security and Equilibrium Existence in Quasi-Symmetric Discontinuous Games*

Christian Ewerhart**

June 11, 2022

Abstract. Payoff security combined with reciprocal upper semicontinuity is sufficient for better-reply security, and consequently for the existence of a pure strategy Nash equilibrium in compact, quasiconcave games by Reny's (1999) theorem. Analogously, diagonal payoff security combined with upper semicontinuity of the diagonal payoff function has been widely understood to be sufficient for diagonal better-reply security, and consequently for the existence of a symmetric pure strategy Nash equilibrium in compact, diagonally quasiconcave, quasi-symmetric games. We show by example that this is incorrect. Specifically, diagonal better-reply security may fail to hold, and a symmetric pure strategy equilibrium may fail to exist, if some player's payoff function lacks lower semicontinuity, with respect to the opponents' symmetric strategy profile, at all strategy profiles reached from a non-equilibrium profile on the diagonal by a unilateral better response of that player. These difficulties disappear, both in the game and in its mixed extension, if the lower bound on a player's payoff in the definition of diagonal payoff security is raised to reflect the higher levels that arbitrary better responses may achieve. We also discuss the relationship between our strengthened condition and diagonal payoff security.

Keywords. Discontinuous games · Equilibrium existence · Quasi-symmetric games · Diagonal payoff security

JEL-Codes. C62 (Existence and Stability Conditions of Equilibrium), C72 (Non-cooperative Games)

*) Phil Reny provided valuable feedback on an earlier draft of this paper.

***) Department of Economics, University of Zurich, Schönberggasse 1, CH-8001 Zurich, Switzerland; christian.ewerhart@econ.uzh.ch.

1. Introduction

The topological approach to equilibrium existence, initiated by Nash (1950, 1951), formulates general conditions sufficient for the existence of pure and mixed strategy Nash equilibria in discontinuous non-cooperative games. Standard conditions for the existence of a pure strategy Nash equilibrium, derived by Glicksberg (1952), Fan (1952), and Debreu (1952), require strategy spaces to be compact (as well as convex and non-empty), and each player's payoff function to be quasiconcave in that player's own strategy, as well as continuous in the overall strategy profile.¹ But, as emphasized by Dasgupta and Maskin (1986), especially the continuity requirement is overly restrictive in many economic applications. Therefore, more flexible concepts such as *transfer-continuity* (Baye et al., 1993) and *better-reply security* (Reny, 1999) have been developed so as to replace the continuity assumptions on compact quasiconcave games without losing the conclusion of equilibrium existence. Moreover, sufficient conditions have been derived to make the application of such concepts even more straightforward. Most prominently, Reny introduced two conditions on discontinuous games, *payoff security* and *reciprocal upper semicontinuity*, and showed that they jointly imply better-reply security.²

Conditions for equilibrium existence can be further relaxed in the special case of *quasi-symmetric* games, which is a class of games strictly nesting the class of symmetric games. To this end, Reny defined suitable variants of quasiconcavity and better-reply security for quasi-symmetric games. These are referred to as *diagonal quasiconcavity* and *diagonal better-reply security*. He showed that, if a quasi-symmetric game is compact, diagonally quasiconcave, and diagonally better-reply secure, then a symmetric pure strategy Nash equilibrium exists. This result allows numerous applications, in particular to the analysis of Bertrand competition,

¹Analogous results for symmetric games have been obtained by Nash (1951, Thm. 2) and Moulin (1986, p. 115), in particular.

²Notably, the availability of sufficient conditions remains desirable even in view of the important further generalizations obtained by McLennan et al. (2011), Borelli and Meneghel (2013), Reny (2016), and others. Cf. Olszewski and Siegel (2022), who take a similar perspective.

location choice, brand loyalty, and multi-unit auctions.³

In analogy to Reny’s development of sufficient conditions in the case of general discontinuous games, it has been widely understood that two conditions known as *diagonal payoff security* and *upper semicontinuity of the diagonal payoff function* jointly imply diagonal better-reply security and, hence, are sufficient for the existence of a symmetric pure strategy Nash equilibrium in compact, diagonally quasiconcave, quasi-symmetric games.⁴ We present below an example to show that this is incorrect. In the example, diagonal better-reply security does not hold, and all pure strategy Nash equilibria are asymmetric, while the game satisfies diagonal payoff security and upper semicontinuity of the diagonal payoff function, in addition to being compact, diagonally quasiconcave, and quasi-symmetric.

In the example, problems arise because the payoff function of some player lacks lower semicontinuity in her opponent’s strategy at *all* strategy profiles reached from a non-equilibrium strategy profile on the diagonal via a unilateral deviation of that player to some better response. But if, for at least one such better response, the deviating player could, up to any given $\epsilon > 0$, secure her payoff at the corresponding strategy profile, the difficulties would disappear. In particular, suppose we strengthened the requirement of diagonal payoff security to what will be called *strong diagonal payoff security*, wherein we raise the lower bound on a player’s payoff in the definition of diagonal payoff security to reflect the higher levels that arbitrary better responses may achieve. Then, the two conditions, strong diagonal payoff security and upper semicontinuity of the diagonal payoff function, jointly imply diagonal better-reply security. In particular, these two conditions are sufficient for the existence of a symmetric pure strategy equilibrium in any compact, diagonally quasiconcave, quasi-symmetric game.

³Also this result has been generalized in the meanwhile. See, in particular, Reny’s (2020) discussion of Bich and Laraki (2012).

⁴Cf. Reny (1999, Prop. 4.2, Cor. 4.3, Ex. 4.1, and the proof of the second part of Cor. 5.3). These claims have indeed been the basis of further analysis, especially more recently. See, for example, Ciardiello (2010), Van den Berg and Bos (2017), Matakos and Xefteris (2017), Plan (2017), Au and Kawai (2020), Eguia and Xefteris (2021), and Petrikaitė (2022).

A formal elaboration of this idea may be found below. The necessary background on quasi-symmetric discontinuous games is provided in Section 2. Section 3 concerns diagonal payoff security in quasi-symmetric games: We show by example that diagonal payoff security and upper semicontinuity of the diagonal payoff function will not, in general, jointly imply diagonal better-reply security. Using the same example, we show that these two conditions will neither, in general, jointly imply the existence of a symmetric pure strategy Nash equilibrium in compact, diagonally quasiconcave, quasi-symmetric games. In Section 4, we introduce the notion of strong diagonal payoff security and show that, combined with the upper semicontinuity of the diagonal payoff function, it implies diagonal better-reply security. In Section 5, we apply those findings to derive conditions sufficient for the existence of symmetric pure and mixed strategy Nash equilibria in discontinuous quasi-symmetric games. Section 6 discusses the relationship between diagonal payoff security and strong diagonal payoff security. In particular, it is shown that the two concepts are equivalent if each player's payoff function is lower semicontinuous in her opponents' symmetric strategy profile at all strategy profiles reached from the diagonal via a unilateral deviation by that player. Finally, we illustrate the usefulness of this observation by applying it to models of price competition that do not satisfy the usual quasiconcavity assumption. In an Appendix, we clarify the relationship between payoff security and strong diagonal payoff security.

2. Background on quasi-symmetric discontinuous games

We follow Reny (1999), which should be consulted for a fuller exposition. For a given game in strategic form, $G = (X_i, u_i)_{i=1}^N$, let $i \in \{1, \dots, N\}$ denote an arbitrary player, X_i the set of player i 's pure strategies, and $u_i : X_1 \times \dots \times X_N \rightarrow \mathbb{R}$ player i 's payoff function, respectively. We shall assume throughout that $G = (X_i, u_i)_{i=1}^N$ is a *compact* game meaning that, for each player i , the strategy set X_i is a non-empty compact subset of some topological vector space and the payoff function u_i is bounded.

We say that $G = (X_i, u_i)_{i=1}^N$ is *quasi-symmetric* if the following two conditions are satisfied: (i) For all players i, j , $X_i = X_j$; (ii) $u_1(x, y, \dots, y) = u_2(y, x, y, \dots, y) = \dots = u_N(y, \dots, y, x)$ for all $x, y \in X$, where $X = X_1 = \dots = X_N$ denotes the common strategy space. Note that a symmetric game is quasi-symmetric, but the converse does not hold in general (unless $N = 2$). For $x, y \in X$, we denote by $u_i(y, \dots, x, \dots, y)$ the function u_i evaluated at the strategy profile in which player i chooses x while all others choose y . If $G = (X_i, u_i)_{i=1}^N$ is quasi-symmetric, its *diagonal payoff function* $v : X \rightarrow \mathbb{R}$ is defined as $v(x) = u_1(x, \dots, x) = \dots = u_N(x, \dots, x)$ for every $x \in X$. As usual, a strategy profile $(x_1, \dots, x_N) \in X_1 \times \dots \times X_N$ is said to *lie on the diagonal*, or alternatively, to be *symmetric*, if $x_1 = \dots = x_N$.

Let $G = (X_i, u_i)_{i=1}^N$ be a quasi-symmetric game. Given $\alpha \in \mathbb{R}$ and $x \in X$, we shall say that player i can *secure a payoff of α along the diagonal at (x, \dots, x)* , if there exists $\bar{x} \in X$ such that $u_i(x', \dots, \bar{x}, \dots, x') \geq \alpha$ for all x' in some open neighborhood U of x . The game G is called *diagonally payoff secure* if, for every $x \in X$ and every $\epsilon > 0$, each player i can secure a payoff of $u_i(x, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) . Further, G is *diagonally better-reply secure* if, whenever $(x^*, u^*) \in X \times \mathbb{R}$ is in the closure of the graph of its diagonal payoff function and (x^*, \dots, x^*) is not an equilibrium, some player i can secure a payoff strictly above u^* along the diagonal at (x^*, \dots, x^*) . Finally, G is *diagonally quasiconcave* if X is convex, and for every player i , all $x^1, \dots, x^m \in X$, and all $\bar{x} \in \text{co}\{x^1, \dots, x^m\}$,

$$u_i(\bar{x}, \dots, \bar{x}) \geq \min_{1 \leq n \leq m} u_i(\bar{x}, \dots, x^n, \dots, \bar{x}). \quad (1)$$

We shall make use of the following result due to Reny (1999, Thm. 4.1).

Proposition 2.1 *If $G = (X_i, u_i)_{i=1}^N$ is quasi-symmetric, compact, diagonally quasiconcave, and diagonally better-reply secure, then it possesses a symmetric pure strategy Nash equilibrium.*

Proof. See Reny (1999, p. 1042).

Q.E.D.

3. An example

Consider the following game G_0 . There are $N = 2$ players. Each player $i \in \{1, 2\}$ chooses a strategy from the unit interval $X = X_1 = X_2 = [0, 1]$. Player 1's payoff function is given by

$$u_1(x, y) = \begin{cases} x & \text{if } y \in [0, \frac{1}{2}) \\ 1 - x & \text{if } y \in [\frac{1}{2}, 1]. \end{cases} \quad (2)$$

Player 2's payoff function is given by $u_2(x, y) = u_1(y, x)$, for all $x, y \in X$. Clearly, G_0 is quasi-symmetric and compact. Further properties of G_0 are collected in the following proposition.

Proposition 3.1

- (i) G_0 is diagonally payoff secure;
- (ii) the diagonal payoff function in G_0 is upper semicontinuous;
- (iii) however, G_0 is not diagonally better-reply secure.

Proof. (i) Take some $x \in [0, 1]$ and $\epsilon > 0$. We claim that each player $i \in \{1, 2\}$ can secure a payoff of $u_i(x, x) - \epsilon$ along the diagonal at (x, x) . Because G_0 is quasi-symmetric, it suffices to prove the claim for player 1. There are three cases. Suppose first that $x \in [0, \frac{1}{2})$. Then, $u_1(x, x) = x$, and we merely need to show that player 1 can secure a payoff of $x - \epsilon$ along the diagonal at (x, x) . But by using her best response $\bar{x} = 1$, player 1 can in fact secure a payoff of 1. Indeed, $u_1(1, x') = 1$ for all x' in an open neighborhood of x . Since $1 \geq x - \epsilon$, this proves the claim in the first case. Suppose next that $x = \frac{1}{2}$. Then, $u_1(x, x) = \frac{1}{2}$. In this case, by playing $\bar{x} = \frac{1}{2}$, player 1 can secure a payoff of $\frac{1}{2} - \epsilon$. Indeed,

$$u_1(\frac{1}{2}, x') \geq \min\{x', 1 - x'\} \geq \frac{1}{2} - \epsilon \quad (3)$$

for all x' in an open neighborhood of $x = \frac{1}{2}$. Hence, we have verified the claim also in this case. Suppose, finally, that $x \in (\frac{1}{2}, 1]$. Then, $u_1(x, x) = 1 - x$, but by playing

her best response $\bar{x} = 0$, player 1 can again secure a payoff of 1. Since $1 > 1 - x - \epsilon$, this proves the claim in the final case as well. Thus, G_0 is indeed diagonally payoff secure. (ii) The diagonal payoff function in G_0 is given as

$$v(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}) \\ 1 - x & \text{if } x \in [\frac{1}{2}, 1]. \end{cases} \quad (4)$$

Clearly, v is continuous. In particular, v is upper semicontinuous, as has been claimed. (iii) Let $x^* = \frac{1}{2}$ and $u^* = \frac{1}{2}$. Since $v(x^*) = u^*$, the pair (x^*, u^*) is an element of the graph of the diagonal payoff function in G_0 , which is easily seen to be identical to its closure (since v is continuous). Moreover, (x^*, x^*) is not an equilibrium, because player 1, for instance, might deviate to $x = 0$, thereby raising her payoff to $u_1(0, x^*) = 1 > \frac{1}{2} = u_1(x^*, x^*)$. To prove that G_0 is not diagonally better-reply secure, it suffices to show that player 1 cannot secure a payoff strictly above u^* along the diagonal at (x^*, x^*) . For this, let $\bar{x} \in [0, 1]$ be *any* strategy of player 1, and let U be *any* open neighborhood of $x^* = \frac{1}{2}$. Then,

$$u_1(\bar{x}, x') = \begin{cases} x' & \text{if } x' \in [0, \frac{1}{2}) \\ 1 - x' & \text{if } x' \in [\frac{1}{2}, 1]. \end{cases} \quad (5)$$

Since the open neighborhood U has a nonempty intersection with the interval $[0, \frac{1}{2})$, there exists some $x' \in U$ such that $u_1(\bar{x}, x') = x' < \frac{1}{2} = u^*$. Therefore, player 1 cannot even secure a payoff of u^* (let alone a payoff strictly above u^*) along the diagonal at (x^*, x^*) . This proves the last claim, and hence, the proposition. Q.E.D.

It follows from Proposition 3.1 that diagonal payoff security and upper semicontinuity of the diagonal payoff function do not, in general, jointly imply diagonal better-reply security. The next proposition relates this observation to the existence of a symmetric equilibrium in pure strategies.

Proposition 3.2

(i) G_0 is diagonally quasiconcave;

(ii) however, G_0 does not possess a symmetric pure strategy Nash equilibrium.

Proof. (i) Clearly, $X = [0, 1]$ is convex. Take $x^1, \dots, x^m \in X$ and let $\bar{x} \in \text{co}\{x^1, \dots, x^m\}$. We wish to show that

$$u_1(\bar{x}, \bar{x}) \geq \min_{1 \leq n \leq m} u_1(x^n, \bar{x}) \tag{6}$$

$$u_2(\bar{x}, \bar{x}) \geq \min_{1 \leq n \leq m} u_2(\bar{x}, x^n). \tag{7}$$

As G_0 is quasi-symmetric, it suffices to verify relationship (6). Recall that

$$u_1(\bar{x}, \bar{x}) = \begin{cases} \bar{x} & \text{if } \bar{x} \in [0, \frac{1}{2}) \\ 1 - \bar{x} & \text{if } \bar{x} \in [\frac{1}{2}, 1]. \end{cases} \tag{8}$$

Suppose first that $\bar{x} \in [0, \frac{1}{2})$. Since $\bar{x} \in \text{co}\{x^1, \dots, x^m\}$, and X is one-dimensional, there exists $n \in \{1, \dots, m\}$ such that $x^n \leq \bar{x}$. Therefore, $u_1(x^n, \bar{x}) = x^n \leq \bar{x} = u_1(\bar{x}, \bar{x})$, and hence, inequality (6) holds true in this case. Suppose next that $\bar{x} \in [\frac{1}{2}, 1]$. Clearly, there exists n such that $x^n \geq \bar{x}$. Therefore, $u_1(x^n, \bar{x}) = 1 - x^n \leq 1 - \bar{x} = u_1(\bar{x}, \bar{x})$, and hence, inequality (6) holds true also in this case. Since both cases have been covered, it follows that G_0 is indeed diagonally quasiconcave. (ii)

The proof is by contradiction. Suppose that (x^*, x^*) is a symmetric pure strategy Nash equilibrium in G_0 . There are two cases. Suppose first that $x^* \in [0, \frac{1}{2})$. Then, player 1's unique best response to x^* is given by $x = 1$. Therefore, $x^* = 1$, in conflict with the hypothesis. Suppose next that $x^* \in [\frac{1}{2}, 1]$. Then, similarly, player 1's unique best response to x^* is $x = 0$, hence $x^* = 0$, which is again impossible. The claim follows. Q.E.D.

The proposition above adds to the discussion that G_0 is diagonally quasiconcave, yet does not possess a symmetric Nash equilibrium in pure strategies. In particular,

it follows that the two beforementioned conditions, diagonal payoff security and upper semicontinuity of the diagonal payoff function, do not in general imply the existence of a symmetric Nash equilibrium in a compact, diagonally quasiconcave, quasi-symmetric game.

Proposition 3.2 relates to a result in Amir et al. (2010) for submodular games on the square with strictly quasiconvex payoffs. Such games necessarily possess a pure strategy Nash equilibrium as a consequence of Topkis' fixed point theorem.⁵ Amir et al. note that any player whose payoff function is strictly quasiconvex in her own strategy will optimally choose an extremal strategy in response to a pure strategy. Under additional assumptions ensuring that the strategy profiles $(0, 0)$ and $(1, 1)$ are not equilibria, all pure strategy Nash equilibria must be asymmetric. We note here that their result extends to payoff functions that are strictly monotone in own strategy (i.e., either strictly increasing or strictly decreasing) for any given strategy of the opponent. Consistent with this logic, the submodular game G_0 admits precisely two asymmetric pure strategy Nash equilibria, viz. $(0, 1)$ and $(1, 0)$, but no symmetric pure strategy Nash equilibrium.

4. Strong diagonal payoff security

We start with a definition.

Definition 4.1 *A quasi-symmetric game $G = (X_i, u_i)_{i=1}^N$ is **strongly diagonally payoff secure** if for each $i \in \{1, \dots, N\}$, every $x \in X$, every $\hat{x}_i \in X_i$, and every $\epsilon > 0$, player i can secure a payoff of $u_i(x, \dots, \hat{x}_i, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) .*

Strong diagonal payoff security is a variant of diagonal payoff security (cf. Section 2). It is obtained by tracing through the proof of Reny (1999, Prop. 3.2) for general games. The condition in the original definition saying that player i can secure a payoff of $u_i(x, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) is replaced by the

⁵For this, the natural ordering of the strategy interval is reversed for one of the two players. See Milgrom and Roberts (1990) and Vives (1990).

stronger condition that, for *every* strategy $\hat{x}_i \in X_i$, player i can secure a payoff of $u_i(x, \dots, \hat{x}_i, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) . By reflecting the potentially higher payoff level $u_i(x, \dots, \hat{x}_i, \dots, x)$ that player i might obtain from a unilateral deviation to $\hat{x}_i \in X_i$, the concept tightens.⁶ Therefore, any strongly diagonally payoff secure game is, in particular, diagonally payoff secure.

However, the converse implication does not hold in general, not even for $N = 2$. Indeed, recall from Proposition 3.1 that G_0 is diagonally payoff secure. But G_0 is not strongly diagonally payoff secure. To understand why, note that with $x = \frac{1}{2}$, $\hat{x}_1 = 0$, and $\epsilon = \frac{1}{4}$, player 1 cannot secure a payoff of $\alpha = u_1(\hat{x}_1, x) - \epsilon = \frac{3}{4}$ along the diagonal at (x, x) . This is so because for any $\bar{x}_1 \in [0, \frac{1}{2})$, it holds that $u_1(\bar{x}_1, x') = \bar{x}_1 < \frac{1}{2} < \alpha$ for any $x' \in [0, \frac{1}{2})$, while the interval $[0, \frac{1}{2})$ necessarily has a nonempty intersection with any open neighborhood of $x = \frac{1}{2}$. Similarly, for any $\bar{x}_1 \in [\frac{1}{2}, 1]$, it holds that $u_1(\bar{x}_1, x) = 1 - \bar{x}_1 \leq \frac{1}{2} < \alpha$, even though x is necessarily contained in any open neighborhood of x . Thus, G_0 is indeed not strongly diagonally payoff secure, which shows that a diagonally payoff secure game need not, in general, be strongly diagonally payoff secure. This clarifies the relationship between Definition 4.1 and the notion of diagonal payoff security.⁷

The following result proposes a set of conditions sufficient for diagonal better-reply security in quasi-symmetric discontinuous games.

Proposition 4.1 *If a quasi-symmetric game $G = (X_i, u_i)_{i=1}^N$ is strongly diagonally payoff secure and its diagonal payoff function v is upper semicontinuous, then G is diagonally better-reply secure.*

Proof. The following proof is a straightforward adaption of Reny (1999, proof of Prop. 3.2). Suppose that (x^*, u^*) is in the closure of the graph of v , and that (x^*, \dots, x^*) is not an equilibrium. Then, by upper semicontinuity, either $v(x^*) > u^*$

⁶Indeed, it is not hard to see that G is strongly diagonally payoff secure iff for each i , every $x \in X$, and every $\epsilon > 0$, player i can secure a payoff of $\sup_{\hat{x}_i \in X_i} u_i(x, \dots, \hat{x}_i, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) . But $\sup_{\hat{x}_i \in X_i} u_i(x, \dots, \hat{x}_i, \dots, x) \geq u_i(x, \dots, x)$.

⁷Another question is the relationship between Definition 4.1 and payoff security. As this issue is not crucial for our analysis, it has been relegated to an Appendix.

or $v(x^*) = u^*$. In the latter case, because (x^*, \dots, x^*) is not an equilibrium, some player i has a better response than x^* , say $\hat{x}_i \in X_i$, such that $u_i(x^*, \dots, \hat{x}_i, \dots, x^*) > u_i(x^*, \dots, x^*) = u^*$. Consequently, in either case there is a player $i \in \{1, \dots, N\}$ and a strategy $\hat{x}_i \in X_i$ (equal to x^* in the former case) such that $u_i(x^*, \dots, \hat{x}_i, \dots, x^*) > u^*$. Fix now this player i . Choose $\epsilon > 0$ so that $u_i(x^*, \dots, \hat{x}_i, \dots, x^*) > u^* + \epsilon$. Because G is strongly diagonally payoff secure, player i has a strategy \bar{x}_i such that $u_i(x', \dots, \bar{x}_i, \dots, x') \geq u_i(x^*, \dots, \hat{x}_i, \dots, x^*) - \epsilon$ for all x' in some open neighborhood of x^* . Since $u_i(x^*, \dots, \hat{x}_i, \dots, x^*) - \epsilon > u^*$, this implies that player i can secure a payoff strictly above u^* along the diagonal at (x^*, \dots, x^*) . Thus, G is diagonally better-reply secure, as has been claimed. Q.E.D.

5. Conditions for equilibrium existence

This section combines Reny's main existence theorem for quasi-symmetric games (cf. Section 2) with Proposition 4.1 to derive conditions sufficient for the existence of symmetric pure and mixed strategy Nash equilibria in quasi-symmetric discontinuous games. We start with the case of pure strategy Nash equilibrium.

Corollary 5.1 *If $G = (X_i, u_i)_{i=1}^N$ is quasi-symmetric, compact, diagonally quasi-concave, and strongly diagonally payoff secure, and v is upper semicontinuous, then G possesses a symmetric pure strategy Nash equilibrium.*

Proof. Immediate from Propositions 2.1 and 4.1. Q.E.D.

Example 5.1 Consider a compact, quasi-symmetric, strongly diagonally payoff secure game $G = (X_i, u_i)_{i=1}^N$ with $X = [0, 1]$, and such that v is upper semicontinuous on $[0, 1]$. If for each player $i \in \{1, \dots, N\}$ and for all $\bar{x} \in [0, 1]$, the function $u_i(\bar{x}, \dots, x, \dots, \bar{x})$ is either:

- (i) nondecreasing in x on $[0, \bar{x}]$, or
- (ii) nonincreasing in x on $[\bar{x}, 1]$,

then G possesses a symmetric pure strategy Nash equilibrium.

This follows directly from Corollary 5.1 since conditions (i) and (ii) imply that G is diagonally quasiconcave.

Next, we turn to the case of mixed strategy Nash equilibria. Let $G = (X_i, u_i)_{i=1}^N$ be a compact Hausdorff game such that u_i is measurable in the strategy profile for each player $i \in \{1, \dots, N\}$. Letting M_i denote the space of probability measures on the Borel subsets of X_i , each M_i is compact in the weak*-topology. In the *mixed extension* of G , denoted by \bar{G} , each player i chooses a probability measure $\mu_i \in M_i$. Player i 's payoff in \bar{G} is determined as the Lebesgue integral $\bar{u}_i = \int_{X_1 \times \dots \times X_N} u_i d(\mu_1 \times \dots \times \mu_N)$, where $\mu_1 \times \dots \times \mu_N$ denotes the product measure on $X_1 \times \dots \times X_N$. Let $G = (X_i, u_i)_{i=1}^N$ be a game such that $X_1 = \dots = X_N = X$. Then, the common strategy space of \bar{G} is $M = M_1 = \dots = M_N$. As noted by J. Duggan, quasi-symmetry of G does not imply that \bar{G} is quasi-symmetric, so that a stronger requirement on G is needed.⁸ Following Plan (2017), we will say that G is *weakly symmetric* if for any two players $i, j \in \{1, \dots, N\}$, there exists a permutation σ of the set $\{1, \dots, N\}$ such that $\sigma(i) = j$, $\sigma(j) = i$, and $u_i(x_1, \dots, x_N) = u_j(x_{\sigma(1)}, \dots, x_{\sigma(N)})$ for all $x_1, \dots, x_N \in X$. Any symmetric game is weakly symmetric, and likewise any weakly symmetric game is quasi-symmetric, but the reverse implications do not hold in general.

Corollary 5.2 *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a weakly symmetric, compact, Hausdorff game. Then G possesses a symmetric mixed strategy Nash equilibrium if its mixed extension, $\bar{G} = (M_i, \bar{u}_i)_{i=1}^N$, is better-reply secure along the diagonal. Moreover, \bar{G} is better-reply secure along the diagonal if it is strongly diagonally payoff secure and each $\bar{u}_i(\mu, \dots, \mu)$ is upper semicontinuous as a function of μ on M .*

Proof. By Plan (2017, Thm. 10), $\bar{G} = (M_i, \bar{u}_i)_{i=1}^N$ is quasi-symmetric. Accounting for this fact, the first claim corresponds to the first part of Reny (1999, Cor. 5.3). The second claim follows then from Corollary 5.1 above. Q.E.D.

⁸I am very much indebted to Phil Reny for sharing this valuable information, thereby correcting an error in an earlier draft of this paper.

Milgrom and Weber (1985) defined randomized strategies for Bayesian games by assuming that each player chooses a probability distribution over the product of her type space and her action space. Provided that all type and action spaces are compact Hausdorff, a straightforward application of Tychonoff's theorem, combined with the elementary fact that the product of two Hausdorff spaces is again Hausdorff, shows that Corollary 5.2 extends, with identical proof, to their setting as well.

6. Discussion

What is the relationship between diagonal payoff security and strong diagonal payoff security? The following result addresses this question by providing a condition on a diagonally payoff secure game sufficient to be strongly diagonally payoff secure.

Proposition 6.1 *Suppose that the quasi-symmetric game $G = (X_i, u_i)_{i=1}^N$ is diagonally payoff secure, and that $u_i(x, \dots, x_i, \dots, x)$ is lower semicontinuous in x on $X \setminus \{x_i\}$ for each player i and all $x_i \in X_i$. Then, G is strongly diagonally payoff secure.*

Proof. Fix some player $i \in \{1, \dots, N\}$, and take some $x \in X$, $\hat{x}_i \in X_i$, $\epsilon > 0$. We claim that player i can secure a payoff of $u_i(x, \dots, \hat{x}_i, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) . There are two cases. Suppose first that $\hat{x}_i = x$. Then, clearly, diagonal payoff security implies that player i can secure a payoff of $u_i(x, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) . This proves the claim if $\hat{x}_i = x$. Suppose next that $\hat{x}_i \neq x$. Then, $x \in X \setminus \{\hat{x}_i\}$, and $u_i(x', \dots, \hat{x}_i, \dots, x')$ is lower semicontinuous in x' at x . Thus, there is an open neighborhood U of x such that $u_i(x', \dots, \hat{x}_i, \dots, x') \geq u_i(x, \dots, \hat{x}_i, \dots, x) - \epsilon$ for any $x' \in U$. In particular, by letting $\bar{x}_i = \hat{x}_i$, it is seen that $u_i(x', \dots, \bar{x}_i, \dots, x') \geq u_i(x, \dots, \hat{x}_i, \dots, x) - \epsilon$ for any x' sufficiently close to x . It follows that, also in the case $\hat{x}_i \neq x$, player i can secure a payoff of $u_i(x, \dots, \hat{x}_i, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) . Since the claim has been established in both cases, the proposition follows. Q.E.D.

The proposition says that diagonal payoff security implies strong diagonal payoff security if each player's payoff function is lower semicontinuous in her opponents' symmetric strategy profile at all strategy profiles that are reached from the diagonal via a unilateral deviation of that player.

The lower semicontinuity assumption is crucial, even for $N = 2$. In G_0 , which satisfies diagonal payoff security, one can easily verify that, for instance, $u_1(x, y)$ lacks lower semicontinuity in y at any strategy profile that is reached from the symmetric profile $(\frac{1}{2}, \frac{1}{2})$ via a unilateral deviation by player 1 to some $x \in [0, \frac{1}{2})$.⁹ And indeed, as noted above, the conclusion of Proposition 6.1 does not hold for G_0 .

Proposition 6.1 implies that diagonal payoff security and strong diagonal payoff security are equivalent in all games whose payoff discontinuities lie on the diagonal only. This case includes some applications of economic interest. For instance, the Bertrand duopoly with continuous demand is both diagonally payoff secure and diagonally quasiconcave. Since price competition creates discontinuities on the diagonal only, Proposition 6.1 implies that games in this class are even strongly diagonally payoff secure, and hence, using Corollary 5.1, possess a symmetric pure strategy Nash equilibrium. This is remarkable since the firm's profit functions may fail to satisfy the usual quasiconcavity assumption. Analogous considerations are feasible in models of brand loyalty, where a firm charging a price from its customer base that strictly exceeds the competitor's price incurs a fixed cost $c > 0$ (cf. Baye et al., 1993). Thus, the concepts of diagonal payoff security and strong diagonal payoff security turn out to be equivalent in economically meaningful examples of quasi-symmetric discontinuous games, and this observation proves useful for establishing the existence of a symmetric pure strategy Nash equilibrium in those games.

⁹For a formal argument, let $\epsilon > 0$ such that $x < 1 - x - \epsilon$. Then, $u_1(x, \frac{1}{2} - \delta) = x < 1 - x - \epsilon = u_1(x, \frac{1}{2}) - \epsilon$ for any small $\delta > 0$. Hence, $u_1(x, y)$ is not lower semicontinuous in y at $y = \frac{1}{2}$.

A. Appendix

In this Appendix, we briefly discuss the relationship between payoff security and strong diagonal payoff security. Let $G = (X_i, u_i)_{i=1}^N$ be a discontinuous game, as before. Recall the following definitions from Reny (1999, Sec. 3). Given $\alpha \in \mathbb{R}$ and $(x_1, \dots, x_N) \in X_1 \times \dots \times X_N$, player i can *secure* a payoff of α at (x_1, \dots, x_N) if there exists $\bar{x}_i \in X_i$, such that

$$u_i(x'_1, \dots, x'_{i-1}, \bar{x}_i, x'_{i+1}, \dots, x'_N) \geq \alpha \quad (9)$$

for all $(x'_1, \dots, x'_{i-1}, x'_{i+1}, \dots, x'_N) \in X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_N$ in some open neighborhood of $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$. The game G is called *payoff secure* if for every $(x_1, \dots, x_N) \in X_1 \times \dots \times X_N$ and every $\epsilon > 0$, each player i can secure a payoff of $u_i(x_1, \dots, x_N) - \epsilon$ at (x_1, \dots, x_N) .

Proposition A.1 *Let $G = (X_i, u_i)_{i=1}^N$ be a quasi-symmetric game. If G is payoff secure, then it is strongly diagonal payoff secure. Conversely, if G is strongly diagonal payoff secure and $N = 2$, then it is payoff secure.*

Proof. Immediate from the definitions.

Q.E.D.

It is not hard to verify that strong diagonal payoff security does not, in general, imply payoff security. To be sure, consider the following example. There are $N \geq 3$ players. Each player $i \in \{1, \dots, N\}$ chooses a strategy x_i from the unit interval $[0, 1]$. The payoff of player i is given as

$$u_i(x_1, \dots, x_N) = \begin{cases} 0 & \text{if } x_1 = \dots = x_{i-1} = x_{i+1} = \dots = x_N \\ -1 & \text{otherwise.} \end{cases} \quad (10)$$

Clearly, the thereby defined game G_1 is quasi-symmetric and strongly diagonally payoff secure. However, G_1 is not payoff secure, because for $\epsilon = \frac{1}{2}$, player 1 cannot secure a payoff of $u_1(0, \dots, 0) - \epsilon = -\frac{1}{2}$ at $(0, \dots, 0) \in [0, 1]^N$. Indeed, since

$N \geq 3$, any open neighborhood of $(0, \dots, 0) \in [0, 1]^{N-1}$ necessarily contains a point $(x'_1, \dots, x'_{i-1}, x'_{i+1}, \dots, x'_N)$ such that $x'_j \neq x'_k$ for some players $j, k \in \{1, \dots, N\} \setminus \{i\}$. But then, $u_i(x'_1, \dots, x'_{i-1}, x'_{i+1}, \dots, x'_N) = -1 < u_1(0, \dots, 0) - \epsilon$.

References

- Amir, R., Garcia, F., Knauff, M. (2010). Symmetry-breaking in two-player games via strategic substitutes and diagonal nonconcavity: A synthesis. *Journal of Economic Theory*, 145(5): 1968–1986.
- Au, P. H., Kawai, K. (2020). Competitive information disclosure by multiple senders. *Games and Economic Behavior*, 119: 56–78.
- Baye, M. R., Tian, G., Zhou, J. (1993). Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs. *Review of Economic Studies*, 60(4): 935–948.
- Barelli, P., Meneghel, I. (2013). A note on the equilibrium existence problem in discontinuous games. *Econometrica*, 81(2): 813–824.
- Bich, P., Laraki, R. (2012). A unified approach to equilibrium existence in discontinuous strategic games. *HAL Open Science*, halshs-00717135.
- Ciardello, F. (2010). A more refined convexity idea for Nash equilibria. *Journal of Fixed Point Theory and Applications*, 8(2): 365–393.
- Dasgupta, P., Maskin, E. (1986). The existence of equilibrium in discontinuous economic games, I: Theory. *Review of Economic Studies*, 53(1): 1–26.
- Debreu, G. (1952). A social equilibrium existence theorem. *Proceedings of the National Academy of Sciences*, 38(10): 886–893.
- Eguia, J. X., Xefteris, D. (2021). Implementation by Vote-Buying Mechanisms. *American Economic Review*, 111(9): 2811–2828.

- Fan, K. (1952). Fixed-point and minimax theorems in locally convex topological linear spaces. *Proceedings of the National Academy of Sciences*, 38(2): 121–126.
- Glicksberg, I. L. (1952). A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. *Proceedings of the American Mathematical Society*, 3(1): 170–174.
- Matakos, K., Xefteris, D. (2017). When extremes meet: Redistribution in a multiparty model with differentiated parties. *Journal of Theoretical Politics*, 29(4): 546–577.
- McLennan, A., Monteiro, P. K., Tourky, R. (2011). Games with discontinuous payoffs: A strengthening of Reny’s existence theorem. *Econometrica*, 79(5): 1643–1664.
- Milgrom, P., Roberts, J. (1990). Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica*, 58(6): 1255–1277.
- Milgrom, P., Weber, R. (1985). Distributional strategies for games with incomplete information. *Mathematics of Operations Research*, 10(4): 619–632.
- Moulin, H. (1986). *Game Theory for the Social Sciences*, NYU Press.
- Nash, J. F. (1950). Equilibrium points in n -person games. *Proceedings of the National Academy of Sciences*, 36(1): 48–49.
- (1951). Non-cooperative games. *Annals of Mathematics*, 2(54): 286–295.
- Olszewski, W., Siegel, R. (2022). Equilibrium existence in games with ties. *Theoretical Economics*, forthcoming.
- Petrikaitė, V. (2022). Escaping search when buying. *International Journal of Industrial Organization*, 82: 102828.

Plan, A. (2017). Symmetric n -player games, University of Arizona Economics Working Paper, 17-08.

Reny, P. J. (1999). On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica*, 67(5): 1029–1056.

— (2016). Nash equilibrium in discontinuous games. *Economic Theory*, 61(3): 553–569.

— (2020). Nash equilibrium in discontinuous games. *Annual Review of Economics*, 12: 439–470.

Van den Berg, A., Bos, I. (2017). Collusion in a price-quantity oligopoly. *International Journal of Industrial Organization*, 50: 159–185.

Vives, X. (1990). Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics*, 19(3): 305–321.