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Evolutionary Selection against Iteratively Weakly Dominated Strategies*

Axel Bernergård[†] Erik Mohlin[‡]

November 12, 2018

Abstract

This paper provides sufficient conditions under which regular payoff monotonic evolutionary dynamics (a class of imitative dynamics that includes the replicator dynamic) select against strategies that do not survive a sequence of iterated elimination of weakly dominated strategies. We apply these conditions to Bertrand duopolies and first-price auctions. Our conditions also imply evolutionary selection against iteratively strictly dominated strategies.

Keywords: Iterated elimination of weakly dominated strategies; Iterated admissibility; Payoff monotonicity; Convex monotonicity; Evolutionary dynamics; Replicator dynamic.

JEL code: C72, C73.

1 Introduction

The foundations of iterated elimination of *strictly* dominated strategies are relatively well-understood from the perspectives of both evolutionary and epistemic game-theory. Similarly, the *epistemic* foundations of iterated elimination of *weakly* dominated strategies have been explored in depth.¹ In contrast, little is known about the *evolutionary*

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¹Epistemic foundations of iterated strict dominance have been provided by Bernheim (1984), Pearce (1984), and Tan and Werlang (1988). For the case of weak dominance the situation is more involved, see the contrasting perspectives of Brandenburger et al. (2008) and Asheim and Dufwenberg (2003).

underpinning of iterated weak dominance. In the current paper we present results that narrow this knowledge gap.

A number of results deal with deterministic evolutionary selection against strategies that fail to survive iterated elimination of *strictly* dominated strategies (IESDS).² Samuelson and Zhang (1992) (see also Nachbar 1990) show that, starting from any interior initial state, *pure* strategies that are iteratively strictly dominated by *pure* strategies vanish asymptotically along the solution trajectory of any (*payoff*) *monotonic* dynamic. A dynamic is monotonic if it satisfies the condition that one strategy has a higher growth rate than another strategy if and only if the former strategy earns a higher payoff than the latter. Samuelson and Zhang (1992) also establish that *mixed* strategies that are iteratively strictly dominated by *mixed* strategies vanish under any *aggregate monotonic* evolutionary dynamic. This class of dynamics includes the replicator dynamic (Taylor and Jonker 1978). Hofbauer and Weibull (1996) consider evolutionary dynamics that satisfy *convex monotonicity*. They find that, starting from any interior initial state, *pure* strategies that are iteratively strictly dominated by *mixed* strategies vanish asymptotically under any *convex monotonic* dynamic. Viossat (2015) completes the picture by showing that under *concave monotonic* dynamics, *mixed* strategies that are iteratively strictly dominated by *pure* strategies vanish asymptotically.³ All of the above classes of dynamics are *imitative*, meaning that strategies that are currently absent from the population remain absent forever. Hofbauer and Sandholm (2011) show that a large class of evolutionary dynamics that are *non-imitative* fail to eliminate strictly dominated strategies in some games. For this reason we restrict attention to imitative dynamics when developing our results in this paper.

When it comes to weak dominance, it is well known that in some games some strategies that are eliminated by iterative elimination of *weakly* dominated strategies (IEWDS) do *not* vanish under the replicator dynamic and related dynamics (e.g. games G^2 , G^4 , and G^6 below). In other games strategies that do not survive iterative elimination of weakly dominated strategies do become extinct under the replicator dynamic (e.g. games G^1 , G^3 , and G^5 below). A few papers establish evolutionary selection against iteratively weakly dominated strategies in particular (important) games, such as Cressman (1996) for the finitely repeated Prisoner's Dilemma game, and Ponti (2000) for the Centipede game.

The present paper is motivated by an interest in finding out whether there are any general characteristics of games that imply that iteratively weakly dominated strategies are eliminated by a large class of evolutionary dynamics. Given the abundance of results

²There is also a literature on elimination of weakly dominated strategies (though not iterated elimination) under stochastic evolutionary dynamics, e.g. Samuelson (1994) and Kuzmics (2011).

³Since aggregate monotonic dynamics are both convex monotonic and concave monotonic, the results of Hofbauer and Weibull (1996) and Viossat (2015) imply that for aggregate monotonic dynamics both *pure* strategies that are iteratively strictly dominated by *mixed* strategies, and *mixed* strategies that are iteratively strictly dominated by *pure* strategies, vanish asymptotically.

on iterated strict dominance, the scarcity of results on iterated weak dominance is intriguing. On a more practical note, games with iteratively weakly dominated strategies occur in many applications, some of which we discuss below.

We focus on a class of *regular payoff monotonic* dynamics.⁴ In addition to being payoff monotonic such dynamics have the crucial property that if one strategy yields a higher payoff than another strategy, then the ratio of the growth-rate difference to the payoff difference is bounded away from zero. The replicator dynamic is regular payoff monotonic as are all aggregate monotonic and all convex monotonic dynamics.

We define three properties on sequences of pure strategies dominated by pure strategies in symmetric two-player games, called (a) monotonicity, (b) pairwise weak dominance, and (c) local strict transitivity. For finite symmetric two-player games, we show that if a sequence of IEWDS has these three properties, then any regular payoff monotonic dynamic, starting from any interior initial state, is guaranteed to asymptotically eliminate all pure strategies that are iteratively weakly dominated by pure strategies in that IEWDS.

To the best of our knowledge we are the first to provide general conditions under which imitative dynamics select against strategies that fail to survive IEWDS.⁵ Any sequence of iterated elimination of pure strategies that are *strictly* dominated by pure strategies has the three key properties. Thus, *from the perspective of imitative evolutionary dynamics, the well-established distinction between iterated strict and iterated weak dominance seems less important than the hitherto neglected distinction between different kinds of iterated weak dominance.*

We apply our result to a discretised versions of Bertrand duopoly and a first-price auction. In these games all strategies except one can be removed by a sequence IEWDS that satisfies our conditions. As the grid is made finer this strategy converges to the Nash equilibrium of the game with continuous strategy sets. Hence our result guarantees that any regular payoff monotonic dynamic selects the Nash equilibrium in these games.

The rest of the paper is organised as follows. Section 2 provides basic notation and definitions. Section 3 contains (3.1) the main result and (3.2) illustration and motivation of the different components of our sufficient condition. Section 4 contains all proofs, including (4.1) a summary, (4.2) proofs of lemmata, and (4.3) proof of the main theorem. Applications are considered in section 5. Section 6 concludes.

⁴We thank Bill Sandholm for suggesting this terminology.

⁵An earlier version of this paper was included in Bernergård's PhD thesis (Bernergård 2014). A very limited precursor was also included in the online appendix to Mohlin (2012). Laraki and Mertikopoulos (2013) introduce the higher-order replicator dynamic and show that it performs one round of elimination of weakly dominated strategies. Marx (1999) defines a belief-based adaptive learning process, similar to that of Milgrom and Roberts (1991), and shows that *if* it converges then players only put positive probability on strategies that survive IEWDS.

2 Notation and Definitions

Let G be a two-player normal-form symmetric game with a finite pure-strategy set $S = \{1, \dots, m\}$, strategy-profile set S^2 , and a payoff function $u : S^2 \rightarrow \mathbb{R}$. The mixed strategy set is denoted by Δ , and $\pi : \Delta^2 \rightarrow \mathbb{R}$ is the mixed payoff function. For $s \in S$, let $e^s \in \Delta$ denote the unit vector with 1 in the s -th position.

Definition 1 *A pure strategy $s \in S$ is weakly dominated if there exists a mixed strategy $x \in \Delta$ such that $\pi(x, y) \geq \pi(e^s, y)$ for all $y \in \Delta$, and $\pi(x, y) > \pi(e^s, y)$ for some $y \in \Delta$.*

Iterative weak dominance is not well defined in the sense that which strategies survive can depend on the order of elimination, and in the sense that there might not be any strategy which survives all iterated eliminations. We will look at specific cases of iterative elimination of weakly dominated strategies where each eliminated strategy is eliminated by a *pure* strategy:

Definition 2 *An IEWDS for the symmetric two-player game G is a sequence $((s_k, d_k))_{k=1}^{m'}$ such that for $k = 1, \dots, m'$, $s_k \in S \setminus \{s_{k-1}, s_{k-2}, \dots, s_1\}$ is weakly dominated by $d_k \in S \setminus \{s_{k-1}, s_{k-2}, \dots, s_1\}$ in the game that is constructed from G by restricting the strategy set to $S \setminus \{s_{k-1}, s_{k-2}, \dots, s_1\}$ for both players.*

It is important to note that according to this definition of IEWDS a strategy is removed from *both* players' strategy set at the same time. This is natural given that we only consider single population dynamics as specified below. We will sometimes write “a sequence of IEWDS” instead of “an IEWDS” to refer to a sequence $((s_k, d_k))_{k=1}^{m'}$ for linguistic reasons or to emphasize that an IEWDS is a sequence.

Individuals from an infinite population are randomly matched to play the game G . Each individual is programmed to play one of the m pure strategies. A population state is a point $x = (x_1, \dots, x_m) \in \Delta$. The expected payoff of an s -strategist, at state x is $\pi_s(x) = \sum_i u(s, i) x_i$. The average payoff in the population is $\bar{\pi}(x) = \sum_i \pi_i(x) x_i$. Evolution of the fraction of s -players is governed by an evolutionary dynamic of the form

$$\dot{x}_s = g_s(x) x_s,$$

where the growth-rate functions g_1, \dots, g_m from Δ to \mathbb{R} are Lipschitz continuous and satisfy $\sum_s g_s(x) x_s = 0$. This defines a vector field $\varphi : \Delta \rightarrow \mathbb{R}^m$, such that $\dot{x} = \varphi(x)$. By standard arguments the system has a unique solution $\xi(\cdot, x^0) : \mathbb{R} \rightarrow \Delta$ through any initial condition x^0 , such that $\xi(0, x^0) = x^0$ and $\frac{\partial}{\partial t}(\xi(t, x^0)) = g(\xi(t, x^0))\xi(t, x^0)$ for all t .

We will assume that dynamics are monotonic, and in addition that there is uniform bound on how small the growth-rate difference can be relative to the payoff difference.⁶

⁶The reader who is familiar with uniform monotonicity defined by Cressman (2003) will recognise this condition as half of the requirements for uniform monotonicity.

Definition 3 Let sgn denote the sign function. The dynamic defined by g_1, \dots, g_m is monotonic if $\text{sgn}(g_i(x) - g_j(x)) = \text{sgn}(\pi_i(x) - \pi_j(x))$ for all $i, j \in S$ and all $x \in \Delta$.

Definition 4 The dynamic defined by g_1, \dots, g_m is regular monotonic if there is a positive constant $\underline{\lambda}$ such that $g_i(x) - g_j(x) \geq \underline{\lambda}(\pi_i(x) - \pi_j(x))$ for all $i, j \in S$ and all $x \in \Delta$ with $\pi_i(x) \geq \pi_j(x)$. Equivalently, the dynamic defined by g_1, \dots, g_m is regular monotonic if it is monotonic and there is a positive constant $\underline{\lambda}$ such that $|g_i(x) - g_j(x)| \geq \underline{\lambda}|\pi_i(x) - \pi_j(x)|$ for all $i, j \in S$ and all $x \in \Delta$.

We note that monotonic dynamics are such that strategies grow at the same rate if and only if they yield the same expected payoff. We will use this property frequently to determine the limiting behavior of ratios of the form x_i/x_j .

All aggregate monotonic dynamics (Samuelson and Zhang 1992) are evidently regular payoff monotonic since if a dynamic is aggregate monotonic, then there is a positive and continuous function λ such that $g_i(x) - g_j(x) = \lambda(x)(\pi_i(x) - \pi_j(x))$ and we can set $\underline{\lambda} = \min_{x \in \Delta} \lambda(x)$. Hofbauer and Weibull (1996) show that any convex monotonic dynamic can be written as

$$g_s(x) = \lambda(x)f(\pi_s(x)) + \mu(x),$$

for a positive and continuous function λ , a convex and strictly increasing function f , and a real-valued function μ . So, if $\pi_i(x) \geq \pi_j(x)$, then by the convexity of f ,

$$g_i(x) - g_j(x) = \lambda(x)[f(\pi_i(x)) - f(\pi_j(x))] \geq \lambda(x) \cdot (\min \Delta f) \cdot [\pi_i(x) - \pi_j(x)],$$

where $\min \Delta f = f(\min_{s \in S, y \in \Delta} \pi_s(y)) - f(\min_{s \in S, y \in \Delta} \pi_s(y) - 1)$. Hence there is a positive constant $\underline{\lambda} = \min_{y \in \Delta} \lambda(y) \cdot (\min \Delta f)$, such that if $\pi_i(x) \geq \pi_j(x)$, then

$$g_i(x) - g_j(x) \geq \underline{\lambda}(\pi_i(x) - \pi_j(x)), \tag{1}$$

for all $i, j \in S$ and all $x \in \Delta$ with $\pi_i(x) \geq \pi_j(x)$; and thus convex monotonicity implies regular payoff monotonicity.

3 Result

3.1 Properties of IEWDS and Main Result

The following definition introduces three properties that a sequence of IEWDS may satisfy

Definition 5 Consider an IEWDS $((s_k, d_k))_{k=1}^{m'}$. Define three properties that the IEWDS may satisfy as follows:

(a) **Monotonicity:** For any $k \in \{1, 2, \dots, m' - 1\}$, the strategy d_k that eliminates strategy s_k in step k is either eliminated in step $k + 1$, i.e. $s_{k+1} = d_k$, or is used to eliminate another strategy in step $k + 1$, i.e. $d_{k+1} = d_k$.

(b) **Pairwise weak dominance:** For any $k \in \{1, 2, \dots, m'\}$, $u(d_k, s_k) > u(s_k, s_k)$ or $u(d_k, d_k) > u(s_k, d_k)$.

(c) **Local strict transitivity:** For any $k \in \{1, 2, \dots, m' - 1\}$ and $\tilde{k} \in \{k + 1, \dots, m'\}$, if $s_{\tilde{k}}$ is the strategy that eliminates s_k , i.e. $s_{\tilde{k}} = d_k$, so that $d_{\tilde{k}}$ is the strategy that eliminates the strategy that eliminates s_k , then $u(d_{\tilde{k}}, s_{\tilde{k}}) = u(d_{\tilde{k}}, d_k) > u(s_k, d_k)$.

Remark 1 If $s_{\tilde{k}} = d_k$, then by the definition of iterated weak dominance $u(d_{\tilde{k}}, s_{\tilde{k}}) \geq u(s_{\tilde{k}}, s_{\tilde{k}}) = u(d_k, d_k) \geq u(s_k, d_k)$. Hence the clause “ $u(d_{\tilde{k}}, d_k) > u(s_k, d_k)$ ” in (c) is equivalent to “ $u(d_{\tilde{k}}, s_{\tilde{k}}) > u(s_{\tilde{k}}, s_{\tilde{k}})$ or $u(d_k, d_k) > u(s_k, d_k)$ ”. Thus local strict transitivity is satisfied if $u(d_k, s_k) > u(s_k, s_k)$ holds for all $k = 1, \dots, m'$, or if $u(d_k, d_k) > u(s_k, d_k)$ holds for all $k = 1, \dots, m'$.

The first condition, *monotonicity*, says that the strategy d_k which is used to eliminate strategy s_k in step k of elimination is either eliminated or eliminates one more strategy in step $k + 1$ of elimination. The second condition, *pairwise weak dominance*, requires that for each step in the given order of elimination, the weakly dominant strategy d_k earns strictly more against the strategy s_k that it weakly dominates than what the weakly dominated strategy earns against itself, or the weakly dominant strategy d_k earns strictly more against itself than what the strategy s_k earns against d_k . The third condition, *local strict transitivity*, requires that if d_k eliminates s_k and $d_{\tilde{k}}$ eliminates d_k then either d_k earns strictly more than s_k against d_k or $d_{\tilde{k}}$ earns strictly more than d_k against d_k .

For any given game G there may be several sequences of IEWDS that satisfy these three properties. Furthermore, which strategies that remain when no more eliminations can be made can depend on the order of elimination. However, there will always be some strategies that survive all such sequences of IEWDS, and the following theorem shows that only those strategies can survive evolution in the long run if the dynamic is regular monotonic.

Theorem 1 Consider a regular payoff monotonic dynamic with induced solution mapping $\xi(t, x^0)$. Let $E \subset S$ denote the set of all $s \in S$ for which there exists an IEWDS $(s_k, d_k)_{k=1}^{m'}$ with $s = s_k$ for some $k = 1, \dots, m'$ that satisfies (a) monotonicity, (b) pairwise weak dominance, and (c) local strict transitivity. Then $\lim_{t \rightarrow \infty} \xi_s(t, x^0) = 0$ for all $s \in E$ and all interior initial states x^0 .

Before proving this theorem we will motivate regular payoff monotonicity and the three properties that together constitute our sufficient condition.⁷

⁷There is a literature that has developed conditions for order indepenence of iterated elimination of weakly dominated strategies, e.g. Marx and Swinkels (1997). There appears to be no substantial connection between these conditions and our condition.

3.2 Motivation of the Jointly Sufficient Conditions and of Regular Monotonicity

In this section we motivate our jointly sufficient conditions (a) monotonicity, (b) pairwise weak dominance, and (c) local strict transitivity. For each condition we provide an example of a game in which there is a sequence of IEWDS which does not satisfy the condition in question, and we show that some strategies eliminated by that IEWDS are not eliminated by evolution. Moreover, we motivate the restriction to regular monotonic dynamics by presenting a game with a strategy that is eliminated by a sequence of IEWDS satisfying (a)-(c), but which may remain in the population forever if the evolutionary dynamic is non-regular monotonic.

3.2.1 (b) Pairwise Weak Dominance

Consider the following two games:

$$\begin{array}{cc} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ G^1 & G^2 \end{array}$$

In both games there is a sequence of IEWDS where strategy 2 eliminates 1 and then strategy 3 eliminates strategy 2. Formally, $((s_k, d_k))_{k=1}^2$ with $(s_1, d_1) = (1, 2)$ and $(s_2, d_2) = (2, 3)$. In G^1 this is the only possible order of elimination – the only other IEWDS is the trivial one in which strategy 2 eliminates 1 and the process is then halted. In G^1 any regular payoff monotonic evolutionary dynamic, starting from any interior initial state, will asymptotically eliminate strategies 1 and 2. Thus, evolution selects the only profile that survives all IEWDS in G^1 . Figure 1a illustrates this for the replicator dynamic.⁸ By contrast, Figure 1b illustrates the replicator dynamic in game G^2 . In this game strategy 2 is not always eliminated by evolution, despite not surviving all IEWDS. The set of all states at which strategy 1 is eliminated by evolution constitutes an asymptotically stable

⁸All figures were created using Dynamo software (Sandholm and Dokumaci 2007).

set (Thomas 1985), represented by the thick black line segment in Figure 1b.

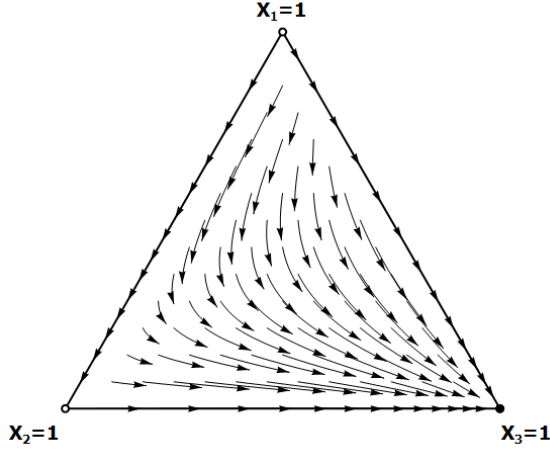


Figure 1a. G^1

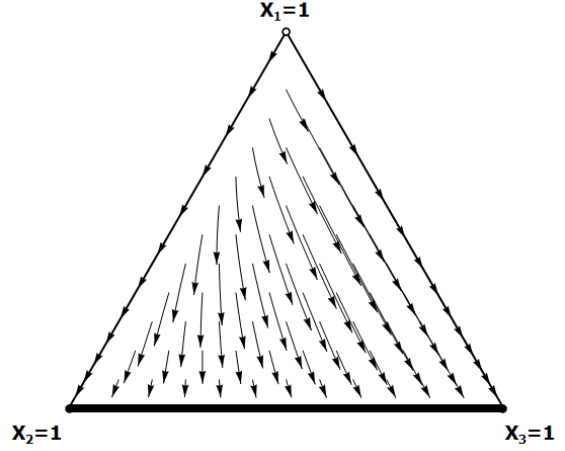


Figure 1b. G^2

We note the following property, which the suggested IEWDS satisfies in G^1 but not in G^2 : strategy 2, which eliminates strategy 1, earns strictly more than strategy 1 against strategy 1. Likewise, strategy 3, which eliminates strategy 2, earns strictly more than strategy 2 against strategy 2. Formally, if we consider the IEWDS $((s_k, d_k))_{k=1}^2$ with $(s_1, d_1) = (1, 2)$ and $(s_2, d_2) = (2, 3)$, then in game G^1 it holds that $u(d_k, s_k) > u(s_k, s_k)$, whereas in G^2 this is not the case.

Next consider the following two games.

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ G^3 & G^4 \end{array}$$

In G^3 , $(s_1, d_1) = (1, 2)$ and $(s_2, d_2) = (2, 3)$ is an IEWDS, and this is the only possible order of elimination. Any regular payoff monotonic evolutionary dynamic, starting from any interior initial state, will asymptotically eliminate strategies 1 and 2, as illustrated for the replicator dynamic in Figure 2a. The sequence $(1, 2), (2, 3)$ does not satisfy the property that $u(d_k, s_k) > u(s_k, s_k)$ for all k , but we do have that $u(d_k, d_k) > u(s_k, d_k)$ for all k . In G^4 , $(s_1, d_1) = (3, 2)$ is an IEWDS which does not satisfy $u(d_k, d_k) > u(s_k, d_k)$. It turns out that strategy 3 is not always asymptotically eliminated by regular payoff monotonic evolutionary dynamics. The set of all states at which strategy 1 is eliminated constitutes an asymptotically stable set in game G^4 , as illustrated for the replicator

dynamic in Figure 2b.

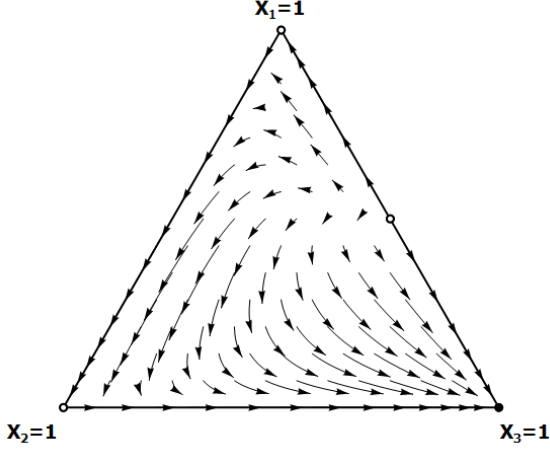


Figure 2a. G^3

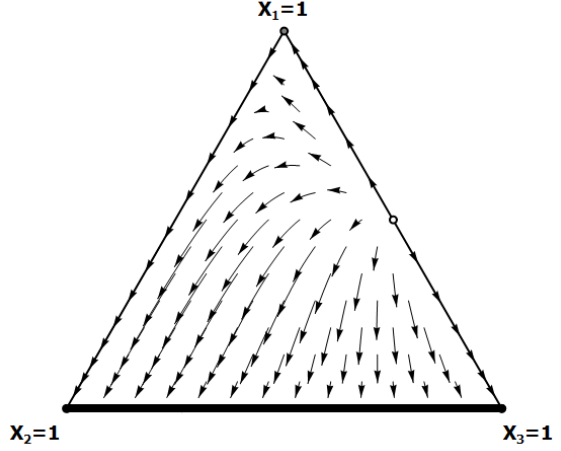


Figure 2b. G^4

3.2.2 (c) Local Strict Transitivity

Above we showed examples of IEWDS with $u(d_k, s_k) > u(s_k, s_k)$ or $u(d_k, d_k) > u(s_k, d_k)$ where strategies which were eliminated by such an IEWDS also were asymptotically eliminated by regular payoff monotonic evolutionary dynamics. The following two games demonstrate that this property is *not* sufficient to guarantee selection against strategies that fail to survive IEWDS.

$$\begin{array}{cc} \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ G^5 & G^6 \end{array}$$

In both G^5 and G^6 , $(1, 2), (2, 3)$ is an IEWDS, and this is the only possible order of elimination. As illustrated in Figure 3, the replicator dynamic, starting from any interior initial state, will asymptotically eliminate strategies 1 and 2 in G^5 , while in game G^6 strategy 2 is not always eliminated by evolution. In fact, if initially $x_1 \geq x_3$ then *all* monotonic dynamics (not necessarily regular monotonic) converge to the state where only strategy 2 remains in the population.⁹ The thin diagonal line in Figure 3b represents the

⁹Notice that strategies 1 and 3 have the same payoff against all strategies s with $s \neq 1, 3$. In the Appendix we show that therefore $u(1, 3) > u(3, 3)$ is enough to ensure that, under a monotone dynamic, $x_3(t) \rightarrow 0$ if $x_1(0)/x_3(0)$ is sufficiently large.

set of states at which $x_1/x_3 = 1/2$.

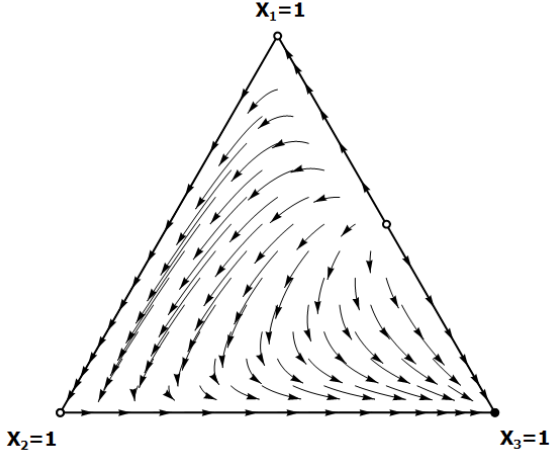


Figure 3a. G^5

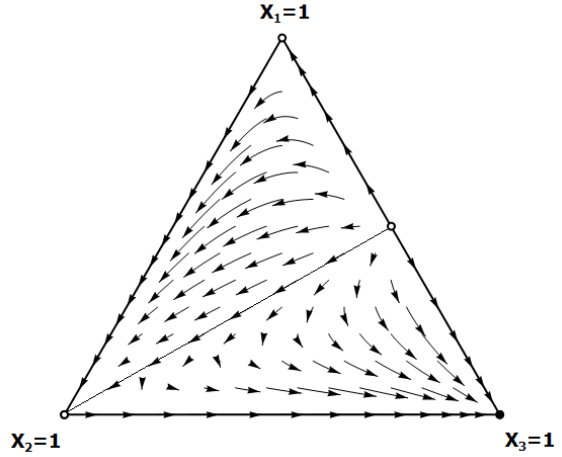


Figure 3b. G^6

We note the fact that in G^5 it is the case that since $d_1 = 2 = s_2$ it holds that if $d_k = s_{\tilde{k}}$, then $\pi(d_{\tilde{k}}, d_k) > \pi(d_k, d_k)$ or $\pi(d_k, d_k) > \pi(s_k, d_k)$. This is not the case in game G^6 .

3.2.3 (a) Monotonicity

Above we have studied the properties (b) pairwise weak dominance and (c) local strict transitivity. In games with more than three strategies these two properties are not strong enough for our purposes. To see this consider the game

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} .$$

G^7

This game is similar to G^6 but we have added a fourth strategy, which is identical to strategy 2, against which all strategies earn 0. In G^7 , $(1, 4), (2, 3)$ is an IEWDS in which strategy 2 is eliminated. Furthermore, both (b) pairwise weak dominance and (c) local strict transitivity are satisfied by this IEWDS (local strict transitivity is trivially satisfied since there are no k , and \tilde{k} for which $d_k = s_{\tilde{k}}$). But, as we show in the Appendix, G^7 is such that all monotonic dynamics converge to states where strategy 1 and 3 are absent, provided that $x_1 \geq x_3$ initially.

3.2.4 Regular Monotonicity

The following game G^8 has a sequence of IEWDS where strategy 2 eliminates strategy 1, and then strategy 4 eliminates strategy 2 and 3.

$$G^8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & -1 \\ -2 & 2 & 0 & 0 \end{pmatrix}.$$

This sequence of IEWDS satisfies our three properties (a)-(c), and thus all regular monotonic dynamics are such that starting from any interior initial condition $x_4(t) \rightarrow 1$. An implication of this is that if we start from an interior initial condition $x(0)$ with $x_1(t) > x_2(t) > x_4(t)$, then $x_2(t)$ must eventually become larger than $x_1(t)$ because otherwise strategy 4 would never start growing. For dynamics that are monotonic but not regular monotonic this does not necessarily happen because the growth rate difference between strategy 2 and strategy 1 can freeze to 0 too fast as we get closer and closer to $x_3 = 1$. In the Appendix we give an example of a monotonic dynamic for which $x_3(t) \rightarrow 1$ starting from some interior initial conditions $x(0)$ with $x_1(0) > x_2(0) > x_4(0)$. We have not performed any complete investigation into the differences in results for regular monotonic dynamics and monotonic dynamics but this example shows that our theorem fails if the extra requirement of regular monotonicity is removed.

4 Proofs

4.1 Example

Before proving Theorem 1 we prove the result for just the game G^5 to convey an idea of how the proof works. For G^5 , $(1, 2)$, $(2, 3)$ is an IEWDS. Since strategy 2 weakly dominates strategy 1, and since strategy 2 is strictly better than strategy 1 against strategy 2 we can use Lemma 3 below to conclude that either $\int_0^\infty x_2(t)dt < +\infty$ and $x_1(t)/x_2(t)$ converges to a real number r (possibly 0); or $\int_0^\infty x_2(t)dt = +\infty$ and $x_1(t)/x_2(t) \rightarrow 0$.

In the first case, by the comparison theorem for positive integrals, it follows from $\int_0^\infty x_2(t)dt < +\infty$ and $x_1(t)/x_2(t) \rightarrow r$ that $\int_0^\infty x_1(t)dt < +\infty$. By standard arguments, using the uniform continuity of $x_i(t)$, $\int_0^\infty x_1(t)dt < +\infty$ and $\int_0^\infty x_2(t)dt < +\infty$ implies $x_1(t) \rightarrow 0$ and $x_2(t) \rightarrow 0$. So, in this case $x_1(t) \rightarrow 0$, and $x_2(t) \rightarrow 0$ as we wanted to show.

In the second case we can compare the payoffs for strategy 1 and 3 and, since strategy 3 is strictly better against strategy 2, apply Lemma 3 to conclude that $x_1(t)/x_3(t) \rightarrow 0$. After that we can compare the payoffs for strategy 2 and 3 and, since strategy 3 is

strictly better against 3 and only worse against 1, apply Lemma 3 again to conclude that either $\int_0^\infty x_3(t)dt < +\infty$ and $x_2(t)/x_3(t) \rightarrow r$, or $\int_0^\infty x_3(t)dt = +\infty$ and $x_2(t)/x_3(t) \rightarrow 0$. But, $\int_0^\infty x_3(t)dt < +\infty$ and $x_2(t)/x_3(t) \rightarrow r$ is not possible because it would violate the comparison theorem for positive integrals since $\int_0^\infty x_2(t)dt = +\infty$. So, in this case we have that $x_1(t)/x_2(t) \rightarrow 0$ and $x_2(t)/x_3(t) \rightarrow 0$ which clearly implies $x_1(t) \rightarrow 0$, and $x_2(t) \rightarrow 0$.

Proving our general theorem requires going through a number of similar steps, using Lemma 3 and Lemma 4, to show that the order in a sequence of IEWDS that satisfies properties (a) to (c) matches the order of the speed of evolutionary elimination in the sense that $x_{s_k}(t)/x_{d_k}(t) \rightarrow 0$, with the only possible exception to this rule being when $\int_0^\infty x_{s_k}(t)dt$ and $\int_0^\infty x_{d_k}(t)dt$ are both finite and $x_{s_k}(t)/x_{d_k}(t)$ converges to a real number (possibly 0).

4.2 Four Lemmata

The proof of Theorem 1 relies on four lemmata that we prove in this subsection.

Lemma 1 *If the growth-rate functions are Lipschitz continuous, and the dynamic is regular payoff monotonic, then there is a constant $\bar{\lambda}$ such that $g_i(x) - g_j(x) \leq \bar{\lambda}(\pi_i(x) - \pi_j(x))$ for all i, j and all $x \in \Delta$ such that $\pi_i(x) \geq \pi_j(x)$.*

Proof. We prove the lemma by finding a constant $\bar{\lambda}(i, j)$ for an arbitrary pair i and j . The desired result then follows from taking the smallest of these constants for all combinations of strategies i and j . If $x \in \Delta$ is such that $\pi_i(x) = \pi_j(x)$, then regular payoff monotonicity implies $g_i(x) = g_j(x)$ so we do not have to be concerned about such x . So, let $x \in \Delta$ be such that $\pi_i(x) - \pi_j(x) > 0$.

Suppose first that $\pi_i(y) - \pi_j(y) > 0$ for all $y \in \Delta$. Then, by continuity of the growth rate and payoff functions, we can define $\bar{\lambda}(i, j) > 0$ by

$$\bar{\lambda}(i, j) = \frac{\max_{y \in \Delta} (g_i(y) - g_j(y))}{\min_{y \in \Delta} (\pi_i(y) - \pi_j(y))}.$$

Suppose instead that $\pi_i(y) - \pi_j(y) \leq 0$ for some $y \in \Delta$. Let $S_+ = \{s \in S : u(i, s) > u(j, s)\}$ and $S_- = S \setminus S_+ = \{s \in S : u(i, s) \leq u(j, s)\}$. The set S_+ is not empty since $\pi_i(x) - \pi_j(x) > 0$. Similarly, the set S_- is not empty since $\pi_i(y) - \pi_j(y) \leq 0$ for some $y \in \Delta$. Let $x' \in \Delta$ be such $x'_s \leq x_s$ for $s \in S_+$, $x'_s \geq x_s$ for $s \in S_-$, and $\pi_i(x') - \pi_j(x') = 0$. That is, x' is constructed from x by decreasing the fractions for $s \in S_+$ and increasing the fraction for $s \in S_-$ until $\pi_i(x') = \pi_j(x')$. Note that for the 1-norm, $\|x - x'\|_1 = \sum_{s \in S} |x_s - x'_s| = 2 \sum_{s \in S_+} (x_s - x'_s)$. By Lipschitz continuity there is a constant $C > 0$ such that

$$g_i(x) - g_j(x) - (g_i(x') - g_j(x')) = g_i(x) - g_j(x) \leq C \|x - x'\|_1. \quad (2)$$

Let $c = \min_{s \in S_+} (u(i, s) - u(j, s)) > 0$. Since x' is such that $(x_s - x'_s)(u(i, s) - u(j, s)) \geq 0$ for all s ,

$$\begin{aligned} \pi_i(x) - \pi_j(x) &= \pi_i(x) - \pi_j(x) - (\pi_i(x') - \pi_j(x')) = \sum_{s \in S} (x_s - x'_s)(u(i, s) - u(j, s)) \quad (3) \\ &\geq \sum_{s \in S_+} (x_s - x'_s)(u(i, s) - u(j, s)) \geq c \sum_{s \in S_+} (x_s - x'_s) \\ &= \frac{c}{2} \|x - x'\|_1. \end{aligned}$$

Together, (2) and (3) imply

$$g_i(x) - g_j(x) \leq \frac{2C}{c} (\pi_i(x) - \pi_j(x)).$$

■

Lemma 2 *Assume that growth-rate functions are Lipschitz continuous, and that the dynamic is regular payoff monotonic. Consider a trajectory x . Suppose that $s, d \in S$ and $T \in \mathbb{R}_+$ are such that there are integrable functions $a(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $b(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\int_{t=0}^{\infty} b(t)dt < +\infty$ and*

$$\pi_d(x(t)) - \pi_s(x(t)) > a(t) - b(t) \quad (4)$$

for all $t > T$. Then $x_s(t)/x_d(t)$ converges to a real number which is 0 if $\int_{t=0}^{\infty} a(t)dt = +\infty$.

Proof of Lemma 2. It follows from our assumption of regular monotonicity and Lemma 1 that there are constants $\underline{\lambda}$ and $\bar{\lambda}$ such that

$$\underline{\lambda} |\pi_d(x(t)) - \pi_s(x(t))| \leq |g_d(x(t)) - g_s(x(t))| \leq \bar{\lambda} |\pi_d(x(t)) - \pi_s(x(t))| \quad (5)$$

for all $t > T$. Together (4) and (5) imply that $g_d(x(t)) - g_s(x(t)) > \underline{\lambda}(a(t) - b(t))$ if $a(t) - b(t) \geq 0$; and $g_d(x(t)) - g_s(x(t)) > \bar{\lambda}(a(t) - b(t))$ if $a(t) - b(t) < 0$. Hence

$$g_d(x(t)) - g_s(x(t)) > \underline{\lambda}a(t) - \bar{\lambda}b(t)$$

for all $t > T$. Therefore, since $\int_{t=0}^{\infty} b(t)dt < +\infty$,

$$\int_{t=0}^{\infty} (g_d(x(t)) - g_s(x(t)))dt = \begin{cases} +\infty & \text{if } \int_{t=0}^{\infty} a(t)dt = +\infty, \\ r \in \mathbb{R} \text{ or } +\infty & \text{if } \int_{t=0}^{\infty} a(t)dt < +\infty. \end{cases}$$

Since

$$\ln(x_d(\tau)/x_s(\tau)) - \ln(x_d(0)/x_s(0)) = \int_{t=0}^{\tau} (g_d(x(t)) - g_s(x(t)))dt.$$

the desired result follows. ■

Lemma 3 *Assume that growth-rate functions are Lipschitz continuous, and that the dynamic is regular payoff monotonic. Consider a trajectory x . If $d, j, s \in S$ are such that*

$$(i) \ u(d, j) > u(s, j), \text{ and}$$

$$(ii) \ x_i(t)/x_j(t) \rightarrow 0 \text{ for all } i \text{ such that } u(d, i) < u(s, i) \text{ and } \int_0^\infty x_i(\tau) d\tau = +\infty,$$

then $x_s(t)/x_d(t)$ converges to a real number.

If we also have that

$$(iii) \ \int_0^\infty x_j(\tau) d\tau = +\infty,$$

then $x_s(t)/x_d(t) \rightarrow 0$.

This lemma says that *if* (i) strategy d performs strictly better than strategy s against strategy j , (ii) all strategies against which s is a better reply than d , and which remain in the population for a long time, eventually become infinitely less frequent than strategy j , and (iii) strategy j remains in the population for a long time, *then* strategy s eventually becomes infinitely less frequent than strategy d . If (iii) does not hold then the ratio of strategy s to strategy d may converge to a positive number.¹⁰

Proof of Lemma 3. Let S_0 be the set of strategies $i \neq j$ such that $\int_0^\infty x_i(t) dt < +\infty$ and let $S_1 = S \setminus (S_0 \cup \{j\})$. Set

$$\begin{aligned} a(t) &= \frac{1}{2}x_j(t), \\ b(t) &= \sum_{i \in S_0} x_i(t) |u(d, i) - u(s, i)|, \\ c(t) &= \frac{1}{2}x_j(t) + \sum_{i \in S_1} x_i(t)(u(d, i) - u(s, i)). \end{aligned}$$

Then

$$\pi_d(x(t)) - \pi_s(x(t)) \geq a(t) - b(t) + c(t).$$

Since $x_i(t)/x_j(t) \rightarrow 0$ holds for all $i \in S_1$ with $u(d, i) < u(s, i)$, there is a T such that $c(t) > 0$ for all $t > T$. Therefore Lemma 2 applies and yields the desired conclusion. ■

Lemma 4 *Assume that growth-rate functions are Lipschitz continuous, and that the dynamic is regular payoff monotonic. Consider a trajectory x . If $s, d \in S$ are such*

$$(i) \ u(d, s) > u(s, s)$$

$$(ii) \ \int_0^\infty x_d(\tau) d\tau = +\infty, \text{ and}$$

¹⁰To make the intuition more transparent one may think of Lemma 3 as follows: (i) states that strategy j is "prey" for d . (ii) states that all of the prey for s (d 's "competitor") that survive long enough is infinitely less frequent than j . (iii) states that d 's prey survives long enough. Hence, eventually d becomes infinitely more frequent than s .

(iii) $x_i(t)/x_d(t) \rightarrow 0$ for all i with $u(d, i) < u(s, i)$,

then $x_s(t)/x_d(t) \rightarrow 0$.

Lemma 4 says that if (i) strategy d performs strictly better than strategy s against strategy s , (ii) strategy d remains in the population for a long time, and (iii) all strategies against which s is a better reply than d eventually become infinitely less frequent than strategy d , then strategy s eventually becomes infinitely less frequent than strategy d .¹¹

Claim 1 Let $z : \mathbb{R}_+ \rightarrow \mathbb{R}$ be differentiable. Assume that there exists $\eta > 0$ and $T > 0$ such that for any $t \geq T$, $\dot{z}(t) < 0$ whenever $z(t) = \eta$. Then either $z(t) > \eta$ for all $t \geq T$ or there exists $T' \geq T$ such that $z(t) \leq \eta$ for all $t \geq T'$.

Proof of Claim 1. If $z(s) \leq \eta$ for some $s \geq T$ then we can set $T' = \min\{t \in \mathbb{R}_+ : T \leq t \leq s, z(t) \leq \eta\}$. ■

Proof of Lemma 4. Let $\eta > 0$ be given. To prove Lemma 4 it is sufficient to show that $x_s(t)/x_d(t) < \eta$ for all sufficiently large t . Under the assumptions of Lemma 4, there exists positive constants c_1, c_2 , depending on the payoff function, such that

$$\pi_d(x(t)) - \pi_s(x(t)) \geq c_1 x_s(t) - c_2 x_d(t) \sum_{i: u(d, i) < u(s, i)} \frac{x_i(t)}{x_d(t)}. \quad (6)$$

Since $x_i(t)/x_d(t) \rightarrow 0$ for all i with $u(d, i) < u(s, i)$ there is some T such that the sum in (6) is smaller than $c_1 \eta / (2c_2)$ for all $t \geq T$. Thus

$$\pi_d(x(t)) - \pi_s(x(t)) \geq c_1 x_s(t) - c_1 x_d(t) \eta / 2 = c_1 x_d(t) \left[\frac{x_s(t)}{x_d(t)} - \frac{\eta}{2} \right].$$

for all $t \geq T$. Let $z(t) = x_s(t)/x_d(t)$. For any $t \geq T$, if $z(t) = \eta$, then $\pi_d(x(t)) - \pi_s(x(t)) > 0$; and hence by monotonicity $\dot{z}(t) < 0$. By Claim 1 it follows that either (a) $z(t) > \eta$ for all $t \geq T$; or (b) there exists T' such that $z(t) \leq \eta$ for all $t \geq T'$. If (b) holds then we are done by definition of z . If (a) holds, then

$$\pi_d(x(t)) - \pi_s(x(t)) \geq \eta c_1 x_d(t) / 2$$

for all $t \geq T$ and Lemma 2 applies with $a(t) = \eta c_1 x_d(t) / 2$ and $b(t) = 0$ implying $x_s(t)/x_d(t) \rightarrow 0$.¹² ■

¹¹ Again, to make the intuition more transparent one may think of Lemma 4 as follows: (i) states that strategy s is "prey" for d , (ii) states that s 's competitor d remains in the population for a long time. (iii) states that all of s 's prey becomes much less frequent than strategy d . Hence s becomes extinct.

¹² Case (a) never occurs since $x_s(t)/x_d(t) \rightarrow 0$ contradicts $z(t) > \eta$ for all $t \geq T$.

4.3 Proof of Theorem

Let $((s_k, b_k))_{k=1}^{m'}$ be a given IEWDS that satisfies (a) monotonicity, (b) pairwise weak dominance, and (c) local strict transitivity. We can assume, without loss of generality, that $s_k = k$ for $k = 1, \dots, m'$ and that $b_{m'} = m' + 1$. Let d_1, d_2, \dots, d_n be the increasing list of all $s \in S$ such that $s = b_k$ for some $k \in \{1, \dots, m'\}$. For notational purposes, let $d_0 = 1$. By monotonicity, $(d_k)_{k=1}^n$ is such that for each $k = 1, \dots, n$, d_k is used to eliminate all strategies s with $d_{k-1} \leq s < d_k$ in the sequence of IEWDS. We now prove Theorem 1 by way of *induction* on $(d_k)_{k=1}^n$.

4.3.1 Induction Base

We want to show that

Property 0 $\int_0^\infty x_s(t)dt < +\infty$ and $x_s(t)/x_{d_1}(t) \rightarrow r_s \geq 0$ for all $s \leq d_1$; or

Property 1 $\int_0^\infty x_{d_1}(t)dt = +\infty$ and $x_s(t)/x_{d_1}(t) \rightarrow 0$ for all $s < d_1$.

Set $s_1 = 1$ and $\beta_{s_1} = u(d_1, s_1) - u(d_1, d_1)$, and $\gamma_{s_1} = u(d_1, d_1) - u(s_1, d_1)$. Since d_1 eliminates 1 in the IEWDS we know by (b), pairwise weak dominance, that $\beta_{s_1} > 0$ or $\gamma_{s_1} > 0$.

Suppose $\beta_{s_1} > 0$. Then we can apply Lemma 3 with $j = s_1$, $s = s_1$, and $d = d_1$. (Note that part (ii) of the lemma is satisfied by weak dominance.) If $\int_0^\infty x_{s_1}(t)dt = +\infty$, then $x_{s_1}(t)/x_{d_1}(t) \rightarrow 0$. By the comparison test for improper integrals, $\int_0^\infty x_{s_1}(t)dt = +\infty$ and $x_{s_1}(t)/x_{d_1}(t) \rightarrow 0$ imply that $\int_0^\infty x_{d_1}(t)dt = +\infty$. So, if $\int_0^\infty x_{s_1}(t)dt = +\infty$, then Property 1 is satisfied for $s = s_1$. If $\int_0^\infty x_{s_1}(t)dt < +\infty$, then $x_{s_1}(t)/x_{d_1}(t) \rightarrow r \geq 0$; and if $r > 0$, then we cannot have $\int_0^\infty x_{d_1}(t)dt = +\infty$ since $\int_0^\infty x_{d_1}(t)dt = +\infty$, $\int_0^\infty x_{s_1}(t)dt < +\infty$, and $x_{s_1}(t)/x_{d_1}(t) \rightarrow r > 0$ contradicts the comparison test for improper integrals. So, if $\int_0^\infty x_{s_1}(t)dt < +\infty$, then Property 0 or Property 1 is satisfied for $s = s_1$.

Suppose $\gamma_{s_1} > 0$. Then we can apply Lemma 3 with $j = d_1$, $s = s_1$, and $d = d_1$. (By weak dominance part (ii) of the lemma is satisfied.) If $\int_0^\infty x_{d_1}(t)dt = +\infty$, then $x_{s_1}(t)/x_{d_1}(t) \rightarrow 0$ and Property 1 is satisfied for $s = s_1$. If $\int_0^\infty x_{d_1}(t)dt < +\infty$, then $x_{s_1}(t)/x_{d_1}(t) \rightarrow r \geq 0$. By the comparison test for improper integrals, $\int_0^\infty x_{d_1}(t)dt < +\infty$ and $x_{s_1}(t)/x_{d_1}(t) \rightarrow r \geq 0$ imply that $\int_0^\infty x_{s_1}(t)dt < +\infty$. Thus Property 0 is satisfied for $s = s_1$ if $\int_0^\infty x_{d_1}(t)dt < +\infty$.

We have now examined all possibilities to show that Property 0 or Property 1 holds for $s = s_1 = 1$.

Set $s_2 = s_1 + 1$ and assume that $s_2 < d_1$. Let $\beta_{s_2} = u(d_1, s_2) - u(s_2, s_2)$, and let $\gamma_{s_2} = u(d_1, d_1) - u(s_2, d_1)$. Since d_1 eliminates s_2 in the IEWDS we know by assumption (b) that $\beta_{s_2} > 0$ or $\gamma_{s_2} > 0$. Suppose first that Property 0 holds for $s = s_1$. Then $\int_0^\infty x_{d_1}(t)dt < +\infty$ and thus we do not have to be concerned about the sign of $u(s_2, s_1) - u(d_1, s_1)$ when we

apply Lemma 3. If $\beta_{s_2} > 0$, then Lemma 3 applies with $j = s_2$, and if $\gamma_{s_2} > 0$ then Lemma 3 applies with $j = d_1$. In both cases the implied result is that $x_{s_2}(t)/x_{d_1}(t)$ converges to a real number which in turn, since $\int_0^\infty x_{d_1}(t)dt < +\infty$, by the comparison test for improper integrals implies that $\int_0^\infty x_{s_2}(t)dt < +\infty$. So, Property 0 holds for $s = s_2$ if it holds for $s = s_1$. Suppose instead that Property 1 holds for $s = s_1$. Then $\int_0^\infty x_{d_1}(t)dt = +\infty$ and $x_{s_1}(t)/x_{d_1}(t) \rightarrow 0$ so if $\beta_{s_2} > 0$, then Lemma 4 implies that $x_{s_2}(t)/x_{d_1}(t) \rightarrow 0$. If $\gamma_{s_2} > 0$, then we can apply Lemma 3 with $j = d_1$ to derive the same result. So, Property 1 holds for $s = s_2$ if it holds for $s = s_1$.

Set $s_3 = s_2 + 1$, and assume that $s_3 < d_1$. Then we can repeat the same arguments since we now know that either Property 0 holds for $s < s_3$ or Property 1 holds for $s < s_3$. We can continue using the same argument until we reach $s = d_1 - 1$.

4.3.2 Induction Step

Suppose that for all $\tilde{k} = 1, \dots, k$,

Property A0 $\int_0^\infty x_s(t)dt < +\infty$ and $x_s(t)/x_{d_{\tilde{k}}}(t) \rightarrow r_s \geq 0$ for all $s \leq d_{\tilde{k}}$; or

Property A1 $\int_0^\infty x_{d_{\tilde{k}}}(t)dt = +\infty$ and $x_s(t)/x_{d_{\tilde{k}}}(t) \rightarrow 0$ for all $s < d_{\tilde{k}}$.

Then

Property B0 $\int_0^\infty x_s(t)dt < +\infty$ and $x_s(t)/x_{d_{k+1}}(t) \rightarrow r_s \geq 0$ for all $s \leq d_{k+1}$; or

Property B1 $\int_0^\infty x_{d_{k+1}}(t)dt = +\infty$ and $x_s(t)/x_{d_{k+1}}(t) \rightarrow 0$ for all $s < d_{k+1}$.

To see that this is true suppose first that Property A0 is satisfied for $\tilde{k} = k$. Then $\int_0^\infty x_s(t)dt < +\infty$ for all $s \leq d_k$ and we may employ the same argument as in the proof of the induction base to show that Property B0 or B1 is satisfied since strategies $s \leq d_k$ will not matter when we apply Lemma 3.¹³

If Property A0 is not satisfied for $\tilde{k} = k$, then Property A1 is satisfied for $\tilde{k} = k$ and thus $x_s(t)/x_{d_k}(t) \rightarrow 0$ for all $s < d_k$ and $\int_0^\infty x_{d_k}(t)dt = +\infty$. We first wish to show that $x_{d_k}(t)/x_{d_{k+1}}(t) \rightarrow 0$.

Our sequence of IEWDS is such that d_{k+1} is the strategy that eliminates d_k and thus $u(d_k, d_k) \leq u(d_{k+1}, d_k)$. If $u(d_k, d_k) < u(d_{k+1}, d_k)$, then Lemma 3 applies (set $s = j = d_k$ and $d = d_{k+1}$) and yields $x_{d_k}(t)/x_{d_{k+1}}(t) \rightarrow 0$, as desired. Therefore, assume $u(d_k, d_k) = u(d_{k+1}, d_k)$. Then by (b), pairwise weak dominance, $u(d_{k+1}, d_{k+1}) > u(d_k, d_{k+1})$.

¹³There will be two cases to check. One where $\int_0^\infty x_{d_{k+1}}(t)dt < +\infty$, in which case Property B0 will be satisfied; and one where $\int_0^\infty x_{d_{k+1}}(t)dt = +\infty$ in which case Property B1 will be satisfied.

Consider $s = d_k - 1$ (which is eliminated by d_k) and examine the ratio $x_{d_k-1}(t)/x_{d_{k+1}}(t)$. Since the IEWDS is such that d_{k+1} eliminates d_k and d_k eliminates $d_k - 1$,

$$u(d_{k+1}, i) \geq u(d_k, i) \geq u(d_k - 1, i) \text{ for all } i \geq d_k.$$

Thus, $u(d_{k+1}, i) \geq u(d_k - 1, i)$ for all $i \geq d_k$ and $x_i(t)/x_{d_k}(t) \rightarrow 0$ for all $i < d_k$. Furthermore, by (c), local strict transitivity, the inequality $u(d_{k+1}, i) \geq u(d_k - 1, i)$ is strict at $i = d_k$. Therefore Lemma 3 applies and yields $x_{d_k-1}(t)/x_{d_{k+1}}(t) \rightarrow 0$.

Now consider $s = d_k - 2$ and assume $d_k - 2$ is eliminated by d_k and not by $d_k - 1$. Since d_{k+1} eliminates d_k , and since d_k eliminates $d_k - 2$,

$$u(d_{k+1}, i) \geq u(d_k, i) \geq u(d_k - 2, i) \text{ for all } i \geq d_k,$$

and, by local strict transitivity, at least one of these inequalities is strict at $i = d_k$, and so $u(d_{k+1}, d_k) > u(d_k - 2, d_k)$. Since $x_i(t)/x_{d_k}(t) \rightarrow 0$ for all $i < d_k$, Lemma 3 applies and yields $x_{d_k-2}(t)/x_{d_{k+1}}(t) \rightarrow 0$.

By repeating the argument we can conclude that $x_s(t)/x_{s_{k+1}}(t) \rightarrow 0$ for all s with $d_{k-1} \leq s \leq d_k - 1$, where $d_{k-1} = 1$ if $k = 1$. By the induction hypothesis $x_s(t)/x_{d_{k-1}}(t) \rightarrow r_s$ for all $s \leq d_{k-1}$, and therefore $x_{d_{k-1}}(t)/x_{d_{k+1}}(t) \rightarrow 0$ implies

$$x_s(t)/x_{d_{k+1}}(t) \rightarrow 0 \text{ for all } s < d_k. \quad (7)$$

Recall that $u(d_k, d_{k+1}) < u(d_{k+1}, d_{k+1})$ by (b). It therefore follows from (7) and Lemma 3 that $x_{d_k}(t)/x_{d_{k+1}}(t)$ converges to a real number. Since $\int_0^\infty x_{d_k}(t)dt = +\infty$ it follows from the comparison test for improper integrals that $\int_0^\infty x_{d_{k+1}}(t)dt = +\infty$ and so, by part (iii) of Lemma 3, the real number that $x_{d_k}(t)/x_{d_{k+1}}(t)$ converges to is 0.

We have shown that

$$\int_0^\infty x_{d_{k+1}}(t)dt = +\infty \text{ and } \frac{x_s(t)}{x_{d_{k+1}}(t)} \rightarrow 0 \text{ for all } s \leq d_k. \quad (8)$$

Set $s = d_k + 1$, and assume that $s < d_{k+1}$. If $u(d_{k+1}, d_{k+1}) > u(s, d_{k+1})$, then (8) and Lemma 3 yield $x_s(t)/x_{d_{k+1}}(t) \rightarrow 0$. If $\pi(d_{k+1}, s) > \pi(s, s)$, then (8) and Lemma 4 yield $x_s(t)/x_{d_{k+1}}(t) \rightarrow 0$. Therefore,

$$\int_0^\infty x_{d_{k+1}}(t)dt = +\infty \text{ and } \frac{x_s(t)}{x_{d_{k+1}}(t)} \rightarrow 0 \text{ for all } s \leq d_k + 1.$$

We can repeat this step for $s = s_k + 1, s_k + 2, \dots, s_{k+1} - 1$, and so $x_s(t)/x_{d_{k+1}}(t) \rightarrow 0$ for all such s . We have thus shown that Property B1 is satisfied if Property A1 is satisfied for $\tilde{k} = k$, and this completes the proof of the induction step.

5 Applications

We go through two applications, discretised versions of Bertrand duopoly and a two-player first-price auction, which are both such that it is straightforward to construct a sequence of IEWDS that satisfies the conditions of Theorem 1 and that eliminates all but one strategy. Thus our main result implies that a regular payoff monotonic dynamic selects this strategy, which approaches the Nash equilibrium strategy as the grid is made finer.

5.1 Bertrand Duopoly

Consider the following discretised Bertrand duopoly. Let $1/M$ be the smallest monetary unit. Firm i sets price $p_i \in \{0, \frac{1}{M}, \frac{2}{M}, \dots, \frac{M-1}{M}, 1\}$. Demand is $1 - p$ for the firm with the lowest price. The marginal cost is equal to zero. Thus the profit is

$$u(p_i, p_j) = \begin{cases} p_i (1 - p_i) & \text{if } p_i < p_j \\ \frac{1}{2} p_i (1 - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}.$$

The payoff matrix is

	0	$\frac{1}{M}$	$\frac{2}{M}$	\dots	$\frac{M-1}{M}$	1
0	0	0	0	\dots	0	0
$\frac{1}{M}$	0	$\frac{1}{2} \frac{1}{M} (1 - \frac{1}{M})$	$\frac{1}{M} (1 - \frac{1}{M})$	\dots	$\frac{1}{M} (1 - \frac{1}{M})$	$\frac{1}{M} (1 - \frac{1}{M})$
$\frac{2}{M}$	0	0	$\frac{1}{2} \frac{2}{M} (1 - \frac{2}{M})$	\dots	$\frac{2}{M} (1 - \frac{2}{M})$	$\frac{2}{M} (1 - \frac{2}{M})$
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
$\frac{M-1}{M}$	0	0	0	\dots	$\frac{1}{2} \frac{M-1}{M} (1 - \frac{M-1}{M})$	$\frac{M-1}{M} (1 - \frac{M-1}{M})$
1	0	0	0	\dots	0	0

Since it is better to have a slightly lower price than your opponent than to have the same price, for every $l \in \{0, 1, 2, \dots, M-1\}$,

$$u\left(\frac{l}{M}, \frac{l+1}{M}\right) = \frac{l}{M} \left(1 - \frac{l}{M}\right) > \frac{1}{2} \frac{l+1}{M} \left(1 - \frac{l+1}{M}\right) = u\left(\frac{l+1}{M}, \frac{l+1}{M}\right).$$

Since is also better to have the same price as your opponent than to have a higher price,

$$u\left(\frac{l}{M}, \frac{l}{M}\right) > u\left(\frac{l+1}{M}, \frac{l}{M}\right).$$

Therefore, if we set $s = \frac{l+1}{M}$ and $d = \frac{l}{M}$, then both $u(d, s) > u(s, s)$ and $u(d, d) > u(s, d)$ are satisfied and we can create a sequence of IEWDS that satisfies the conditions of Theorem 1 by eliminating strategies in decreasing order from 1 to $2/M$ and finally

eliminating 0. Then only strategy $1/M$ remains and thus this strategy is selected by evolution. By increasing M evolution selects a strategy profile that is arbitrarily close to 0.

5.2 Two-Player First-Price Auction with Common Values

The structure of a discretised first-price common-value auction with two players is similar to that of a discretised Bertrand duopoly. Let $1/M$ be the smallest monetary unit. Buyer i places a bid $p_i \in \{0, \frac{1}{M}, \frac{2}{M}, \dots, \frac{M-1}{M}, 1\}$. The value of the prize is 1. Thus the payoff is

$$u(p_i, p_j) = \begin{cases} 1 - p_i & \text{if } p_i > p_j \\ \frac{1}{2}(1 - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i < p_j. \end{cases}$$

The payoff matrix is

	0	$\frac{1}{M}$	$\frac{2}{M}$	\dots	$\frac{M-1}{M}$	1
0	$\frac{1}{2}$	0	0	\dots	0	0
$\frac{1}{M}$	$1 - \frac{1}{M}$	$\frac{1}{2}(1 - \frac{1}{M})$	0	\dots	0	0
$\frac{2}{M}$	$1 - \frac{2}{M}$	$1 - \frac{2}{M}$	$\frac{1}{2}(1 - \frac{2}{M})$	\dots	0	0
\vdots	\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
$\frac{M-1}{M}$	$1 - \frac{M-1}{M}$	$1 - \frac{M-1}{M}$	$1 - \frac{M-1}{M}$	\dots	$\frac{1}{2}(1 - \frac{M-1}{M})$	0
1	0	0	0	\dots	0	0

It is better to bid slightly above your opponent than to bid the same so, for every $l \in \{0, 1, 2, \dots, M-1\}$,

$$u\left(\frac{l}{M}, \frac{l-1}{M}\right) = 1 - \frac{l}{M} > \frac{1}{2}\left(1 - \frac{l-1}{M}\right) = u\left(\frac{l-1}{M}, \frac{l-1}{M}\right).$$

It is also better to bid the same as your opponent than to bid below,

$$u\left(\frac{l}{M}, \frac{l}{M}\right) > u\left(\frac{l-1}{M}, \frac{l}{M}\right).$$

Therefore, if we set $s = \frac{l-1}{M}$ and $d = \frac{l}{M}$, then both $u(d, s) > u(s, s)$ and $u(d, d) > u(s, d)$ are satisfied and we can create a sequence of IEWDS that satisfies the conditions of Theorem 1 by eliminating strategies in increasing order from 0 to $(M-2)/M$ and finally eliminating 1. Then only strategy $(M-1)/M$ remains and thus this strategy is selected by evolution. By increasing M evolution selects a strategy profile that is arbitrarily close to 1.

6 Conclusion

To the best of our knowledge we are the first to provide general conditions under which a large class of imitative dynamics select against iteratively weakly dominated strategies. If a sequence of IEWDS satisfies our sufficient conditions, then, starting from any interior initial state, the strategies that are eliminated by that sequence become extinct asymptotically. Our results are proved for the class of regular payoff monotonic dynamics, which includes convex monotonic and aggregate monotonic dynamics such as the replicator dynamic.

A potential shortcoming of our result is that it is not readily applicable to extensive form games, which is an important context in which weakly dominated strategies occur naturally. We note that Cressman (2003) provides some reason to be sceptical about the prospects of finding general conditions for evolutionary selection of iteratively weakly dominated strategies in extensive form games. It should also be pointed out that although our condition ensures that, starting from any interior initial state, strategies that fail iterative elimination of weakly dominated strategies become extinct asymptotically, our condition does *not* imply that the set of strategies that survive iterated weak dominance is Lyapunov stable. See Sandholm (2015), page 752, for a counterexample.

It is well known that reinforcement learning (Arthur 1993, Cross 1973, Erev and Roth 1998) is closely linked with the replicator dynamic in the sense that the latter can be obtained as the mean field approximation of the former, as established by Börgers and Sarin (1997) and Hopkins (2002). In light of this it would be interesting to test whether the convergence results of this paper are borne out in learning environments conducive to reinforcement learning, such as environments where subjects have no knowledge of the game they play and only receive feedback information about their own payoffs (as in Nax et al. 2016). We leave for future work the testing of our theorems in information settings conducive to learning heuristics that induce the replicator dynamic.

References

- Arthur, W. B. (1993), ‘On designing economic agents that behave like human agents’, *Journal of Evolutionary Economics* **3**(1), 1–22.
- Asheim, G. B. and Dufwenberg, M. (2003), ‘Admissibility and common belief’, *Games and Economic Behavior* **42**(2), 208–234.
- Bernergård, A. (2014), *Best replying, to yourself and others*, Stockholm School of Economics, Stockholm. PhD Thesis.
- Bernheim, B. D. (1984), ‘Rationalizable strategic behavior’, *Econometrica* **52**(4), 1007–1028.

- Börger, T. and Sarin, R. (1997), ‘Learning through reinforcement and replicator dynamics’, *Journal of Economic Theory* **77**(1), 1–14.
- Brandenburger, A., Friedenberg, A. and Keisler, H. J. (2008), ‘Admissibility in games’, *Econometrica* **76**(2), 307–352.
- Cressman, R. (1996), ‘Evolutionary stability in the finitely repeated prisoner’s dilemma game’, *Journal of Economic Theory* **87**(68), 234–248.
- Cressman, R. (2003), *Evolutionary Dynamics and Extensive Form Games*, The MIT Press, London.
- Cross, J. G. (1973), ‘A stochastic learning model of economic behavior’, *The Quarterly Journal of Economics* **87**(2), 239–266.
- Erev, I. and Roth, A. E. (1998), ‘Predicting how people play games: Reinforcement learning in experimental games with unique, mixed strategy equilibria’, *American Economic Review* **88**(4), 848–881.
- Hofbauer, J. and Sandholm, W. H. (2011), ‘Survival of dominated strategies under evolutionary dynamics’, *Theoretical Economics* **6**(3), 341–377.
- Hofbauer, J. and Weibull, J. W. (1996), ‘Evolutionary selection against dominated strategies’, *Journal of Economic Theory* **71**(2), 558–573.
- Hopkins, E. (2002), ‘Two competing models of how people learn in games’, *Econometrica* **70**(6), 2141–2166.
- Kuzmics, C. (2011), ‘On the elimination of dominated strategies in stochastic models of evolution with large populations’, *Games and Economic Behavior* **72**(2), 452–466.
- Laraki, R. and Mertikopoulos, P. (2013), ‘Higher order game dynamics’, *Journal of Economic Theory* **148**(6), 2666–2695.
- Marx, L. M. (1999), ‘Adaptive learning and iterated weak dominance’, *Games and Economic Behavior* **26**(2), 253–278.
- Marx, L. M. and Swinkels, J. M. (1997), ‘Order independence for iterated weak dominance’, *Games and Economic Behavior* **18**(2), 219–245.
- Milgrom, P. and Roberts, J. (1991), ‘Adaptive and sophisticated learning in normal form games’, *Games and Economic Behavior* **3**(1), 82–100.
- Mohlin, E. (2012), ‘Evolution of theories of mind’, *Games and Economic Behavior* **75**(1), 299–312.

- Nachbar, J. H. (1990), ‘Evolutionary selection dynamics in games: Convergence and limit properties’, *International journal of game theory* **19**(1), 59–89.
- Nax, H. H., Burton-Chellew, M. N., West, S. A. and Young, H. P. (2016), ‘Learning in a black box’, *Journal of Economic Behavior & Organization* **127**, 1–15.
- Pearce, D. G. (1984), ‘Rationalizable strategic behavior and the problem of perfection’, *Econometrica* **52**(4), 1029–1050.
- Ponti, G. (2000), ‘Cycles of learning in the centipede game’, *Games and Economic Behavior* **30**(1), 115–141.
- Samuelson, L. (1994), ‘Stochastic stability in games with alternative best replies’, *Journal of Economic Theory* **64**(1), 35–65.
- Samuelson, L. and Zhang, J. (1992), ‘Evolutionary stability in asymmetric games’, *Journal of Economic Theory* **57**(2), 363–391.
- Sandholm, W. H. (2015), Population games and deterministic evolutionary dynamics, in ‘Handbook of game theory with economic applications’, Vol. 4, Elsevier, pp. 703–778.
- Sandholm, W. H. and Dokumaci, E. (2007), ‘Dynamo: Phase diagrams for evolutionary dynamics (software suite)’, Available at <http://www.ssc.wisc.edu/whs/dynamo>.
- Tan, T. C.-C. and Werlang, S. R. d. C. (1988), ‘The bayesian foundations of solution concepts of games’, *Journal of Economic Theory* **45**(2), 370–391.
- Taylor, P. D. and Jonker, L. B. (1978), ‘Evolutionary stable strategies and game dynamics’, *Mathematical Biosciences* **40**(1-2), 145–156.
- Thomas, B. (1985), ‘On evolutionarily stable sets’, *Journal of Mathematical Biology* **22**(1), 105–115.
- Viossat, Y. (2015), ‘Evolutionary dynamics and dominated strategies’, *Economic Theory Bulletin* **3**(1), 91–113.

7 Appendix

7.1 (c) Local Strict Transitivity

In section 3.2.2 we claimed that for G^6 and under any monotonic dynamic, $x_2(t) \rightarrow 1$ if $x_1(0) \geq x_3(0)$. This is a consequence of the following more general result.¹⁴

Lemma 5 *Assume that growth-rate functions are Lipschitz continuous, and that the dynamic is payoff monotonic. Suppose that (i) $u(1, s) = u(3, s)$ for any $s \notin \{1, 3\}$; and (ii) $u(1, 1) > u(3, 1)$. If $x(0) \in \Delta$ is such that $x_1(0) > 0$ and the ratio $x_1(0)/x_3(0)$ large enough to imply that $\pi_1(x(0)) > \pi_3(x(0))$, then $x_3(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Proof of Lemma 5. Let $z(t) = x_3(t)/x_1(t)$ and $z(0) = \eta$. Since $\pi_1(x(0)) > \pi_3(x(0))$, and since the dynamic is monotonic, we have that $\dot{z}(0) < 0$ and assumptions (i) and (ii) then imply that $\dot{z}(t) < 0$ and $\pi_1(x(t)) > \pi_3(x(t))$ for all $t > 0$. Thus, by monotonicity, $g_1(x(t)) - g_3(x(t)) > 0$ for all $t > 0$.

If $x_3(t) \not\rightarrow 0$, then there exists $\varepsilon_1 > 0$ and $t_1 < t_2 < \dots$ such that $t_k \rightarrow \infty$ and $x_3(t_k) \geq 2\varepsilon_1$ for all k . Let $\varepsilon_2 > 0$ be such that $\pi_1(x) - \pi_3(x) > \varepsilon_2$ if $\varepsilon_1 \leq x_3 \leq \eta x_1$. Note that $\eta x_1(t) \geq x_3(t)$ since $z(0) = \eta$ and z decreases with time. By uniform continuity of $x_3(\cdot)$, which is implied by Lipschitz continuity of the growth-rate functions, there is a $\kappa > 0$ such that $x_3(t) \geq \varepsilon$ for all $t \in (t_k - \kappa, t_k + \kappa)$ for all k . We can assume, without loss of generality, that $t_{k+1} - t_k > \kappa$ for all k . Since $\pi_1(x) - \pi_3(x) \geq \varepsilon_2$ for all $t \in (t_k - \kappa, t_k + \kappa)$, there exists $\varepsilon_3 > 0$ such that $g_1(x) - g_3(x) \geq \varepsilon_3$ for all $t \in (t_k - \kappa, t_k + \kappa)$.¹⁵ Thus

$$\int_{t=0}^{\infty} (g_1(x(t)) - g_3(x(t))) dt \geq \sum_{k=1}^{\infty} \varepsilon_3 \kappa = +\infty$$

which implies $x_3(t)/x_1(t) \rightarrow 0$. This, in turn, implies $x_3(t) \rightarrow 0$. This completes the proof. Alternatively, it can be shown that Lemma 5 is implied by Proposition 4.1 in Ponti (2000). ■

For G^6 we have that $\pi_2(x(0)) - \pi_1(x(0)) = x_1(0)$. An argument similar to the one we just used to prove Lemma 5 then shows that $x_1(t) \rightarrow 0$ as $t \rightarrow \infty$ from all interior initial conditions, and for all monotonic dynamics. (If $x_1(t) \not\rightarrow 0$, then $x_1(t)/x_2(t) \rightarrow 0$ by uniform continuity of $x_1(\cdot)$.) Also, G^6 is such that it follows from Lemma 5 that $x_3(t) \rightarrow 0$ as $t \rightarrow \infty$ if $x_1(0) < x_3(0)$. If $x_1(0) = x_3(0)$, then strategies 1 and 3 earn the same and monotonicity implies that they both shrink towards 0 at the same rate and thus $x_3(t) \rightarrow 0$ in this case too.

¹⁴We are grateful to an anonymous referee for suggestions that lead us to the following analysis.

¹⁵We can find ε_3 by minimizing the continuous function $x \mapsto g_1(x) - g_3(x)$ on the compact set of $x \in \Delta$ with $\pi_1(x) - \pi_3(x) \geq \varepsilon_2$. Since the growth rate functions are monotonic this minimum will be positive.

7.2 (a) Monotonicity

In section 3.2.3 we claimed that for G^7 and under any monotonic dynamic $x_1(t) \rightarrow 0$ and $x_3(t) \rightarrow 0$ if $x_1(0) \geq x_3(0)$. Since strategy 4 is identical to strategy 2, and since all strategies earn 0 against strategy 4, this follows from what we have already proven for game G^6 .

7.3 Regular Monotonicity

Our goal is to define a monotonic dynamic and find an interior initial condition $x(0)$ such that $x_3(t) \rightarrow 1$ for the game G^8 , introduced in section 3.2.4. Let the strategies be named 1, 2, 3, and 4 according to row order. Write $s_1(x)$ for the strategy that has the highest payoff at the population state x , and $s_2(x)$ for the strategy that has the second highest payoff and so on. When two or more strategies have the same payoff at a population state x then the ties can be broken arbitrarily, for example by giving priority to the strategy whose name is the lower number. To simplify the notation we write s_i instead of $s_i(x)$ and define growth rate functions by setting

$$\begin{aligned} g_{s_1}(x) &= (\pi_{s_1}(x) - \pi_{s_2}(x))(x_{s_2} + x_{s_3} + x_{s_4}) \\ &\quad + (\pi_{s_2}(x) - \pi_{s_3}(x))^n(x_{s_3} + x_{s_4}) + (\pi_{s_3}(x) - \pi_{s_4}(x))x_{s_4}, \\ g_{s_2}(x) &= g_{s_1}(x) - (\pi_{s_1}(x) - \pi_{s_2}(x)), \\ g_{s_3}(x) &= g_{s_2}(x) - (\pi_{s_2}(x) - \pi_{s_3}(x))^n, \\ g_{s_4}(x) &= g_{s_3}(x) - (\pi_{s_3}(x) - \pi_{s_4}(x)), \end{aligned} \tag{9}$$

for some positive integer n . For $n = 1$ we get the replicator dynamic since then $g_{s_i}(x) = \pi_{s_i}(x) - \bar{\pi}(x)$. The growth rate functions defined by (9) are well defined for all positive integers k in the sense that $\sum_i g_{s_i}(x)x_{s_i} = 0$.¹⁶ They are also Lipschitz continuous because each g_{s_i} is a degree n polynomial of payoff differences with x_1, x_2, x_3, x_4 as coefficients, and if there is a point x^* which is a boundary point of both the set of x with $s_i(x) = j$ and the set of x with $s_i(x) = k$, then $\pi_j(x^*) = \pi_k(x^*)$ and $g_j(x^*) = g_k(x^*)$. They are also monotonic since $g_{s_1}(x) \geq g_{s_2}(x) \geq g_{s_3}(x) \geq g_{s_4}(x)$, with strict inequality wherever payoffs are not equal. However, for $n > 1$, regular monotonicity is violated if there exists $x^* \in \Delta$ such that $\pi_{s_2}(x^*) = \pi_{s_3}(x^*)$, and for every $\delta > 0$ there is an $x \in \Delta$ with $\|x^* - x\| < \delta$ such that $\pi_{s_2}(x) \neq \pi_{s_3}(x)$. To see this, note that for any sequence x^1, x^2, \dots from Δ that converges to x^* , with $\pi_{s_3}(x^t)(x^t) \neq \pi_{s_3}(x^t)(x^t)$ for all t , we have

$$\frac{g_{s_2}(x^t)(x^t) - g_{s_3}(x^t)(x^t)}{\pi_{s_2}(x^t)(x^t) - \pi_{s_3}(x^t)(x^t)} = (\pi_{s_2}(x^t)(x^t) - \pi_{s_3}(x^t)(x^t))^{n-1} \rightarrow 0$$

¹⁶To see this note that $\sum_i g_{s_i}(x)x_{s_i} = g_{s_1}(x) - (\pi_{s_1}(x) - \pi_{s_2}(x))(x_{s_2} + x_{s_3} + x_{s_4}) - (\pi_{s_2}(x) - \pi_{s_3}(x))^n(x_{s_3} + x_{s_4}) - (\pi_{s_3}(x) - \pi_{s_4}(x))x_{s_4}$

as $t \rightarrow \infty$ if $n > 1$. For the game G^8 regular monotonicity is violated in a neighborhood of $x^* = (0, 0, 1, 0)$ if $n > 1$ and we can use for example the sequence $x^t = \frac{1}{10t}(3, 2, 10t - 6, 1)$ to show it.

From now on, assume that $n = 2$. For G^8 , consider the subset X of Δ where $x_1 \geq x_2 \geq 3x_4$ and $x_3 \geq 3/4$. For all $x \in X$ we have that $s_1(x) = 3, s_2(x) = 2, s_3(x) = 1, s_4(x) = 4$; and, importantly, the only way for a solution to our differential equations to leave X is for $x_2(t)$ to outgrow $x_1(t)$. Setting $s_1(x) = 3, s_2(x) = 2, s_3(x) = 1, s_4(x) = 4$ yields

$$\begin{aligned} g_2(x) &= (\pi_{s_1}(x) - \pi_{s_2}(x))(-x_{s_1}) + (\pi_{s_2}(x) - \pi_{s_3}(x))^2(x_{s_3} + x_{s_4}) + (\pi_{s_3}(x) - \pi_{s_4}(x))x_{s_4} \\ &= (2x_1 + x_2 - x_4)(-x_3) + x_2^2(x_1 + x_4) + (2x_1 - 2x_2)x_4. \end{aligned}$$

Since $x_1 \geq x_2 \geq 3x_4$ and $x_3 \geq 3/4$ for $x \in X$ we can put a simple upper bound on $g_2(x)$ for $x \in X$:

$$\begin{aligned} g_2(x) &= (2x_1 + x_2 - x_4)(-x_3) + x_2^2(x_1 + x_4) + (2x_1 - 2x_2)x_4 \\ &< -3(2x_1 + x_2 - x_4)/4 + x_2^2(x_1 + x_4) + (2x_1 - 2x_2)x_4 \\ &= -x_2/2 - (x_2/4 - 3x_4/4) - x_1(6/4 - x_2^2 - 2x_4) - x_4(2x_2 - x_2^2) \\ &< -x_2/2. \end{aligned}$$

It follows that $\dot{x}_2(t) = g_2(x(t))x_2(t) \leq -x_2(t)^2/2$.

The differential equation $\dot{z}(t) = -z(t)^2/2$ is separable and has the solution $z(t) = 2/(t + 2/z(0))$. Since $\dot{x}_2(t) \leq \dot{z}(t)$ whenever $x_2(t) \geq z(t)$, we can conclude that if $z(0) = x_2(0)$, then $x_2(t) \leq z(t)$ for all t , which implies

$$\begin{aligned} \int_0^\infty x_2^2(t)dt &\leq \int_0^\infty z(t)^2dt = \int_0^\infty \frac{4}{(t + 2/z(0))^2}dt \\ &= \left[\frac{-4}{t + 2/z(0)} \right]_0^\infty = 2z(0) = 2x_2(0). \end{aligned}$$

So, for solution mappings that never leave X we have established an upper bound for $\int_0^\infty x_2^2(t)dt$ that depends on $x_2(0)$. Because of how our growth rate functions are defined with $n = 2$ we can use this to establish an upper bound for $x_2(t)/x_1(t)$. Pick any $x(0) \in X$ such that $x_2(0) = 1/100$ and $x_1(0) = e^2/100$.¹⁷ Then, for all T such that $x(t) \in X$ for all $t \leq T$,

$$\begin{aligned} \ln \left(\frac{x_2(T)}{x_1(T)} \right) &= \int_0^T (g_2(x(t)) - g_1(x(t)))dt + \ln \left(\frac{x_2(0)}{x_1(0)} \right) = \int_0^T x_2^2(t)dt + \ln \left(\frac{x_2(0)}{x_1(0)} \right) \\ &\leq 2x_2(0) + \ln \left(\frac{x_2(0)}{x_1(0)} \right) = \frac{2}{100} - 2 \ln e < -1. \end{aligned}$$

¹⁷Here e^2 denotes Euler's number squared, not a unit vector in Δ .

It follows that $x_2(T)/x_1(T) \leq e^{-1}$ for all such T . Since $x_2(t)/x_1(t)$ changes continuously with time, it follows that $x_2(t) < x_1(t)$ for all $t \geq 0$ and thus $x(t) \in X$ for all $t \geq 0$. Since the solution mapping never leaves X , strategy 3 grows forever and strategies 1, 2, 4 decrease forever. To rule out that x_1, x_2, x_4 converge to something other than 0 we can look at $g_3(x) - g_1(x)$ which satisfies $g_3(x(t)) - g_1(x(t)) \geq x_1(t)$ for $x \in X$. Hence, if $x_1(t)$ converged to something other than 0, then $x_1(t)/x_3(t)$ would converge to 0, which contradicts that $x_1(t)$ converges to something other than 0. Thus $x_1(t) \rightarrow 0$, and since $x_1 \geq x_2 \geq x_4$ for $x \in X$ it follows that $x_2(t)$ and $x_3(t)$ converge to 0 as well. The reason that this happens for $n = 2$ is that for $n = 2$, $\int_0^\infty (g_2(x(t)) - g_1(x(t)))dt$ can be finite even if $\int_0^\infty x_2(t)dt$ is infinite.¹⁸ For $n = 1$, i.e., for the replicator dynamic, this is not possible.

¹⁸We did not prove that $\int_0^\infty x_2(t)dt$ is infinite for the $x(0) \in X$ we picked but it is since $g_2(x(t)) > -(2x_1(t) + x_2(t)) > -(2e^2 + 1)x_2(t)$ for all t .