

Fischer, Thomas

Working Paper

Thomas Piketty and the Rate of Time Preference

Working Paper, No. 2017:1

Provided in Cooperation with:

Department of Economics, School of Economics and Management, Lund University

Suggested Citation: Fischer, Thomas (2017) : Thomas Piketty and the Rate of Time Preference, Working Paper, No. 2017:1, Lund University, School of Economics and Management, Department of Economics, Lund

This Version is available at:

<https://hdl.handle.net/10419/260210>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

Working Paper 2017:1

Department of Economics
School of Economics and Management

Thomas Piketty and the Rate of Time Preference

Thomas Fischer

January 2017



LUND
UNIVERSITY

Thomas Piketty and the Rate of Time Preference

Thomas Fischer*

Lund University

Knut Wicksell Centre of Financial Studies

thomas.fischer@nek.lu.se

Lund, January 2017

Using a standard model where the individual consumption path is computed solving an optimal control problem, we investigate central claims of Piketty (2014). Rather than $r > g$ (confirmed in the data) $r - \rho > g$ - with ρ being the rate of time preference - matters. If this condition holds and the elasticity of substitution in the production function is larger than one, the capital share converges to one in the long run. Nevertheless, this does not have major impact on the distribution of wealth. The latter, however, converges to maximum inequality for heterogeneous time preferences or rates of interest (either persistent or stochastic).

JEL classification: D31 - E21 - C63

key words: wealth inequality - optimal control path - dynamic efficiency

1. Introduction

Capital in the 21st century is, without a doubt, the most important economic book of the last years. While it might not be the most-read book - owing to its extensive length - it is definitively the most discussed book. In particular, in the USA (the English translation of) the book perfectly captured the *Zeitgeist* and the political debate about inequality. In the book, Thomas Piketty not only summarises central results of his own work (including important work with co-authors such as Emmanuel Saez, Gabriel Zucman, or Anthony Atkinson), but also presents novel claims. In summary, the book

*I would like to thank Thomas Piketty and Günther Rehme for helpful remarks.

consists of three major parts: (i) a very long-run historical record of empirical evidence regarding inequality, (ii) predictions regarding the future evolution of inequality based upon theoretical ideas, and (iii) policy recommendations in order to combat inequality. While part (i) is widely praised¹, critics mostly focused on the theoretical predictions (ii) and the policy recommendations (iii) - in particular the global wealth tax - Piketty (2014) derives from his theoretical predictions.²

In fact, many economists disagreed with several (radical) answers proposed in Piketty (2014). Even if one disagrees with the answers proposed by Piketty (2014), the book is highly important as it poses important questions - which sometimes are even more important than the answers - providing the ground for politically relevant research questions in the economic fields of growth and distribution. The aim of this paper is therefore to discuss several findings and claims made by Piketty (2014) in a standard theoretical model.

Piketty (2014) claims to have found a *central contradiction of capitalism* with the inequality $r > g$. He argues that the latter condition ensures ever increasing wealth inequality. As, however, put forward amongst others by Mankiw (2015) this is a standard condition in macroeconomic theory that ensures *dynamic efficiency* (Abel et al., 1989). It is important to point out that the critique presented in Mankiw (2015) only identified the condition $r > g$ as a natural consequence in economics. Nevertheless, it did not feature a statement about its impact on inequality. In fact, the statement made in Mankiw (2015) might even strengthen the point made in Piketty (2014) implying that the presence of the relation $r > g$ is an inherent instability of the capitalist system.

We investigate the relationship $r - g$ in a standard optimal control set-up and emphasise the role of the rate of time preference ρ . This formal framework is not only the standard starting point in this type of analysis but is also employed by Thomas Piketty himself in a more recent and more formalised reasoning (Piketty and Zucman, 2015). Using this model we discuss several central predictions put forward Piketty (2014) and also compare it to the empirical evidence. We formalise several claims of Piketty (2014) and render more precisely some (non-standard) assumptions to derive them.

Firstly, we show that the condition $r > g$ always emerges in a model for which Ponzi-schemes are ruled out. The latter, however, does not, per se, make a statement about the evolution of wealth as a part of the wealth is consumed. For the special case of log-utility considered in the model, the marginal propensity to consume out of wealth equals the rate of time preference $c_w = \rho$. The condition for wealth to grow without bounds is therefore not $r > g$ but $r > g + \rho$ (respectively $r - g > \rho$). In fact, the case proposed by Thomas Piketty is nested within this case for the (unrealistic) assumption of a time preference of zero ($\rho = 0$) implying no consumption out of wealth.³ This is the case in

¹Auerbach and Hassett (2015) argue that the indicated trends tend to disappear once higher frequency data is considered rather than the data that averages in time.

²A detailed and very critical discussion especially covering the latter parts of the books is presented in Blume and Durlauf (2015).

³Note that a Krusell and Smith (2015) make a similar argument, however, focusing on the role of depreciations being an involuntary form of consumption for which capital *eats up itself*. In particular, they argue that its imprecise treatment in Piketty (2014) leads to the wrong conclusion of a non-

an economy that is dynamically efficient and not subject to overaccumulation of wealth. Recent data for evolved economies identifies this as the plausible case. In a model, in which the return to wealth evolves according to a Constant Elasticity production function with an elasticity larger than one or a production function of the AK-type the latter condition leads to the fact that in the long run the capital share converges to one. This assumption, however, does not allow to make strong statements about the personal distribution of wealth. We show that despite $r - g > \rho$ wealth inequality converges to a level as low as consumption inequality as the disequalizing effect of inheritance dilutes in the long run. Wealth inequality, however, becomes extreme once we allow for heterogeneous rates of time preference ρ_i as well as heterogeneous returns r_i . For the latter, it is irrelevant if they are of a temporary stochastic nature or persistent in both cases leading to maximum wealth inequality.

The remainder of this paper is organised as follows. Section 2 presents our very standard model, which is discussed in section 3, in which we emphasise the role of different relations between $r - g$ and ρ . We consider both aggregate macroeconomic effects (in a representative agent version in section 3.1) and the personal distribution of economic measures in section 3.2. Section 4 critically discusses the model and wraps up.

2. The Model

This section presents the underlying model. For the modelling technique we decided to stick to a very standard approach and show its strong implications for the given subject. Piketty (2014) contains barely any economic modelling. This, however, is also due to the fact that the author tries to address a general audience uninterested in formal economic modelling. The book, however, also has a detailed mathematical appendix (available on the author's website) in which he provides some formal rationale. Moreover, in a more recent survey, Piketty and Zucman (2015) also present a more formalised argument. In both cases, a similar model starting from an Euler-equation in continuous time is considered making it a natural starting point for this paper. The model is kept deliberately simple in order to allow for closed-form solutions that are able to capture the underlying rationale precisely. The predominant existing literature aimed at explaining wealth inequality⁴ is highly complex and requires numerical simulations.

The wealth accumulation results from a life-cycle motive. We assume an infinitely-lived dynasty that aims at distributing consumption in time in order to maximise a given utility function depending only on consumption. The objective function subject to maximization is therefore given as follows:

$$\int_0^{\infty} \exp(-\rho t) U(C(t)) dt, \quad (1)$$

convergence of the capital output ratio for the zero growth case. Note that in our case the ratio only explodes for $r > \rho$ in the case of $g = 0$ and also not mechanically in any case with $g = 0$.

⁴ The models are often labelled *Bewley-type* models. Excellent recent surveys can, for example, be found in Quadrini and Ríos-Rull (2015) or DeNardi (2015).

for which $\rho > 0$ represents the discount rate. Current labour income Y can be stored using a storage technology that yields an exogenously given return r . One might also consider this as a representation of a *fruit-tree* economy (Lucas, 1978). The *fruit* is the current labour income Y , whereas the stock of wealth represents the *trees*. The latter can be retained using a storage technology that yields a constant annual return of $\frac{\dot{W}}{W} = r$. As a result, the basic model is not closed by a production function usually assumed to account for the behaviour of firms. In the subsequent section, however, a general Constant Elasticity of Substitution (CES) type function is introduced in order to discuss some claims made by Piketty (2014) about the functional distribution.

The maximization problem is subject to the flow constraint:

$$\dot{W} = rW(t) - C(t) + Y(t). \quad (2)$$

It is important to acknowledge that $W(t)$ is not bound to positive values. In fact, in the model, agents can have negative wealth, which implies that they hold debt ($D(t) \equiv -W(t) > 0$). Finally we assume an exogenously given labour income following an exponential process:⁵

$$Y(t) = Y_0 \exp(gt). \quad (3)$$

One key aim of this paper is to derive a dynamic path for a given point of time t rather than just steady states which are usually in the center of this type of analysis.

We assume a utility of the isoelastic Constant Relative Risk Aversion (CRRA) type with a unit elasticity:⁶

$$U(C(t)) = \lim_{\gamma \rightarrow 1} \frac{C(t)^{1-\gamma} - 1}{1-\gamma} = \ln(C(t)) \quad (4)$$

The problem can be written in the form of a Hamiltonian:⁷

$$\mathcal{H} = \exp(-\rho t)U(C) + \lambda(Y + rW - C). \quad (5)$$

We have to compute the first-order conditions with respect to the control (here the level of consumption C) and the state (here the level of wealth W). The optimality conditions are as follows:

$$\frac{\partial \mathcal{H}}{\partial C} \stackrel{!}{=} 0 \rightarrow \exp(-\rho t)U'(C) = \lambda, \quad (6)$$

and

$$-\frac{\partial \mathcal{H}}{\partial W} = -\lambda r \stackrel{!}{=} \dot{\lambda}, \quad (7)$$

as well as the standard transversality condition:

$$\lim_{T \rightarrow \infty} \lambda(T)W(T) \stackrel{!}{=} 0. \quad (8)$$

⁵We can also rewrite this as a differential equation $\dot{Y} = gY$ with $Y(0) = Y_0$.

⁶In this case income (predominant for $\gamma > 1$) and substitution effects (predominant for $\gamma < 1$) on consumption for a change of the interest rate perfectly cancel out each other. In the isoelastic case an increase in the interest rate decreases the value of human capital and thereby decreases consumption due to a pure wealth effect (Bertola, 2000).

⁷Note that for the sake of readability we spare the time indexes.

Differentiating equation 6 and equating it with equation 7 yields:

$$-\lambda r = -\exp(-\rho t)U'(C)r \stackrel{!}{=} \dot{\lambda} = -\rho \exp(-\rho t)U'(C) + \exp(-\rho t)U''(C)C', \quad (9)$$

which can be solved, leading to:

$$-\frac{U''(C)C'}{U'(C)} = r - \rho, \quad (10)$$

for which $C' = \dot{C}$. For the assumed isoelastic case with $-\frac{U''(C)}{U'(C)} = \frac{1}{C}$ this implies the following differential equation:

$$\frac{\dot{C}}{C} = r - \rho, \quad (11)$$

with the closed-form solution:

$$C(t) = C_0 \exp([r - \rho]t). \quad (12)$$

Reinserting this result in the flow-constraint (equation 2) results in a first-order linear differential equation:

$$\dot{W} - rW = -C_0 \exp([r - \rho]t) + Y_0 \exp(gt). \quad (13)$$

The latter is a standard first-order linear differential equation with constant coefficients which solves as follows:

$$W = C_1 \exp(rt) + \frac{C_0}{\rho} \exp([r - \rho]t) - \frac{Y}{r - g} \exp(gt). \quad (14)$$

In this case C_0 and C_1 represent constant values that have to be determined. To completely solve this model we have to define initial and end conditions, for which the latter is given by the transversality condition. This is frequently shifted to a footnote and treated as a mere technicality, yet is of utter importance in the given context.

$$\lim_{T \rightarrow \infty} \lambda(T)W(T) \stackrel{!}{=} 0, \quad (15)$$

which, using the closed-form solution of the shadow price $\lambda(t)$ resulting from equation 7, can be written as:⁸

$$\lim_{T \rightarrow \infty} \lambda(0) \exp(-rT)W(T) = \lim_{T \rightarrow \infty} U'(C(0)) \exp(-rT)W(T) \stackrel{!}{=} 0. \quad (16)$$

As initial consumption $C_0 > 0$ has a non-zero value, marginal utility is finite, leading to the following more simple condition:

$$\lim_{T \rightarrow \infty} \exp(-rT)W(T) \stackrel{!}{=} 0. \quad (17)$$

⁸Using equation 6 this can also be written $\lim_{T \rightarrow \infty} \exp(-\rho T)U'(C(T))W(T) \stackrel{!}{=} 0$. Using the solution of the differential equation with undetermined coefficients and the relation $U'(C(t)) = \frac{1}{C(t)}$ this yields:

$\lim_{T \rightarrow \infty} \frac{W(T)}{C(T)} \exp(-\rho T) = \lim_{T \rightarrow \infty} \frac{C_1}{C_0} + \frac{1}{\rho} \exp(-\rho T) - \frac{Y_0}{(r-g)C_0} \exp([g - r]T) \stackrel{!}{=} 0$ only holding for $C_1 = 0$ and $r > g$ identical to the other derivation.

Assume that wealth grows in a geometric process with a constant rate g_W ($\frac{\dot{W}}{W} = g_W \leftrightarrow W(t) = W_0 \exp(g_W t)$). To ensure stability the growth rate of wealth should be smaller than the rate of interest implying the following necessary condition:

$$r > g_W. \quad (18)$$

In order to test the condition we insert the result of equation 14 into the transversality condition:

$$\begin{aligned} \lim_{T \rightarrow \infty} \exp(-rT) \left[C_1 \exp(rT) + \frac{C_0}{\rho} \exp([r - \rho]T) - \frac{Y_0}{r - g} \exp(gT) \right] = \\ \lim_{T \rightarrow \infty} \left[C_1 - \frac{Y_0}{r - g} \exp([g - r]T) \right] \stackrel{!}{=} 0. \end{aligned} \quad (19)$$

To ensure the condition we require both $r > g$ and $C_1 = 0$.

To finalise the computation we require an initial condition of the dynasty⁹ being endowed with a given amount of non-labour wealth $W(0) = W_0$:

$$W_0 \stackrel{!}{=} W(0) = \frac{C_0}{\rho} - \frac{Y}{r - g} \rightarrow C_0 = \rho \left(W_0 + \frac{Y_0}{r - g} \right). \quad (20)$$

As a result consumption is given by:

$$C(t) = \rho \left(W_0 + \frac{Y_0}{r - g} \right) \exp([r - \rho]t), \quad (21)$$

whereas wealth evolves according to:

$$W(t) = \left(W_0 + \frac{Y_0}{r - g} \right) \exp([r - \rho]t) - \frac{Y_0}{r - g} \exp(gt). \quad (22)$$

The latter two equations can also be summarised in the following manner:

$$C(t) = \rho(W(t) + H(t)), \quad (23)$$

with $H(t) = \frac{Y_0}{r - g} \exp(gt)$ being human wealth (as derived in appendix A), making consumption depend on both financial and human wealth (also cf. Blanchard (1985), Weil (1989), Buiter (1988)).

We can normalise the results with growth of labour income ($\exp(gt)$). Normalised results are present with smaller case letters which for wealth leads to:

$$w(t) = \frac{W(t)}{\exp(gt)} = \left(W_0 + \frac{Y_0}{r - g} \right) \exp([r - \rho - g]t) - \frac{Y_0}{r - g}, \quad (24)$$

⁹Note that while the dynasty lives infinitely it still requires being born by a parent generation at time $t = 0$. This parent generation inherits a certain amount of wealth to their offspring.

whereas consumption follows:

$$c(t) = \frac{C(t)}{\exp(gt)} = \rho \left(W_0 + \frac{Y_0}{r - g} \right) \exp([r - \rho - g]t). \quad (25)$$

This result derives from the solution of the abstract Euler equation. It is, however, easy to translate to standard consumption function with constant marginal propensities to consume out of wealth c_w and income c_y :

$$C(t) = c_y \cdot Y(t) + c_w \cdot W(t) \leftrightarrow c(t) = c_y Y_0 + c_w w(t). \quad (26)$$

By inserting equation 3¹⁰ and equation 24 into equation 26 and comparing coefficients it is easy to show that:

$$c_w = \rho, \quad (27)$$

and:

$$c_y = \frac{\rho}{r - g}. \quad (28)$$

This also gives a more tangible interpretation to the somewhat foggy notion of the rate of time preference ρ as it is equal to the marginal propensity to consume out of wealth. The marginal propensity to consume out of the flow size labour income is large than out of stock $\frac{c_y}{c_w} = \frac{1}{r - g} > 1$ for reasonable values of r and g .¹¹ The latter, however, also implies that the consumption out of physical capital w and human capital (being $\frac{Y_0}{r - g}$) is identical. The essential practical difference between the two is that while physical wealth can be traded continuously in asset markets - in the absence of slavery - human capital cannot be exchanged entirely at any time. Nevertheless, it generates a constant flow of income, being the wage compensation.

3. A Test of Thomas Piketty's Predictions in the Model

In this section, we take the model presented so far and discuss aggregate implications (cf. section 3.1) and distributional effects (cf. section 3.2), as considered in Piketty (2014), by means of the model. In particular, we differentiate between three cases regarding the relation of $r - g$ and ρ that lead to different outcomes.

3.1. Aggregate Implications

In his work, Piketty (2014) emphasised the importance of the sign of $r - g$. In particular, he argues that, if this level is positive (i.e. the inequality $r - g > 0 \leftrightarrow r > g$ holds), this implies a high level of inequality. In this work, we present another inequality by considering the sign of $r - g - \rho$. In fact, we show that the case presented Piketty (2014) is nested in our model for the case of $\rho = 0$.

¹⁰Note that due to the normalization we have $y(t) = \frac{Y(t)}{\exp(gt)} = Y_0$.

¹¹Formally, it requires $1 + g > r$ which always holds for $r < 1$.

As put forward by Samuelson (1975) (and reemphasized in the discussion of Piketty (2014) by Mankiw (2015)) the condition $r > g$ is always satisfied in a dynamic efficient economy. In a steady state discussion in equilibrium there is $r = g + \rho$ ¹² and thereby $r - g = \rho > 0$ for reasonable values of ρ that are positive.¹³ In our model, $r - g$ and ρ do not have necessarily to equate, yet we require $r > g$. The latter results concretely from the transversality condition.

For the sake of illustration let us consider the opposing situation of $g > r$. If the latter held, human capital would not take a finite value. In fact, for the case of $g > r$ labour income would always grow in excess of capital income (Domar, 1944).¹⁴ The latter makes Ponzi-schemes possible. Moreover, it implies that a Pay-As-You-Go scheme in which the internal rate of return equals g is always superior to a capital based with a return of r as emphasised in the Overlapping Generations (OLG) literature (e.g. Diamond (1965)) for which the latter can emerge.¹⁵ The case of $g > r$ is therefore also often referred to as a situation of dynamic inefficiency. In a dynamic inefficient economy Pareto-improvements are possible making some agents better off without decreasing the welfare of others.

In some models - in particular the ones stemming from the OLG-type literature (cf. for example Samuelson (1958)) - the case of $r > g$ is possible.¹⁶ Therefore, finally, the question of which scenario prevails has to be answered empirically. Abel et al. (1989) test this condition for OECD countries from 1929 (after the Great Depression) until 1989 (publication date of their paper) and show that dynamic efficiency prevails. They use a simple test to identify whether profits exceed investments that requires them to compare the savings ratio s and the capital share $\alpha = \frac{rK}{Y}$. The condition for dynamic efficiency is:¹⁷

$$s < \alpha. \quad (29)$$

This method is very useful as it circumvents the tricky issue of measuring the real rate of interest r .

In figure 1 we show more recent data from several OECD countries. It is interesting to observe that at the moment all inspected countries can be considered dynamically efficient by means of the condition $\alpha > s$. Eventually, before the mid 1980s the opposite

¹²This case is also referred to as the *Modified Golden-rule level* (Blanchard and Fischer, 1989). In this case, the economy accumulates capital below the Golden-rule level ($r = g$) due to a preference for current consumption. In particular, economies with higher levels of ρ indicating lower patience accumulate less capital, yet always remain efficient from a dynamic perspective.

¹³This argument was already modified to hold in the context of our model. Mankiw (2015) also considers redistribution by means of a tax on capital income $\tau > 0$ and the role of an Intertemporal Elasticity of Substitution σ (which in our case is one resulting from the assumption of log-utility) making $g = \sigma(r - \tau - \rho)$.

¹⁴A formal derivation of the latter is presented in appendix A.

¹⁵The so-called *Bewley-type* models, in which heterogeneous agents accumulate wealth in order to ensure against idiosyncratic risks, also lead to the case in which $r - g < \rho$ (as discussed in the preceding stanza) (Aiyagari, 1994). As shown in Moll et al. (2015) - in a non-growing economy $g = 0$ - the difference between $\rho - r$ increases with the number of collateral constraint households.

¹⁶Note that in the type of model discussed in this paper, we always will have $r > g$ for the transversality condition holding.

¹⁷Formally it requires: $I = S = sY < \pi = rK \Leftrightarrow s < \frac{rK}{Y} = \alpha$.

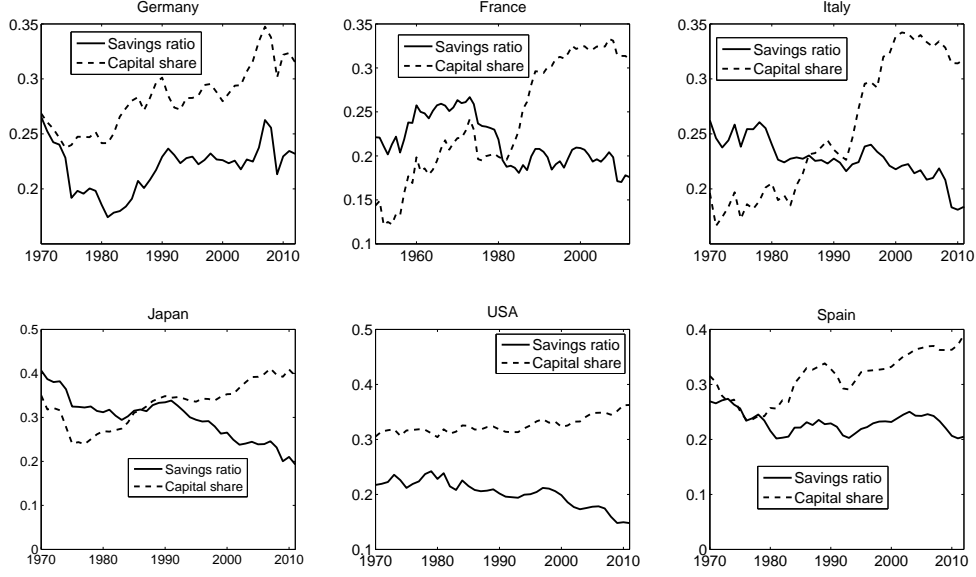


Figure 1: Capital share α and savings ratio s in several OECD countries (Data source: OECD (2012))

was the case for many countries. This is furthermore in line with the rationale of Piketty et al. (2014), who argue that the 1980s were a time of a paradigm change to lower redistribution and more deregulation, also manifesting itself in a stronger capital share. It is also interesting to point out that the USA always exhibited a significantly positive gap between the capital share and the savings ratio ($\alpha - s > 0$), indicating that dynamic efficiency was always prevailing. Moreover, the gap eventually even grew. Similar behaviour can also be observed for Japan and Spain.

As a result, dynamic inefficiency is theoretically possible, but not of interest regarding the empirical evidence.

The following proposition summarises the evolution of consumption and wealth for the different scenarios.

Proposition 3.1 (Consumption and Wealth)

1. $r - g < \rho$: Normalised consumption c is decreasing in time $\dot{c} < 0$ and $\lim_{t \rightarrow \infty} c(t) = 0$ and households accumulate debt equal to their human capital $\lim_{t \rightarrow \infty} w(t) = -\frac{Y_0}{r-g} < 0$.
2. $r - g = \rho$: Normalised consumption and wealth are constant in time and wealth is only determined by initial wealth endowments $w(t) = W_0$.
3. $r - g > \rho$: Normalised consumption increases in time $\dot{c} > 0$. Long-run normalised consumption as well as normalised wealth do not converge to a finite level $\lim_{t \rightarrow \infty} c(t) = \lim_{t \rightarrow \infty} w(t) = \infty$.

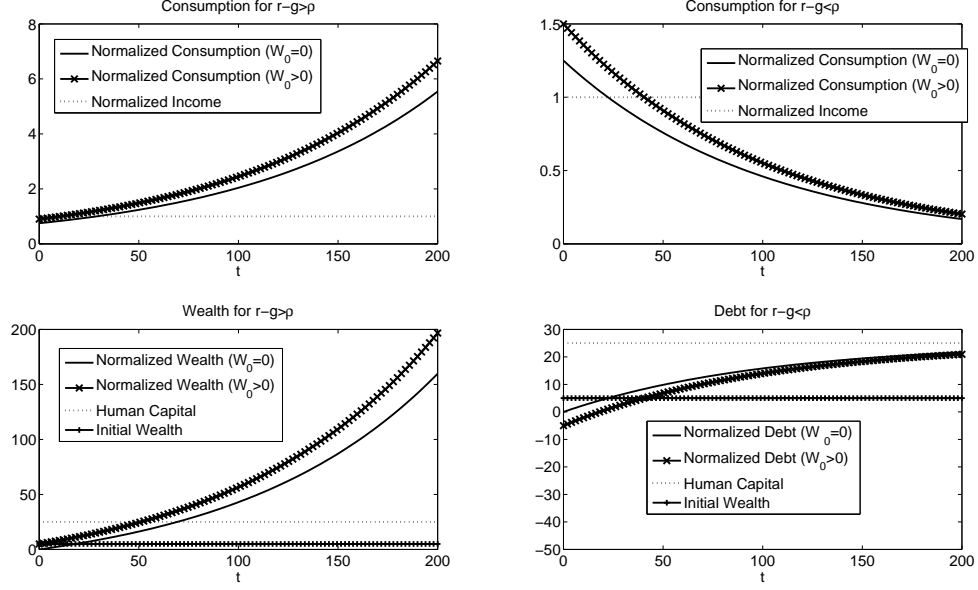


Figure 2: Normalised wealth and consumption for $\rho < r - g$ (left panel) respectively $\rho > r - g$ (right panel)

Let us discuss the three cases in detail. We start of with the case for which $c_w = \rho > r - g$, implying that agents have a strong preference for current consumption. As a result in the short run current consumption exceeds current income. This case is displayed in the right panels of figure 2 showing normalised figures.¹⁸ Agents can do so by incurring debt ($w(t) < 0$). Following from the transversality condition, their debt capacity is given by the level of their human capital $-w_\infty = h = \frac{Y_0}{r-g}$. The short-run violation of the budget constraint is traded-off for a long-run consumption of zero ($\lim_{t \rightarrow \infty} C(t) = 0$). In the long run, we have:

$$\begin{aligned}
 \lim_{t \rightarrow \infty} [\dot{W}(t) - rW(t)] &\stackrel{!}{=} \lim_{t \rightarrow \infty} Y(t) \\
 \Leftrightarrow \lim_{t \rightarrow \infty} \left[\frac{\dot{W}(t)}{\exp(gt)} - r \frac{W(t)}{\exp(gt)} \right] &= \lim_{t \rightarrow \infty} \left[\frac{\dot{W}(t)}{\exp(gt)} - r w(t) \right] \stackrel{!}{=} \lim_{t \rightarrow \infty} \frac{Y(t)}{\exp(gt)} = Y_0 \quad (30) \\
 &\Leftrightarrow -g \frac{Y_0}{r-g} + r \frac{Y_0}{r-g} = Y_0
 \end{aligned}$$

This implies that in the long run in a growing economy debt still increases (i.e. wealth decreases $\dot{W} < 0$) and another part of the income is employed to pay interest on ac-

¹⁸For the specific values we assume $r = 7\%$ (which can be computed from Piketty and Zucman (2014) using the relationship $r = \frac{\alpha}{K/Y}$ and their values for $\frac{K}{Y}$ being rather constant in time). Auerbach and Hassett (2015) - considering the USA - compute a similar value using the given database and also confirm little time variation. Moreover, we assume $g = 3\%$ (Jones, 2015b), $Y_0 = 1$, and $W_0 = 5$. In the right panels we assume $\rho_1 = 5\% > r - g = 4\%$, whereas in the left panel we have $\rho_2 = 3\% < r - g = 4\%$.

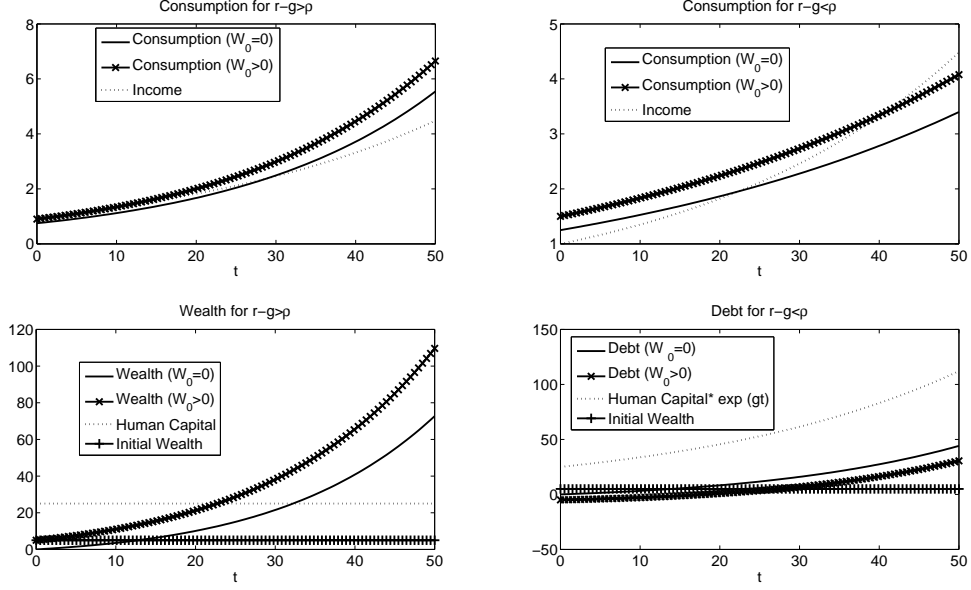


Figure 3: Wealth and consumption for $\rho > r - g$ (left panel) respectively $\rho < r - g$ (right panel)

cumulated debt ($rD = -rW$). In the case without growth ($g = 0$) nested within this model, in the long run debt is not increasing (i.e. $\dot{W} = 0$) and all current income is employed to pay interest on accumulated debt ($Y_0 = rD = -rW = -r\frac{Y_0}{r}$).

Moreover, it is important to point out that even in this scenario consumption $C(t)$ grows as displayed in figure 3. The pace of consumption growth, however, falls back relative to income growth as a large amount of income is employed in order to serve interest on accumulated debt.

Regarding the long run evolution of debt/wealth, it is also important to point out that the initial endowment W_0 does not matter as $\lim_{t \rightarrow \infty} W(t) \neq f(W_0)$. As displayed in figure 2, in the short run a higher initial endowment (e.g. in the form of a larger inheritance) does matter and allows for higher consumption and lower debt.¹⁹ Nevertheless, in the long run this effect diminishes as all initially endowed wealth ends up being consumed.

The second case presented in proposition 3.1 is the knife-edge case $r - g = \rho$. In this case, at any point of time t we have:

$$c(t) = \rho \left(W_0 + \frac{Y_0}{r - g} \right) = \rho W_0 + Y_0, \quad (31)$$

for consumption respectively wealth:

$$w(t) = W_0. \quad (32)$$

¹⁹Note that in contrast to figure 2 this figure only displays the first 50 simulation periods, especially to emphasise at which point of time for the case $r - g < \rho$ income exceeds consumption.

Agents consume all of their flow income Y_0 making $c_y = 1$. The stock level of normalised wealth is also constant as the growth rate of normalised wealth ($r - g$) equals its consumption ratio ($c_w = \rho$). In fact, all capital gains (also being a flow income) are directly consumed. As a result the wealth level is only given by the initial endowment of wealth. Therefore, in this case the initial endowment with wealth ($W_0 > 0$) matters as agents with higher endowment are wealthier in the long run. In fact, the normalised wealth is given as follows:

$$W(t) = W_0 \exp(gt) = W_0 \exp([r - \rho]t). \quad (33)$$

At first sight this seems to be at odds with the transversality condition requiring a finite (present) value of wealth in the long run. Notice, however, that in the model at hand wealth grows with a pace of:

$$g_W = \frac{\dot{W}}{W} = r - \rho, \quad (34)$$

which due to the presence of the consumption out of wealth $c_w \equiv \rho > 0$, always satisfies the condition $g_W < r$ required by the transversality condition.

Finally, we come to the third case presented in proposition 3.1. For this case we have a marginal propensity to consume out of flow income smaller than one ($c_y = \frac{\rho}{r-g} < 1 \leftrightarrow \rho < r - g$). In the short run individuals consume less than their current income. As a mirror image of the first-case this leads to infinite growth of long run consumption as well as wealth. This increased consumption is financed by means of interest payments on accumulated wealth.

The intuition behind this result is also straightforward: in our fruit economy more fruits are harvested than consumed by the agents. In the absence of depreciation (wealth does not get moldy) a larger stock of fruits is accumulated.

Once again, for this case the value of initially inherited wealth ($W_0 > 0$) matters leading to both higher long-run wealth as well as higher consumption (cf. figure 2 and 3).

So far, we have presented the results from a normative framework implying the optimal reaction of agents and distinguished between three cases. Theoretically, all cases are possible in the given model. To identify the most realistic case, we must, however, consider the empirical evidence. One of the key insights of Piketty and Zucman (2014) (also widely referred to in Piketty (2014)) is the growing ratio of wealth to income setting in after World War II, especially in Europe. This ratio in our model could be captured by $\frac{w(t)}{Y_0}$. As already shown, the increasing ratio can be rationalised in a scenario in which $c_w = \rho < r - g$, reflecting the third case of proposition (3.1).

To make a statement about the relationship between $r - g$ and ρ we have to measure the rate of time preference ρ . The latter is hard to measure. In particular it is frequently assumed that it equals the differences between $r - g \equiv \rho$ which - as already discussed - is the standard steady state. In contrast to r and g , which particularly depend on technology, the latter is a behavioural parameter.²⁰

²⁰A recent cross-country study presented substantial differences for this parameter (Wang et al., 2011). They measure patience (and thereby low values for ρ) for Germanic/Nordic as well as Anglo-Saxon countries and high time preference for Latin American, Latin Europe as well as African countries.

We propose a very simple estimation for the ratio of time preference using the aggregate savings ratio. As already shown, the marginal propensity to consume out of income is given by $c_y = \frac{\rho}{r-g}$ and can be related to the savings ratio in the following manner:

$$c_y = \frac{\rho}{r-g} \approx 1 - s, \quad (35)$$

implying:

$$0 < s = 1 - \frac{\rho}{r-g} \Leftrightarrow \rho < r - g, \quad (36)$$

for the realistic case of a positive savings ratio as documented in figure 1. Using this savings ratio and realistic parameters for r and g ²¹ the rate of time preference ranges between:²²

$$2.4\% < \rho < 3.6\%, \quad (37)$$

implying a value of $\rho \approx 3\%$ as employed in the left panels of figure 2 as well as figure 3. Despite the recent increase in the savings ratio (as a result of the crisis), there is a long-run decrease in the savings ratio (in all documented countries but Germany) implying an increase in the rate of time preference ρ .

By focusing on the relationship $r-g$ Piketty (2014) (implicitly) assumes $\rho = 0$, thereby disregarding consumption also implying $r - g > \rho = 0$ of ever increasing consumption and wealth as presented in proposition 3.1.

Using the model, we can make a statement about the functional distribution of income. The latter has been in the focus of classical economists such as Marx or Ricardo. In the neoclassical school of thought the latter is considered a given stylised fact not subject to time variation. The latter is motivated by empirical evidence of Kaldor (1957) and (frequently tacitly) assumed by using the elegant Cobb-Douglas production function (Cobb and Douglas, 1928). More recent research - in particular Karabarbounis and Neiman (2013) - documents an increase in the capital share in many developing countries and rationalise this by a CES production function with an elasticity of substitution larger than one. In fact, the functional distribution maps different sources of income to the different production factors labour and capital. In a pure two-agent world, this is identical to mapping it to the social group of workers and capitalists. While this might have been a good description of the situation in the early days of the industrial revolution, during which Karl Marx penned his writings, this is definitely not an accurate description of the modern society. Not only do even low income agents hold capital, but high income individuals receive their primary share of income from labour. Atkinson and Piketty (2007) document the increased share of salaries in the composition of the top-income share and argue that the *working rich* have overtaken the *coupon-clipping rich*. Nevertheless, the functional distribution of income is still interesting as it offers a picture of the relation between human work and dead capital. Furthermore, it

The authors attribute this effect to cultural differences and are able to disentangle it from a wealth effect.

²¹Refer to footnote 18 for details.

²²This assumes the range of $0.1 < s < 0.4$ as reported in figure 1.

can easily be described using the established system of national accounts. In contrast to that, the personal distribution of income requires detailed micro-data.

We want to investigate this relation using the model at hand. The capital share is defined as:

$$\alpha(t) = \frac{rW(t)}{rW(t) + Y(t)} = \left(1 + \frac{Y_0}{rw(t)}\right)^{-1}, \quad (38)$$

which for the given case is:

$$\alpha(t) = \frac{-rY_0 + r(W_0(r - g) + Y_0) \exp([r - g - \rho]t)}{-gY_0 + r(W_0(r - g) + Y_0) \exp([r - g - \rho]t)}. \quad (39)$$

In the short run we have:

$$\alpha(0) = \frac{rW_0(r - g)}{(r - g)(Y_0 + rW_0)} = \left(1 + \frac{Y_0}{W_0 r}\right)^{-1} = f(Y_0[-], W_0[+], r[+]), \quad (40)$$

for which terms in brackets determine the marginal effect of each factor. Not surprisingly, the capital share increases for a high ratio of the stock level of wealth to the flow level of income ($\frac{W_0}{Y_0}$) and - in line with the argument made in Piketty (2014) - with a high level of the interest rate r .

If we consider the capital share $\alpha(t)$ in the long run, we can - once again - distinguish three cases:

Proposition 3.2 (Functional Distribution)

1. $r - g < \rho$: In the very long run $\lim_{t \rightarrow \infty} \alpha(t) = \frac{r}{g}$.
2. $r - g = \rho$: There are no dynamics in the capital share and $\alpha(t) = \alpha(0)$ for any t .
3. $r - g > \rho$: In the long run all income falls to capital $\lim_{t \rightarrow \infty} \alpha(t) = 1$.

In the first case agents accumulate debt to the value of their human capital $w_\infty = -h = \frac{Y_0}{r-g}$. This implies the relationship:

$$\lim_{t \rightarrow \infty} \alpha(t) = \frac{rw_\infty}{rw_\infty + Y_0} = \frac{-\frac{rY_0}{r-g}}{-\frac{rY_0}{r-g} + Y_0} = \frac{-rY_0}{-rY_0 + Y_0(r - g)} = \frac{r}{g}. \quad (41)$$

This is also the result following the simple intuition when considering the nominator growing at a rate of r and the denominator at g . As already documented in the data, the case of a society that aggregates debt in the aggregate is not realistic.

In the knife-edge case there are no dynamics and the capital share is constant. Moreover - and as already put forward - it is directly and positively related to inherited wealth W_0 . The latter case is also the steady-state result discussed in Mankiw (2015).

For the third case, in the long run all income is captured by capital. This is also the case put forward in the dystopian vision of Piketty (2014). In his original work (which he slightly cushions in Piketty (2015)) this was a result of $r > g$. To discuss

this rationale in detail, we firstly want to recapitulate the reasoning used in the original work (Piketty, 2014). Here the evidence of an increasing capital to income ratio $\frac{k}{y}$ is presented. As the nominator is assumed to grow with r , whereas the denominator grows with g , the relationship $r > g$ implies no long-run convergence in this ratio. The capital share α - measuring functional distribution - is assumed to follow a *simple accounting identity* (Piketty, 2014) with $\alpha = r \cdot \frac{k}{y}$. For a given rate of interest this also does not converge, leading to an ever-increasing share capitalists in the total income distribution and thereby non-converging inequality.

We partly followed this argument by (implicitly) assuming that the return to capital r does not change in the process of capital accumulation. To be more precise, we now want to introduce an aggregate production function of the Constant Elasticity of Substitution (CES) type. The latter is analytically very appealing as several cases are nested within this function. Moreover, it played a major role in the debate of Piketty (2014) - in particular the value of elasticity was subject of an economic debate. The general CES function is given by:

$$Y = A \left(\bar{\alpha} K^{-\eta} + (1 - \bar{\alpha}) N^{-\eta} \right)^{-\frac{1}{\eta}}, \quad (42)$$

with the inputs capital K and labour N as well as a pseudo-capital share of $0 < \bar{\alpha} < 1$. We can normalise this function with capital and income per capita ($k = \frac{K}{N}$ respectively $y = \frac{Y}{N}$) leading to:

$$y \equiv f = A \left(\bar{\alpha} k^{-\eta} + (1 - \bar{\alpha}) \right)^{-\frac{1}{\eta}}, \quad (43)$$

for which $\eta \geq -1$ can be transformed to the elasticity of substitution $\sigma = \frac{1}{\eta+1} \geq 0 \leftrightarrow \eta = \frac{1-\sigma}{\sigma}$. The capital share is determined as follows:

$$\alpha = \frac{k f'(k)}{f}, \quad (44)$$

which - using the Constant Elasticity of Substitution (CES) production function - results in:

$$\alpha = A \frac{\bar{\alpha}}{\bar{\alpha} + (1 - \bar{\alpha}) k^{\eta}}. \quad (45)$$

Following Piketty and Zucman (2015) it is easy to verify that the latter can also be written as:

$$\alpha = A \bar{\alpha} \left(\frac{k}{y} \right)^{\frac{\sigma-1}{\sigma}} = A \bar{\alpha} \left(\frac{k}{y} \right)^{-\eta}. \quad (46)$$

The pleasant feature of the CES function is that several - more standard - cases are nested within this function. In particular for $\eta = 0$ ($\sigma = 1$), the function boils down to the standard Cobb-Douglas production function ($Y = K^{\bar{\alpha}} N^{1-\bar{\alpha}}$), for which the capital share is constant ($\alpha = A \bar{\alpha}$) and does not vary with the level of capital ($\frac{\partial \alpha}{\partial k} = 0$) (Cobb and Douglas, 1928). This is the workhorse model in basically any macroeconomic model and implies the strong assumption that the functional distribution is a *stylised fact* not varying in time or with changes of the level of capital.

So far we have considered the other extreme case of $\sigma \rightarrow \infty$ ($\eta = -1$) for which the function boils down to a linear production function $Y = A\bar{\alpha}K + (1 - \bar{\alpha})N \leftrightarrow y = A\bar{\alpha}k + (1 - \bar{\alpha})$. In this case there are no scale effects and the return to capital is constant ($r = f'(k) = A\bar{\alpha}$). Moreover, the capital share increases with capital ($\frac{\partial \alpha}{\partial k} > 0$). One could interpret this as a totally robotised world.

As formally discussed in Jones (2015a) the latter can also be rationalised by an AK-type production function given by:

$$y = Ak, \quad (47)$$

with a constant level of $r = f_k = A = \text{const}$ irrespective of the level of the capital stock k .²³ This function does not entail labour in the production function thereby having an (implicit) elasticity of substitution of $\sigma = \infty$. Formally, this is a CES-type production function with $\eta = -1$ and $\bar{\alpha} = 1$.

As put forward by Piketty (2014), this extreme (and certainly unrealistic) case is not necessary for deriving a positive relation between the capital/income ratio and the capital share. In fact, this prevails for an elasticity of substitution larger than one ($\sigma > 1$ respectively $-1 < \eta < 0$) - implying an elastic demand function. In this case, the volume effect dominates and an increase in capital leads to an increase in the capital share ($\frac{\partial \alpha}{\partial k} > 0$). In fact, Karabarbounis and Neiman (2013) - explaining the evidence of a long-run decrease in the labour share by a relative price decrease of capital - estimate a level of $\sigma = 1.25 > 1$. This case is sometimes also referred to as the *capital-biased* case (Giovannoni, 2014) for which capital and labour are substitutes.

In summary, we do not have to go as far as $\sigma \rightarrow \infty$ to come to the result that the aggregate capital share converges to one. In fact the condition of $\sigma > 1$ is sufficient for the case for which $r - g - \rho > 0$.²⁴ As also pointed out by Stiglitz (2015), in his book, Piketty (2014) seems to (implicitly) assume $\rho = 0$. As already shown, the latter implies the unrealistic notion of a consumption rate of zero $c_y = c_w = 0$. For the latter condition $r - g > \rho$ simplifies to $r > g$. In combination with $\sigma > 1$ indicating a capital-biased production, the ever increasing share in the functional distribution holds ($\lim_{t \rightarrow \infty} \alpha(t) = 1$).

The inverse case would emerge for $0 < \sigma < 1$ ($0 < \eta < \infty$), presenting an inelastic demand function for which the price effect prevails and increase in the level of capital would decrease the functional share of capital ($\frac{\partial \alpha}{\partial k} < 0$). This case can be labelled the *labour-biased* case, for which capital and labour are complements (Giovannoni, 2014). Most of the empirical literature confirms this case (Acemoglu, 2003). In fact, the Cobb-

²³In this type of literature - also labeled endogenous growth theory - the interest rate is exogenous and constant, whereas the growth rate varies and is endogenous in the model (Romer, 1986). In fact, this literature is diametrical to the Solow-Swan model, for which g is exogenous and r is determined within the model. We (implicitly) assume that both r and g are exogenous.

²⁴A similar argument is presented in Zamparelli (2014) arguing in a Pasinetti-type model (Pasinetti, 1962) in which the combination of $\sigma > 1$ with $s_c > s_w$ also leads to a convergence of the functional distribution to $\alpha = 1$. The underlying mechanism driving this result is the fact that the savings ratio of the capitalists s_c is larger than the savings ratio of the workers s_w . The latter is necessary as $s_c = s_w$ would imply euthanasia of the rentier as the latter does not have access to labour income (Taylor, 2014).

Douglas case is just the extreme isoelastic case ($\sigma = 1$) for which price and volume effects exactly offset each other. For the given model the case $0 < \sigma < 1$ eventually leads to the diametrical result. The high amount of accumulated wealth lowers the return on capital, making the capital share converge to zero. Or more formally, $\lim_{k \rightarrow \infty} \alpha = 0$. In this case there is *euthanasia of the rentier*, for which the capitalist cannibalises itself.²⁵

For our realistic values ($r = 7\%$, $g = 3\%$, $\rho = 3\%$ ²⁶, and $W_0 = 5$) we obtain some reasonable results. The initial ratio of capital to income is $\frac{W_0}{Y_0} = 5$, while the initial ratio of capital to total income (including capital income) is given by $\frac{W_0}{Y_0 + rW_0} \approx 3.7$. In the very long run this ratio is given by:

$$\lim_{t \rightarrow \infty} \frac{W(t)}{Y(t) + rW(t)} = \lim_{t \rightarrow \infty} \frac{\alpha(t)}{r} = \frac{1}{r} \approx 14.3. \quad (48)$$

Piketty and Zucman (2014) report a capital income ratio that increased from 2 to 3.5 to a range of approx. 4 to 7 between 1970 and 2010, implying an annual growth rate of approx. 1.7%. For the calibrated values in the model (displayed in figure 3) the ratio grows from 3.7 to approx. 9 within 50 years implying an annual growth of approx. 1.8%. The initial capital share is given by $\alpha(0) \approx 0.26$, broadly in line with the data presented in figure 1.

While the changes in the functional distribution - though existing (also cf. figure 1) - are rather modest, Piketty (2014) also summarises the recent research on the personal distribution of income showing a strong increase in both wealth and income inequality. We investigate the latter with the given model in the following section.

3.2. Distributional Consequences

So far, we have only considered aggregate sizes (respectively means). The aim of Piketty (2014) was it to address distributional issue (i.e. higher moments). We slightly extend the model for the most simple formulation with $N = 2$ agents, allowing for intuitive closed-form solutions. The latter is, however, enhanced by numerical simulations with $N > 2$ agents.

For the analytical discussion of the $N = 2$ agents case we follow Fischer (2015) and consider the ratio of economic figures as a measure of inequality. In the model we have three economic figures that can be compared: consumption, income, and wealth. Empirical evidence confirms that - of the three measures - consumption (which is also the usual measure of economic well-being imputed into the utility function) is most equally distributed, whereas the stock size wealth is the most unequally distributed resource. We start off with the inequality of income. We define the ratio:

$$z_{Y,t} = \frac{Y_{i,t}}{Y_{j,t}}, \quad (49)$$

²⁵This *bon mot* dates back to Keynes (1936) and was also employed by Piketty (2014). In a recent work, Taylor (2014) discusses this effect more thoroughly.

²⁶These value implies $r - g > \rho$.

as a simple scalar measure of inequality. For total equality we have $z_t = 1$, whereas for total inequality we either have $z_{Y,t} = \infty$ (agent j receives no income) respectively $z_{Y,t} = 0$ (agent i receives no income). If we define agent i as being the richer of the two, the measure is also easy to relate to the Gini-coefficient being the most common measure of inequality:²⁷

$$Gini_t(Y) = \frac{z_{Y,t} - 1}{z_{Y,t} + 1}, \quad (50)$$

for $z_{Y,t} > 1$. Thereby, we have $Gini(z_{Y,t} = \infty) = 1$ as well as $Gini(z_{Y,t} = 1) = 0$.

We assume that initially income inequality is exogenously given by $z_{Y,0} = \frac{Y_{1,0}}{Y_{2,0}}$. We sustain the assumption of income following an exponential growth-process²⁸ $Y_i(t) = Y_{i,0} \exp(g_i t)$ leading to a ratio of:

$$z_{Y,t} = \frac{Y_{i,0} \exp(g_i t)}{Y_{j,0} \exp(g_j t)} = \frac{Y_{i,0}}{Y_{j,0}} \exp([g_i - g_j]t) = z_{Y,0} \exp([g_i - g_j]t), \quad (51)$$

at a specific time t . If the income of both agents grows at the same pace ($g_i = g_j$), the inequality does not change in time. In the case in which one individual always grows at a different pace ($g_i \neq g_j$) inequality does not come to a halt and in the long run we have $\lim_{t \rightarrow \infty} Gini_t(Y) = 1$.

The case of income inequality is the most uninteresting in this model as it solely depends on exogenous assumptions. In contrast to this, consumption and wealth result from an optimised reaction given the income process. Let us first consider inequality of consumption. Using our scalar measure of inequality for normalised consumption $z_{c,t}$ we have:

$$z_{c,t} = \frac{c_{i,t}}{c_{j,t}} = \frac{\rho \left(W_{i,0} + \frac{Y_{i,0}}{r-g} \right) \exp([r-g-\rho]t)}{\rho \left(W_{j,0} + \frac{Y_{j,0}}{r-g} \right) \exp([r-g-\rho]t)}. \quad (52)$$

We want to assume the values of r , g , and ρ are identical for all agents. As a result, consumption inequality is constant in time. Moreover, it is determined by a different share of self-earned to inherited income. The latter is defined as:

$$\omega_{inher,i} = \frac{W_{i,0}}{W_{i,0} + \frac{Y_{i,0}}{r-g}}. \quad (53)$$

In fact, the inequality of consumption is given by:²⁹

$$z_{c,t} = z_c = \frac{W_{i,0} + \frac{Y_{i,0}}{r-g}}{W_{j,0} + \frac{Y_{j,0}}{r-g}} = \frac{\frac{1}{r-g} \left(Y_{i,0} \frac{\omega_{inher,i}}{1-\omega_{inher,i}} + Y_{i,0} \right)}{\frac{1}{r-g} \left(Y_{j,0} \frac{\omega_{inher,j}}{1-\omega_{inher,j}} + Y_{j,0} \right)} = z_{Y,0} \frac{1 - \omega_{inher,j}}{1 - \omega_{inher,i}}, \quad (54)$$

²⁷The latter is easy to prove if we keep in mind the definition of the finite size version of the Gini-coefficient $Gini_t(Y) = -\frac{N+1}{N-1} + \frac{2 \sum_{i=1}^N Y_i \cdot i}{(N-1) \sum_{i=1}^N Y_i}$ which for the $N = 2$ case equals: $Gini_t(Y) = -3 +$

$\frac{4Y_{i,t} + 2Y_{j,t}}{Y_{i,t} + Y_{j,t}} = \frac{Y_{i,t} - Y_{j,t}}{Y_{i,t} + Y_{j,t}} = \frac{z_{Y,t} - 1}{z_{Y,t} + 1}$.

²⁸The latter is also in line with empirical evidence (Jones, 2015b).

²⁹This is easy to verify as we know that $W_{i,0} = \frac{Y_{i,0}}{r-g} \frac{\omega_{inher,i}}{1-\omega_{inher,i}}$ and results directly from equation 53.

which is larger than the initial income inequality $z_{Y,0}$ for $\omega_{inher,i} > \omega_{inher,j}$ being the case for which high income households also have a higher share of inherited income. In contrast to this, consumption inequality is lower than labour income inequality for the opposing case.

A similar case prevails for wealth inequality. For the case of identical interest rates r , growth rates g and rates of time preference ρ the ratio of normalised wealth is given as follows:

$$z_{w,t} = \frac{\left(W_{i,0} + \frac{Y_{i,0}}{r-g}\right) \exp([r - \rho - g]t) - \frac{Y_{i,0}}{r-g}}{\left(W_{j,0} + \frac{Y_{j,0}}{r-g}\right) \exp([r - \rho - g]t) - \frac{Y_{j,0}}{r-g}}. \quad (55)$$

Let us start with the case for which the inheritance share for both agents is identical ($\omega_{inher,i} = \omega_{inher,j} \equiv \omega_{inher}$), leading to:

$$z_{w,t} = \frac{Y_{i,0} \frac{1}{1-\omega_{inher}} \exp([r - \rho - g]t) - 1}{Y_{j,0} \frac{1}{1-\omega_{inher}} \exp([r - \rho - g]t) - 1} = z_{Y,0} \frac{\frac{1}{1-\omega_{inher}} \exp([r - \rho - g]t) - 1}{\frac{1}{1-\omega_{inher}} \exp([r - \rho - g]t) - 1} = z_{Y,0}. \quad (56)$$

In this case wealth inequality would be equal to income inequality regardless of the relationship between $r - g$ and ρ (being identical for all agents). This is strongly at odds with empirical evidence documenting a far higher level of inequality for the stock measure of wealth as compared to the flow measure of income (cf. e.g. Davies and Shorrocks (2000)). One explanation for the latter would be the heterogeneous share of inherited wealth $\omega_{inher,i} \neq \omega_{inher,j}$. In the short run ($t = 0$) we have:

$$z_{w,0} = z_{Y,0} \frac{\omega_{inher,i}(1 - \omega_{inher,j})}{\omega_{inher,j}(1 - \omega_{inher,i})}, \quad (57)$$

implying a higher wealth than income inequality if high income individuals ($Y_{i,0} > Y_{j,0}$) inherit more than low income individuals ($\omega_{inher,i} > \omega_{inher,j}$). In this case wealth inequality (described in the preceding equation) is also higher than consumption inequality (as computed by equation 54). For the long run ($t \rightarrow \infty$) the case is more complicated. In fact, in this case we have to - once again - distinguish between three cases.

Proposition 3.3 (Long-run inequality of wealth for heterogeneous inheritance shares)

1. $r - g < \rho$: In the long run wealth inequality converges to income inequality.
2. $r - g = \rho$: In the long run wealth inequality is only given by inherited wealth.
3. $r - g > \rho$: Long-run wealth inequality is determined by both inherited and self-earned wealth. It is larger than income inequality if $\omega_{inher,i} > \omega_{inher,j}$.

First and foremost, it is important to acknowledge that wealth inequality converges to a finite level regardless of the relation between r , g and ρ . The concrete distribution, however, depends on the concrete relation between ρ and $r - g$.

Let us start with the first case considered in proposition 3.3. As already shown in proposition 3.1, in the first case agents accumulate debt according to their individual human capital $w_{i,\infty} = -\frac{Y_{i,0}}{r-g}$. As a result the distribution of wealth (or more precisely debt) is given by the human capital which is only determined by the income. As a result, wealth inequality is only determined by income inequality. More formally this can also be shown using:

$$\lim_{t \rightarrow \infty} z_{w,t} = z_{Y,0} \lim_{t \rightarrow \infty} \frac{\frac{1}{1-\omega_{inher,i}} \exp([r-\rho-g]t) - 1}{\frac{1}{1-\omega_{inher,j}} \exp([r-\rho-g]t) - 1}, \quad (58)$$

which for the case of $r-g < \rho$ leads to:

$$\lim_{t \rightarrow \infty} z_{w,t} = z_{Y,0} \frac{-1}{-1} = z_{Y,0}. \quad (59)$$

For the knife-edge case ($r-g = \rho$) we have the following result:

$$\lim_{t \rightarrow \infty} z_{w,t} = z_{Y,0} \lim_{t \rightarrow \infty} \frac{\frac{1}{1-\omega_{inher,i}} \exp([r-\rho-g]t) - 1}{\frac{1}{1-\omega_{inher,j}} \exp([r-\rho-g]t) - 1} = z_{Y,0} \frac{\omega_{inher,i}(1-\omega_{inher,j})}{\omega_{inher,j}(1-\omega_{inher,i})} = z_{w,0}. \quad (60)$$

In effect the inequality of wealth is only determined by the unequal endowment given in the form of inheritances. The latter is also straight forward from intuition. For the case of $r-g = \rho$ we have $c_y = 1$, meaning that individuals consume all income in a hand-to-mouth manner. The only difference in wealth can therefore occur due to differences in initial wealth $W_{i,0}$. The latter result also has an interesting policy implication. A policy maker who shows discontent with the distribution of wealth only has to interfere one single time. This is at period $t = 0$ (e.g. in the form of a one-time inheritance tax) to achieve a desired level of wealth inequality.

Finally, we have the third case, which we consider to be the case empirically relevant given a positive savings ratio lower than one. Moreover this case is (implicitly) assumed in Piketty (2014), when assuming a dynamic efficient economy $r > g$ without time preference $\rho = 0$. For this case we have:

$$\lim_{t \rightarrow \infty} z_{w,t} = z_{Y,0} \lim_{t \rightarrow \infty} \frac{\frac{1}{1-\omega_{inher,i}} - \exp([\rho+g-r]t)}{\frac{1}{1-\omega_{inher,j}} - \exp([\rho+g-r]t)} = z_{Y,0} \frac{(1-\omega_{inher,j})}{(1-\omega_{inher,i})}, \quad (61)$$

which is higher than $z_{Y,0}$ when high income individuals also have a higher labour income $\omega_{inher,i} > \omega_{inher,j}$. It is also interesting to compare this result with $z_{w,0} = z_{Y,0} \frac{\omega_{inher,i}(1-\omega_{inher,j})}{\omega_{inher,j}(1-\omega_{inher,i})}$. Even for the case of $r-g > \rho$ inequality of wealth decreases if the following condition holds:³⁰

$$\omega_{inher,j} < \omega_{inher,i}. \quad (62)$$

³⁰To derive this we require $\lim_{t \rightarrow \infty} z_{w,t} < z_{w,0} \Leftrightarrow \frac{(1-\omega_{inher,j})}{(1-\omega_{inher,i})} < \frac{\omega_{inher,i}(1-\omega_{inher,j})}{\omega_{inher,j}(1-\omega_{inher,i})}$.

The latter is always required to have an inequality of wealth higher than inequality of income. In fact, we have $\lim_{t \rightarrow \infty} z_{w,t} = z_{c,t}$, implying that inequality of wealth converges to a lower level identical to the level of consumption inequality. The underlying economic rationale is that the relevance of inherited capital diminishes with increasing life-time.

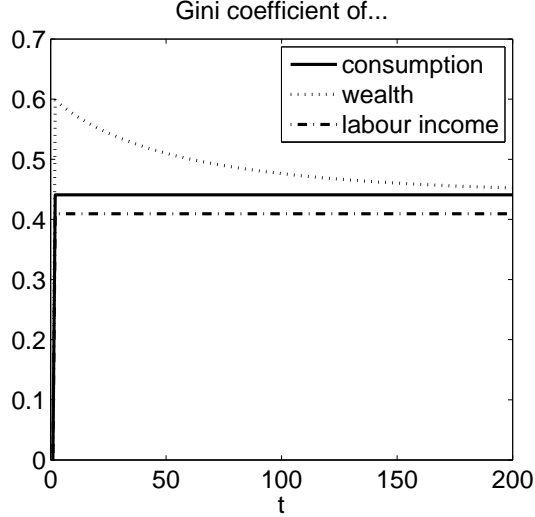


Figure 4: Gini-coefficient for consumption, income, and wealth for $\rho > r - g$

Figure 4 shows exemplary simulation results for the case of $\rho > r - g$.³¹ We assume that both inherited wealth as well as income follow a log-normal distribution. In this case - and in line with the empirical literature - it is assumed that inherited wealth is more unequally distributed.³² Each agent is assigned a random draw from the income and the distribution of initial wealth. For the case presented in figure 4, there is a negative correlation (amounting to -0.08) between income and self-earned wealth, implying that high income individuals also inherit a large amount. If there was positive correlation, wealth would be more equally distributed than income. As formally derived, wealth inequality in the long run decreases to the level of consumption inequality, which is more unequal than inequality of labour income due to the positive correlation between wealth and income. It is also interesting to reconcile this result with the result of the previous section, in which we showed that for $r - g > \rho$ and a CES-function with an elasticity of substitution $\sigma > 1$, in the long-run capital constitutes the complete income share ($\lim_{t \rightarrow \infty} \alpha(t) = 1$). This result seemed very alarming. In contrast to this, the inequality of wealth is rather modest in the given scenario. In fact, in the long run it is as low as the inequality of consumption. Thus, the inequality as measured by the functional distribution and the personal distribution can highly diverge.

In essence, we found that the inequality of consumption and especially wealth that emerges in this model is fairly low if we sustain the assumption of homogeneous interest

³¹For the concrete values we chose those also applied for the simulations presented in figures 2 and 3, i.e. $\rho = 3\%$, $g = 3\%$, and $r = 7\%$. The simulation was run for $N = 100$ agents.

³²For the latter case the standard deviation of wealth must be higher than that of income. In a first-order approximation $Gini(x) \approx \frac{\sigma_x}{\sqrt{\pi}}$ holds.

r , growth g , and time preference ρ rates. Let us now consider a heterogeneous rate of return $r_i \neq r_j$. The latter is also at the heart of the argument in Piketty (2014), in which he documents higher yields for larger funds. To stretch this point he compares the returns of heterogeneous university funds. This is certainly not directly transferable to the evolution of private household wealth. Nevertheless, there is some empiric evidence (e.g. Yitzhaki (1987)) that confirms that large wealth also earns larger returns. The theoretic rationale is also straight-forward: there are economics of scale in managing large wealth portfolios. Moreover, certain high yield investments (e.g. hedge funds) are only available for a certain (very high) minimum deposit and therefore only available to the richest individuals. As already shown, different rates of returns do not impact on the inequality of labour income. It matters, however, for consumption as well as wealth.

Inequality of consumption (both normalised c and non-normalised C) for heterogeneous $r_i \neq r_j$ is given as follows:

$$z_{c,t} = z_{Y,0} \frac{r_j - g}{r_i - g} \frac{1 - \omega_{inher,j}}{1 - \omega_{inher,i}} \exp([r_i - r_j]t) \equiv z_{C,t}. \quad (63)$$

The given measure does not converge for a persistence difference of $r_i - r_j \neq 0$. In fact, for $r_i > r_j$ household i is responsible for the complete consumption share in the long run. In the opposite case the reverse is true. Regardless of heterogeneous labour income and heterogeneous inheritance, the household with the higher return to wealth always outpaces the other households in the long run.

If the growth rates also differ, we have:

$$z_{c,t} = z_{Y,0} \frac{r_j - g_j}{r_i - g_i} \frac{1 - \omega_{inher,j}}{1 - \omega_{inher,i}} \exp([(r_i - r_j) - (g_i - g_j)]t), \quad (64)$$

whereas for the more important non-normalised values we still have:

$$z_{C,t} = z_{Y,0} \frac{r_j - g_j}{r_i - g_i} \frac{1 - \omega_{inher,j}}{1 - \omega_{inher,i}} \exp([r_i - r_j]t). \quad (65)$$

Let us abstract heterogeneous interest rates by assuming $r_i = r_j \equiv r$, leading to a value constant in time:

$$z_{C,t} = z_C = z_{Y,0} \frac{r - g_j}{r - g_i} \frac{1 - \omega_{inher,j}}{1 - \omega_{inher,i}}, \quad (66)$$

at any period t . Still, the inequality of consumption can be higher than in the case with homogeneous growth rates as long as:

$$g_i > g_j, \quad (67)$$

implying that the high income household $Y_{i,0} > Y_{j,0}$ has a higher growth rate. In summary, consumption inequality does not diverge for heterogeneous growth rates. Yet, if agents who receive a higher initial income also have a higher growth rate of income, the long run level of inequality of consumption is higher than with homogeneous growth rates (and vice versa for the diametrical case).

Finally, we can discuss the case for heterogeneous rates of time preference $\rho_i \neq \rho_j$. Assuming that growth g and interest r are uniform leads to:

$$z_{C,t} = \frac{\rho_i}{\rho_j} z_{Y,0} \frac{1 - \omega_{inher,j}}{1 - \omega_{inher,i}} \exp([\rho_j - \rho_i]t). \quad (68)$$

This case does not converge, leading to maximum inequality.

As a result consumption inequality only diverges for heterogeneous interest rates and heterogeneous rates of time preference, whereas heterogeneous income growth does not affect the measure of consumption inequality in the long run.

The case of wealth is slightly more complicated. Let us start with an extreme case for which the rates of interest deviate so much that some individuals accumulate debt whereas others hold positive wealth. For a given time preference and growth rate, formally this implies:

$$r_j < \rho + g < r_i, \quad (69)$$

for which individuals with high interest rates r_i aggregate wealth, whereas the others accumulate debt. In this case, there are no limits to wealth inequality. If we measure the inequality by means of the Gini-coefficient, even the technical upper bound of 1 must not be holding. Shorrocks et al. (2013) document Gini-coefficients for wealth inequality and report a value of 1.07 for Denmark.³³ This result seems very odd at first sight. Note, however, that the ordering of the agents according to level only satisfies the condition that the Lorenz-curve is convex. If the input values are negative (implying negative net-worth individuals), the slope is negative. If the amount of negative net worth as well as the share of those individuals holding it is high enough, the Gini-coefficient can exceed the unity threshold.

Figure 5 reports an exemplary simulation in which the Gini-coefficient for wealth exceeds one. This case features interest rates ranging from $2\% \leq r_i \leq 12\%$ which, due to the uniform $\rho + g = 3\% + 3\% = 6\%$, makes some agents indebted and others net claim holders. Nevertheless, in the long run wealth and consumption inequality converge to an identical level.

It is interesting to point out that wealth and consumption inequality move in opposite directions. The underlying rationale is that, while some households incur debt, others take claim positions, increasing inequality of wealth (in this case even beyond the unity threshold). The debt allows some households to (temporarily) increase their consumption making inequality of consumption lower than inequality of wealth. As, however, already shown debt in the short run (implied by $\rho_i > r - g$) also leads to the fact that the consumption path decreases in time. Thus, in the long run (here starting off from approx. period 40) wealth inequality lowers, whereas consumption inequality increases (making them once again move in the opposite direction). For the given simulation, the saver households dominate.³⁴ In effect, the results of the case for which for all agents

³³It is, however, important to point out that in the sample that aims at covering all countries in the world, this is the only case for which a value larger than one is documented.

³⁴Formally this is because we assume that interest rates are linearly distributed implying a larger share of agents for which $r_i > \rho + g = 6\%$.

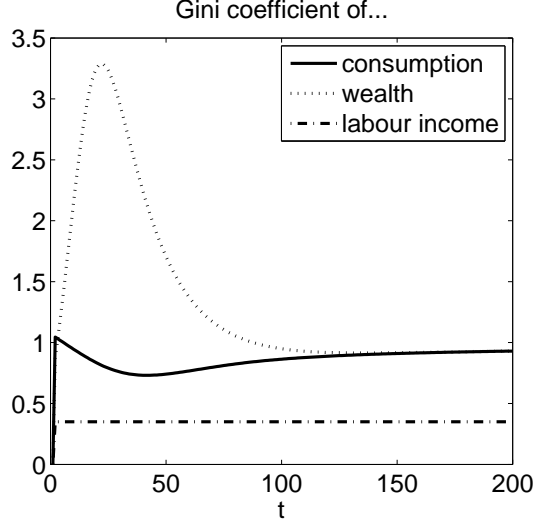


Figure 5: Gini-coefficient for consumption, income, and wealth for heterogeneous rates of r

$r > \rho + g$ holds reemerge; i.e. wealth and consumption inequality converge to a uniform level in the long run.

The case also emerges for which the rates of time preference $\rho_i \neq \rho_j$ diverge satisfying:

$$\rho_i < r - g < \rho_j, \quad (70)$$

for agents i accumulate wealth and of type j hold debt. The same case also applies for heterogeneous growth rates satisfying:

$$g_i < r - \rho < g_j, \quad (71)$$

for which - once again - type i agents have wealth and type j agents hold debt. The latter is somewhat puzzling as the agents with higher growth are more prone to be indebted. The latter, however, increases their human capital, thereby also increasing consumption while lowering financial wealth.

In the following we want to focus on the situation in which all agents satisfy the condition of wealth accumulation $\rho < r - g$.³⁵ We differentiate between the case of heterogeneous growth g_i as well as time preference ρ_i and interest rate r_i .

Proposition 3.4 (Long-run inequality of wealth for heterogeneous interest, growth, and time preference rates)

1. For heterogeneous rates of interest r_i as well heterogeneous rates of time preferences satisfying $\rho_i < r - g$ respectively $r_i > \rho + g$ for all i , there is maximum wealth inequality in the long run ($\lim_{t \rightarrow \infty} \text{Gini}(W_t) = 1$).

³⁵Or, put differently, we do not discuss the unrealistic case of aggregate debt accumulation ($\rho > r - g$). The knife-edge case $\rho = r - g$ also makes no sense in a heterogeneous agent case as the knife-edge condition can not be satisfied for all heterogeneous agents.

2. For heterogeneous growth rates g_i satisfying $g_i < r - \rho$ wealth inequality converges to the level of consumption inequality ($\lim_{t \rightarrow \infty} \text{Gini}(W_t) = \text{Gini}(C_t) < 1$).

The proof is as follows. First, consider the case of heterogeneous growth rates. The scalar measure of wealth inequality (non-normalised) can be written as follows:

$$\begin{aligned} \lim_{t \rightarrow \infty} z_{W,t} &= \lim_{t \rightarrow \infty} \frac{\left(W_{i,0} + \frac{Y_{i,0}}{r-g_i}\right) \exp([r-\rho]t) - \frac{Y_{i,0}}{r-g_i} \exp(g_i t)}{\left(W_{j,0} + \frac{Y_{j,0}}{r-g_j}\right) \exp([r-\rho]t) - \frac{Y_{j,0}}{r-g_j} \exp(g_j t)} \\ &= \lim_{t \rightarrow \infty} \frac{\left(W_{i,0} + \frac{Y_{i,0}}{r-g_i}\right) - \frac{Y_{i,0}}{r-g_i} \exp([g_i - r - \rho]t)}{\left(W_{j,0} + \frac{Y_{j,0}}{r-g_j}\right) - \frac{Y_{j,0}}{r-g_j} \exp([g_j - r - \rho]t)}, \end{aligned} \quad (72)$$

which converges as both $g_i < r + \rho$ and $g_j < r + \rho$ leading to:

$$\lim_{t \rightarrow \infty} z_{W,t} = \frac{\left(W_{i,0} + \frac{Y_{i,0}}{r-g}\right)}{\left(W_{j,0} + \frac{Y_{j,0}}{r-g}\right)} \equiv \lim_{t \rightarrow \infty} z_{C,t}. \quad (73)$$

The opposing is the case for heterogeneous rates of interest $r_i \neq r_j$:

$$\begin{aligned} \lim_{t \rightarrow \infty} z_{W,t} &= \lim_{t \rightarrow \infty} \frac{\left(W_{i,0} + \frac{Y_{i,0}}{r_i-g}\right) \exp([r_i-\rho]t) - \frac{Y_{i,0}}{r_i-g} \exp(g t)}{\left(W_{j,0} + \frac{Y_{j,0}}{r_j-g}\right) \exp([r_j-\rho]t) - \frac{Y_{j,0}}{r_j-g} \exp(g t)} \\ &= \lim_{t \rightarrow \infty} \frac{\left(W_{i,0} + \frac{Y_{i,0}}{r_i-g}\right) - \frac{Y_{i,0}}{r_i-g} \exp([g - r_i - \rho]t)}{\left(W_{j,0} + \frac{Y_{j,0}}{r_j-g}\right) \exp([r_j - r_i]t) - \frac{Y_{j,0}}{r_j-g} \exp([g - r_i - \rho]t)} \\ &= \lim_{t \rightarrow \infty} z_{Y,0} \frac{r_j - g}{r_i - g} \frac{1 - \omega_{inher,i}}{1 - \omega_{inher,j}} \exp([r_i - r_j]t), \end{aligned} \quad (74)$$

which does not converge for $r_i \neq r_j$. Moreover, the larger the difference between $r_j - r_i$, the faster the increase of inequality. A similar rationale applies for the case of heterogeneous rates of time preference $\rho_i \neq \rho_j$.

The latter statement can also be confirmed by simulations. For heterogeneous rates of growth (cf. figure 6(b)) only income inequality converges to maximum inequality. In the exemplary simulation, we assumed $0 \leq g_i \leq 3\%$ all satisfying $g_i < r - \rho = 4\%$. Inequality of consumption does not vary in time, whereas inequality of wealth converges to the very same level. Figure 6(a) presents the case of heterogeneous interest rates $7\% \leq r_i \leq 14\%$ all satisfying the assumption $r_i > \rho + g = 6\%$. As discussed for a constant level of income inequality, both wealth inequality and consumption inequality converge to maximum inequality ($\lim_{t \rightarrow \infty} \text{Gini}_t = 1$). The convergence speed itself increases with the dispersion of interest rates. A similar case applies for heterogeneous rates of time preference.³⁶

³⁶The result of non-converging wealth inequality for the case of heterogeneous time preference rates was already elaborated by Becker (1980). In his case, in the steady state equilibrium without growth

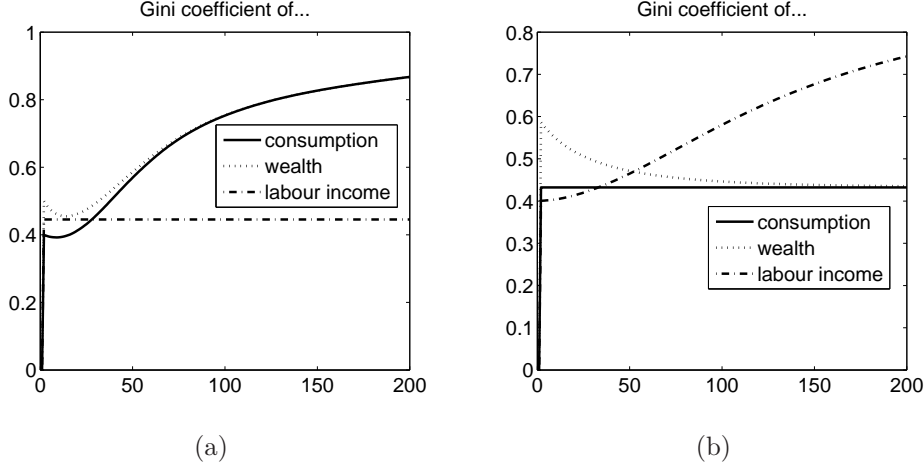


Figure 6: Inequality for heterogeneous interest rates r_i (left panel) and heterogeneous growth rates g_i (right panel)

Finally, we briefly want to touch on stochastic growth and interest rates. Assume that returns on wealth storage are stochastic with a uniform mean but with idiosyncratic stochastic disturbances:

$$r_{i,t} = r + \tilde{r}_{i,t}, \quad (75)$$

with the noise process $\tilde{r}_{i,t}$ is i.i.d. following from a normal distribution with a zero mean:

$$\tilde{r}_{i,t} \sim N(0, \sigma_r). \quad (76)$$

This case is identical to the case with heterogeneous r_i , implying a long-run convergence to total inequality for consumption and wealth. The income process itself is not affected. This result was presented in Fernholz and Fernholz (2014). The key difference between the case discussed before, however, lies in the persistence of the inequality. By assumption, in the first case some agents had persistently higher returns than other $r_i > r_j$ in any period t . In the simple stochastic presentation employed there is no persistence. Technically, this follows from the assumption of the independence of the stochastic shock. This is of utter importance when considering the policy interpretation. In the persistent case, one dynasty accumulates the dominant share of wealth resources. In the stochastic case, there is also total wealth inequality; the concrete dynasty at the top, however, is subject to changes from period to period. The empirical case is somewhere in between these two theoretical extrema. Formally, this could be captured by a more elaborate

the steady interest rate equals the time preference rate of the most patient individual ($r \equiv \rho^* = \min\{\rho_1, \rho_2, \dots, \rho_n\}$). As the model disallows the accumulation of debt all other individuals consume in a hand to mouth manner ($c_y = 1$) and do not aggregate wealth, in contrast to the most patient individual. A more intuitive explanation of the results is that individuals with the lowest propensity to consume (i.e. the lowest value of ρ) will, in the long run, always accumulate more wealth. In the long run, this effect even outdoes other forms of heterogeneity such as unequal labour income or unequal inheritance.

stochastic return process that also features an autocorrelation term. We leave this to future research.

A similar case emerges for growth with idiosyncratic stochastic shocks for which:

$$g_{i,t} = g + \tilde{g}_{i,t}, \quad (77)$$

with the i.i.d. process:

$$\tilde{g}_{i,t} \sim N(0, \sigma_g). \quad (78)$$

This is the case of incomplete insurance markets not allowing agents to hedge against adverse labour market outcomes.

We can confirm the latter using simulations. Figure 7(a) shows the case with stochastic interest rates and with $\sigma_r = 0.001$. While the inequality of income is unaffected, consumption and wealth inequality converge to maximum inequality. Figure 7(b) presents a stochastic growth rate with $\sigma_g = 0.001$. Income inequality converges to maximum inequality, while wealth inequality exceeds the unity threshold as some agents (temporarily) end up being debt holders. Even the inequality of consumption shows (small) variation in time due to the noise shocks (cf. with equation 66).

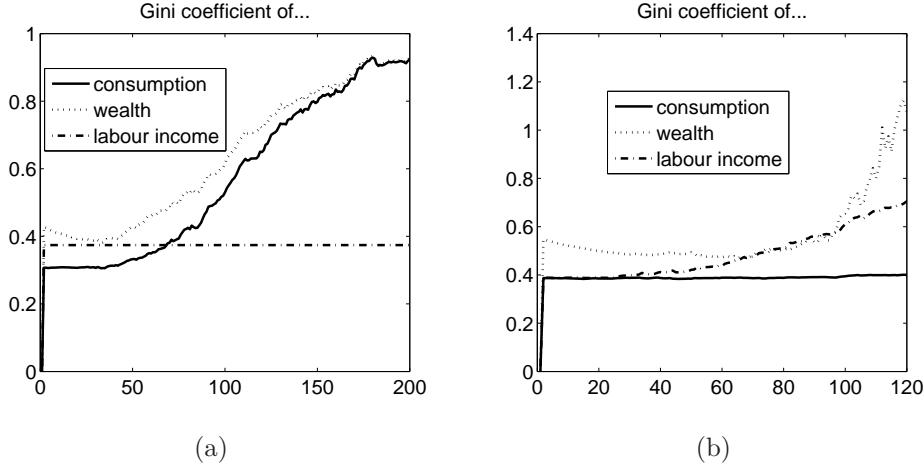


Figure 7: Inequality for stochastic interest rates r_i (left panel) and stochastic growth rates g_i (right panel)

In the standard Bewley-type literature, agents insure against these shocks by accumulating precautionary savings (or more flowery *savings for a rainy day*). This, however, requires knowledge about the nature of the stochastic process. In our (very simple) case agents simply do not know (i.e. *Knightian* uncertainty, Knight (1933)) and refrain from making precautionary savings. More technically, in this case the variance *blows up*. It is also important to point out that the latter once again implies that inequality is highly non persistent and despite maximum inequality agents always change positions on the social ladder.

4. Conclusion and Outlook

The aim of this paper is to discuss some of the key claims made by Piketty (2014) using formalised reasoning. In particular, we present the solution to an infinite horizon consumption smoothing problem in continuous time. We especially focus on the role of $r > g$ as emphasised in the original work of Piketty (2014).

The latter is the condition for dynamic efficiency and - given the empirical evidence presented in this paper - is likely to hold in the developed economies at the current point of time. In this paper we emphasise the role of the rate of time preference, which is implicitly set to zero ($\rho = 0$) in Piketty (2014). We show that the combination of the assumptions (i) $r - g > \rho$ (rather than $r > g$ as in his original work) and (ii) an elasticity of substitution in the production of $\sigma > 1$ leads to the fact that in the long run the capital share in the economy approaches one. The second assumption implies that the volume of effect of capital accumulation (resulting from the assumption $r - g > \rho$) dominates its price effect (*too much capital kills the return on capital*). Despite this high inequality in the functional distribution of income, the personal distribution of income, consumption, and wealth is rather modest in the model. While inequality of income is assumed to be exogenous, inequality of wealth and consumption are jointly determined by the inequality of income and the correlation between income and the individual ratio of human to physical capital. As shown in the model a key to derive realistic distribution lies in the heterogeneity of interest rates and growth rates. We consider two extreme cases of them being (i) persistently heterogeneous at any period or (ii) being subject to i.i.d. noise shocks.

The model is kept substantially simple in order to allow for closed-form solutions that are further confirmed by exemplary numerical simulations. The model lacks two key ingredients usually considered in more complicated models of heterogeneity requiring numerical simulations: (i) finite life time, as emphasised by the Overlapping Generations (OLG) literature and (ii) savings for a rainy day due to idiosyncratic shocks, as stressed in the Bewley-type literature. The first case could be incorporated into the Blanchard (1985) tradition (also considering a continuous time framework) and the latter by allowing the agents to internalise the stochastic nature of the income process in their savings decision. We plan to include this in future research.

All in all, the aim of this paper was to clarify some of the key predictions made in Piketty (2014) within a very common analytical framework. As shown in the paper, to derive some of his predictions some highly unusual assumptions have to be made. On the other hand, the standard model fails by predicting a way too low level of wealth inequality as documented (amongst others by Piketty (2014)) in the empirical evidence. The overall research agenda, therefore, requires substantial extensions to the given model in order to have a realistic and easy-to-understand theoretical toolbox for deriving meaningful policy conclusions.

A. The Value of Human Capital

The value of human capital is easy to verify if we consider the two differential equations governing the evolution of human capital. The flow of labour income evolves as follows:

$$\dot{Y}(t) = gY(t), \quad (79)$$

whereas the stock of human capital H evolves as follows (cf. e.g. Blanchard (1985), Weil (1989), or Buiter (1988)):

$$\dot{H}(t) = rH(t) - Y(t). \quad (80)$$

This equation directly follows from the no-arbitrage equation of asset pricing requiring:

$$1 + r = \frac{H_{t+1} + Y_t}{H_t} \leftrightarrow H_{t+1} = (1 + r)H_t - Y_t, \quad (81)$$

which in continuous time equals:

$$\dot{H}(t) = rH(t) - Y(t). \quad (82)$$

The solution to this system of differential equations is given by:

$$Y(t) = Y_0 \exp(gt), \quad (83)$$

and:

$$H(t) = \frac{Y_0}{r - g} \exp(gt) + K \exp(rt). \quad (84)$$

The solution to the latter is given by:

$$H(t) = \exp(rt) \left(K + \frac{Y_0}{r - g} \exp([g - r]t) \right) = K \exp(rt) + \frac{Y_0}{r - g} \exp(gt), \quad (85)$$

for some K to be determined.

As presented in Becker (2008) for asset-pricing the transversality condition reads:

$$\lim_{T \rightarrow \infty} H(T) \exp(-rT) = \lim_{T \rightarrow \infty} \left[\frac{Y_0}{r - g} \exp([g - r]T) + K \right] \stackrel{!}{=} 0, \quad (86)$$

being identical to the condition presented in equation 19 only holding for both $r > g$ and $K = 0$.

As a result human capital is given by:

$$H(t) = \frac{Y_0}{r - g} \exp(gt), \quad (87)$$

respectively, in normalised terms:

$$h(t) = \frac{Y_0}{r - g}, \quad (88)$$

being a constant value.

References

- Abel, A., G. N. Mankiw, L. H. Summers, and R. Zeckhauser (1989). Assessing Dynamic Efficiency: Theory and Evidence. *Review of Economic Studies* 56(1), 1–19.
- Acemoglu, D. (2003). Labor- And Capital-Augmenting Technical Change. *Journal of the European Economic Association* 1(1), 1–37.
- Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics* 109(3), 659–84.
- Atkinson, A. B. and T. Piketty (Eds.) (2007). *Top Incomes Over the Twentieth Century: A Contrast Between Continental European and English-Speaking Countries*. Number 9780199286881 in OUP Catalogue. Oxford University Press.
- Auerbach, A. J. and K. Hassett (2015). Capital Taxation in the 21st Century. *American Economic Review* 105(5), 38–42.
- Becker, R. A. (1980). On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households. *The Quarterly Journal of Economics* 95(2), 375–82.
- Becker, R. A. (2008). Transversality Condition. In S. N. Durlauf and L. E. Blume (Eds.), *The New Palgrave Dictionary of Economics*. Basingstoke: Palgrave Macmillan.
- Bertola, G. (2000). Macroeconomics of distribution and growth. In A. Atkinson and F. Bourguignon (Eds.), *Handbook of Income Distribution*, Volume 1 of *Handbook of Income Distribution*, Chapter 9, pp. 477–540. Amsterdam: Elsevier.
- Blanchard, O. and S. Fischer (1989). *Lectures on Macroeconomics*. Cambridge, MA: MIT Press.
- Blanchard, O. J. (1985). Debt, Deficits, and Finite Horizons. *Journal of Political Economy* 93(2), 223–47.
- Blume, L. E. and S. N. Durlauf (2015). Capital in the Twenty-First Century: A Review Essay. *Journal of Political Economy* 123(4), 749–777.
- Buiter, W. H. (1988). Death, Birth, Productivity Growth and Debt Neutrality. *Economic Journal* 98(391), 279–293.
- Cobb, C. W. and P. H. Douglas (1928). A Theory of Production. *The American Economic Review* 18(1), 139–165.
- Davies, J. B. and A. F. Shorrocks (2000). The distribution of wealth. In A. Atkinson and F. Bourguignon (Eds.), *Handbook of Income Distribution*, Volume 1 of *Handbook of Income Distribution*, Chapter 11, pp. 605–675. Amsterdam: Elsevier.

- DeNardi, M. (2015). Quantitative Models of Wealth Inequality: A Survey. NBER Working Papers 21106, National Bureau of Economic Research, Inc.
- Diamond, P. (1965). National debt in a neoclassical growth model. *American Economic Review* 55(5), 1126–1150.
- Domar, E. D. (1944). The "burden of the debt" and the national income. *The American Economic Review* 34(4), 798–827.
- Fernholz, R. and R. Fernholz (2014). Instability and concentration in the distribution of wealth. *Journal of Economic Dynamics and Control* 44(C), 251–269.
- Fischer, T. (2015). Incentives, inequality, and mobility. Mimeo, Technische Universität Darmstadt.
- Giovannoni, O. (2014). What Do We Know About the Labor Share and the Profit Share? Part I: Theories. Economics Working Paper Archive 803, Levy Economics Institute.
- Jones, C. I. (2015a). Pareto and Piketty: The Macroeconomics of Top Income and Wealth Inequality. *Journal of Economic Perspectives* 29(1), 29–46.
- Jones, C. I. (2015b). The Facts of Economic Growth. NBER Working Papers 21142, National Bureau of Economic Research, Inc.
- Kaldor, N. (1957). A model of economic growth. *The Economic Journal* 67(268), pp. 591–624.
- Karabarbounis, L. and B. Neiman (2013). The Global Decline of the Labor Share. *The Quarterly Journal of Economics* 129(1), 61–103.
- Keynes, J. M. (1936). *The General Theory of Employment Interest and Money*. London: Macmillan.
- Knight, F. H. (1933). *Risk, uncertainty and profit*. Boston, MA: Hart, Schaffner & Marx; Houghton Mifflin Co.
- Krusell, P. and A. A. Smith (2015). Is Piketty’s “Second Law of Capitalism” Fundamental? *Journal of Political Economy* 123(4), 725–748.
- Lucas, R. E. (1978). Asset Prices in an Exchange Economy. *Econometrica* 46(6), 1429–45.
- Mankiw, N. G. (2015). Yes, $r > g$. So What? *American Economic Review* 105(5), 43–47.
- Moll, B., J. Han, J.-M. Lasry, and P.-L. Lions (2015). Heterogeneous Agent Models in Continuous Time. mimeo, Princeton University.

- OECD (2012). *Statistical database of the Organisation for Economic Co-Operation and Development* (last accessed on 11/11/2014).
- Pasinetti, L. L. (1962). Rate of profit and income distribution in relation to the rate of economic growth. *The Review of Economic Studies* 29(4), 267–279.
- Piketty, T. (2014). *Capital in the twenty-first century*. Boston: Harvard University Press.
- Piketty, T. (2015). Putting Distribution Back at the Center of Economics: Reflections on Capital in the Twenty-First Century. *Journal of Economic Perspectives* 29(1), 67–88.
- Piketty, T., E. Saez, and S. Stantcheva (2014). Optimal Taxation of Top Labor Incomes: A Tale of Three Elasticities. *American Economic Journal: Economic Policy* 6(1), 230–71.
- Piketty, T. and G. Zucman (2014). Capital is back: Wealth-income ratios in rich countries 1700–2010. *The Quarterly Journal of Economics* 129(3), 1255–1310.
- Piketty, T. and G. Zucman (2015). Wealth and Inheritance in the Long Run. In A. Atkinson and F. Bourguignon (Eds.), *Handbook of Income Distribution*, Volume 2 of *Handbook of Income Distribution*, Chapter 8, pp. 1303–1368. Elsevier.
- Quadrini, V. and J.-V. Ríos-Rull (2015). Inequality in Macroeconomics. In A. Atkinson and F. Bourguignon (Eds.), *Handbook of Income Distribution*, Volume 2 of *Handbook of Income Distribution*, Chapter 8, pp. 1229–1302. Elsevier.
- Romer, P. M. (1986). Increasing Returns and Long-run Growth. *Journal of Political Economy* 94(5), 1002–37.
- Samuelson, P. A. (1958). An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money. *Journal of Political Economy* 66, 467–482.
- Samuelson, P. A. (1975). Optimum social security in a life-cycle growth model. *International Economic Review* 16(3), 539–544.
- Shorrocks, A., J. B. Davies, and R. Lluberas (2013). *Global Wealth Databook 2013*. Research institute, Credit Suisse.
- Stiglitz, J. E. (2015). New theoretical perspectives on the distribution of income and wealth among individuals: Part ii: Equilibrium wealth distributions. Working Paper 21190, National Bureau of Economic Research.
- Taylor, L. (2014). The Triumph of the Rentier? Thomas Piketty vs. Luigi Pasinetti & John Maynard Keynes. SCEPA working paper series. SCEPA’s main areas of research are macroeconomic policy, inequality and poverty, and globalization. 2014-7, Schwartz Center for Economic Policy Analysis (SCEPA), The New School.

- Wang, M., M. O. Rieger, and T. Hens (2011). How Time Preferences Differ: Evidence from 45 Countries. Discussion Papers 2011/18, Department of Finance and Management Science, Norwegian School of Economics.
- Weil, P. (1989). Overlapping families of infinitely-lived agents. *Journal of Public Economics* 38(2), 183–198.
- Yitzhaki, S. (1987). The Relation between Return and Income. *The Quarterly Journal of Economics* 102(1), 77–96.
- Zamparelli, L. (2014). Wealth Distribution, Elasticity of Substitution, and Piketty: an anti-dual Pasinetti Economy. Working Papers 1/15, Sapienza University of Rome, DISS.