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Sequential Rules for House Allocation with Price Restrictions

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Sequential Rules for House Allocation with Price Restrictions*

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Abstract

Abstract. This paper considers a housing market with price restrictions. On such market, price equilibrium may be excluded for certain preference profiles. However, the existence of a unique minimal rationing price equilibrium has previously been established on a general preference domain that contains “almost all” preference profiles. This type of equilibrium has been demonstrated to be an important ingredient in a direct and strategy-proof allocation mechanism for housing markets with price restrictions. The main contribution of this paper is to provide a finite ascending price sequence that terminates to a minimal rationing price equilibrium. This sequence is demonstrated to play a key-role in an Iterative English Auction Rule for housing markets with price restrictions.

JEL Classification: C78; D44; D45.

Keywords: Rationing Price Equilibrium; (Extended) English Price Sequences; Iterative English Auction Rule.

1 Introduction

This paper considers the problem of allocating a finite set of indivisible items, called houses, to a finite set of agents in the absence of property rights and when monetary transfers are feasible

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but bounded from below and above by exogenously given price restrictions. One example of such a market is a housing market with a legislated rent control, i.e., a market where rents are bounded to belong to an exogenously given set that specifies the rent ceilings as well as the minimal acceptable rent for the landlord. Another example is a labor market with legislated minimum wages where wages are bounded to belong to an exogenously given set that specifies the minimum wages as well as the employers maximum willingness to pay for the service of the workers. This class of problems has recently been considered by, e.g., Andersson and Svensson (2014, 2016), Herings (2015), Talman and Yang (2008), and Zhu and Zhang (2011).

Because price restrictions exclude price equilibrium for certain preference profiles, the concept of a Rationing Price Equilibrium (RPE, henceforth) is adopted. Such equilibrium always exists and Andersson and Svensson (2014) demonstrated that there exists a minimal RPE price vector for each preference profile in a general preference domain that contains all rational preference profiles satisfying a monotonicity and a continuity assumption. Moreover, a minimal RPE price vector is unique for each profile on a restricted preference domain that, in a mathematical meaning, contains “almost all” preference profiles.¹ Andersson and Svensson (2014) used this unique vector to define a strategy-proof allocation rule. However, their strategy-proof rule is a direct mechanism, and a dynamic rule that converges to a minimal RPE price vector is lacking in the literature.²

It is important to identify a dynamic rule also for the house allocation problem with price restrictions since dynamic rules often are preferred to their direct counterparts (see, e.g., Cramton, 1998; Engelberecht-Wiggans and Kahn, 1991). One reason for this is that dynamic rules not necessarily require full preference revelation. If, for example, a number of heterogenous items are being allocated using a direct mechanism, agents’ always have to reveal all information about their demand for all admissible prices. If, on the other hand, a dynamic mechanism is adopted, full demand revelation will only prevail in an extreme case while agents’, in most cases, only have to reveal partial information about their demand at some specific prices (see Section 3 for an example). Another reason is that dynamic rules are typically more transparent than their direct counterparts and it is therefore more likely that agents’ play weakly dominant strategies (if such strategies exists) under a dynamic mechanism than under a direct mechanism. This has been observed in laboratory experiments (see, e.g., Harstad, 2000; Kagel and Levin, 1993; Kagel et al., 1987) and it has been discussed in the theoretical literature (see, e.g., Ausubel, 2004; Li, 2016).

The main part of this paper is devoted to defining and analyzing a sequence of increasing prices, called an English Price Sequence (EPS, henceforth). An EPS is finite and the end point

¹More specifically, Andersson and Svensson (2016) provide a measure on subsets of preference profiles in the general preference domain, and prove that the subset of profiles that are excluded from the general domain has measure zero.

²Note that there exists dynamic rules for housing markets with price restrictions (Herings, 2015; Talman and Yang, 2008; Zhu and Zhang, 2011) but none of them generally converge to an RPE and none of them have been demonstrated to satisfy any good incentive properties (not even in their direct versions, in contrast to the direct mechanism of Andersson and Svensson, 2014, 2016).

price vector of an EPS can always be chosen to be an RPE price vector. To make sure that the EPS always converges to an RPE price vector, each price vector in the increasing price sequence is supported by a “temporary state”. These states may contain non-feasible assignments of the houses, but are useful for obtaining a rigorous definition of excess demand in form of over-demanded sets. To move from one price vector to the next in the EPS, all prices in over-demanded sets, and only those, are increased. In the end, an RPE price vector can be obtained. Note also that an EPS is well-defined for all preference profiles in a general preference domain, but there may be several price paths consistent with the sequence, and all price paths need not necessarily converge to the same RPE price vector (see Example 4). On the other hand, if the domain restriction adopted by Andersson and Svensson (2014, 2016) is considered, then all possible price paths terminate to the unique minimal RPE price vector for any given preference profile in the restricted preference domain.

An extended version of the EPS can be seen as the outcome of an auction rule which implements a minimal price vector. This auction rule is called the Iterative English Auction Rule and it is a combination of the different steps of the extended EPS, $t = 1, \dots, T$, and a number of direct mechanisms used to identify the prices between steps t and $t + 1$. In this dynamic procedure, agents reveal only partial preference information and a measure of maximal preference revelation is provided. In fact, arbitrary little information needs to be revealed by adding more steps to the extended EPS (i.e., by increasing T). The considered iterative auction rule is also demonstrated to contain, e.g., the Queue Allocation Mechanism (Svensson, 1994) and the Exact Auction Mechanism (Demange et al., 1986) as special cases.

The paper also provides sufficient conditions for reported demand sets to be consistent with rational preferences. These conditions are important because if agents are required to report in accordance with these conditions, then Theorem 1 in Andersson and Svensson (2016) implies that truthful bidding is an ex post Nash equilibrium in the Iterative English Auction Rule.

Even if the main results of this paper only are valid on a restricted preference domain, the considered domain contains, in a mathematical meaning, almost all preference profiles that are rational and satisfy a monotonicity and a continuity assumption (see the Appendix in Andersson and Svensson, 2016). This is in contrast to previous auction rules that almost exclusively are defined on the domain where preferences are represented by quasi-linear utility functions (see, e.g., Ausubel, 2004, 2006; Demange et al., 1986; Gul and Staccetti, 2000; Kelso and Crawford, 1982; Mishra and Parkes, 2007, 2009). There are good reasons for considering a larger preference domain than the quasi-linear. Namely, as argued by, e.g., Alaei et al. (2016), Baisa (2017), Morimoto and Serizawa (2015) and Zhou and Serizawa (2016), agents may be risk averse, budget constrained, or experience wealth effects.

The model with price restrictions, considered in this paper, contains many previously investigated models as special cases, including, e.g., a model where no agent has the property right over any of the houses and where monetary transfers are (a) infeasible (e.g., Balinski and Sönmez, 1999; Hylland and Zeckhauser, 1979; Svensson, 1994), or (b) feasible but bounded from

below (e.g., Shapley and Shubik, 1972; Demange and Gale, 1985; Demange et al., 1986). There is, however, a fundamental difficulty when identifying a dynamic rule for the more general model with price restrictions. Namely, if monetary transfers are infeasible, a dynamic rule needs only to solve the problem of allocating the houses among the agents (i.e., prices are irrelevant), whereas an dynamic rule, for the case when monetary transfers are feasible but bounded from below, needs only to solve the problem of identifying an equilibrium price vector as the price vector can be used as a tool to allocate the houses among the agents. For the problem considered in this paper, however, the dynamic rule needs simultaneously to consider the allocation of the houses as well as the identification of a price vector. Note also that the price vector in the considered problem is not, a priori, restricted to belong to a finite set of price vectors as in the standard matching with contracts framework (e.g., Echenique, 2012; Hatfield and Kojima, 2010; Hatfield and Milgrom, 2005; Herings, 2015).³ These observations coupled with the assumption of weak preferences suggest that it is a non-trivial task to find a dynamic rule for a housing market with price restrictions.

The remaining part of the paper is outlined as follows. Section 2 contains the formal model and some basic definitions. A preview of the English Price Sequence and the Iterative English Auction Rule can be found in Section 3 where an extended example is provided to illustrate a few basic ideas and principles in some of the coming definitions. The English Price Sequence is formally defined and analysed in Section 4. The corresponding Iterative English Auction Rule and its strategic properties as well as its relation to other dynamic rules in the literature are investigated in Section 5. All proofs are relegated to the Appendix.

2 The Model and Basic Definitions

The set of *houses* and *agents* are denoted by $H = \{1, \dots, m\}$ and $A = \{1, \dots, n\}$, respectively. No agent in A has the property right to any house in H , and each agent in A wish to acquire at most one house in H . Agents also have an option not to buy, or rent, a house at all. This outside option is formally represented by a *null house*, denoted by 0, the supply of which is unlimited.

An *assignment* is a mapping $\mu : A \rightarrow H \cup \{0\}$. An assignment is *feasible* if for $a \neq a'$, $\mu_a = \mu_{a'}$ only when $\mu_a = 0$, i.e., two distinct agents in A cannot be assigned the same house in H at a feasible assignment. Let the set μ_0 contain all houses not assigned to any agent at

³The problem studied by Andersson and Svensson (2014, 2016) has some special characteristics compared to the four mentioned matching with contracts papers. For example, all of the other papers analyze a framework with a finite number of contracts where agents are assumed to have strict preferences and where the divisible good cannot take arbitrary values (Herings, 2015, do not assume strict preferences but indifference relations play no significant role in his Adjustment Process as agents are assumed to choose an arbitrary contract from their choice sets in the case when the cardinality of the choice sets is greater than one), whereas Andersson and Svensson (2014, 2016) do not restrict the analysis to a finite set of contracts, preferences are weak, and the divisible good can take an arbitrary number on the (extended) real line. The models do, however, share the property that, e.g., the models of Demange et al. (1986) and Kelso and Crawford (1982) among others, can be included as special cases of the frameworks.

assignment μ , i.e.:

$$\mu_0 = \{h \in H; \mu_a \neq h \text{ for all } a \in A\} \cup \{0\}.$$

The vector $p = (p_0, \dots, p_m) \in \mathbb{R}_+^{m+1}$ is a *price vector*. A coordinate in p is denoted by p_h and represents the price of house $h \in H \cup \{0\}$. Price vectors are restricted by exogenously given lower and upper bounds, denoted by $\underline{p} \in \mathbb{R}_+^{m+1}$ and $\bar{p} \in \overline{\mathbb{R}}_+^{m+1}$, respectively, where $\underline{p}_h \leq \bar{p}_h$ for all $h \in H$ and $\underline{p}_0 = \bar{p}_0 = 0$.⁴ Here, $\overline{\mathbb{R}}_+ = [0, \infty]$ represents the non-negative part of the extended real line. The *price space* is given by:

$$\mathbb{P} = \{p \in \mathbb{R}_+^{m+1} : \underline{p}_h \leq p_h \leq \bar{p}_h \text{ for each } h \in H \cup \{0\}\}.$$

To simplify notation, let $(h, p) \equiv (h, p_h)$, i.e., (h, p) means house h at price p_h at the price vector p .

The *preferences* of agent $a \in A$ are denoted by R_a , and are represented by a complete and transitive binary relation on the set of houses and prices $(H \cup \{0\}) \times \mathbb{R}_+$. The corresponding strict and indifference relations are denoted by P_a and I_a , respectively. For each agent $a \in A$, preferences are assumed to be strictly *monotonic*, i.e., $(h, p)P_a(h', p')$ if $p_h < p'_h$. Preferences are also assumed to be *continuous*, i.e., the sets $\{p_h \in \mathbb{R}_+ : (h, p)R_a(h', p')\}$ and $\{p_h \in \mathbb{R}_+ : (h', p')R_a(h, p)\}$ are closed for each $a \in A$ and all $h, h' \in H$ and all $p'_h \in \mathbb{R}_+$. Finally, the preferences are assumed to satisfy a *finiteness* condition stating that for each $a \in A$ and $h \in H$ there is a “sufficiently large” $p_h \in \mathbb{R}$ such that $(0, 0)P_a(h, p_h)$. All preference relations R_a satisfying the above properties for agent $a \in A$ are gathered in the set \mathcal{R}_a . A (preference) *profile* is a list $R = (R_1, \dots, R_n)$ that belongs to the set $\mathcal{R} = \mathcal{R}_1 \times \dots \times \mathcal{R}_n$.

Because price vectors are restricted to belong to the set \mathbb{P} , a price equilibrium need not exist for some profiles in \mathcal{R} . Consequently, a weaker equilibrium notion is needed to analyze the model. This weakening must contain some kind of rationing mechanism as prices alone cannot solve the allocation problem. Here, it is assumed that the rationing mechanism is based on a *priority-order*, denoted by π . Formally, $\pi : A \rightarrow \{1, \dots, n\}$ is a bijection where the highest-ranked agent $a \in A$ is the agent with $\pi_a = 1$, the second highest ranked agent a' has $\pi_{a'} = 2$, and so on.⁵

In the remaining part of this paper, an arbitrary but fixed profile $R \in \mathcal{R}$ and an arbitrary but fixed priority structure π are considered. An *economy* \mathcal{E} is, therefore, defined by the fixed concepts A, H, \mathbb{P}, R , and π .

Definition 1. For a given economy \mathcal{E} , a *state* $x = (\mu, p)$ is a pair where μ is an assignment and

⁴In most of the analysis, it is assumed that $\underline{p} \neq \bar{p}$, i.e., that $\underline{p}_h < \bar{p}_h$ for some $h \in H$. In the case when $\underline{p}_h = \bar{p}_h$ for all $h \in H$, the Queue Allocation Mechanism (Svensson, 1994) can be applied to solve the allocation problem (see Section 5.2).

⁵In Andersson and Svensson (2014), a more general priority structure is considered. In this paper, a less general structure is considered for notational simplicity and without loss of generality.

$p \in \mathbb{P}$. The state $x = (\mu, p)$ is feasible if the assignment μ is feasible. A state x may also be written as a collection of n bundles of type $x_a = (\mu_a, p)$, i.e., $x = (x_1, \dots, x_n)$.

A state x is priority respecting if there is no agent $a \in A$ that strictly prefers some bundle $x_{a'}$ to his own assigned bundle x_a and bundle $x_{a'}$ is assigned to some other agent $a' \in A$ that has lower priority than agent a , and, furthermore, all agents weakly prefer their assigned bundle to any bundle containing an unassigned house.

Definition 2. For a given economy \mathcal{E} , a state $x = (\mu, p)$ is *priority respecting* if for all $a, a' \in A$: (i) $x_{a'} P_a x_a$ only if $\pi_{a'} < \pi_a$, and (ii) $x_a R_a(h, p)$ if $h \in \mu_0$.⁶

For any two given states, x and x' , a trading cycle is a sequence of distinct agents, say (a_1, \dots, a_k) , where, at state x' , each agent in the sequence, except that last, is assigned the house that was assigned to the subsequent agent in the sequence at state x (e.g., at state x' , agent a_1 is assigned the house that was assigned to agent a_2 at state x , and so on). A trading cycle is closed if the last agent in the sequence, at state x' , is assigned the house that was assigned to the first agent in the sequence at state x , and open if the last agent in the sequence, at state x' , is assigned a house that was unassigned as state x .

Definition 3. For a given economy \mathcal{E} , let $x = (\mu, p)$ and $x' = (\mu', p')$ be two distinct states and $(a_j)_{j=1}^k$ a sequence of distinct agents where:

- (i) $\mu'_{a_j} = \mu_{a_{j+1}}$ for all $1 \leq j < k$, and;
- (ii) $\mu'_a = \mu_a$ for all $a \neq a_j$ and all $1 \leq j \leq k$.

The sequence $(a_j)_{j=1}^k$ is called a *trading cycle* from μ to μ' . A trading cycle is *closed* if $\mu'_{a_k} = \mu_{a_1}$ and *open* if $\mu'_{a_k} \in \mu_0$.

A state $x = (\mu, p)$ is constrained efficient if the assignment μ is feasible and if there is no feasible trading cycle that respects priorities and, in addition, makes all agents in the cycle weakly better off and at least one agent strictly better off.

Definition 4. For a given economy \mathcal{E} , a trading cycle from μ to μ' is a (Pareto) *improvement of μ* if μ and μ' are priority respecting and $x'_a R_a x_a$ for all $a \in A$ and $x'_a P_a x_a$ for some $a \in A$. A state $x = (\mu, p)$ is *constrained efficient* if μ is feasible and there is no improvement of μ to a feasible assignment μ' .

A state x is a rationing price equilibrium if three conditions are satisfied. The first guarantees that the state is constrained efficient. The second condition is based on Drèze (1975) and a series of subsequent papers (where Herings, 2015; Talman and Yang, 2008, are the most closely related

⁶In Balinski and Sönmez (1999), condition (i) of Definition 2 is called *fairness* and condition (ii) is the combination of *individual rationality* and *non-wastefulness*.

to this study) where it is argued that rationing should only be put to use when the price of a house equals its upper price bound, or, equivalently, when prices alone cannot solve the allocation problem. Note that this means that whenever an agent is assigned a house with a price strictly below the upper price bound, the agent is also assigned his most preferred house at the given prices. The last condition is the standard property that the price of any unassigned house must equal its lower price bound (see, e.g., Demange and Gale, 1985).

Definition 5. For a given economy \mathcal{E} , a state $x = (\mu, p)$ is a *rationing price equilibrium* (RPE) if the following conditions hold:

- (i) x is a constrained efficient,
- (ii) for all $h \in H$, $p_h = \bar{p}_h$ if $(h, p)P_a(\mu_a, p)$ for some $a \in A$, and;
- (iii) for all $h \in H$, $p_h = \underline{p}_h$ if $h \in \mu_0$.

A price vector p is an RPE price vector if there is an assignment μ such that the state (μ, p) is an RPE (note that there may be several assignments that are consistent with the price vector p). For a given economy \mathcal{E} , the set of RPE states is denoted by Σ and the corresponding set of price vectors is denoted by Π , i.e., $\Pi = \{p \in \mathbb{P} : (\mu, p) \in \Sigma \text{ for some assignment } \mu\}$.

Definition 6. For a given economy \mathcal{E} , a price vector $p^* \in \Pi$ is a *minimal RPE price vector* if, for $p \in \Pi$, $p \leq p^*$ only if $p = p^*$. A *minimal RPE state* at profile $R \in \mathcal{R}$ is a feasible state $x = (\mu, p)$ such that p is a minimal RPE price vector in Π .

3 Example and Preview

This section contains an example that illustrates some of the main features of the dynamic price mechanisms that are formally introduced and analyzed in Sections 4 and 5. The purpose of the example is to point at a few basic ideas in some of the coming definitions and to roughly illustrate how the price adjustment rule and the assignment rule work.

Let $A = \{1, 2, 3\}$ and $H = \{1, 2, 3\}$ be the sets of agents and houses, respectively, and let $\pi_a = a$ for all $a \in A$. Suppose further that the lower and upper price bounds are given by $\underline{p} = (0, 0, 0)$ and $\bar{p} = (3, 7, 7)$, respectively. Preferences over consumption bundles (h, p) are represented by a quasi-linear utility function $u_{ah}(p) = v_{ah} - p_h$ where $v_{a0} = 0$, and:

$$(v_{ah}) = \begin{pmatrix} 13 & 15 & 1 \\ 20 & 15 & 1 \\ 20 & 15 & 1 \end{pmatrix}.$$

The English Price Sequence (EPS, henceforth) consists of a price adjustment rule and an assignment rule, and these rules jointly define a finite sequence of (supporting temporary) states where

the last price vector in the sequence is a minimal RPE price vector. In this dynamic process, prices as well as assignments are adjusted starting at the lower price bound. The basic rule is to increase the prices for all houses in a specific (minimal) set of over-demanded houses. Due to the upper price bounds, this basic rule is not always sufficient to eliminate all over-demanded sets. In such cases, the priority-order will determine the (temporary) assignment of the houses.

Table 1 provides a step-by-step description of the dynamic process. At each price vector p^j , a subeconomy is considered. Such economy contains all houses with prices strictly below the upper price bound and all agents that not have been assigned a house with a price equal to the upper price bound. The set $D_a(p^j)$ contains all houses in the considered subeconomy that, in addition, belong to the demand set of agent a at prices p^j , and the set $\text{MOD}(p^j)$ contains the (minimal) set of over-demanded houses in the considered subeconomy at prices p^j .

The dynamic process starts at the lower price bound $p^1 = (0, 0, 0)$, and the subeconomy now contains all houses and all agents since no agent yet have been assigned a house and $p^1 < \bar{p}$. Given price vector p^1 , a temporary state is defined as a state where all agents in the subeconomy are assigned their most preferred house in the subeconomy at the given prices (if it is possible to assign all agents distinct houses, the process will terminate, but this is not possible here). Hence, the temporary state is given by (μ^1, p^1) where $\mu^1 = (2, 1, 1)$. This state is not feasible since house 1 is assigned to both agent 2 and agent 3. This also means that house 1 is over-demanded, and its price is increased by one unit and the new price vector $p^2 = (1, 0, 0)$ is obtained. The assignment for the temporary state at prices p^2 is now given by $\mu^2 = (2, 1, 1)$, and house 1 is again over-demanded. Consequently, $p^3 = (2, 0, 0)$ and, by repeating the arguments, it also follows that $p^4 = (3, 0, 0)$, and the price of house 1 has now reached its upper price bound $\bar{p}_1 = 3$.

Table 1: Illustration of the price adjustment rule and the assignment rule

Step j	p^j	$D_1(p^j)$	$D_2(p^j)$	$D_3(p^j)$	μ^j	$\text{MOD}(p^j)$
1	(0, 0, 0)	{2}	{1}	{1}	(2, 1, 1)	{1}
2	(1, 0, 0)	{2}	{1}	{1}	(2, 1, 1)	{1}
3	(2, 0, 0)	{2}	{1}	{1}	(2, 1, 1)	{1}
4	(3, 0, 0)	{2}	{1}	{2}	(2, 1, 2)	{2}
5	(3, 1, 0)	{2}	{1}	{2}	(2, 1, 2)	{2}
6	(3, 2, 0)	{2}	{1}	{2}	(2, 1, 2)	{2}
7	(3, 3, 0)	{2}	{1}	{2}	(2, 1, 2)	{2}
8	(3, 4, 0)	{2}	{1}	{2}	(2, 1, 2)	{2}
9	(3, 5, 0)	{1, 2}	{2}	{2}	(1, 2, 2)	{2}
10	(3, 6, 0)	{1}	{2}	{2}	(1, 2, 2)	{2}
11	(3, 7, 0)	{1}	{2}	{3}	(1, 2, 3)	\emptyset

Since the price of house 1 has reached its upper price bound, it is impossible to eliminate over-demand by means of price increases. In this situation, house 1 is temporarily assigned to the highest ranked agent that demands house 1, and the remaining agents and houses constitute the

new subeconomy. Because agents 2 and 3 only demand house 1 at prices p^4 and $\pi_2 < \pi_3$, it follows that agent 2 temporarily is assigned house 1. Given that house 1 (temporarily) has been assigned to agent 2, the subeconomy is defined by the houses in $\{2, 3\}$ and the agents in $\{1, 3\}$. In this subeconomy and at prices $p^4 = (3, 0, 0)$, agents 1 and 3 both demand house 2, and because the agents are assigned their most preferred house in the subeconomy, the assignment is given by $\mu^4 = (2, 1, 2)$. This also means that house 2 is over-demanded at prices p^4 . The price of house 2 is therefore increased by one unit. By repeating the arguments, the price vector $p^9 = (3, 5, 0)$ will eventually be reached.

At prices $p^9 = (3, 5, 0)$, agent 1 is indifferent between houses 1 and 2. Because the price of house 1 has reached its upper price bound and agent 1 now is the highest ranked agent that demands house 1 (i.e., $\pi_1 < \pi_2 < \pi_3$), agent 1 is temporarily assigned house 1 and a new subeconomy, containing only the houses in $\{2, 3\}$ and the agents in $\{2, 3\}$, is considered. In this subeconomy and at prices p^9 , agents 2 and 3 only demand house 2 and, consequently, the price of house 2 is increased by one unit. By repeating the arguments, the price vector p^{11} will eventually be reached and, at these prices, the price of house 2 equals its upper bound, i.e., $p_2^{11} = \bar{p}_2 = 7$. Since house 2 is the only house in the subeconomy that is demanded by agents 2 and 3 at prices p^{11} and $\pi_2 < \pi_3$, house 2 is temporarily assigned to agent 2. Note now that the prices of house 1 and house 2 equal their corresponding upper price bounds and that they have been temporarily assigned to agents 1 and 2, respectively. This also means that a new subeconomy containing only house 3 and agent 3 needs to be considered. In this subeconomy, agent 3 is temporarily assigned house 3 since house 3 is strictly preferred to the null house. Hence, all agents have (temporary) been assigned a house and because the temporary assignment is feasible, it now becomes a permanent assignment. Therefore, the process terminates at state (μ^{11}, p^{11}) . This state is a minimal RPE state.

Note that the above process requires 11 steps to converge to a minimal RPE state. In the (Extended) English Price Sequence and the Iterative English Auction Rule, however, not all these steps are explicitly written out since they are based on so-called (small) price regimes and, more precisely, on the supremum of these price regimes. The relevant steps of the considered dynamic processes are the ones where prices as well as temporary assignments change. Hence, the English Price Sequence and the Iterative English Auction Rule are described by the sequences (p^1, p^4, p^9, p^{11}) and (x^1, x^4, x^9, x^{11}) , respectively, where $x^j = (\mu^j, p^j)$ for $j \in \{1, 4, 9, 11\}$. These sequences can fairly easily be identified in the above example because of the assumptions that utility functions are quasi-linear and valuations as well as prices are non-negative integers. The iterative auction rule defined in this paper is, however, defined on a general preference domain and its functionality is not restricted to utility functions and price spaces with the above properties. For this reason, more sophisticated techniques are needed. For example, Morimoto and Serizawa (2015) study the case with continuous price paths and non-quasi-linear preferences, and Section 5 describes how prices discretely can be increased along the price path for general preferences satisfying a monotonicity and a continuity assumption.

A final remark is related to the difference in revealed information about preferences in the above dynamic mechanism compared to a static (direct) mechanism. One way of classifying this difference is to look at the exact information that an agent has to reveal under the two different formats. In the direct mechanism, agent 1 has to reveal complete information about preferences, i.e., that $(v_{10}, v_{11}, v_{12}, v_{13}) = (0, 13, 15, 1)$. In the dynamic mechanism, on the other hand, agent 1 only reveals that $v_{12} > v_{11} > v_{13}$ and $v_{12} - v_{11} = 2$. An alternative estimate of the degree of preference revelation is to count the number of alternatives that an agent has to rank. In this case, the direct mechanism requires an agent to rank exactly $1 \times 4 \times 8 \times 8 = 256$ alternatives (one for each possible price vector in the price space) whereas the dynamic mechanism only requires an agent to rank 11 alternatives (one for each price vector in the sequence). Section 5 defines a measure on the upper bound of revealed preference information. In the above example, this upper bound is given by 15 meaning that, in worst case, an agent has to rank houses at 15 out of the 256 possible price vectors in the price space.

4 The English Price Sequence

This section defines and analyses a finite sequence of increasing price vectors called the English Price Sequence (EPS, henceforth). However, to define this sequence, the concepts of over-demanded sets, temporary states, and price regimes need to be introduced.

Let $p \in \mathbb{P}$ be a price vector and $D_a(p) = \{h \in H \cup \{0\} : (h, p) R_a(h', p) \text{ for all } h' \in H \cup \{0\}\}$ the demand set of agent $a \in A$. A set $H' \subset H$ of houses is *over-demanded* at a price vector $p \in \mathbb{P}$ if:

$$|H'| < |\{a \in A : D_a(p) \subseteq H'\}|.$$

A set $H' \subset H$ is a *minimal over-demanded* (MOD, henceforth) set if H' is over-demanded and there is no over-demanded set $H'' \subset H'$ ($H'' \neq H'$).

For any $S \subset H$, the *reduced demand* is defined as $D_a(p) \cap S$. A set $H' \subset H$ is *over-demanded with respect to reduced demand* if:

$$|H'| < |\{a \in A : D_a(p) \cap S \subseteq H'\}|.$$

The definition of a minimal over-demanded set of houses with respect to reduced demand is almost identical to the above definition of a minimal over-demanded set, and the only difference is that reduced demand sets are considered instead of demand sets.

Definition 7. For a given economy \mathcal{E} , a sequence $(H_j)_{j=1}^J$ is a *partition of minimal over-demanded sets of houses* (PMOD, henceforth), at a price vector $p \in \mathbb{P}$, if $H \cup \{0\} = \cup_{j=1}^J H_j$ is a partition of H such that:

- (i) for all $j < J$, H_j is a minimal over-demanded set in $S_j = H \setminus \cup_{i=1}^{j-1} H_i$ with respect to the reduced demand $D_a(p) \cap S_j$, and;
- (ii) there is no minimal over-demanded set in H_J .

The over-demanded part of H is defined as $H_{od} = \cup_{j=1}^{J-1} H_j$.

Note that if $J = 1$, then $H_{od} = \emptyset$. Furthermore, a PMOD can always be found recursively by finding an arbitrary MOD set H_1 in H , by finding an arbitrary MOD set H_2 in $H \setminus H_1$ with respect to the reduced demand $D_a(p) \cap (H \setminus H_1)$, and so on. Because this sequence terminates when it is not possible to find any more MOD sets and because H is finite, this sequence must be finite.

Example 1. Consider the example from Section 3 and the price vector $p^1 = (0, 0, 0)$. It is clear that the set of houses $\{1\}$ is over-demanded since agents 2 and 3 only demand house 1 at these prices, i.e., $D_2(p^1) = D_3(p^1) = \{1\}$. It is also a minimal over-demanded set since the set $\{1\}$ is a singleton, i.e., no subset of it can be over-demanded by definition. Note also that the set $\{1\}$ is the only minimal over-demanded set at prices p^1 , i.e., the PMOD is given by $H = H_1 \cup H_2$ where $H_1 = \{1\}$ and $H_2 = \{2, 3\}$. Moreover, $H_{od} = H_1$. \square

Consider now a given state in an economy with strictly more agents than houses (i.e., $|A| > |H|$) where the set containing all houses H is minimal over-demanded and, in addition, all agents are assigned some house from their demand sets and each house is assigned to some agent. In such situation, a fixed but arbitrary agent can be removed from the economy and it will still be possible to assign all remaining agents some house from their demand sets and each house can still be assigned to some remaining agent.

Proposition 1. Let \mathcal{E} be an economy and $x = (\mu, p)$ a state in \mathcal{E} . Suppose that $|A| > |H|$, H is a minimal over-demanded set at price vector $p \in \mathbb{P}$, and $\mu : A \rightarrow H$ is an assignment such that $\mu_a \in D_a(p)$ for all $a \in A$ and $H = \{h \in H : h = \mu_a \text{ for some } a \in A\}$. Consider now a fixed but arbitrary agent $a' \in A$. Then there is an assignment $\mu' : A \setminus \{a'\} \rightarrow H$ such that $\mu'_a \in D_a(p)$ for all $a \in A \setminus \{a'\}$ and $H = \{h \in H : h = \mu'_a \text{ for some } a \in A \setminus \{a'\}\}$.

Let \mathcal{E} be an economy and $x = (\mu, p)$ a state in \mathcal{E} . Given \mathcal{E} and x , a *subeconomy*, \mathcal{E}^x , contains the houses in $H^p = \{h \in H : p_h < \bar{p}_h\} \cup \{0\}$ and the agents in $A^x = \{a \in A : \mu_a \in H^p\}$. Note also that the preferences and the priority-order are the same in \mathcal{E}^x and \mathcal{E} , while the demand set for each agent $a \in A^x$ in the subeconomy \mathcal{E}^x is given by the *constrained demand* defined as $D_a^x(p) = \{h \in H^p : (h, p) R_a(h', p) \text{ for all } h' \in H^p\}$.

Example 2. Consider the example from Section 3, the price vector $p^9 = (3, 5, 0)$, and note that the price of house 1 is the only price that equals its upper price bound. This also means that $H^{p^9} = \{2, 3\}$. Note next that, at these prices, all three agents demand house 1, but because agent

1 is the agent with the highest priority (i.e., $\pi_1 < \pi_2 < \pi_3$) it follows that agent 1 is assigned house 1. Hence, the subeconomy is defined by the sets $H^{p^9} = \{2, 3\}$ and $A^{x^9} = \{2, 3\}$. In this subeconomy, the constrained demand sets for the agents in A^{x^9} are given by $D_2^{x^9}(p^9) = D_3^{x^9}(p^9) = \{2\}$. \square

In the subeconomy \mathcal{E}^x , let H_{od}^p be the over-demanded part of H^p as defined in Definition 7 but now with respect to constrained demand. To define a temporary state in \mathcal{E} , a stronger notion of a priority respecting state needs to be introduced.

Definition 8. For any economy \mathcal{E} , a state $x = (\mu, p)$ in \mathcal{E} is *strongly priority respecting* if:

- (i) the state x is priority respecting,
- (ii) $\mu_a = \mu_{a'}$ and $\mu_a \in H \setminus H^p$ only if $a' = a$, and;
- (iii) there are no two distinct agents a and a' such that $\mu_a \in H \setminus H^p$, $\mu_{a'} \in H_{od}^p$, $x_{a'} I_{a'} x_a$, and $\pi_{a'} < \pi_a$.

The second condition of the definition essentially states that the assignment μ must be feasible for the set of houses of which the prices equal the upper price bound, whereas the last condition states that the houses with a price equal to the upper bound should be allocated to the agents that demand them and according to the priority-order (of course, this need not be the case at an RPE, as any RPE state must be constrained efficient, but it will be convenient to work with strongly priority respecting states before reaching the end state of the EPS to be introduced below).

Definition 9. For any economy \mathcal{E} , a state $x = (\mu, p)$ in \mathcal{E} is a *temporary state* if:

- (i) the state x is strongly priority respecting,
- (ii) there is no improvement of μ with a strongly priority respecting assignment μ' ,
- (iii) for all $h \in H$, $p_h = \underline{p}_h$ if $h \in \mu_0$, and;
- (iv) for all $h \notin H_{od}^p$, $\mu_a = \mu_{a'} = h$ (for $a \neq a'$) only if $h \in \mu_0$.

Since the assignment μ , at a temporary state $x = (\mu, p)$, not necessarily is feasible, each agent that is assigned a house in H^p will also be assigned his most preferred house in H^p by condition (ii) of Definition 9. Note also that temporary states are not necessarily constrained efficient but possess a weaker form of efficiency, and that a temporary state exists for each $p \in \mathbb{P}$ and each profile $R \in \mathcal{R}$. Note, finally, that an assignment μ can be chosen such that condition (iv) of Definition 9 is satisfied whenever conditions (i)–(iii) are satisfied. This is a direct consequence of Proposition 1.

Definition 10. Let \mathcal{E} be an economy and $x = (\mu, p)$ a state in \mathcal{E} . Let \mathcal{E}^x be the corresponding subeconomy with H^p and A^x being the sets of houses and agents, respectively, and let H_{od}^p be the over-demanded part of H^p in \mathcal{E}^x . A set of prices $\Pi^x \subset \mathbb{P}$, called a *price regime*, is then defined by $p' \in \Pi^x$ if and only if there is a temporary state $x' = (\mu', p')$ such that:

- (i) $p_h \leq p'_h < \bar{p}_h$ if $h \in H_{od}^p$ and $p'_h = p_h$ if $h \notin H_{od}^p$,
- (ii) $\mu'_a = \mu_a$ if $\mu_a \notin H_{od}^p$,
- (iii) $\mu'_a \in H^p$ if and only if $\mu_a \in H^p$, and;
- (iv) $H_{od}^{p'} = H_{od}^p$, where $H_{od}^{p'}$ is the over-demanded part of H^p at the price vector p' .

Hence, the prices in the vector $p' \in \Pi^x$ has (weakly) increased only for the over-demanded houses compared to the price vector p . Moreover, there is a temporary state $x' = (\mu', p')$ such that the over-demanded set of houses is identical at states x' and x , and the set of agents that are assigned an over-demanded house is the same at states x' and x .

Example 3. Consider the example from Section 3 and the state $x^9 = (\mu^9, p^9)$ where $\mu^9 = (1, 2, 2)$ and $p^9 = (3, 5, 0)$. From Example 2, it is known that the subeconomy is defined by $H^{p^9} = \{2, 3\}$ and $A^{x^9} = \{2, 3\}$. In this subeconomy, house 2 is minimal over-demanded and, consequently, $H_{od}^9 = \{2\}$. The price regime $\Pi^{x^9} \subset \mathbb{P}$ is then defined by:

$$\Pi^{x^9} = \{p' \in \mathbb{P} : p'_1 = 3, 5 \leq p'_2 < 7, \text{ and } p'_3 = 0\}.$$

This follows because all conditions of Definition 10 are satisfied for any price vector $p' \in \Pi^{x^9}$ at the temporary state $x' = (\mu', p')$ whenever $\mu' = (1, 2, 2)$. \square

Let now $x = (\mu, p)$ be a temporary state and let $\xi(x) = \sup \Pi^x$. Note that $\xi(x)$ need not be a singleton, i.e., it may be a set (see Example 4, below).

Definition 11. A sequence of price vectors $(p^t)_{t=1}^T$ constitutes an *English Price Sequence* (EPS, henceforth) if there is a supporting sequence $(x^t)_{t=1}^T$ of temporary states, with $x^t = (\mu^t, p^t)$, such that $p^{t+1} \in \xi(x^t)$. The starting point is $p^1 = \underline{p}$. The English Price Sequence terminates at Step T if $p^T \neq p^{T-1}$ and $p^{T+1} = p^T$.

Note that there may be several English Price Sequences consistent with a specific profile in \mathcal{R} . This insight is formally illustrated in the following example.

Example 4. Let $A = \{1, 2, 3, 4, 5\}$ and $H = \{1, 2, 3, 4, 5\}$ be the sets of agents and houses, respectively, and let $\pi_a = a$ for all $a \in A$. Suppose further that prices are non-negative integers and that the lower and upper price bounds are given by $\underline{p} = (0, 0, 0, 0, 0)$ and $\bar{p} = (2, 2, 2, 2, 2)$,

respectively. Preferences over bundles (h, p) are represented by a quasi-linear utility function $u_{ah}(p) = v_{ah} - p_h$ where $v_{a0} = 0$, and:

$$(v_{ah}) = \begin{pmatrix} 10 & 10 & 2 & 2 & 2 \\ 10 & 2 & 2 & 5 & 2 \\ 2 & 10 & 2 & 2 & 5 \\ 2 & 2 & 2 & 10 & 2 \\ 2 & 2 & 2 & 2 & 10 \end{pmatrix}.$$

The starting point of the EPS is the lower price bound $p^1 = \underline{p} = (0, 0, 0, 0, 0)$. To find a supporting temporary state $x^1 = (p^1, \mu^1)$, note that all agents can be assigned their most preferred house at prices p^1 since $H^{p^1} = H$. Hence, it must be the case that $\mu^1 = (1, 1, 2, 4, 5)$ or $\tilde{\mu}^1 = (2, 1, 2, 4, 5)$. Independently of which assignment that is chosen, the set $\{1, 2\}$ is the unique minimal over-demanded set and, consequently, $p^2 = \xi(x^1) = (2, 2, 0, 0, 0)$. Now, the price of houses 1 and 2 have reached their corresponding upper price bounds and all agents have the same demand sets at prices p^2 as they have at prices p^1 . Because agent 1 is the agent with the highest priority, agent 1 will be assigned either house 1 or house 2, and the subeconomy for the next step of the EPS is dependent on this choice. More specifically, if agent 1 is assigned house 1, then agent 3 is assigned house 2, and if agent 1 is assigned house 2, then agent 2 is assigned house 1. In the former case, the subeconomy is defined by $H^{p^2} = \{3, 4, 5\}$ and $A^{x^2} = \{2, 4, 5\}$, and in the latter case the subeconomy is defined by $H^{p^2} = \{3, 4, 5\}$ and $A^{x^2} = \{3, 4, 5\}$. This choice will also give rise to two different EPS. These two sequences will next be analyzed separately from each other and the main insights are summarized in Tables 2 and 3.

English Price Sequence 1. Let the subeconomy be defined by $H^{p^2} = \{3, 4, 5\}$ and $A^{x^2} = \{2, 4, 5\}$. In this subeconomy, the constrained demand sets for the agents in A^{x^2} are given by $D_2^{x^2}(p^2) = D_4^{x^2}(p^2) = \{4\}$ and $D_5^{x^2}(p^2) = \{5\}$. Hence, $\mu^2 = (1, 4, 2, 4, 5)$ and the set $\{4\}$ is the unique minimal over-demanded set in the subeconomy. Consequently, $p^3 = \xi(x^2) = (2, 2, 0, 2, 0)$. Now, the price of house 4 has reached its upper price bound and house 4 is the only house in the constrained demand sets of agents 2 and 4. Because $\pi_2 < \pi_4$, house 4 is assigned to agent 2 and a new subeconomy defined by $H^{p^3} = \{3, 5\}$ and $A^{x^3} = \{4, 5\}$ must be considered. Because the constrained demand sets now are given by $D_4^{x^3}(p^3) = \{3, 5\}$ and $D_5^{x^3}(p^3) = \{5\}$, it is possible to assign each agent in the subeconomy distinct houses from their demand sets by setting $\mu^3 = (1, 4, 2, 3, 5)$. The vector p^3 is the end point price vector of this EPS. Hence, the EPS is described by (p^1, p^2, p^3) .

English Price Sequence 2. Let the subeconomy be defined by $H^{p^2} = \{3, 4, 5\}$ and $A^{x^2} = \{3, 4, 5\}$. By repeating the above arguments, it follows that $\mu^2 = (2, 1, 5, 4, 5)$ and, consequently, that $p^3 = \xi(x^2) = (2, 2, 0, 0, 2)$ and $\mu^3 = (2, 1, 5, 4, 3)$. The vector p^3 is the end point price vector of this EPS. \square

Table 2: Illustration of English Price Sequence 1 in Example 4.

Step j	p^j	μ^j	$\text{MOD}(p^j)$
1	(0, 0, 0, 0, 0)	μ^1 or $\tilde{\mu}^1$	{1, 2}
2	(2, 2, 0, 0, 0)	(1, 4, 2, 4, 5)	{4}
3	(2, 2, 0, 2, 0)	(1, 4, 2, 3, 5)	\emptyset

Table 3: Illustration of English Price Sequence 2 in Example 4.

Step j	p^j	μ^j	$\text{MOD}(p^j)$
1	(0, 0, 0, 0, 0)	μ^1 or $\tilde{\mu}^1$	{1, 2}
2	(2, 2, 0, 0, 0)	(2, 1, 5, 4, 5)	{5}
3	(2, 2, 0, 0, 2)	(2, 1, 5, 4, 3)	\emptyset

One can verify that both end point price vectors identified in Example 4 are minimum RPE price vectors. The fact that a minimum RPE price vector needs not be unique for all profiles in \mathcal{R} has been established earlier in the literature as reported in the following proposition.

Proposition 2. (Andersson and Svensson, 2014) For any economy \mathcal{E} and any $R \in \mathcal{R}$, there is a minimal, but not necessarily unique, Rationing Price Equilibrium price vector $p^* \in \Pi$.

To obtain uniqueness of a minimal RPE price vector, some profiles have to be excluded from the domain \mathcal{R} . The domain restriction that will be employed in this paper is a subset $\tilde{\mathcal{R}} \subset \mathcal{R}$, where $\tilde{\mathcal{R}}$ is the set of profiles such that no two houses are “connected by indifference” at any price vector $p \in \mathbb{P}$.

Definition 12. For a given economy \mathcal{E} , two houses, h_1 and h_t , in $H \cup \{0\}$, are *connected by indifference* if there is a price vector $p \in \mathbb{P}$, a sequence of distinct agents (a_1, \dots, a_{t-1}) , and a sequence of distinct houses (h_1, \dots, h_t) such that:

- (i) $p_{h_1} = \bar{p}_{h_1}$, and $p_{h_t} = \underline{p}_{h_t}$ or $p_{h_t} = \bar{p}_{h_t}$, and;
- (ii) $(h_j, p)I_{a_j}(h_{j+1}, p)$ for $1 \leq j \leq t - 1$.

Note that the domain $\tilde{\mathcal{R}}$, in a mathematical meaning, contains almost all profiles in \mathcal{R} (see the Appendix in Andersson and Svensson, 2016).

Proposition 3. (Andersson and Svensson, 2014) For any economy \mathcal{E} where $R \in \tilde{\mathcal{R}}$, a minimal Rationing Price Equilibrium price vector $p^* \in \Pi$ is unique.

The next three results contain some fundamental properties of an EPS. The first states that an EPS is finite for any profile $R \in \mathcal{R}$. The other two results are valid on the reduced domain $\tilde{\mathcal{R}}$ and demonstrate that the end state of an EPS always can be chosen to be an RPE and that the end point price vector p^T of an EPS is the minimal RPE price vector.

Proposition 4. For a given economy \mathcal{E} where $R \in \mathcal{R}$, if $(p^t)_{t=1}^T$ is an English Price Sequence, $(p^t)_{t=1}^T$ is weakly increasing and T is finite.

Proposition 5. For a given economy \mathcal{E} where $R \in \tilde{\mathcal{R}}$, if $(p^t)_{t=1}^T$ is an English Price Sequence and $(x^t)_{t=1}^T$ is a supporting sequence of temporary states, then $x^T = (\mu^T, p^T)$ can be chosen to be a rationing price equilibrium.

Theorem 1. For a given economy \mathcal{E} where $R \in \tilde{\mathcal{R}}$, if $(p^t)_{t=1}^T$ is an English Price Sequence, then $p^T = p^*$ where p^* is the minimal RPE price vector.

Proposition 4 shows that there is only a finite number of steps in an EPS. Theorem 1 demonstrates that, on the restricted domain $\tilde{\mathcal{R}}$, the endpoint of the EPS is a minimal RPE price vector. An extended version of an EPS can be seen as the outcome of an auction rule which implements the minimal price vector. This auction rule is called the Iterative English Auction Rule (see Section 5) and it is a combination of a dynamic procedure, the different steps $t = 1, \dots, T$, and a number of direct mechanisms, the steps from t to $t + 1$. In the dynamic procedure, agents reveal only partial preference information. In fact, as will be demonstrated in Section 5, the extended EPS can be adjusted so that arbitrary little preference information is revealed by introducing more steps in the extended EPS (i.e., a larger T). For certain classes of preferences, e.g., quasi-linear preferences, it is also possible to define a dynamic procedure between the various steps and obtain a complete iterative auction rule. The alternative is to let the agents report their complete ranking in (arbitrarily small) price regimes which are iteratively defined and then, given p^t , calculate p^{t+1} . In principle, the latter alternative is the same as the one used in a dynamic procedure based on quasi-linear preferences since quasi-linear preferences are completely revealed by a finite number of parameters.

Remark 1. Note that an EPS is unique on the restricted domain $\tilde{\mathcal{R}}$ and this need not be the case on the full preference domain \mathcal{R} as previously illustrated in Example 4. It is then natural to investigate if any minimal RPE price vector is achievable via an EPS for the profiles in \mathcal{R} where a minimal RPE price vector not is unique. We conjecture that this is the case even if we have been unable to find a formal proof. \square

In the remaining part of this section, the above mentioned extended version of an EPS is considered. This extension is based on “small price regimes” and within each small price regime some prices are constant while some prices are discretely increased at most by a factor $\delta > 0$. The smaller δ is, the less preference information has to be revealed to the auctioneer (as will be explained in Section 5). To formalize this, let $\delta > 0$ be any positive real number and:

$$\begin{aligned} \delta\mathbb{N}^m &= \{\alpha \in \mathbb{R}_+^m : \alpha_j = \delta k_j \text{ for some } k_j \in \mathbb{N}\}, \\ S_\alpha &= \times_{h \in H} [\alpha_h, \alpha_h + \delta) \text{ for each } \alpha \in \delta\mathbb{N}^m. \end{aligned}$$

For each $\delta > 0$, Definition 10 can now be reformulated in the following way.

Definition 13. Let \mathcal{E} be an economy, $x = (\mu, p)$ a state in \mathcal{E} , and $p \in S_\alpha$ (α unique). Let \mathcal{E}^x be the corresponding subeconomy with H^p and A^x being the sets of houses and agents respectively, and let H_{od}^p be the over-demanded part of H^p in \mathcal{E}^x . A set of prices $\Pi^x \subset \mathbb{P}$, called a *small price regime*, is then defined by $p' \in \Pi^x$ if and only if there is a temporary state $x' = (\mu', p')$ such that:

- (i) $\alpha_h \leq p_h \leq p'_h < \min[\alpha_h + \delta, \bar{p}_h]$ if $h \in H_{od}^p$ and $p'_h = p_h$ if $h \notin H_{od}^p$,
- (ii) $\mu'_a = \mu_a$ if $\mu_a \notin H_{od}^p$,
- (iii) $\mu'_a \in H^p$ if and only if $\mu_a \in H^p$, and;
- (iv) $H_{od}^{p'} = H_{od}^p$, where $H_{od}^{p'}$ is the over-demanded part of H^p at the price vector p' .

Note that the only difference between the above definition and Definition 10 is that prices can increase at most by δ in Definition 13.

Definition 14. A sequence of price vectors $(p^t)_{t=1}^T$ constitutes an *Extended English Price Sequence* (EEPS, henceforth) if there is a supporting sequence $(x^t)_{t=1}^T$ of temporary states, with $x^t = (\mu^t, p^t)$, such that $p^{t+1} \in \xi(x^t)$. The starting point is $p^1 = \underline{p}$. The EEPS terminates at Step T if $p^T \neq p^{T-1}$ and $p^{T+1} = p^T$.

The proof of the following theorem is omitted as it is identical to the proof of Theorem 1.

Theorem 1'. For a given economy \mathcal{E} where $R \in \tilde{\mathcal{R}}$, if $(p^t)_{t=1}^T$ is an EEPS, then $p^T = p^*$ where p^* is the minimal RPE price vector.

5 The Iterative English Auction Rule

Any iterative auction rule requires agents to reveal demand information sequentially. This information is then used to increase prices discretely or continuously. As will be explained in this section, the (Extended) English Price Sequence can be seen as the outcome of an iterative auction rule called the Iterative English Auction Rule. The auction rule is a mix of the different steps of the EEPS, $t = 1, \dots, T$, and a number of direct mechanisms used to identify the prices between steps t and $t + 1$.

A first observation is that when agents reveal their demand information sequentially, there is no guarantee that the revealed preference information is consistent with some preference relation in \mathcal{R}_a . Sufficient conditions for this will be provided in Section 5.1, but for now it is assumed that reported demand information can be derived from some preference relation in \mathcal{R}_a . To formalize this, it will initially be assumed that preferences are replaced by demand sets.

Consider now an economy \mathcal{E} where preferences are replaced by demand sets, i.e., for each price vector $p \in \mathbb{P}$ and each agent $a \in A$, there is a demand set $d_a^p \subset H \cup \{0\}$.⁷ If a demand set

⁷The notation d_a^p for reported demand sets is, in the remaining part of the paper, used instead of the previously introduced notation $D_a(p)$ since d_a^p , a priori, is not derived from some preference ordering in \mathcal{R}_a .

d_a^p is derived from a profile $R \in \mathcal{R}$ and $p \in \mathbb{P}$, then points 1 – 3., below, called *demand ranking*, are satisfied, i.e., there is a complete ranking r_a^p of the set d_a^p such that:

1. if $h, h' \in d_a^p$ and $h \notin H^p$ and $h' \in H^p$, then $hr_a^p h'$,
2. if $h, h' \in d_a^p$ and $h, h' \in H^p$, then $hr_a^p h'$ and $h'r_a^p h$ (indifference), and;
3. if $h \in d_a^p$ and $h' \notin d_a^p$, then not $h'r_a^p h$.

The demand ranking condition is implied by preferences $R_a \in \mathcal{R}_a$ and the condition is also sufficient to construct an EEPS. Given a complete ranking r_a^p , a temporary state $x = (\mu, p)$ can be constructed and the set H_{od}^p is well-defined. Note also that not all information related to demand is needed to construct a small price regime Π^x . In fact, only the demands of the agents a with $\mu_a \in H_{od}^p$ in the domain $\alpha_h \leq p_h \leq p'_h < \min[\alpha_h + \delta, \bar{p}_h]$, $h \in H_{od}^p$, and α chosen so that $p \in S_\alpha$ are needed.

Suppose now that the demand sets are consistent with rational preferences $R \in \mathcal{R}$, i.e., a reported demand set d_a^p can be derived from some preference ordering $R_a \in \mathcal{R}_a$. Then the EEPS can be seen as the outcome of an auction rule defined as follows.

The Iterative English Auction Rule. Initialize the price vector to $p^1 = \underline{p}$. For each Step $t := 1, \dots, T$:

1. Each agent $a \in A$ reports his demand set $d_a^{p^t}$ at prices p^t .
2. Calculate a supporting temporary state $x^t = (\mu^t, p^t)$ and define a subeconomy \mathcal{E}^{x^t} .
3. All agents in $A_{od}^{p^t} = \{a \in A : \mu_a^t \in H_{od}^{p^t}\}$ report their demand at prices p for $\alpha_h \leq p_h \leq p'_h < \min[\alpha_h + \delta, \bar{p}_h]$ where $p^t \in S_\alpha$.
4. Define a small price regime Π^{x^t} and calculate $p^{t+1} \in \xi(x^t)$.
5. If $p^{t+1} = p^t$, stop. Otherwise, set $t := t + 1$ and continue. □

The Iterative English Auction Rule will produce an EEPS $(p^t)_{t=1}^T$ where, for all t and all $h \in H$, it holds that $p_h^t \leq p_h^{t+1} \leq p_h^t + \delta$. An EEPS in itself contains a finite sequence $(p^t)_{t=1}^T$ of small price adjustments. Within a subeconomy, the price vectors p^t and p^{t+1} can be connected by a continuous path if demand is continuous. An explicit rule to obtain a continuous price path with general preferences is provided by, e.g., Morimoto and Serizawa (2015).

In a direct mechanism, agents report preferences over the entire price space \mathbb{P} , while only demand in a subset of the price space $\mathbb{P}_\delta \subset \mathbb{P}$ are reported when the Iterative English Auction Rule is adopted. To investigate the degree of preference revelation in the dynamic process in more detail, a measure (e.g., the Lebesgue measure) of \mathbb{P}_δ , denoted by $\mathbf{m}\mathbb{P}_\delta$, is next defined and it is demonstrated that this measure converges to zero when $\delta \rightarrow 0$.

The measure of S_α is defined to be $\mathbf{m}S_\alpha = \delta^m$, where m is the cardinality of the set H . Moreover, $\mathbb{P}_\delta \subset \cup_{j=1}^J S_{\alpha^j}$, where $\alpha^j \in \delta\mathbb{N}^m$ and $\alpha^j \leq \alpha^{j+1}$ for all j , and $\mathbf{m}\mathbb{P}_\delta \leq \mathbf{m}(\cup_{j=1}^J S_{\alpha^j}) = J\delta^m$. However, J depends on δ so we have to find an upper bound on J that does not depend on δ . For this purpose, for $h \in H$ let $l_h - 1$ be the integer part of $(\bar{p}_h - \underline{p}_h)/\delta$. Then $J < \sum_{h \in H} l_h$. Hence:

$$\mathbf{m}\mathbb{P}_\delta \leq J\delta^m < \sum_{h \in H} l_h \delta^m \leq \sum_{h \in H} ((\bar{p}_h - \underline{p}_h)/\delta + 1)\delta^m.$$

The expression to the far right in the above condition defines an upper bound on the measure $\mathbf{m}\mathbb{P}_\delta$. Clearly, $\mathbf{m}\mathbb{P}_\delta \rightarrow 0$ as $\delta \rightarrow 0$ (if $m > 1$), i.e., the smaller δ is, the less preference information has to be revealed to the auctioneer.

5.1 Incentives

In a given economy \mathcal{E} and for a given profile $R \in \mathcal{R}$, the direct mechanism $f(R) = x$, where $x = (\mu, p)$ is a minimal RPE state, is strategy-proof on the domain $\tilde{\mathcal{R}} \subset \mathcal{R}$ (Andersson and Svensson, 2016, Theorem 1). If the reported demand sets d_a^p in the Iterative Auction Rule are derived from the true preference profile R , then the outcome is $f(R)$. Consequently, an agent a cannot manipulate the outcome by reporting any demand set d_a^p that is consistent with rational preferences $R_a \in \mathcal{R}_a$. This also means that bidding truthfully is an ex post Nash equilibrium. Next, sufficient conditions on reported demand sets d_a^p such that d_a^p can be derived from some profile $R_a \in \mathcal{R}_a$ are provided.⁸ For this purpose, consider a continuous increasing price path $p^t \in \mathbb{P}$ (where $t \in [0, T] \subset \mathbb{R}_+$ and $T < \infty$), and the following conditions on the reported demand sets d_a^p :

- *Monotonicity*: If $h, h' \in d_a^{p^t}$, $h \in d_a^{p^{t'}}$, $p_h^t = p_h^{t'}$, $p_{h'}^{t'} > p_{h'}^t$, then $h' \notin d_a^{p^{t'}}$.
- *Continuity*: For each $h \in H$, the set $\{t \in [0, T] : h \in d_a^{p^t}\}$ is closed.
- *Demand ranking*: Existence of a complete ranking r_a^p of the demand set d_a^p .
- *Non-emptiness*: For each agent $a \in A$ and any prices p^t , $d_a^{p^t} \cap (H^{p^t} \cup \{0\}) \neq \emptyset$.

Theorem 2. If the reported demand sets $d_a^{p^t}$ satisfy Monotonicity, Continuity, Demand ranking and Non-emptiness for all t , then the demand sets $d_a^{p^t}$ can be derived from some preference relation $R_a \in \mathcal{R}_a$.

⁸Such conditions are sometimes referred to as “activity rules” and they are commonly adopted to prove that truthful bidding is an ex post Nash equilibrium, see, e.g., Harsha et al. (2010), Mishra and Parkes (2007, 2009) or de Vries et al. (2007).

5.2 Relation to Previous Literature

It is next demonstrated that two well-known mechanisms from the literature can be regarded as special cases of the Iterative English Auction Rule, but that none of them generally can solve the house allocation problem with price restrictions. The first mechanism is a fixed price mechanism (Svensson, 1994) and the second is a standard English auction with no upper price bound (Demange et al., 1986).

Consider first the bounding case when the upper and lower price bounds coincide, i.e., when $\underline{p} = \bar{p}$ and, consequently, when $\mathbb{P} = \{\underline{p}\}$. To avoid to introduce too much notation, it is, in this bounding case, assumed that all agents have strict preferences at prices \underline{p} , i.e., $(h, \underline{p})P_a(h', \underline{p})$ or $(h', \underline{p})P_a(h, \underline{p})$ for all distinct houses $h, h' \in H$ and all agents $a \in A$.⁹ As will be explained in the below, the main conclusion holds also for the more general case with weak preferences. The assumption of strict preferences means that the restricted demand set $D_a(\underline{p}) \cap H'$ contains exactly one house for each (nonempty) subset $H' \subseteq H$. Consider now the Iterative English Auction Rule, and recall that the starting point is $p^1 = \underline{p}$. The first objective is to find a temporary state $x^1 = (\mu^1, p^1)$ such that $p^2 = \xi(x^1)$. Because $\mathbb{P} = \{\underline{p}\}$, it is clear that $p^1 = p^2$. Hence, the rule terminates at Step 2 and it, therefore, only remains to find the temporary state $x^1 = (\mu^1, p^1) = (\mu^1, \underline{p})$. Because $\mathbb{P} = \{\underline{p}\}$, it follows that $H^p = \emptyset$, and, consequently, that a strongly priority respecting state is a feasible priority respecting state. But then, any feasible priority respecting state x that is not Pareto dominated by some other feasible priority respecting state x' is a temporary state by Definition 9. Such state can be identified using the following mechanism from Svensson (1994).

Queue Allocation Mechanism. Consider a given profile $R \in \mathcal{R}$, and let prices p be fixed and equal to \underline{p} and set $H^1 = H$. For each Step $t = 1, \dots, n$:

Step t. Consider the agent with $\pi_t = t$. Let $\mu_t = D_a(\bar{p}) \cap H^t$, define $H^{t+1} = H^t \setminus \{\mu_t\}$, and go to Step $t + 1$.

The proof that the outcome of the Queue Allocation Mechanism is Pareto efficient and a feasible priority respecting state can be found in Svensson (1994, Theorem 1).¹⁰ In fact, Svensson (1994) proved that this indeed also is true when preferences are weak by introducing choice sets for the agents in the various steps of the mechanism. Hence, in the bounding case when prices are fixed, the Iterative English Auction Rule recommends the same outcome as the Queue Allocation Mechanism. It is clear that the Queue Allocation Mechanism generally cannot be used to identify a minimal RPE state since it, by definition, only works in economies with no prices or fixed prices (i.e., when $\underline{p} = \bar{p}$). In this sense, the Queue Allocation Mechanism needs only to solve the problem of allocating the houses among the agents and it cannot handle price increases.

⁹This is a standard assumption when monetary transfers not are feasible, see, e.g., Abdulkadiroğlu and Sönmez (1998, 1999, 2003), Balinski and Sönmez (1999), Gale and Shapley (1962), or Roth (1982).

¹⁰In Svensson (1994), a feasible priority respecting state is called a *weakly fair allocation*.

Consider now the bounding case when $\bar{p}_h = \infty$ for all $h \in H$. In this case, $H^p = H$ and, consequently, $H \setminus H^p = \emptyset$. Two important implications follow from this assumption. First, for a given economy \mathcal{E} and a given state x , the subeconomy \mathcal{E}^x is always identical to \mathcal{E} . Second, a priority respecting state also is strongly priority respecting. Given these two observations, consider now Step t of the Iterative English Auction Rule and the supporting temporary state x^t . Because the requirement on x^t is that there should be no (strongly) priority respecting state x' that Pareto dominates x , it is clear that all agents in A must temporarily be assigned one of their most preferred houses in $H^{p^t} = H$ in each Step t . In this sense, the process of temporarily assigning agents to houses in each step of the rule is unnecessary, and the only relevant part of the rule is to identify $p^{t+1} \in \xi(x^t)$ based on the set $H_{od}^{p^t}$. The special case when $H_{od}^{p^t}$ is given by an arbitrary minimal over-demanded set at prices p^t and preferences are quasi-linear was considered by Demange et al. (1986) in their Exact Auction Mechanism.

Exact Auction Mechanism. Initialize the price vector to $p^1 := \underline{p}$. For each Step $t = 1, \dots, T$:

Step t . Collect the demand set $D_a(p^t)$ of each agent $a \in A$. If there is no over-demanded set at p^t , terminate the auction. Else, select a minimal over-demanded set $H_{od}^{p^t} \subseteq H$, and increase the prices of all houses in $H_{od}^{p^t}$ uniformly until the set $H_{od}^{p^t}$ no longer is over-demanded while keeping the prices of the houses in $H \setminus H_{od}^{p^t}$ unchanged. Denote the generated price vector by p^{t+1} and go to Step $t + 1$.

That the Exact Auction Mechanism generally cannot be used to identify a minimal RPE state is easy to see. Imagine, for example, an economy with only two agents and two houses (called 1 and 2). Suppose further that both agents demand only house 1 at all prices in \mathbb{P} . In this case, the Exact Auction Mechanism will increase the price for house 1 until the upper price bound, \bar{p}_1 , is reached but will then be unable to solve the remaining allocation problem because it has no built-in rationing system, i.e., it relies on market prices to solve the allocation problem. Even if one would add a rationing system to the Exact Auction Mechanism, it is not clear how the mechanism would work as the economy \mathcal{E} is fixed in all steps of the mechanism. In, e.g., the example from Section 3, it is not clear how to continue after Step 4 since it is impossible to increase the prices of the only minimal over-demanded set (i.e., the set $\{1\}$). A more general remark is that the Exact Auction Mechanism needs only to solve the problem of identifying an equilibrium price vector in each step of the mechanism as the price vector identified in the final step of the mechanism can be used as a tool to allocate the houses among the agents. The Iterative English Auction Rule, on the other hand, needs to simultaneously consider the allocation of the houses as well as the identification of a price vector in each step of the rule. As a consequence, the Exact Auction Mechanism cannot generally be used to identify a minimal RPE state.

Appendix: Proofs for Specific Economies

The key result in this Appendix is Lemma 4. This lemma will be used in the succeeding Appendix to prove the main results of the paper. To prove this lemma, some additional lemmas are needed. All these lemmas are valid in some specific economies.

Consider first an economy, called \mathcal{E}^0 , with strictly more agents than houses (i.e., $n > m$) and without any null houses. The set of feasible price vectors in \mathcal{E}^0 is $\mathbb{P} = \{\mathbb{R}_+^m : \underline{p}_h \leq p_h \leq \bar{p}_h \text{ for all } h \in H\}$. Given an economy \mathcal{E}^0 , a corresponding *extended economy* \mathcal{E}^e is defined as a double replica economy in the following way:

- (i) Let $A^m = \{a_{ij} : a_{ij} = a_i \text{ for } j = 1, \dots, m \text{ and } a_i \in A\}$ be the agents in \mathcal{E}^e ,
- (ii) Let $H^n = \{h_{ij} : h_{ij} = h_i \text{ for } j = 1, \dots, n \text{ and } h_i \in H\}$ be the houses in \mathcal{E}^e ,
- (iii) A price vector p is feasible in \mathcal{E}^e if and only if $p \in \mathbb{P}^n$, where p_{ij} is the price of house $h_{ij} \in H^n$,
- (iv) An assignment $\mu : A^m \rightarrow H^n$ is feasible if and only if μ is bijective,
- (v) Preferences in \mathcal{E}^e are the same as in \mathcal{E}^0 , i.e., $R_{a_{ij}} = R_{a_i}$ for all $a_{ij} \in A^m$ and $a_i \in A$.

Hence, the extended economy \mathcal{E}^e is defined to be an economy where there are m copies of each agent in \mathcal{E}^0 and n copies of each house in \mathcal{E}^0 . As a result, the number of agents and houses coincides in \mathcal{E}^e and is equal to nm . An *equilibrium state* in \mathcal{E}^e is defined to be a feasible state (μ, p) where $x_a R_a(h, p)$ for all $a \in A^m$ and all $h \in H^n$, i.e., a feasible state where each agent is assigned a house from his demand set at the given prices p .

Lemma 1. For a given extended economy \mathcal{E}^e where $R \in \mathcal{R}$, let $x = (\mu, p)$ be an equilibrium state in \mathcal{E}^e . Then $p_{ij} = p_{ij'}$ for all j, j' .

Proof. Suppose that $p_{ij} < p_{ij'}$ for some i, j, j' . Since houses h_{ij} and $h_{ij'}$ are copies of the same house $h_i \in H$, by construction of \mathcal{E}^e , no agent will demand house $h_{ij'}$ at prices p . But then the assignment μ cannot be feasible since $|A^m| = |H^n|$ by construction, which contradicts that x is an equilibrium state. \square

A consequence of the above lemma is that one can change the notation of an equilibrium in \mathcal{E}^e . More explicitly, if the state $x = (\mu, p)$ is an equilibrium in \mathcal{E}^e and $p \in \mathbb{P}$, then p_i can be used to denote the common price of all identical houses $h_{ij} \in H^n$.

It is well-known that in an economy \mathcal{E}^0 with an equal number of agents and houses, a state $x = (\mu, p)$ is an equilibrium if and only if no subset $S \subset H$ is over-demanded (see, e.g., Hall, 1935). As will be demonstrated in the next lemma, a corresponding result holds for the extended economy \mathcal{E}^e . Let $S \subset H$, $p \in \mathbb{P}$, $A_S^m = \{a \in A^m : D_a(p) \subset S\}$, and $S^n = \times_1^n S$. Here, A_S^m

contains the agents in \mathcal{E}^e that only demand houses in S . Since houses in \mathcal{E}^e are copies of houses in \mathcal{E}^0 , the demand sets of the agents are identical in \mathcal{E}^e and \mathcal{E}^0 . Note also that the set $S^n \subset H^n$ is the product n times of the set $S \subset H$.

Lemma 2. For a given extended economy \mathcal{E}^e where $R \in \mathcal{R}$, a state $x = (\mu, p)$ in \mathcal{E}^e is an equilibrium if and only if $|A_S^m| \leq n|S|$ for all $S \subset H$.

Proof. From Hall (1935) it is known that a state $x = (\mu, p)$ in \mathcal{E}^0 is an equilibrium if and only if:

$$|A_S| \leq |S| \text{ for all } S \subset H. \quad (1)$$

By the construction of the replica economy \mathcal{E}^e and Lemma 1, it follows that if $h_i \in D_a(p)$ for some agent $a \in A$, then $h_{ij} \in D_a(p)$ for all $j = 1, \dots, n$. But then $D_a(p) \subset S$ for some $S \subset H$ implies that $D_a(p) \subset S^n$ for some $S^n \subset H^n$. It now follows that condition (1) is equivalent to $|A_{S^n}^m| \leq |S^n|$, i.e., equivalent to $|A_S^m| \leq n|S|$ for all $S \subset H$. \square

Lemma 3. Consider economy \mathcal{E}^0 where $|A| > |H|$ and H is minimal over-demanded at price vector $p \in \mathbb{P}$. Then there is an assignment $\mu : A^m \rightarrow H^n$ such that $x = (\mu, p)$ is an equilibrium in the extended economy \mathcal{E}^e .

Proof. Suppose that H is minimal over-demanded at price vector $p \in \mathbb{P}$, and let S be an arbitrary but proper subset of H . Because no proper subset S of H is over-demanded, by definition, it follows that $|A_S| \leq |S|$, and, consequently, that $n|A_S| \leq n|S|$. Because $D_a(p) \subset S$ for some $S \subset H$ implies that $D_a(p) \subset S^n$ for some $S^n \subset H^n$, it is clear that $|A_S^m| \leq n|A_S|$. Hence, $|A_S^m| \leq n|S|$. Because the set S is arbitrarily selected, it follows from Lemma 2 that there is an assignment μ such that $x = (\mu, p)$ is an equilibrium in the extended economy \mathcal{E}^e . \square

Lemma 4. For a given economy \mathcal{E}^0 , suppose that H is a minimal over-demanded set of houses at price vector $p \in \mathbb{P}$ with $\underline{p} < p < \bar{p}$. Then, in each neighborhood of p , there are price vectors $p', p'' \in \mathbb{P}$ such that $p'' < p < p'$ and $H = H_{od}$ at p'' as well as at p' .¹¹

Proof. Because H is a minimal over-demanded set, there is an assignment such that $x = (\mu, p)$ is an equilibrium in \mathcal{E}^e by Lemma 3. But then, by the Perturbation Lemma in Alkan et al. (1991), there are, in each neighborhood of p , price vectors $p'', p' \in \mathbb{P}$ and assignments μ'' and μ' such that $p'' < p < p'$ and $x'' = (\mu'', p'')$ as well as $x' = (\mu', p')$ are equilibria in \mathcal{E}^e .

Using the above finding and Lemma 1, it needs to be demonstrated that $H = H_{od}$ at p'' as well as at p' to complete the proof. As the arguments are symmetric in these two cases, the result is only proved for p' . For this purpose, consider allocation x' and let $H = H_{od} \cup H_J$ be PMOD

¹¹Note that H is a minimal over-demanded set at p but at p' and p'' the set H is over-demanded but not necessarily a minimal over-demanded set. Note also that $p'' < p < p'$ means that $p''_h < p_h < p'_h$ for all $h \in H$.

at p' and $A_{od} = \{a \in A : D_a(p') \subset H_{od}\}$. Then, the following holds as μ' is bijective and as $H_{od} \neq \emptyset$:

$$n|H_J| = |\{a \in A^m : \mu'_a \in H_J\}| \leq nm - |A_{od}^m|.$$

Note next that the following holds by construction:

$$nm - |A_{od}^m| = nm - m|A_{od}| = m|A \setminus A_{od}| \leq m|H_J|.$$

Adding the above two inequalities, it is clear that they can hold only if $H_J = \emptyset$, i.e., only if $H = H_{od}$, as $n > m$ in the extended economy \mathcal{E}^e . \square

This Appendix ends with the proof of Proposition 1. Note that even if the proposition is valid for an economy \mathcal{E} , it is essentially an economy \mathcal{E}^0 that is considered due to the restrictions that $|A| > |H|$ and no agent is assigned a null house.

Proposition 1. Let \mathcal{E} be an economy and $x = (\mu, p)$ a state in \mathcal{E} . Suppose that $|A| > |H|$, H is a minimal over-demanded set at price vector $p \in \mathbb{P}$, and $\mu : A \rightarrow H$ is an assignment such that $\mu_a \in D_a(p)$ for all $a \in A$ and $H = \{h \in H : h = \mu_a \text{ for some } a \in A\}$. Consider now a fixed but arbitrary agent $a' \in A$. Then there is an assignment $\mu' : A \setminus \{a'\} \rightarrow H$ such that $\mu'_a \in D_a(p)$ for all $a \in A \setminus \{a'\}$ and $H = \{h \in H : h = \mu'_a \text{ for some } a \in A \setminus \{a'\}\}$.

Proof. For each set $S \subseteq H$, let $A_S = \{a \in A : D_a(p) \subset S\}$. Because H is a minimal over-demanded set, by assumption, it follows from the definition of a minimal over-demanded set that $|A_H| > |H|$ and $|A_S| \leq |S|$ for all $S \subset H$ ($S \neq H$). Let now $A' \subset A$ be such that $a' \notin A'$ and $|A'| = |H|$. Consider the economy with houses H and the reduced set of agents A' . Also let $A'_S = A_S \cap A'$ for all subsets $S \subset H$.

Now $|A'_H| = |H|$ and $|A'_S| \leq |A_S| \leq |S|$ for all $S \subset H$ ($S \neq H$). It then follows from Hall (1935) that there is an assignment $\mu'' : A' \rightarrow H$ such that $\mu''_a \in D_a(p)$ for all $a \in A'$ and $H = \{h \in H : h = \mu''_a \text{ for some } a \in A'\}$. But then the proof is completed by defining μ' as $\mu'_a = \mu''_a$ for $a \in A' \cup \{a'\}$ and $\mu'_a = \mu_a$ for $a \in A \setminus A' \cup \{a'\}$. \square

Appendix: Proofs of the Main Results

This Appendix contains the proofs of the main results of the paper and they are all valid in an economy \mathcal{E} where the houses belong to the set $H \cup \{0\}$ and for an arbitrary relation between the number of agents and houses.

Proposition 4. For a given economy \mathcal{E} where $R \in \mathcal{R}$, if $(p^t)_{t=1}^T$ is an English Price Sequence, $(p^t)_{t=1}^T$ is weakly increasing and T is finite.

Proof. The sequence $(p^t)_{t=1}^T$ is clearly weakly increasing since $p^{t+1} \in \xi(x^t) = \sup \Pi^{x^t}$ and $p^t \in \Pi^{x^t}$, by construction.

To prove that the sequence $(p^t)_{t=1}^T$ also is finite, suppose, to obtain a contradiction, that $T = \infty$. Because $(p^t)_{t=1}^T$ is an EPS, there is a supporting sequence $(x^t)_{t=1}^T$ of temporary states. For each t , let $H^{p^t} = \{h \in H \cup \{0\} : p_h^t < \bar{p}_h\}$. Then $H^{p^{t+1}} \subset H^{p^t}$ for all t since the sequence $(p^t)_{t=1}^T$ is weakly increasing. Moreover, since H is a finite set, there is a t_0 such that $H^{p^t} = H^{p^{t_0}}$ for all $t \geq t_0$. Let now t_0 be so large that for all $t \geq t_0$, $p_h^t > \underline{p}_h$ if $h = \mu_a^t \in H_{od}^{p^t}$ for some $a \in A$. Such t_0 exists since $T = \infty$ by assumption. Note also that the assumption $T = \infty$ implies that $H^{p^t} \setminus \{0\} \neq \emptyset$ for all $t \geq t_0$. To prove the result, it will be demonstrated that, for some t_1 , $H_{od}^{p^t} = H_{od}^{p^{t_1}}$ for all $t \geq t_1 \geq t_0$, and, consequently, that prices cannot increase beyond t_1 . This contradicts the assumption that $T = \infty$.

Note next that, for each step t of the EPS, the sets $\{h \in H : h = \mu_a^t \text{ for some } a \in A\}$ and $\{a \in A : \mu_a^t = 0\}$ contain the houses in H that are assigned to some agent and the agents that are assigned a null house, respectively. Based on these two sets, it is possible to define an number θ^t , at each step $t \geq t_0$, defined as follows:

$$\theta^t = |\{h \in H : h = \mu_a^t \text{ for some } a \in A\}| + |\{a \in A : \mu_a^t = 0\}|.$$

Consider now an arbitrary t where $t + 1 > t \geq t_0$, and define the price vectors p and p' by $p = p^t$ and $p' = p^{t+1}$. Let also $x = (\mu, p)$ and $x' = (\mu', p')$ be supporting temporary states at prices p and p' , respectively. Define now $H_1 = H^{p^t} \setminus H_{od}^p$, $H'_1 = H^{p'} \setminus H_{od}^{p'}$, $S = H'_1 \cap H_{od}^p$, and $G = H_{od}^{p'} \cap H_1$. Then, $H'_1 \setminus S = H_1 \setminus G$ and $H'_{od} \cup S = H_{od} \cup G$. It is first proved that θ^t is weakly increasing in t for $t \geq t_0$. To see this, note that:

- Because $H'_{od} \cup S = H_{od} \cup G$, it also follows that $|H'_{od} \cup S| = |H_{od} \cup G|$. Moreover, for each $h \in H_{od} \cup G$ there is an agent $a \in A$ such that $\mu_a = h$. Similarly, for each $h' \in H'_{od} \cup S$ there is an agent $a' \in A$ such that $\mu'_{a'} = h'$. But then:

$$|\{a \in A : \mu_a \in G\} \cup \{h \in H : h \in H_{od}\}| = |H_{od} \cup G|, \quad (2)$$

$$|\{a \in A : \mu'_{a'} \in S\} \cup \{h \in H : h \in H'_{od}\}| = |H'_{od} \cup S|. \quad (3)$$

- Note that if $\mu_a \in H_1 \setminus G$, then $\mu'_{a'} \in H_1 \setminus G$. To see this, suppose that $\mu_a \in H_1 \setminus G$ but $\mu'_{a'} \notin H_1 \setminus G$. Then $\mu'_{a'} \in H_{od}^{p'}$ or $\mu'_{a'} \in G$. But $\mu'_{a'} \in H_{od}^{p'}$ cannot be the case because of monotonicity. Further, if $\mu'_{a'} \in G$ then $(\mu'_{a'}, p') P_a(\mu_a, p)$ since $G \subset H_{od}^{p'}$, and then also $(\mu'_{a'}, p) P_a(h, p)$, which is a contradiction to the equilibrium condition for the state x . Hence, if $\mu_a \in H_1 \setminus G$, then $\mu'_{a'} \in H_1 \setminus G$. This means that:

$$|\{a \in A : \mu_a \in H_1 \setminus G\}| \leq |\{a \in A : \mu'_{a'} \in H_1 \setminus G\}|. \quad (4)$$

From conditions (2)–(4), it now follows that $\theta^t \leq \theta^{t+1}$, i.e., that θ^t is weakly increasing in t for $t \geq t_0$.

Because θ^t is weakly increasing in t for $t \geq t_0$, there is a $t_1 \geq t_0$ such that $\theta^t = \theta^{t_1}$ for all $t \geq t_1$. Then, for all $t \geq t_1$, $|\{a \in A : \mu_a \in H_1 \setminus G\}| = |\{a \in A : \mu'_a \in H_1 \setminus G\}|$ and then, $\mu'_a \in H'_{od} \cup S$ if $\mu_a \in H_{od}$. Hence, if $S = \emptyset$, the proof is completed since this, by the above conclusion, means that $\mu'_a \in H'_{od}$ if $\mu_a \in H_{od}$ and, consequently, that $H^{p^t}_{od} \subset H^{p^{t+1}}_{od}$ for all $t \geq t_1$. But then $H^{p^t}_{od} = H^{p^{t_1}}_{od}$, and there are no further prices increases, contradicting the assumption that $T = \infty$. To demonstrate that $S = \emptyset$, consider the set S and the state x' . Then it must be the case that:

$$\text{if } \mu'_a \notin S \text{ then } x'_a P_a(h, p') \text{ for all } h \in S. \quad (5)$$

This follows for all a with $\mu_a \in H_1$ by monotonicity, and for a with $\mu'_a \in H^{p'}_{od}$ from the definition of the set $H^{p'}_{od}$. Because condition (5) holds, there is a temporary state $x'' = (\mu'', p'')$, satisfying the conditions in Definition 10, with $p'' \in \Pi^x$ and $p'' \leq p'$, $p'' \neq p'$, and so close to p' that for all $\mu'_a \notin S$, $x''_a P_a(h, p'')$ for all $h \in S$. Note that if $\mu_a \in H^{p'}_{od}$, then $\mu'_a \in H^{p'}_{od} \cup S$, and, hence, if also $\mu'_a \notin S$ then $x''_a P_a(h, p'')$ for all $h \in S$. This also means that S is not over-demanded at prices p'' , which contradicts that $S \subset H^{p''}_{od} = H^{p'}_{od}$. Hence, $S = \emptyset$, and the conclusion follows by the above arguments. \square

Note next that Lemma 4 from the preceding Appendix is used in the proof of the next lemma and that the former lemma is valid in economy \mathcal{E}^0 . Because Lemma 5 considers a neighborhood of a price vector p , the results for economy \mathcal{E}^0 extends to the more general economy \mathcal{E} .

Lemma 5. For a given economy \mathcal{E} where $R \in \mathcal{R}$, and for each temporary state $x = (\mu, p)$, if H^{p}_{od} is the over-demanded part of H^p at the price vector p in the subeconomy \mathcal{E}^x and $H^{p}_{od} \neq \emptyset$, then, in each neighborhood of p , there is a price vector $p' \in \Pi^x$ such that $p'_h > p_h$ for all $h \in H^{p}_{od}$ and $H^{p'}_{od}$ is the over-demanded part of H^p at the price vector p' .

Proof. Let \mathcal{E}^x be the subeconomy in \mathcal{E} defined by the temporary state $x = (\mu, p)$ and let H^{p}_{od} be the over-demanded part of H^p . Further, let $H^{p}_{od} = \cup_{j=1}^{J-1} H_j^p$ be defined by a PMOD sequence $(H_j^p)_{j=1}^J$ according to Definition 7, and recall that $H^{p}_{od} \neq \emptyset$ by assumption. The set H_1^p is a minimal over-demanded set in \mathcal{E}^x by construction. But then it holds that $x_a P_a(h, p)$ for all $h \notin H_1^p \cup (H \setminus H^p)$ for all agents a with $\mu_a \in H_1^p$. By applying Lemma 4 on the set H_1^p , it is possible to increase p_h for all $h \in H_1^p$ by an arbitrarily small amount to obtain a price vector p' such that H_1^p is an over-demanded set in \mathcal{E}^x at p' . Moreover, when p' is “sufficiently close” to p , all conditions in Definition 10 will be respected. Hence, $p' \in \Pi^x$.

The above procedure can be repeated by applying Lemma 4 to H_2^p , and, again, the prices p_h for $h \in H_2^p$ can be increased by an arbitrary small amount, and a new price vector $p' \in \Pi^x$ can be obtained where $p' \geq p$ and $p'_h > p_h$ for $h \in H_1^p \cup H_2^p$. By repeating this procedure a finite number of times, a price vector $p' \in \Pi^x$ such that $p'_h > p_h$ for all $h \in H^{p}_{od}$ will be obtained. \square

Lemma 6. For a given economy \mathcal{E} where $R \in \tilde{\mathcal{R}}$, let $\tilde{x} = (\tilde{\mu}, \tilde{p})$ be a temporary state and $p \in \xi(\tilde{x})$. Then there is an assignment μ such that $x = (\mu, p)$ is a temporary state.

Proof. Since $\tilde{x} = (\tilde{\mu}, \tilde{p})$ is a temporary state and $p \in \xi(\tilde{x})$, there is an assignment μ' such that the state $x' = (\mu', p)$ is priority respecting and $p_h = \underline{p}_h$ for all $h \in H$ if $h \in \mu'_0$. If $x = (\mu', p)$ also is a temporary state, then the proof follows by setting $\mu = \mu'$. Suppose, therefore, in the remaining part of the proof, that $x = (\mu', p)$ is not a temporary state.

To complete the proof, it will be demonstrated that there always is a way to modify μ' into some other assignment μ such that $x = (\mu, p)$ is a temporary state. To make such modification, recall that, by assumption, $\tilde{x} = (\tilde{\mu}, \tilde{p})$ is a temporary state while $x = (\mu', p)$ is not. From the definition of a temporary state (Definition 9), it then follows that the state \tilde{x} is strongly priority respecting, and that there is no improvement of $\tilde{\mu}$ with a strongly priority respecting assignment $\hat{\mu}$. Even if these two properties are satisfied at state $\tilde{x} = (\tilde{\mu}, \tilde{p})$, they are not necessarily satisfied at state $x = (\mu', p)$.

It is next demonstrated how the assignment μ' can be modified step-by-step to obtain a strongly priority respecting assignment while respecting the property that the price of an unassigned house always is at the lower price bound. Starting with the assignment μ' , a new assignment μ'' is defined such that $\mu''_a = \mu'_a$ for all but one agent called a' . The step from μ' to μ'' can be taken in two different ways called (1) and (2). Let $H' = \{h \in H : p_h = \bar{p}_h\}$.

- (1) If there are agents $a, a' \in A$, with $\pi_{a'} > \pi_a$, and $\mu'_a = \mu'_{a'} \in H'$, and $\pi_{a'} > \pi_a$ for all agents a with $\mu'_a = \mu'_{a'}$, then define $\mu''_{a'}$ according to:
 - (a) If there is $h' \in H'$ such that $\mu_{a''} = h'$, with $\pi_{a'} < \pi_{a''}$, and $(h', p) R_{a'}(h, p)$ for all $h \in H \setminus H'$, let $\mu''_{a'} = h'$.
 - (b) If there is no h' according to (a), let $h' \in H \setminus H'$ and (h', p) be a maximal alternative according to preference $R_{a'}$, and let $\mu''_{a'} = h'$.
- (2) If there are agents $a, a' \in A$ such that $\mu'_a = \mu'_{a'} \in H \setminus H'$ and $x_{a'} I_{a'} x_{a''}$, where $\pi_{a'} < \pi_{a''}$, and $\mu'_{a''} \in H'$, then let $\mu''_{a'} = \mu'_{a''}$.

Note now that the states $x' = (\mu', p)$ and $x'' = (\mu'', p)$ almost have the same properties; the state $x'' = (\mu'', p)$ is priority respecting and for all $h \in H$, $p_h = \underline{p}_h$ if $h \in \mu''_0$. However, if $\nu'_h = \min \{\pi_a : \mu'_a = h \in H'\}$ and $\nu''_h = \min \{\pi_a : \mu''_a = h \in H'\}$, then in (1), $\nu''_h \leq \nu'_h$ with some strict inequality. If $\nu'_h = \max \{\pi_a : \mu'_a = h \in H'\}$ and $\nu''_h = \max \{\pi_a : \mu''_a = h \in H'\}$, then in (2), $\nu''_h \leq \nu'_h$ with some strict inequality. This means that the step from x' to x'' can be repeated only finitely many times. If (1) and (2) are applied as many times as possible, a new assignment μ'' is generated. This assignment is injective, strongly priority respecting, and respects the property that the price of an unassigned house always is at the lower price bound.

Suppose now that μ' is a final assignment that is generated after the above iterations. If there is no improvement of μ' with a strongly priority respecting assignment μ'' , one can, again, set $\mu = \mu'$ and the proof is complete. Suppose instead that there is an improvement of μ' , i.e., that there is a trading cycle $(a_j)_{j=1}^k$ from μ' to a strongly priority respecting assignment μ'' . Suppose first that the trading cycle is open. This also means that $x''_{a_i} P_{a_i} x'_{a_i}$ for some agent a_i in the open trading cycle and, consequently, that $\mu''_{a_i} \in H'$. Let l be the largest such index, i.e., $x''_{a_l} P_{a_l} x'_{a_l}$ and $x''_{a_i} I_{a_i} x'_{a_i}$ for all $l < i \leq k$. Such index exists since the trading cycle is open by assumption. Now, $p_{\mu''_{a_l}} = \bar{p}_{\mu''_{a_l}}$ and $p_{\mu''_{a_k}} = \underline{p}_{\mu''_{a_k}}$ since $\mu''_{a_l} \in H'$ and $\mu''_{a_k} \in \mu'_0$, respectively. But then houses μ''_{a_l} and μ''_{a_k} are connected by indifference, which contradicts that $R \in \tilde{\mathcal{R}}$. Hence, the trading cycle cannot be open.

Suppose now that the trading cycle is closed, and let μ'' be an assignment where there are no more possible improvements of μ' . Because all trading cycles are closed, it also follows that $p_h = \underline{p}_h$ if $h \in \mu''_0$ since $p_h = \underline{p}_h$ if $h \in \mu'_0$ by the above conclusions. But then $x = (\mu, p)$ is a temporary state for $\mu = \mu''$. \square

Lemma 7. For a given economy \mathcal{E} where $R \in \tilde{\mathcal{R}}$, let $x = (\mu, p)$ be a temporary state and $x' = (\mu', p')$ an RPE state. Then there is no house $h \in H$ and two distinct agents $a, a' \in A$ such that $\mu_a = \mu_{a'} = h$ and $p_h > p'_h$.

Proof. To obtain a contradiction, suppose there is a house $h \in H$ and two distinct agents $a, a' \in A$ such that $\mu_a = \mu_{a'} = h$ and $p_h > p'_h$. To prove the result, it will, in a series of steps, be demonstrated that there is a state $\tilde{x} = (\tilde{\mu}, \tilde{p})$ which is an improvement of $x = (\mu, p)$, contradicting the assumption that $x = (\mu, p)$ is a temporary state.

Consider now a trading cycle $(a_i)_{i=1}^k$ from μ to μ' such that $a_1 = h$ and $\mu'_{a_k} \in \mu_0$. Such a trading cycle exists since μ' is injective. Further, let:

$$\begin{aligned} S_1 &= \{h \in H : p'_h < p_h\}, \\ S_2 &= \{h \in H : p'_h = p_h\} \cup \{0\}, \\ S_3 &= H \setminus (S_1 \cup S_2). \end{aligned}$$

From the assumption that $\mu_a = \mu_{a'} = h$ and $p_h > p'_h$, it is clear that $S_1 \neq \emptyset$ and that agents a and a' belong to the trading cycle $(a_i)_{i=1}^k$. Next, the trading cycle will be decomposed into a number of sequences of agents of the type $(a_i)_{i=l}^q$ for $l \geq 1$ and $q \leq k$.

1. Consider a sequence of agents $(a_i)_{i=l}^q$ where $\mu_{a_i} \in S_1$ for all $l \leq i \leq q$. Because $x' = (\mu', p')$ is an RPE state, $\mu_{a_i} \in S_1$, $\mu'_{a_i} \neq \mu_{a_i}$ and $p_h > p'_h$, it follows that $x'_{a_i} P_{a_i} x_{a_i}$ for all $l \leq i \leq q$.
2. Consider now an agent a_i in the sequence $(a_i)_{i=1}^k$ with $\mu_{a_i} \in S_1$ and $\mu'_{a_i} = \mu_{a_{i+1}} \notin S_1$. Then $\mu'_{a_i} = \mu_{a_{i+1}} \in S_2$. This follows because if $\mu'_{a_i} = \mu_{a_{i+1}} \in S_3$, then $x_{a_i} R_{a_i}(\mu'_{a_i}, p) P_{a_i}(\mu'_{a_i}, p')$

by monotonicity and the assumptions that $\mu'_{a_i} = \mu_{a_{i+1}} \in S_3$ and x is a temporary state. But $x_{a_i} P_{a_i} x'_{a_i}$ is a contradiction to 1., in the above, since $\mu_{a_i} \in S_1$.

3. Let $\mu_{a_l} \in S_1$ and $\mu_{a_i} \in S_2$ for all $l < i \leq q$, and $\mu'_{a_q} \notin S_2$. Then for all $l < i \leq q$:

- (i) $x'_{a_i} P_{a_i} x_{a_i}, p_{\mu_{a_{i+1}}} = \bar{p}_{\mu_{a_{i+1}}}$ and $\pi_{a_{i+1}} < \pi_{a_i}$,
- (ii) $x'_{a_i} I_{a_i} x_{a_i}$ and $p_{\mu_{a_{i+1}}} < \bar{p}_{\mu_{a_{i+1}}}$, or;
- (iii) $x'_{a_i} R_{a_i} x_{a_i}$ and $\mu'_{a_i} \in S_1 \cup \mu_0$.

Moreover, $\mu'_{a_q} \in S_1$ if $\mu'_{a_q} \notin \mu_0$. To see that these conclusions hold, note first that because $\mu_{a_l} \in S_1$ and $\mu'_{a_l} \in S_2$, it follows by monotonicity and $x'_{a_l} P_{a_l} x_{a_l}$ (i.e., 1. from the above) that $p_{\mu_{a_{l+1}}} = \bar{p}_{\mu_{a_{l+1}}}$, and, consequently, that $\pi_{a_{l+1}} < \pi_{a_l}$ since x is a temporary state. But then $x'_{a_{l+1}} R_{a_{l+1}} x_{a_{l+1}}$ since x' is an RPE state. If $x'_{a_{l+1}} P_{a_{l+1}} x_{a_{l+1}}$, it must, by the same arguments as in the above, be the case that $p_{\mu_{a_{l+2}}} = \bar{p}_{\mu_{a_{l+2}}}$ and $\pi_{a_{l+2}} < \pi_{a_{l+1}}$. If $x'_{a_{l+1}} I_{a_{l+1}} x_{a_{l+1}}$, it cannot be the case that $p_{\mu_{a_{l+2}}} = \bar{p}_{\mu_{a_{l+2}}}$ as this would mean that houses $\mu_{a_{l+1}}$ and $\mu_{a_{l+2}}$ are connected by indifference at profile R which contradicts the assumption that $R \in \tilde{\mathcal{R}}$, i.e., if $x'_{a_{l+1}} I_{a_{l+1}} x_{a_{l+1}}$ then $p_{\mu_{a_{l+2}}} < \bar{p}_{\mu_{a_{l+2}}}$. By repeating the arguments it is clear that (i) or (ii) holds for all $l < i < q$.

Finally, consider agent q and suppose that $x_{a_q} P_{a_q} x'_{a_q}$. This is equivalent to $x'_{a_{q-1}} P_{a_q} x'_{a_q}$ since $\mu'_{a_{q-1}} = \mu_{a_q}$ and $\mu_{a_q} \in S_2$ by assumption. But then it must be the case that $p_{\mu'_{a_{q-1}}} = p_{\mu_{a_q}} = \bar{p}_{\mu_{a_q}}$ since x' is an RPE state. But this contradicts that x' is an RPE state as $\pi_{a_q} < \pi_{a_{q-1}}$ by (i) in the above. Hence, $x'_{a_q} R_{a_q} x_{a_q}$. Now, it is clear that (iii) holds for agent q . If $\mu'_{a_q} \in \mu_0$, the conclusion follows directly, and if $\mu'_{a_q} \notin S_2 \cap \mu_0$, it follows from monotonicity that $\mu'_{a_q} \in S_1$.

Consider again the trading cycle $(a_i)_{i=1}^k$. Since $\mu'_{a_k} \in \mu_0$ by assumption, there is a house $h \in H$ and two agents $a, a_l \in A$ such that $\mu_a = \mu_{a_l} = h$ and $p_{h_l} > p'_{h_l}$. Moreover, from the above decomposition of the trading cycle, it is clear that the index l can be chosen so that $\mu_{a_l} \in S_1$ and $\mu_{a_i} \in S_2$ for all $i > l$. Consider now assignment $\tilde{\mu}$ where $\tilde{\mu}_{a_i} = \mu'_{a_i}$ for all $i > l$ and $\tilde{\mu}_a = \mu_a$ for the remaining agents $a \in A$. The state \tilde{x} is a temporary state by, construction, and it is an improvement of $x = (\mu, p)$. This contradicts the assumption that x is a temporary state. \square

Proposition 5. For a given economy \mathcal{E} where $R \in \tilde{\mathcal{R}}$, if $(p^t)_{t=1}^T$ is an English Price Sequence and $(x^t)_{t=1}^T$ is a supporting sequence of temporary states, then $x^T = (\mu^T, p^T)$ can be chosen to be a rationing price equilibrium.

Proof. It needs to be demonstrated that $x^T = (\mu^T, p^T)$ can be chosen so that conditions (i)–(iii) of Definition 5 are satisfied.

For condition (i) of Definition 5 to be satisfied, it needs to be demonstrated that the state x^T can be chosen to be constrained efficient, i.e., that μ^T can be chosen to be feasible and

that there is no improvement of μ^T by a feasible assignment μ' . To prove this, note first that $p^T \neq p^{T-1}$ and $p^{T+1} = p^T$, by Definition 11, as p^T , by Proposition 4, is the finite endpoint of $(p^t)_{t=1}^T$. But then there cannot be any over-demanded sets in H^{p^T} . This follows because if $H_{od}^{p^T} \neq \emptyset$, then there is a price vector $p^{T+1} \neq p^T$ where $p^{T+1} \in \xi(x^T)$, by Lemma 5, and then a temporary state $x^{T+1} = (\mu^{T+1}, p^{T+1})$ by Lemma 6. This is a contradiction to $p^T = p^{T+1}$. Since there are no over-demanded sets at p^T , the assignment μ^T can be chosen to be feasible (see Hall, 1935). Moreover, because $H_{od}^{p^T} = \emptyset$, the assignment μ^T cannot be improved by any other priority respecting assignment μ' , by Definition 9(ii), as x^T is a temporary state. But then the assignment μ^T cannot be improved by any other feasible assignment μ' by Definition 4. Hence, the state x^T can be chosen to be constrained efficient.

To see that x^T also condition (ii) of Definition 5 is satisfied, recall that $x^T = (\mu^T, p^T)$ is a temporary state. Hence, μ^T is strongly priority respecting and, consequently, also priority respecting. From Definitions 2(i) and 8(ii), it then follows that $x_a^T P_a x_a^T$ implies that $\pi_{a'} < \pi_a$ and, consequently, that $\mu_{a'}^T \in H \setminus H^{p^T}$ and $p_{\mu_{a'}^T}^T = \bar{p}_{\mu_{a'}^T}^T$. But then condition (ii) of Definition 5 is satisfied.

Finally, condition (iii) of Definition 5 is satisfied since x^T is a temporary state. \square

Theorem 1. For a given economy \mathcal{E} where $R \in \tilde{\mathcal{R}}$, if $(p^t)_{t=1}^T$ is an English Price Sequence, then $p^T = p^*$ where p^* is the minimal RPE price vector.

Proof. Suppose that $(p^t)_{t=1}^T$ is an EPS and that $R \in \tilde{\mathcal{R}}$. Let t be the largest index such that $p^t \leq p^*$, and note that such index always exists as the starting point of the EPS is $p^1 = \underline{p}$ and $\underline{p} \leq p^*$. Consider next the temporary state $x^t = (\mu^t, p^t)$, and let $H_{od}^{p^t} \subset H^{p^t}$ be the set of over-demanded houses at prices p^t in the subeconomy \mathcal{E}^{x^t} . Note first that if $H_{od}^{p^t} = \emptyset$, then $p^t = p^*$ by construction, and the proof of the theorem follows from the construction of the EPS and Proposition 5. Suppose, therefore, that $H_{od}^{p^t} \neq \emptyset$. This assumption together with the definition of the subeconomy \mathcal{E}^{x^t} means that $p_h^t \leq p_h^* < p_h^{t+1}$ for some $h \in H_{od}^{p^t}$, and $p_h^t = p_h^{t+1} \leq p_h^*$ for all $h \in H \setminus H_{od}^{p^t}$. But since $p^{t+1} \in \sup \Pi^{x^t}$, there is also a temporary state $x = (\mu, p)$ in the subeconomy \mathcal{E}^{x^t} such that $p \leq p^{t+1}$ and $p_h^* < p_h$ for all $h \in H_{od}^{p^t}$ with $p_h^* < p_h^{t+1}$.

Let $(H_{od}^{p^t} =) H_{od}^p = \cup_{j=1}^{J-1} H_j$ be the union of minimal over-demanded sets with respect to the reduced demand as defined in Definition 7. The assignment μ can be chosen in various ways, but, according to Proposition 1, in each minimal over-demanded set H_j and for any $h \in H_j$, the assignment μ can be chosen so that there are two distinct agents, a and a' , such that $\mu_a = \mu_{a'} = h$. Since for some $h \in H_{od}^p$, $p_h > p_h^*$ and $\mu_a = \mu_{a'} = h$ (by choice of the assignment μ), a contradiction to Lemma 7 is obtained. \square

Theorem 2. If the reported demand sets $d_a^{p^t}$ satisfy Monotonicity, Continuity, Demand ranking and Non-emptiness for all t , then the demand sets $d_a^{p^*}$ can be derived from some preference relation $R_a \in \mathcal{R}_a$.

Proof. It will be demonstrated that, for any agent $a \in A$, the four conditions in the statement of the theorem are sufficient to construct rational preferences $R_a \in \mathcal{R}_a$ represented by continuous utility functions $u_{ah}(p)$ where $u_{ah}(p)$ denotes agent a 's utility of house $h \in H \cup \{0\}$ at prices p_h and, in addition, $u_{a0}(p)$ is a constant while for $h \neq 0$, $u_{ah}(p)$ is strictly decreasing in p_h .

A fixed but arbitrary agent $a \in A$ is considered in the proof. Let also p^t , for $t \in [0, T]$, be a continuous and increasing price path of time t and let \tilde{v}_a^t be a utility representation of the assignment μ_a^t in the reported demand set $d_a^{p^t}$ at time t . Then \tilde{v}_a^t can be chosen to be a decreasing (by monotonicity) and piecewise continuous (by continuity and demand ranking) function of t . Along the price path, agent a is temporarily assigned different houses (possibly the null house) and \tilde{v}_a^t represents the utility of the particular house, μ_a^t , that agent a is assigned. At certain points of time, say \bar{t}_h , the price path may reach a price bound $p_h^{\bar{t}_h} = \bar{p}_h$, and if $h \in d_a^{\bar{t}_h}$ then it may not be possible to assign house h to agent a if there is another agent a' with higher rank than agent a (according to the priority-order) and $h \in d_{a'}^{\bar{t}_h}$. But by the non-emptiness condition, there is always another house $h' \in d_a^{\bar{t}_h}$, $h' = 0$ or $h' \in H^{p^{\bar{t}_h}}$, that can be assigned to agent a . This also means that the utility function \tilde{v}_a^t may have a discontinuity at time $t = \bar{t}_h$.

To define a utility $u_{ah}(p)$ of the bundle (h, p_h) , a utility v_{ah}^t at time t for each house $h \in H \cup \{0\}$ is first defined. The utility $u_{ah}(p)$ can then be derived from v_{ah}^t as will be shown below.

In the remaining part of the proof, consider a fixed but arbitrary house $h \in H \cup \{0\}$. If $h \in H$ and $p_h^{\bar{t}_h} = \bar{p}_h$ for some $\bar{t}_h \leq T$, then let $\bar{t}_h \leq T$ be the point of time where $p_h^t < \bar{p}_h$ if and only if $t < \bar{t}_h$. If $p_h^t < \bar{p}_h$ for all $t \leq T$, then let $\bar{t}_h = T$. Also let $\bar{t}_0 = T$. Moreover, let S be the set of points of time where house h is indifferent, according to $r_a^{p^t}$, to the assigned house μ_a^t at time t , i.e., $S = \{t \in [0, \bar{t}_h] : h \in d_a^{p^t} \text{ and } \mu_a^t r_a^{p^t} h\}$.

Since reported demand is continuous, the set $S \cup \{\bar{t}_h\}$ is closed and, hence, the union of a countable number of closed intervals. Consider four points of time, $t_1 \leq t_2 < t_3 \leq t_4$, and three intervals, $S' = [t_1, t_2]$, $S'' = (t_2, t_3)$ and $S''' = [t_3, t_4]$ such that $S' \cup S''' \subset S$ while $S'' \cap S = \emptyset$. For $t \in S' \cup S'''$ let $v_{ah}^t = \tilde{v}_a^t$ while for $t \in S''$ let v_{ah}^t be any decreasing function of t such that:

- (i) $v_{ah}^t < \tilde{v}_a^t$,
- (ii) v_{ah}^t is constant if and only if p_h^t is constant, and;
- (iii) $\lim_{t \rightarrow t_2+} v_{ah}^t = \tilde{v}_a^{t_2}$ and $\lim_{t \rightarrow t_3-} v_{ah}^t = \tilde{v}_a^{t_3}$.

Note that $\tilde{v}_a^{t_2} > \tilde{v}_a^{t_3}$. Since the set $S \cup \{\bar{t}_h\}$ is partitioned into a countable number of sets of types S' , S'' and S''' (the last S''' may be half-closed, $[t_3, \bar{t}_h)$), the v_{ah}^t function is defined for all $t \in [0, \bar{t}_h]$ (with $v_{ah}^{\bar{t}_h} = \lim_{t \rightarrow \bar{t}_h-} v_{ah}^t$). Note also that if $h = 0$, then $S = S' = [t_1, T]$ or $S = \emptyset$. In the first case let $v_{ah}^t = \tilde{v}_a^{t_1}$ and in the second case let $v_{ah}^t = \tilde{v}_a^T - 1$.

The entire time interval $[0, T]$ is partitioned into intervals of type S' and S''' , where house h is indifferent to the assigned house μ_a^t to agent a according to the demand ranking r_a^p , and intervals of type S'' , where no house h is indifferent to a house assigned to agent a . In each of these intervals, the function v_{ah}^t is defined. Moreover, if $t = 0$ is not in an S' -type interval, there

is some S'' -type interval with $t_2 = 0$. Also in this case, the definition of v_{ah}^t above is relevant with $v_{ah}^0 = \lim_{t \rightarrow 0^+} v_{ah}^t$. The same is true for $t \geq \bar{t}_h$ with $v_{ah}^{\bar{t}_h} = \lim_{t \rightarrow \bar{t}_h^+} v_{ah}^t$. Note also that the definition of v_{ah}^t can be used even if $S = \emptyset$.

To define the utility function $u_{ah}(p)$, note first that v_{ah}^t is defined for all $t \in [0, T]$. Hence, for each bundle (h, p_h) , $0 \leq p_h \leq p_h^T$, the number v_{ah}^t is well defined. Then let $u_{ah}(p) = v_{ah}^t$, where t is chosen so that $p_h^t = p_h$. For $p_h > p_h^T$, $u_{ah}(p)$ can be chosen arbitrarily such that it is continuous and strictly decreasing in p_h , and such that $u_{ah}(p^T) = v_{ah}^T$.

From the construction of the utility function $u_{ah}(p)$, it follows directly that it is a continuous and strictly decreasing function of p_h (for $p_h \geq 0$) and, hence, u_a is a representation of rational preferences $R_a \in \mathcal{R}_a$. Moreover, from the construction of u_a , it also follows that reported demand d_a^t is the same as demand $D_a(p^t)$ derived from preferences R_a . \square

References

- Abdulkadiroğlu, A. and Sönmez, T. (1998). Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems. *Econometrica*, 66:689–701.
- Abdulkadiroğlu, A. and Sönmez, T. (1999). House Allocation with Existing Tenants. *Journal of Economic Theory*, 88:233–260.
- Abdulkadiroğlu, A. and Sönmez, T. (2003). School Choice: A Mechanism Design Approach. *American Economic Review*, 93:727–747.
- Alaei, S., Kamal, J., and Azarakhsh, M. (2016). Competitive Equilibrium in Two Sided Matching Markets with General Utility Functions. *Operations Research*, 64:638–645.
- Alkan, A., Demange, G., and Gale, D. (1991). Fair Allocation of Indivisible Goods and Criteria of Justice. *Econometrica*, 59:1023–1039.
- Andersson, T. and Svensson, L.-G. (2014). Non-Manipulable House Allocation with Rent Control. *Econometrica*, 82:507–539.
- Andersson, T. and Svensson, L.-G. (2016). Strategy-Proof House Allocation with Price Restrictions. *Journal of Economic Theory*, 163:167–177.
- Ausubel, L. M. (2004). An Efficient Ascending-bid Auction for Multiple Objects. *American Economic Review*, 94:1452–1475.
- Ausubel, L. M. (2006). An Efficient Dynamic Auction for Heterogeneous Commodities. *American Economic Review*, 96:602–629.
- Baisa, B. (2017). Auction Design without Quasilinear Preferences. *Theoretical Economics*, 12:53–78.

- Balinski, M. and Sönmez, T. (1999). A Tale of Two Mechanisms: Student Placement. *Journal of Economic Theory*, 84:73–94.
- Cramton, P. (1998). Ascending Auctions. *European Economic Review*, 42:745–746.
- de Vries, S., Schummer, J., and Vohra, R. V. (2007). On Ascending Vickrey Auctions for Heterogeneous Objects. *Journal of Economic Theory*, 132:95–118.
- Demange, G. and Gale, D. (1985). The Strategy Structure of Two-Sided Matching Markets. *Econometrica*, 53:873–888.
- Demange, G., Gale, D., and Sotomayor, M. (1986). Multi-Item Auctions. *Journal of Political Economy*, 94:863–872.
- Drèze, J. (1975). Existence of an Exchange Economy Under Price Rigidities. *International Economic Review*, 16:301–320.
- Echenique, F. (2012). Contracts versus Salaries in Matching. *American Economic Review*, 102:594–601.
- Engelberecht-Wiggans, R. and Kahn, C. M. (1991). Protecting the Winner: Second-price versus Oral Auctions. *Economics Letters*, 35:243–248.
- Gale, D. and Shapley, L. S. (1962). College Admissions and the Stability of Marriage. *American Mathematical Monthly*, 69:9–15.
- Gul, F. and Staccetti, E. (2000). The English Auction with Differentiated Commodities. *Journal of Economic Theory*, 92:66–95.
- Hall, P. (1935). On Representatives of Subsets. *Journal of London Mathematical Society*, 10:26–30.
- Harsha, P., Barnhart, C., Parkes, D., and Zhang, H. (2010). Strong Activity Rules for Iterative Combinatorial Auctions. *Computers and Operations Research*, 37:1271–1284.
- Harstad, R. M. (2000). Dominant Strategy Adoption and Bidders' Experience with Pricing Rules. *Experimental Economics*, 3:261–280.
- Hatfield, J. W. and Kojima, F. (2010). Substitutes and Stability for Matching with Contracts. *Journal of Economic Theory*, 145:1704–1723.
- Hatfield, J. W. and Milgrom, P. R. (2005). Matching with Contracts. *American Economic Review*, 95:913–935.

- Herings, J.-J. (2015). Equilibrium and Matching under Price Controls. *Research Memorandum 2015:001, Maastricht University Graduate School of Business and Economics (GSBE)*.
- Hylland, A. and Zeckhauser, R. (1979). The Efficient Allocation of Individuals to Positions. *Journal of Political Economy*, 97:293–314.
- Kagel, J. H., Harstad, R. M., and Levin, D. (1987). Information Impact and Allocation Rules in Auctions with Affiliated Private Values: A Laboratory Study. *Econometrica*, 55:1275–1304.
- Kagel, J. H. and Levin, D. (1993). Independent Private Value Auctions: Bidder Behavior in First-, Second- and Third Price Auctions with Varying Number of Bidders. *Economic Journal*, 103:868–879.
- Kelso, A. S. and Crawford, V. P. (1982). Job Matching, Coalition Formation, and Gross Substitutes. *Econometrica*, 50:1483–1504.
- Li, S. (2016). Obviously Strategy-Proof Mechanisms. *Working Paper, Stanford University*.
- Mishra, D. and Parkes, D. (2007). Ascending Price Vickrey Auctions for General Valuations. *Journal of Economic Theory*, 132:335–366.
- Mishra, D. and Parkes, D. C. (2009). Multi-Item Vickrey-Dutch auctions. *Games and Economic Behavior*, 66:326–347.
- Morimoto, S. and Serizawa, S. (2015). Strategy-Proofness and Efficiency with Non-Quasi-Linear Preferences: A Characterization of Minimum Price Walrasian Rule. *Theoretical Economics*, 10:445–487.
- Roth, A. E. (1982). The Economics of Matching: Stability and Incentives. *Mathematics of Operations Research*, 7:617–628.
- Shapley, L. S. and Shubik, M. (1972). The Assignment Game I: The Core. *International Journal of Game Theory*, 1:111–130.
- Svensson, L.-G. (1994). Queue Allocation of Indivisible Goods. *Social Choice and Welfare*, 11:323–330.
- Talman, D. and Yang, Z. (2008). A Dynamic Auction for Differentiated Items Under Price Rigidities. *Economics Letters*, 99:278–281.
- Zhou, Y. and Serizawa, S. (2016). Strategy-proofness and Efficiency for Non-quasi-linear Common-tiered-object Preferences: Characterization of Minimum Price Rule. *The Institute of Social and Economic Research Discussion Paper No. 971*.

Zhu, J. and Zhang, D. (2011). Dynamic Auction for Efficient Competitive Equilibrium Under Price Rigidities. *Advances in Artificial Intelligence*, 7106:809–818.