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Transferring Ownership of Public Housing to Existing Tenants: A Market Design Approach

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Transferring Ownership of Public Housing to Existing Tenants: A Market Design Approach*

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Abstract

This paper explores a housing market with an existing tenant in each house and where the existing tenants initially rent their houses. The idea is to identify equilibrium prices for the housing market given the prerequisite that a tenant can buy any house on the housing market, including the one that he currently is possessing, or continue renting the house he currently is occupying. The main contribution is the identification of an individually rational, equilibrium selecting, and group non-manipulable price mechanism in a restricted preference domain that contains almost all preference profiles. In this restricted domain, the identified mechanism is the equilibrium selecting mechanism that transfers the maximum number of ownerships to the existing tenants. We also argue that the theoretical model represents an extension and an improvement of the U.K. Housing Act 1980 whose main objective is to transfer the ownership of the houses to the existing tenants.

JEL Classification: C71; C78; D71; D78.

Keywords: Existing tenants; equilibrium; minimum equilibrium prices; maximum trade; group non-manipulability; dynamic price process.

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1 Introduction

Matching theory has provided fundamental tools for solving a variety of house allocation problems. Examples include procedures for allocating unoccupied apartments among a set of potential tenants (Gale and Shapley, 1962), methods for reallocating apartments among a group of existing tenants (Shapley and Scarf, 1974), mechanisms for determining the rent on competitive housing markets (Shapley and Shubik, 1971), and rules for setting rents on housing markets with legislated rent control (Andersson and Svensson, 2014). This literature is not only appealing because the suggested allocation rules typically satisfy a number of desirable properties, like non-manipulability, individual rationality and various efficiency notions, but also since there is a relatively short step between theory and practice. This paper contributes to this literature by investigating a new type of matching model with a real-life application. Before discussing the application in detail, the model and our theoretical contributions will be related to the existing literature.

The formal model considers a market with a finite number of houses and the same number of agents, called the existing tenants. Our main interpretation of the model is that the existing tenants are renting the house they are living in, but an alternative interpretation could be the that they own the house instead of renting it. The idea is to identify equilibrium prices for this market, given the prerequisite that a tenant can buy any house in the market, including the one that he currently is possessing, or continue renting the house he currently is occupying. In the model, a fixed lower bound for the equilibrium prices is defined (i.e., reservation prices for the owner of the houses), and in case an existing tenant buys the particular house he currently is living in, the tenant pays only the reservation price. The reservation price can be interpreted as a “personalized” (reduced) price for the existing tenant as the price for all other tenants of that specific house is given by the equilibrium price which is endogenously determined by the preferences of all agents. This also means that any agent has *two* outside options, namely, the “Right to Stay and Rent” and the “Right to Buy His Current House” (for the reservation price).

To solve the above described house allocation problem, it is natural to search for a price mechanism that is individually rational, equilibrium selecting, and non-manipulable. This type of mechanism guarantees that no tenant can lose from participating, that no further rationing of the houses is needed, and that the reported information is reliable. Given the interest in these three specific axioms, the perhaps most natural candidate for an allocation mechanism is based on a “minimum equilibrium price vector” as this type of mechanism previously has been demonstrated to satisfy these specific axioms in a variety of different contexts, including, e.g., single-item auction environments (Vickrey, 1961), assignment markets (Demange and Gale, 1985; Leonard, 1983), and housing markets with rent control (Andersson and Svensson, 2014). Another natural price mechanism for the above described housing market is an individually rational, equilibrium selecting, and trade maximizing mechanism. The latter axiom guarantees that the maximum number of houses are transferred to the tenants in equilibrium, which, often is a specific policy goal for this type of housing market as will be explained below.

Even if one would believe that the above two mechanisms always recommend the same

selection, as lower prices intuitively should increase trade, it turns out that this is generally not the case. In fact, it is not even clear what any of the above two mechanisms recommend due to the non-uniqueness of a minimum equilibrium price vector and the non-uniqueness of an allocation that maximizes trade. The non-uniqueness property of the two mechanisms is somewhat unexpected, given what we know from previous literature (e.g., Demange and Gale, 1985; Leonard, 1983; Shapley and Shubik, 1971), and it is a direct consequence of the fact that agents can block the trade of a house through the “Right to Stay and Rent” or the “Right to Buy His Current House” option. This also highlights an important difference between the considered model and previous literature (e.g., Demange and Gale, 1985; Miyagawa, 2001). The multiplicity of a minimum equilibrium price vector has the severe consequence that *any* allocation mechanism based on minimum equilibrium prices is manipulable in the full preference domain. Similarly, the multiplicity of an allocation that maximizes trade has the consequence that *any* allocation mechanism based on maximum trade is manipulable in the full preference domain. However, by considering a preference domain that contains almost all preference profiles, it turns out that a minimum equilibrium price vector is unique for each preference profile in the restricted domain, and that it is possible to base an individual rational, equilibrium selecting, and group non-manipulable allocation mechanism on this unique price vector. In addition, this mechanism turns out to be a maximum trade mechanism in the restricted domain, and it, therefore, guarantees that the maximum number of houses are transferred from the owner to the existing tenants.

1.1 Application: The U.K. Housing Act 1980

To explain how the model and results of this paper can be adopted in order to extend and improve the U.K. Housing Act 1980, we first need to give a short background.

Public housing is a common form of housing tenure in which the property is owned by a local or central government authority. This type of tenure has traditionally referred to a situation where the central authority lets the right to occupy the units to tenants. In the last 30–35 years, however, many European countries have experienced important changes in their policies. In the United Kingdom, for example, the “Right to Buy” was implemented in the U.K. Housing Act 1980. As its name indicates, the legislation gave existing tenants the *right* to buy the houses they were living in. A tenant could, however, choose to remain in the house and pay the regulated rent.¹ The U.K. Secretary of State for the Environment in 1979, Michael Heseltine, stated that the main motivation behind the Act was:

“... to give people what they wanted, and to reverse the trend of ever increasing dominance of the state over the life of the individual.”²

In other words, the main objective of the Act was to transfer the ownership of the houses from the local or central authority to the existing tenants. To cope with this objective,

¹In the United Kingdom, local authorities have always had legal possibilities to sell public houses to tenants, but until the early 1970s such sales were extremely rare.

²From “Housing Bill – Provisions and Enactment” in Keesing’s Contemporary Archives v.27, January, 1981, p.30644.

the Act also specified that tenants could buy the houses at prices significantly below the market price and that resale is not allowed within five years after purchasing the houses. In the latest update of the Act from 2012, this “discount” was specified to a maximum of £75000 or 60 percent of the house value (70 percent for an apartment) depending on which is lower.³ As can be expected, the result of the Act was that the proportion of public housing in United Kingdom fell from 31 percent in 1979 to 17 percent in 2010.⁴ Similar legislations, with similar effects, have been passed in several European countries, including Germany, Ireland, and Sweden, among others.

The formal model considered in this paper can be applied to the above type of housing market, and its main practical implication is an extension of the U.K. Housing Act 1980 and its European equivalents. The significant difference between the U.K. Housing Act 1980, and its European equivalents, and the model considered in this paper is that we extend the situation from the case where *only* an existing tenant can buy the house that he currently is occupying to a situation where houses can be reallocated among *all* existing tenants in a pre-specified neighborhood. This extension is realistic as “mutual exchange” between tenants already is allowed in the U.K. and many other European countries with a legislated “Right to Buy” option.⁵

A relevant question is of course why the extension of the current U.K. system, proposed in this paper, should be of any interest to a policy maker. As we will argue, there are, at least, three good reasons for this. First, as a tenant in our framework always can choose to continue renting the house they live in or to buy it at the reservation price, exactly as in the prevailing U.K. system, but have the opportunity to buy some other house in the neighborhood, all tenants are weakly better off in the considered model compared to the current U.K. system. Second, because all sold houses in the current U.K. system are sold at the reservation prices, but all sold houses in the considered model are sold at prices weakly higher than the reservation prices, the public authority generates a weakly higher revenue in the considered model compared to the current U.K. system.⁶ Third, the considered model captures the idea that housing needs may change over time, e.g., a family has more children or some children move out. In such cases, a tenant may want to change house but not its housing area. These tenants are given permission to participate in our housing market, but they are not allowed to participate in the prevailing U.K. system as resale not is allowed within five years after purchasing the house.

In summary, the theoretical model considered in this paper can be seen as a realistic extension and an improvement of the current U.K. system for transferring the ownership of

³“Reforming the Right to Buy in 2012 and 2013”, Commons Library Standard Note, U.K. Parliament (retrieved June 12, 2013).

⁴“Housing Europe Review 2012 – The nuts and bolts of European social housing systems”, European Federation of Public, Cooperative and Social Housing (Brussels).

⁵Mutual exchange refers to a situation where two (or more) tenants in the public housing sector “swap” their houses when renting from a public authority. Typical requirements for a mutual exchange to take place are that none of the tenants included in the swap owes rent, is in the process of being evicted, and is moving to a home that the landlord believes is too big or small for their circumstances.

⁶This argument is, of course, only valid if the houses that are sold in the current U.K. system also are sold in the considered model. This is always the case as later illustrated in Proposition 7.

public housing to existing tenants, and the main theoretical finding is a mechanism, that satisfies a number of desirable properties, which can be adopted in order to transfer the ownership.

1.2 Related Literature

The main results of the paper contribute to the existing matching literature in a deeper sense. More explicitly, and to the best of our knowledge, this paper is the first to provide an individually rational, equilibrium selecting, trade maximizing, and non-manipulable allocation mechanism for a housing market with initial endowments (initial ownership) where monetary transfers are allowed. The main results presented in this paper are non-trivial extensions of similar and previously known results because most of the previous literature either consider the case with no initial endowments and where monetary transfers are allowed, or initial endowments but where monetary transfers are not allowed. Furthermore, we are also not aware of *any* matching model that can deal with the two types of outside options that we consider in this paper. In the following, we explain in more detail how the theoretical findings of this paper contribute to the existing and related literature.

The idea of using a minimum equilibrium price vector as a key ingredient in an individually rational, equilibrium selecting, and non-manipulable allocation mechanism was first advocated by Vickrey (1961) in his single-unit sealed-bid second-price auction. This principle was later generalized by Demange and Gale (1985) and Leonard (1983) to the case when multiple heterogeneous houses are sold.⁷ The main differences between this paper and Demange and Gale (1985) and Leonard (1983) is that those consider a two-sided market. Hence, their model cannot handle the case of existing tenants or initial ownership, and, consequently, not the case when buyers can block the trade of a house through the “Right to Stay and Rent” or the “Right to Buy His Current House” option. However, the allocation mechanism considered in this paper and the one in Demange and Gale (1985) share the property that the outcome is an “equilibrium” in the sense that each agent is assigned his weakly most preferred consumption bundle from his consumption set once the prices are determined by the mechanism.

A model with existing tenants and prices is studied by Miyagawa (2001). In his setting, each tenant owns precisely one house and is a seller as well as a buyer, so money and houses are reallocated through a price mechanism exactly as in this paper. Unlike this paper (and, e.g., Vickrey, 1961; Demange and Gale, 1985; Leonard, 1983), Miyagawa (2001) requires the mechanism to be “non-bossy” (Satterthwaite and Sonnenschein, 1981) which roughly means that no tenant can change the assignment for some other tenant without changing the assignment for himself. The assumption of non-bossiness has dramatic consequences on any non-manipulable allocation mechanism in this setting (see also Schummer, 2000). Namely, any individually rational, non-bossy, and non-manipulable allocation mechanism is a fixed price mechanism, and, therefore, the outcome is generally not an “equilibrium”. Because we do not require non-bossiness, our allocation mechanism is not a fixed price

⁷See also Demange, Gale and Sotomayor (1986) or Sun and Yang (2003).

mechanism and it always selects an equilibrium outcome.

Individually rational and non-manipulable mechanisms for housing markets with existing tenants have been considered previously in the literature when monetary transfers are not allowed. Most notably is Gale's Top-Trading Cycles Mechanism (Shapley and Scarf, 1974), that has been further investigated by, e.g. Ma (1994), Postlewaite and Roth (1977), Roth (1982) and Miyagawa (2002). This mechanism always selects an outcome in the core. However, the allocation mechanisms in these papers will, in similarity with Miyagawa (2001), not generally select an equilibrium outcome, neither can they handle the case with monetary transfers.

Finally, a recent and intermediate proposal is due to Andersson and Svensson (2014) where houses are allocated among a set of potential tenants on a two-sided market where prices are bounded from above by price ceilings imposed by the government or a local administration. They define an individual rational, equilibrium selecting and group non-manipulable allocation mechanism that, in its two limiting cases, selects that same outcome as the mechanism in Demange and Gale (1985) and the Deferred Acceptance Algorithm (Gale and Shapley, 1962). However, even if their mechanism is based on a minimal (rationing) equilibrium price vector, their allocation mechanism cannot handle existing tenants and, unlike the mechanism considered in this paper, a rationing rule is typically needed to solve the allocation problem.

1.3 Outline of the Paper

The remaining part of this paper is organized as follows. Section 2 introduces the formal model and some of basic definitions that will be used throughout the paper. The allocation mechanisms are introduced in Section 3 where also the main existence and non-manipulability results of the paper are stated. Section 4 provides a dynamic process for identifying the outcome of the considered allocation mechanism. Section 5 contains some concluding remarks. All proofs are relegated to the Appendix.

2 The Model and Basic Definitions

The *agents* and the *houses* are gathered in the finite sets $A = \{1, \dots, n\}$ and $H = \{1, \dots, n\}$, respectively, where $n = |A|$ is a natural number. Note that the number of agents and houses coincide as we assume that there is an existing tenant in each house. Agent a is the existing tenant of house h if $h = a$.

Each house $h \in H$ has a price $p_h \in \mathbb{R}_+$. These prices are gathered in the price vector $p \in \mathbb{R}_+^n$ which is bounded from below by the *reservation prices* $\underline{p} \in \mathbb{R}_+^n$ of the owner. The reservation price of house h is \underline{p}_h . The reservation prices are arbitrary but fixed and define

a feasible set of prices Ω according to:⁸

$$\Omega = \{p \in \mathbb{R}_+^n : p \geq \underline{p}\}.$$

Agent $a \in A$ can continue renting house $h = a$ at the given fixed rent (“Right to Stay and Rent”) or buy the house he currently is living in at the owner’s reservation price \underline{p}_a (“Right to Buy His Current House”). Agent a can also buy house $h \neq a$ at price p_h . For notational simplicity, and without loss of generality, the fixed rents will not be introduced in the formal framework. Formally, each agent $a \in A$ consumes exactly one (consumption) bundle, x_a , in his *consumption set* $X_a = (H \times \mathbb{R}_+) \cup \{a\}$ where:

$$x_a = \begin{cases} a & \text{if agent } a \text{ continues to rent house } h = a, \\ (a, \underline{p}_a) & \text{if agent } a \text{ buys house } h = a \text{ at price } \underline{p}_a, \\ (h, p_h) & \text{if agent } a \text{ buys house } h \text{ at price } p_h. \end{cases}$$

Note that each agent $a \in A$ has two *outside options*, i.e., the “Right to Stay and Rent”, $x_a = a$, and the “Right to Buy His Current House” (to the reservation price), $x_a = (a, \underline{p}_a)$.⁹ For convenience, we often denote the outside option of agent a simply by house a , i.e., $x_a = a$ will mean that either agent a continues renting *or* buys the house he currently is living in at its reservation price (whichever is better). The agent’s choice between these two options does not affect other agents. For technical reasons, an agent a can also buy his own house at the price p_a , but that will not be the choice of a utility maximizing agent if $p_a > \underline{p}_a$. For simplicity, a bundle of type (h, p_h) will often be written as (h, p) , i.e., $(h, p) \equiv (h, p_h)$. It is then understood that (h, p) means house $h \in H$ with price p_h at the price vector p .

Each agent $a \in A$ has *preferences* over bundles. These preferences are denoted by R_a and are represented by a complete preorder on X_a . The strict and indifference relations are denoted by P_a and I_a , respectively. Preferences are assumed to be *continuous*¹⁰, *strictly monotonic*, and *boundedly desirable*. Strict monotonicity means that agents strictly prefer a lower price to a higher price on any given house, i.e., $(h, p_h) P_a (h, p'_h)$ if $p_h < p'_h$ for any agent $a \in A$ and any house $h \in H$. Bounded desirability means that if the price of a house is “sufficiently high”, the agents will strictly prefer to keep the house they currently are living in rather than buying some other house, i.e., $a P_a (h, p_h)$ for each agent $a \in A$ and for each house $h \in H$ for p_h “sufficiently high”. All preference relations R_a satisfying the above properties for agent $a \in A$ are gathered in the set \mathcal{R}_a . A (preference) *profile* is a list $R = (R_1, \dots, R_n)$ of the agents’ preferences. This list belongs to the set $\mathcal{R} = \mathcal{R}_1 \times \dots \times \mathcal{R}_n$. We also adopt the notational convention of writing a profile $R \in \mathcal{R}$ as $R = (R_C, R_{-C})$ for some nonempty subset $C \subseteq A$.

⁸One can think of the fixed reservation prices as exogenously given and specified in the law as explained in the Introduction.

⁹In an alternative interpretation of the formal model, each agent owns the house he is living in. The price difference $p_h - \underline{p}_h$ can then be interpreted as a tax on trade imposed by a government. In that case $x_a = a$ and $x_a = (a, \underline{p}_a)$ are the same option, and it can be interpreted as “Not Selling” (or “No Tax to be Paid”).

¹⁰Continuity means that the weak upper and lower contour sets of R_a are closed in X_a .

A *state* is a triple (μ, ν, p) , where $\mu : A \rightarrow H$ is a mapping assigning agents to houses, $\nu : A \rightarrow \{0, 1\}$ is an assignment indicating if an agent $a \in A$ is renting, $\nu_a = 0$, or buying, $\nu_a = 1$, and $p \in \mathbb{R}_+^n$ is a price vector. If agent $a \in A$ is assigned house $\mu_a \in H$ and $\nu_a = 1$, he pays:

$$p_{\mu_a} = \begin{cases} p_{\mu_a} & \text{if } \mu_a \neq a, \\ \underline{p}_a & \text{if } \mu_a = a. \end{cases}$$

The assignment function μ is a bijection with the restriction $\mu_a = a$ if $\nu_a = 0$. This means that an agent cannot rent a house that currently he is not living in. Agent a is also the only agent that always can buy house $h = a$ at the reservation price. We will use the simplified notation $x = (\mu, \nu, p)$ for a state, where $x \in \times_{a \in A} X_a$ and $x_a = (\mu_a, p)$ if $\nu_a = 1$ and $\mu_a \neq a$, and $x_a = a$ if $\mu_a = a$ and $\nu_a \in \{0, 1\}$. Here, it is understood that $\nu_a = 0$ only if $a P_a(a, \underline{p}_a)$.

The *cardinality* $|\nu|$ of the assignment ν indicates the number of agents who are buying a house at state (μ, ν, p) , and it is defined by $|\nu| = \sum_{a \in A} \nu_a$.

Definition 1. For a given profile $R \in \mathcal{R}$, a price vector $p \in \Omega$ is an *equilibrium price vector* if there is a state $x = (\mu, \nu, p)$ such that the following holds for all agents $a \in A$: (i) $x_a R_a a$, and (ii) $x_a R_a(h, p)$ for all $h \in H$. If, in addition, the cardinality of the assignment ν is maximal among all states with price vector p satisfying (i) and (ii), the state x is an *equilibrium state*.

The first condition of the definition states that any agent weakly prefers his bundle to the house that he currently is renting (*individual rationality*) and to all other houses in his consumption set at the given prices. This can be seen as a fairness condition as all agents are assigned their most preferred consumption bundle from their consumption set at the given prices. The last condition essentially states that trade is maximal. This condition reflects the fact that the owner of the houses prefers to sell the houses rather than keeping existing tenants (recall from the Introduction that this is the main goal of the U.K. Housing Act 1980 and similar legislations).

For a given profile $R \in \mathcal{R}$, the set of equilibrium prices is denoted by Π_R , and the set of equilibrium states is denoted by \mathcal{E}_R . Hence:

$$\Pi_R = \{p \in \mathbb{R}^n : (\mu, \nu, p) \in \mathcal{E}_R \text{ for some assignments } \mu \text{ and } \nu\}.$$

An equilibrium price vector p' is a *minimum equilibrium price vector*, at a given profile $R \in \mathcal{R}$, if (i) $p' \in \Pi_R$ and (ii) $p \leq p'$ and $p \in \Pi_R$ imply $p = p'$. A state (μ', ν', p') is a *minimum price equilibrium*, at a given profile $R \in \mathcal{R}$, if $(\mu', \nu', p') \in \mathcal{E}_R$ and p' is a minimum equilibrium price vector.

Finally, a state (μ, ν, p) is a *maximum trade equilibrium*, at a given profile $R \in \mathcal{R}$, if $(\mu, \nu, p) \in \mathcal{E}_R$ and $|\nu| \geq |\nu'|$ for all $(\mu', \nu', p') \in \mathcal{E}_R$.

3 Manipulability and Non-Manipulability Results

As already explained in the Introduction, it is natural to let a minimum equilibrium price vector be a key ingredient in an allocation mechanism for the considered house allocation

problem as this type of mechanism previously has been demonstrated to be individually rational, equilibrium selecting, and non-manipulable in various economic environments. Another natural price mechanism is based on individual rationality, equilibrium selection, and maximal trade. Here, the latter axiom guarantees that the mechanism maximizes the number of traded houses, i.e., that it achieves the objectives of the U.K. Housing Act 1980 and similar legislations. These two mechanisms are formally defined next.

Let $\mathcal{E} = \bigcup_{R \in \mathcal{R}} \mathcal{E}_R$. A *mechanism* is a function $f : \mathcal{R} \rightarrow \mathcal{E}$ where, for each profile $R \in \mathcal{R}$, the mechanism f selects an equilibrium state $(\mu, \nu, p) \in \mathcal{E}_R$. A mechanism is called a *minimum price mechanism* if it, for each profile $R \in \mathcal{R}$, selects an equilibrium state $(\mu, \nu, p) \in \mathcal{E}_R$ where p is a minimum equilibrium price vector. A mechanism is called a *maximum trade mechanism* if it, for each profile $R \in \mathcal{R}$, selects an equilibrium state $(\mu, \nu, p) \in \mathcal{E}_R$ that, in addition, is a maximum trade equilibrium.

To be certain that the above two mechanisms are well-defined, we need to establish that the equilibrium set \mathcal{E}_R is nonempty for all profiles in \mathcal{R} because in this case, Π_R will also be nonempty for all profiles in \mathcal{R} by definition. The existence of a minimum equilibrium price vector then follows directly as the set of equilibrium prices is bounded from below by \underline{p} and closed since preferences are continuous, and the existence of a maximum trade equilibrium follows by the non-emptiness of the equilibrium set. In addition, any equilibrium with a minimum equilibrium price vector is *weakly efficient*: there does not exist any other state with a feasible price vector which all agents strictly prefer to the equilibrium; and if there is a unique minimum equilibrium price vector, then all agents weakly prefer any minimum price equilibrium to any other equilibrium state.

Proposition 1. Let $R \in \mathcal{R}$ be a profile and \underline{p} be a vector of reservation prices. Then:

- (i) The set of equilibria \mathcal{E}_R is nonempty.
- (ii) Any minimum price equilibrium is weakly efficient.
- (iii) If there is a unique minimum equilibrium price vector, then all agents weakly prefer any minimum price equilibrium to any other equilibrium state.

We will next, in a series of examples and propositions, investigate the similarities and differences between the two mechanisms defined above. The first example demonstrates that the selection of a minimum price mechanism and the selection of a maximum trade mechanism need not be identical in the full preference domain \mathcal{R} . This result is a bit counterintuitive as one would suspect that these mechanisms always recommend the same selection as lower prices intuitively should increase trade.

Example 1. Let $A = \{1, 2, 3\}$ and $H = \{1, 2, 3\}$ be the sets of agents and houses, respectively, where agent $a \in A$ is the existing tenant of house $h = a$. Let $\underline{p} = (0, 0, 0)$. For each agent $a \in A$, preferences over bundles (h, p) are represented by a quasi-linear utility function $u_{ah}(p) = v_{ah} - p_h$ where the values v_{ah} are represented by real numbers. Let the

utility for agent $a \in A$ of renting house $h = a$ formally be represented by v_{a0} , and:

$$V = (v_{ah}) = \begin{pmatrix} -2 & -2 & 3 & 2 \\ -2 & 1 & 1 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

In this case, $p = (0, 0, 0)$ is the unique minimum equilibrium price vector as $x = (\mu, \nu, p)$ is an equilibrium state for $\mu = (2, 1, 3)$ and $\nu = (1, 1, 0)$. Note next that x is the only possible selection of a minimum price mechanism. Note also that agent 3 continues to rent the house that he currently is living in as this is strictly preferred to buying it at the reservation price. The state $x' = (\mu', \nu', p')$ is a possible selection of a maximum trade mechanism for $\mu' = (3, 2, 1)$, $\nu' = (1, 1, 1)$, and $p' = (0, 1, 0)$. Note that at state x' , agent 2 buys house 2 for its reservation price (and not for price $p'_2 = 1$). Somewhat surprisingly, agent 3 continues renting in the minimum price equilibrium but buys house 1 in the maximum trade equilibrium. Hence, $|\nu'| > |\nu|$ which demonstrates that a minimum price mechanism and a maximum trade mechanism need not make identical selections. Furthermore, all agents weakly prefer the minimum price equilibrium x to the maximum trade equilibrium x' as the utilities are given by $(3, 1, 0)$ and $(2, 1, 0)$, respectively. \square

The observation from Example 1 is in fact more general than it appears. This is revealed in the following proposition.

Proposition 2. Let f be a minimum price mechanism. Then f is a maximum trade mechanism if and only if $1 \leq |A| \leq 2$.

We also remark that a maximum trade mechanism need not be a minimum price mechanism for any $|A| \geq 1$ on the domain \mathcal{R} . This can intuitively be understood by considering the special case when all agents always strictly prefer to buy the house they currently are living in to renting it or to buying some other house, i.e., $(a, \underline{p}_a)P_a a$ and $(a, \underline{p}_a)P_a(h, p_h)$ for all $a \in A$, all $h \neq a$, and all $p \geq \underline{p}$. In this case, a minimum price mechanism will have maximal trade, as all agents buy the house they are living in, and the unique minimum equilibrium price vector is given by $\underline{p} = (\underline{p}_1, \dots, \underline{p}_n)$. In this special case, however, maximal trade will be the outcome of any equilibrium selecting mechanism for any price vector $p \geq \underline{p}$, i.e., a maximum trade mechanism need not necessarily recommend the same state as a minimum price mechanism as the price vector may differ between the two mechanisms.

The following example demonstrates that neither a minimum price mechanism nor a maximum trade mechanism need to make a unique price selection on the full preference domain \mathcal{R} .

Example 2. Let $A = \{1, 2, 3, 4\}$ and $H = \{1, 2, 3, 4\}$ be the sets of agents and houses, respectively. Let $\underline{p} = (0, 0, 0, 0)$. As in Example 1, it is assumed that preferences are represented by quasi-linear utility functions where:

$$V = (v_{ah}) = \begin{pmatrix} 0 & 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 2 & -2 & -2 & 1 \\ 0 & -2 & 2 & -2 & 1 \end{pmatrix}.$$

In this case, both $p' = (1, 0, 0, 0)$ and $p'' = (0, 1, 0, 0)$ are minimum equilibrium price vectors (details are provided in the proof of Proposition 3). This follows since both $x' = (\mu', \nu', p')$ and $x'' = (\mu'', \nu'', p'')$ are equilibrium states for $\mu' = (1, 3, 4, 2)$, $\nu' = (1, 1, 1, 1)$, $\mu'' = (3, 2, 1, 4)$, and $\nu'' = (1, 1, 1, 1)$. Note that at state x' , agent 1 buys his house at its reservation price 0, at state x'' agent 2 buys his house at its reservation price 0, and that trade is maximized at states x' and x'' since $|\nu'| = |\nu''| = |A|$. Note that agent 3's utility in x' is 1 whereas agent 3's utility in x'' is 2. Similarly, agent 4's utility in x' is 2 whereas agent 4's utility in x'' is 1. In other words, agents 3 and 4 have opposed preferences over x' and x'' . \square

The multiplicity of a minimum equilibrium price vector and a maximum trade equilibrium is a direct consequence of the fact that agents can block the trade of a house through their outside option “Right to Stay and Rent” or “Right to Buy His Current House” (at the reservation price). This has the severe consequence that *any* minimum price mechanism and *any* maximum trade mechanism is manipulable on the full preference domain, at least if there are more than three houses on the housing market (see Proposition 3). The following notion of group manipulability and (group) non-manipulability is employed.

Definition 2. A mechanism f is *manipulable* at a profile $R \in \mathcal{R}$ by a nonempty group of agents $C \subseteq A$ if there is a profile $R' = (R'_C, R_{-C}) \in \mathcal{R}$, and two states $f(R) = x = (\mu, \nu, p)$ and $f(R'_C, R_{-C}) = x' = (\mu', \nu', p')$ such that $x'_a P_a x_a$ for all $a \in C$.

If the mechanism f is not manipulable by any group $C \subseteq A$ at profile R , then f is *group non-manipulable at R* . Given $\mathcal{R}^* \subseteq \mathcal{R}$, the mechanism f is *group non-manipulable on the domain \mathcal{R}^** if for any profile $R \in \mathcal{R}^*$, f is *group non-manipulable at R* .

If the mechanism f is not manipulable by any group of size one (or any agent) at profile R , then f is *non-manipulable at R* . Given $\mathcal{R}^* \subseteq \mathcal{R}$, the mechanism f is *non-manipulable on the domain \mathcal{R}^** if for any profile $R \in \mathcal{R}^*$, f is *non-manipulable at R* .

Proposition 3. A minimum price mechanism f is non-manipulable on the domain \mathcal{R} if and only if $1 \leq |A| \leq 3$.

The result in Proposition 3 does not generally carry over to maximum trade mechanisms. It is indeed true that any maximum trade mechanism can be manipulated if $|A| > 3$ (see Proposition 4) but due to the large number of possible maximum trade mechanisms, it depends on the specific maximum trade mechanism if it is manipulable or not when $1 \leq |A| \leq 3$. For example, as a minimum price mechanism is a maximum trade mechanism in the interval $1 \leq |A| \leq 2$, by Proposition 2, it follows from Proposition 3 that a maximum trade mechanism that, in addition, is a minimum price mechanism is non-manipulable if $1 \leq |A| \leq 2$. On the other hand, a maximum trade mechanism that avoids to make the same selection as a minimum price mechanism for any profile in \mathcal{R} whenever possible is non-manipulable on the domain \mathcal{R} if and only if $|A| = 1$.¹¹

¹¹The interested reader may consult the following example which is the key in a formal proof. Suppose, as in Example 1, that preferences are represented by quasi-linear utility functions, and that $A = H = \{1, 2\}$,

Proposition 4. A maximum trade mechanism f is manipulable on the domain \mathcal{R} if $|A| > 3$.

The implication from Propositions 3 and 4 is that if one searches for non-manipulable mechanisms on the full preference domain, one cannot search in the class of minimum price mechanisms or the class of maximum trade mechanisms, at least, if one is interested in housing markets containing more than three houses. Of course, there are non-manipulable mechanisms also on the full preference domain, e.g., a mechanism which, for any profile, sets prices “sufficiently high”¹² so that all agents prefer either renting or buying the house they currently occupy. This mechanism always recommends the identical outcome as the mechanism which is currently used in the United Kingdom, and it will not be of any interest to a public authority that aims to transfer *more* houses to tenants compared to the existing U.K. system.

In the following, we will demonstrate that by excluding some profiles from the domain \mathcal{R} and instead consider the reduced preference domain $\tilde{\mathcal{R}} \subset \mathcal{R}$ where no two outside options, or (using our convenience) no two houses, are “connected by indifference” at any price vector $p \in \Omega$, a minimum equilibrium price vector is unique for all profiles in the reduced domain. This unique price vector is then demonstrated to be the key ingredient in an individually rational, equilibrium selecting, and group non-manipulable mechanism, which, in addition, maximizes trade.

Definition 3. For a given profile $R \in \mathcal{R}$, two houses, h and h' , in H are *connected by indifference* if there is a price vector $p \in \Omega$, a sequence of distinct agents (a_1, \dots, a_q) , and a sequence of distinct houses (h_1, \dots, h_{q+1}) for $q \geq 2$ such that:

- (i) $h = h_1 = a_1$, and $h' = h_{q+1} = a_q$,
- (ii) $a_1 I_{a_1}(h_2, p)$, and $a_q I_{a_q}(h_q, p)$,
- (iii) $(h_j, p) I_{a_j}(h_{j+1}, p)$ for $2 \leq j \leq q-1$ if $q > 2$.

The subset of \mathcal{R} where no two houses are connected by indifference is denoted by $\tilde{\mathcal{R}}$.

In Definition 3, house a_1 stands for agent a_1 ’s outside option of continuing renting a_1 or buying a_1 at its reservation price (and similarly, for house a_q and agent a_q). We remark that the above type of domain restriction contains almost all preference profiles.¹³ In fact,

$v_{10} = v_{11} = v_{20} = -2$, $v_{12} = 0$, $v_{21} = 1$, and $v_{22} = 0$. In this case, a maximum trade mechanism that does not make the same selection as a minimum price mechanism (whenever possible) will select the state $x = (\mu, \nu, p)$ where $\mu = (2, 1)$, $\nu = (1, 1)$, and $p = (\alpha, 0)$ for some $0 < \alpha \leq 1$. However, if agent 2 misrepresents and instead reports $\tilde{v}_{21} = 0$, agent 2 will still be assigned house 1 but at price 0. Hence, the gain by misrepresenting is $\alpha > 0$.

¹²Formally, for any $R \in \mathcal{R}$, choose $p \in \mathbb{R}_+$ such that $aP_a(h, p)$ for all $a \in A$ and all $h \in H$ (and such prices exist because preferences are boundedly desirable).

¹³Informally, this can be illustrated in the following way. Let $R \in \mathcal{R}$ be any profile and h' and h'' be any two houses in $H \cup \{0\}$. Further, let $(a_j)_{j=1}^t$ and $(h_j)_{j=1}^{t+1}$ be any two sequences of distinct agents and

one can argue that the restriction of profiles to $\tilde{\mathcal{R}} \subset \mathcal{R}$ is an assumption of the same character as assuming strict preferences in the absence of monetary transfers, as, e.g., Gale and Shapley (1962), Shapley and Scarf (1974), Roth (1982) and Ma (1994) among others, as also this assumption is mild if preferences are chosen randomly since the probability of an indifference then is zero.

The next example demonstrates how to design reservation prices such that no two houses are connected by indifference given that preferences are represented by quasi-linear utility functions where the valuations are rational numbers.

Example 3. Let \mathbb{Q} denote the set of all rational numbers. For any agent $a \in A$, let \mathcal{R}_a^q consist of all quasi-linear utility functions where $v_{ah} \in \mathbb{Q}$ for all $h \in H \cup \{0\}$. Now choose reservation prices \underline{p} such that $\underline{p}_h \in \mathbb{R}_+ \setminus \mathbb{Q}$ for all $h \in H$, and $\underline{p}_h - \underline{p}_{h'} \in \mathbb{R} \setminus \mathbb{Q}$ for any distinct houses $h, h' \in H$. Then it is easy to verify that $\mathcal{R}_1^q \times \cdots \times \mathcal{R}_n^q \subset \tilde{\mathcal{R}}$. This follows because if, for some $p \in \Omega$ and two sequences of distinct agents (a_1, \dots, a_q) and distinct houses (h_1, \dots, h_{q+1}) , we have:

- (i) $h_1 = a_1$ and $h_{q+1} = a_q$,
- (ii) $v_{a_1 h_1} - \underline{p}_{h_1} = v_{a_1 h_2} - p_{h_2}$ and $v_{a_q h_q} - p_{h_q} = v_{a_q h_{q+1}} - \underline{p}_{h_{q+1}}$, and;
- (iii) $v_{a_j h_j} - p_{h_j} = v_{a_j h_{j+1}} - p_{h_{j+1}}$ for $2 \leq j \leq q-1$,

then summing all left-hand sides and all right-hand sides yields:

$$\sum_{j=1}^q (v_{a_j h_j} - v_{a_j h_{j+1}}) = \underline{p}_{h_1} - \underline{p}_{h_{q+1}},$$

which is a contradiction because the left-hand side belongs to \mathbb{Q} and the right-hand side belongs to $\mathbb{R} \setminus \mathbb{Q}$.¹⁴ \square

Note that all our results apply to any subdomain $\mathcal{R}^* \subseteq \tilde{\mathcal{R}}$, i.e., for applying our results one may construct/design such subdomains and define the mechanisms on this restricted subdomain.

The first main result of the paper demonstrates that the set of equilibrium prices Π_R has a unique minimum equilibrium price vector at any profile $R \in \tilde{\mathcal{R}}$. The proof proceeds by showing that from any two equilibrium states we can construct a new equilibrium state

houses, respectively, such that $h' = h_1$ and $h'' = h_{t+1}$. Then, because of monotonicity and continuity of preferences, there is a unique sequence $(p_{h_j})_{j=1}^t$ of prices such that $(h_j, p_{h_j}) I_{a_j} (h_{j+1}, p_{h_{j+1}})$ for all $j < t$. But for houses h_t and h_{t+1} to be connected by indifference, it is required that $(h_t, p_{h_t}) I_{a_t} (h_{t+1}, p_{h_{t+1}})$. This will occur with probability zero in most cases, e.g., if preferences are quasi-linear and represented by utility functions $v_{ah} \in \mathbb{R}$ for $a \in A$ and $h \in H$, and the various utilities are randomly drawn from a bounded interval on \mathbb{R} . In this meaning a limitation to profiles in $\tilde{\mathcal{R}}$ excludes very few profiles in \mathcal{R} .

¹⁴In some sense, this assumption is related to the “genericity assumption” of Acemoglu et al. (2008) whereby the powers of two coalitions are never identical: here the sum of evaluations is never identical to the difference of any two reservation prices.

with taking the minimum of the two equilibrium price vectors. This result also explains why there are two minimum equilibrium price vectors in Example 2 (and the minimum of the two minimum equilibrium price vectors is not an equilibrium price vector). More precisely, houses $h = 1$ and $h' = 3$ are connected by indifference at prices $p' = (1, 0, 0, 0)$. To see this, consider agents 1 and 2 and the sequence of houses $(1, 3, 2)$. In this case, $q = 2$, $a_1 = 1$, $a_2 = a_q = 2$, $1I_1(3, 0)$, and $2I_2(3, 0)$ so all conditions of Definition 3 are satisfied (note that the last condition in the definition is irrelevant because $q = 2$), i.e., houses $h = 1$ and $h' = 3$ are connected by indifference at prices $p' = (1, 0, 0, 0)$.

Theorem 1. There is a unique minimum equilibrium price vector $p^* \in \Pi_R$ for each profile $R \in \tilde{\mathcal{R}}$.

From Theorem 1, it is clear that the non-uniqueness problem, previously illustrated in Example 2, disappears on the restricted domain $\tilde{\mathcal{R}}$. What may not be so obvious, in the light of Proposition 2, is that a minimum price mechanism always is a maximum trade mechanism on the restricted domain $\tilde{\mathcal{R}}$.

Proposition 5. Let f be a minimum price mechanism. Then f is a maximum trade mechanism on the domain $\tilde{\mathcal{R}}$.

The second main result of the paper demonstrates that a minimum price mechanism, defined on the domain $\tilde{\mathcal{R}} \subset \mathcal{R}$, is group non-manipulable for all profiles in $\tilde{\mathcal{R}}$. Note also that this mechanism always selects an individually rational equilibrium outcome that maximizes trade at each profile in $\tilde{\mathcal{R}}$ by the definition of the mechanism and Proposition 5.

Theorem 2. Let f be a minimum price mechanism. Then f is group non-manipulable on the domain $\tilde{\mathcal{R}}$.

Note that in Theorem 2, the minimum price mechanism is defined on the full domain of preference profiles and for any profile in $\tilde{\mathcal{R}}$, any possible deviation of a coalition is allowed, i.e., the resulting profile after deviation does not need to belong to the restricted domain $\tilde{\mathcal{R}}$.

4 A Dynamic Price Process

This section provides a dynamic process for identifying a minimum equilibrium price vector in a finite number of steps, i.e., the dynamic process is an algorithm for computing a minimum price equilibrium. As an arbitrary but fixed profile $R \in \mathcal{R}$ is considered throughout the section, we will for notational simplicity drop the profile notation R in the equilibrium sets \mathcal{E}_R and Π_R , and instead write \mathcal{E} and Π , respectively.

The key in the dynamic process is a sequence of minimum equilibrium price vectors (p^1, \dots, p^T) consistent with a sequence of fixed assignments (ν^1, \dots, ν^T) . This type of sequence is called a “Dutch price sequence”, and to define it in more detail, some additional

notation is introduced. For this purpose, consider a fixed assignment ν (i.e., an assignment where, for each agent, is it exogenously given whether or not they rent a house), and let Π^ν be the set of equilibrium prices consistent with the fixed assignment ν . Such a set of equilibrium prices is called a price regime. Formally, for any fixed assignment ν , a set $\Pi^\nu \subset \Pi$ is defined to be a *price regime* if:

$$\Pi^\nu = \{p \in \Omega : \text{there is an assignment } \mu \text{ such that } p \in \Pi \text{ for } x = (\mu, \nu, p) \in \mathcal{E}\}.$$

For each assignment ν , the price regime Π^ν is a semi-lattice that is closed and bounded below and, hence, has a unique minimum equilibrium price vector $p^{\nu*} \in \Pi^\nu$. Let $x = (\mu, \nu, p) \in \mathcal{E}$ be an equilibrium state and define a price function $\pi : \mathcal{E} \rightarrow \Omega$ according to $\pi(x) = p^{\nu*}$. Hence, $\pi(x)$ is the unique minimum equilibrium price in the price regime defined by the assignment ν at the state x . Further, define a correspondence ξ from minimum price vectors $p^{\nu*}$ to equilibrium states according to:

$$\xi(p^{\nu*}) = \{x' \in \mathcal{E} : x' = (\mu', \nu', p^{\nu*}) \text{ for some assignments } \mu' \text{ and } \nu'\}.$$

Definition 4. A (possibly infinite) price sequence $(p^t)_{t=1}^T$ of equilibrium prices is a *Dutch price sequence* if:

- (i) $p^1 = \pi(x^0)$ for some equilibrium state $x^0 = (\mu^0, \nu^0, p^0)$ with $\mu_a^0 = a$ for all $a \in A$,
- (ii) there is a supporting sequence $(x^t)_{t=1}^T$ of equilibrium states such that for some $x^t \in \xi(p^t)$, it holds that $p^{t+1} = \pi(x^t)$,
- (iii) $p^T = \pi(x^{T-1})$ and for each $x \in \xi(p^T)$ and $p = \pi(x)$, it holds that $p = p^T$ whenever the price sequence $(p^t)_{t=1}^T$ ends at a finite step T .

To get a better understanding of a Dutch price sequence, consider Figure 1 and the price vector p^0 where all agents prefer to keep the house they currently are living in by either renting or buying it. Such a price vector always exists as this always will be the case for “sufficiently high” prices. But then there is an equilibrium state $x^0 = (\mu^0, \nu^0, p^0)$ as defined in Definition 4(i). Recall next that for the price vector p^0 , there is a corresponding price regime Π^{ν^0} with a unique minimum equilibrium price vector. This vector is denoted by $p^1 = \pi(x^0)$. Now, given prices p^1 , there is a corresponding supporting equilibrium state $x^1 = (\mu^1, \nu^1, p^1)$. Note that this state is not necessarily unique. In the left panel of Figure 1, for example, there are two supporting equilibrium states, $\hat{x}^1 = (\dots, \hat{x}_i^1, \hat{x}_j^1, \dots)$ and $\tilde{x}^1 = (\dots, \tilde{x}_i^1, \tilde{x}_j^1, \dots)$, with corresponding minimum equilibrium price vectors $\hat{p}^2 = \pi(\hat{x}^1)$ and $\tilde{p}^2 = \pi(\tilde{x}^1)$. Any of these price vectors may be chosen and the arguments can be repeated in the exact same fashion. In this way, a Dutch price sequence, $(p^1, p^2, \dots, p^t, \dots)$, as illustrated in the right panel of Figure 1, will be obtained (disregard the dotted path that connects the price vector in the Dutch price sequence for the moment, it will be explained later in this section). What may not be so obvious is that such a sequence always contains a finite number of equilibrium price vectors. This is demonstrated in the following proposition.¹⁵

¹⁵Note also that the above arguments can be used to construct a Dutch price sequence for any profile in \mathcal{R} and any vector of reservation prices. Hence, there always exists a Dutch price sequence.

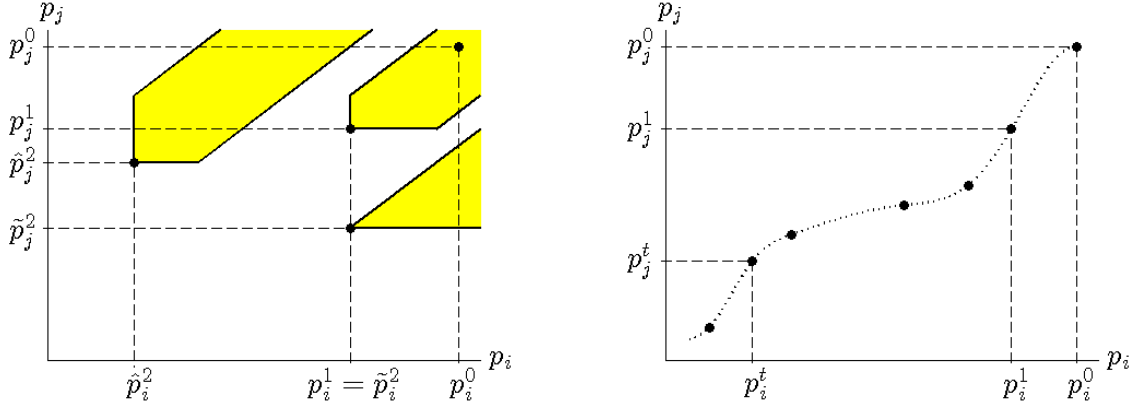


Figure 1: The left panel of the figure illustrates that a Dutch price sequence need not be unique for a given profile in \mathcal{R} , and the right panel of the figure illustrates a Dutch price sequence $(p^1, p^2, \dots, p^t, \dots)$.

Proposition 6. For any given profile $R \in \mathcal{R}$, a Dutch price sequence $(p^t)_{t=1}^T$ contains only a finite number of price vectors, i.e., $T < \infty$.

We next remark that any Dutch price sequence satisfies two important properties. First, $p^{t+1} \leq p^t$ and $p^{t+1} \neq p^t$ for any t in the sequence. This follows since $p^t \in \Pi^{\nu^t}$, and p^{t+1} is the minimum price vector in the price regime Π^{ν^t} . Second, the cardinality of the assignment ν is weakly increasing in the sequence, i.e., if the number of agents that are buying houses at t and $t+1$ are denoted by $|\nu^t|$ with $|\nu^{t+1}|$, respectively, it must be the case that $|\nu^{t+1}| \geq |\nu^t|$. This follows since, at an equilibrium state, the number of buying agents is always maximal at the given price vector. A change from the price regime Π^{ν^t} to the price regime $\Pi^{\nu^{t+1}}$ entails a change of agents that are buying, but the number of buying agents cannot decrease.

As is clear from the above description, there may be several Dutch price sequences for a given profile \mathcal{R} and a given initial start prices p^0 (see, e.g., the left panel of Figure 1). However, if the profile is selected from the domain $\tilde{\mathcal{R}}$ where no two houses are “connected by indifference”, there will be exactly one Dutch price sequence for the given profile. Even more strikingly, the end point of the sequence must be the, by Theorem 1, unique minimum equilibrium price vector $p^* \in \Pi$ for the given profile $R \in \tilde{\mathcal{R}}$. Both these results are formally stated in the last theorem of the paper.

Theorem 3. For any given profile $R \in \tilde{\mathcal{R}}$, a Dutch price sequence $(p^t)_{t=1}^T$ is unique, and p^T is the unique minimum equilibrium price vector in Π_R .

A final observation is that the price vectors in the a unique Dutch price sequence can be identified and connected by applying the “Vickrey-Dutch Auction” (Mishra and Parkes, 2009) in the case when preferences are quasi-linear, and by the Dutch counterpart of the “SA Auction” in Morimoto and Serizawa (2014) in the more general case. More explicitly, for any given profile in $\tilde{\mathcal{R}}$, the unique minimum equilibrium price vector can be obtained by adopting the following dynamic price process.

The Dynamic Price Process. Initialize the price vector to “sufficiently high” prices p^0 in the sense that each agent prefers not to buy any house they currently are *not* living in, i.e., $x_a^0 = a$ for all $a \in A$. Then for each Step $t := 0, \dots, T$:

Step t. For the given p^t and the fixed assignment ν^t , identify the unique minimum equilibrium price vector $p^{t+1} = \pi(x^t)$ in the corresponding price regime Π^{ν^t} . If $p^{t+1} = p^t$, stop. Otherwise for some $x^{t+1} = (\mu^{t+1}, \nu^{t+1}, p^{t+1}) \in \mathcal{E}$, set $t := t + 1$ and continue. \square

If one adopts the Vickrey-Dutch Auction or the Dutch counterpart of the SA Auction to identify the unique minimum equilibrium price vector in the corresponding price regime Π^{ν^t} , it follows directly from the above findings and the convergence results in Mishra and Parkes (2009, Theorem 2), and Morimoto and Serizawa (2014, Proposition 5.1) that the above process is well-defined and that it will converge to the unique minimum equilibrium price vector, for the given profile $R \in \tilde{\mathcal{R}}$, in a finite number of steps. In the right panel of Figure 1, the above procedure will generate the finite path which is represented by the dotted line.

5 Concluding Discussion

This paper has contributed to the matching literature in a deeper sense by considering a new model, and by providing a series of non-trivial extensions of previously known results. We have, however, left a number of interesting research questions for future research, and we believe that two of them stand out more than the others. First, is the domain where no two houses are “connected by indifference” at any price vector, the maximal domain under which a minimum price mechanism is non-manipulable? Second, is the minimum price mechanism the only individual rational, equilibrium selecting, trade maximizing, and non-manipulable mechanism on the domain $\tilde{\mathcal{R}}$? Currently, we do not have an answer to these two questions but it would be interesting to know them, especially since the results of this paper have some very relevant policy implications.

We conclude with a few remarks related to the U.K. Housing Act 1980, previously described in the Introduction. We first remark that in the special case of the formal model where A consists of *one* agent, the situation in the U.K. Housing Act 1980 and its European equivalents are fully reflected in our theoretical framework. That is, the single tenant is given a take-it-or-leave-it offer either to buy the house at a fixed and discounted price, or to continue renting the house. If the agent would report his preferences to the local or central administration, the minimum equilibrium price mechanism would recommend the real-world outcome. Hence, the model considered in this paper can be seen as a representation of the public housing market in United Kingdom today where each tenant is regarded as a “separate housing market”. The compromise proposed in this paper is to merge some of these “separate housing markets” into a new and larger housing market (e.g., all houses in a specific neighborhood). As a tenant always has the option to buy the particular house that he is living in at the fixed reservation price or to continue renting it at the regulated rent, in the formal framework, it is clear that all agents weakly prefer the outcome of the

investigated mechanism to the current U.K. system. In fact, from a revenue point of view, also the public authority weakly prefers the outcome of the minimum equilibrium price mechanism to the prevailing U.K. system as the revenue from the sales always is weakly higher in the suggested mechanism. This result is formally stated in the last proposition of the paper.

Proposition 7. For any profile $R \in \mathcal{R}$, the minimum price mechanism generates a weakly higher revenue to the public authority compared to the current U.K. system.

Appendix: Proofs

The Appendix contains the proofs of all results. It also contains some additional lemmas, definitions, and concepts. All results are proved in the same order as they are presented in the main text of the paper, except Proposition 3 which is proved directly after Theorem 2.

Proposition 1. Let $R \in \mathcal{R}$ be a profile and \underline{p} be a vector of reservation prices.

- (i) The set of equilibria \mathcal{E}_R is nonempty.
- (ii) Any minimum price equilibrium is weakly efficient.
- (iii) If there is a unique minimum equilibrium price vector, then all agents weakly prefer any minimum price equilibrium to any other equilibrium state.

Proof. (i) Since each house is boundedly desirable for each agent $a \in A$ at each profile $R \in \mathcal{R}$, by assumption, there is a price vector $\bar{p} > \underline{p}$ such that $aP_a(h, \bar{p})$ for all $a \in A$ and all $h \in H$. But then, $x = (\mu, \nu, \bar{p})$ constitutes an equilibrium state if $\mu_a = a$ for all $a \in A$, $\nu_a = 1$ if $(a, \underline{p}_a)R_a a$, and $\nu_a = 0$ if $aP_a(a, \underline{p}_a)$. Hence, $x \in \mathcal{E}_R$, and, consequently, $\mathcal{E}_R \neq \emptyset$ for each profile $R \in \mathcal{R}$.

(ii) Let $R \in \mathcal{R}$ and $x = (\mu, \nu, p)$ be an equilibrium state such that p is a minimum price vector. If x is not weakly efficient, then there exists another state $\hat{x} = (\hat{\mu}, \hat{\nu}, \hat{p})$ with a feasible price vector $\hat{p} \geq \underline{p}$ such that $\hat{x}_a P_a x_a$ for all $a \in A$. Because x is an equilibrium state, it must hold that $x_a \bar{R}_a a$ for all $a \in A$. Thus, $\hat{x}_a P_a x_a$ implies that $\hat{x}_a P_a a$ and $\hat{\mu}_a \neq a$. Again, because x is an equilibrium state and $\hat{\mu}_a \neq a$, it follows that $x_a R_a(\hat{\mu}_a, p)$. Thus, $\hat{x}_a P_a x_a$ implies that $\hat{x}_a P_a(\hat{\mu}_a, p)$ and $\hat{p}_{\hat{\mu}_a} < p_{\hat{\mu}_a}$. Because $\hat{\mu}$ is a bijection and \hat{x} is a state, now for all $h \in H$, $p_h > \hat{p}_h \geq \underline{p}_h$. Let $\epsilon = \min_{h \in H} (p_h - \hat{p}_h)$. Note that $\epsilon > 0$ and any agent a 's preference satisfies weak monotonicity: for all $p', p'' \in \mathbb{R}_+^n$ and all $h \in H$, $(h, p')R_a(h, p'')$ if $p'_h < p''_h$ (where $x'_a I_a x''_a$ for any two states x' and x'' for which $x'_a = a = x''_a$). Since Alkan et al. (1991) only require the weak monotonicity property, their Perturbation Lemma applies: there exists another equilibrium state $\tilde{x} = (\tilde{\mu}, \tilde{\nu}, \tilde{p})$ such that $\tilde{p} \leq p$, $\tilde{p} \neq p$, and for all $h \in H$, $p_h - \tilde{p}_h < \epsilon$. Thus, by our choice of ϵ , we have $\tilde{p} \geq \underline{p}$ and \tilde{p} is a feasible equilibrium price vector. Since $p \geq \tilde{p}$ and $p \neq \tilde{p}$, this is a contradiction to the fact that p was a minimum equilibrium price vector.

(iii) Suppose that $x^* = (\mu^*, \nu^*, p^*)$ is a minimum price equilibrium and p^* is the unique minimum equilibrium price vector. Then for any other $p \in \Pi_R$, we have $p \geq p^*$. If for some other equilibrium state $x = (\mu, \nu, p)$ and some agent $a \in A$, $x_a P_a x_a^*$, then from $x_a^* R_a a$ it follows that $\mu_a \neq a$. Because $\mu_a \neq a$ and x^* is an equilibrium, we have $x_a^* R_a(\mu_a, p^*)$. But now by $x_a = (\mu_a, p)$ and $x_a P_a x_a^*$, we have $(\mu_a, p) P_a(\mu_a, p^*)$ and $p_{\mu_a} < p_{\mu_a}^*$ which contradicts the fact that $p \geq p^*$. \square

Proposition 2. Let f be a minimum price mechanism. Then f is a maximum trade mechanism if and only if $1 \leq |A| \leq 2$.

Proof. We first prove that f is a maximum trade mechanism if $1 \leq |A| \leq 2$. For $A = \{1\}$, this follows directly as the single agent is given a take-it-or-leave-it offer to either to buy the house at price p_1 or to continue renting it, and the fact that the agent will continue to rent only if $1P_1(1, p_1)$. Suppose now that $A = \{1, 2\} = \{a_1, a_2\}$ but that f is not a maximum trade mechanism on the domain \mathcal{R} . This means that there exists a profile $R \in \mathcal{R}$ and two states, $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_R$, where the former state is selected by a minimum price mechanism and the latter by a maximum trade mechanism and $|\nu| < |\nu'|$. Because $|\nu| < |\nu'|$, there must be an agent $a_l \in A$ with $\mu_{a_l} = a_l$, $\nu_{a_l} = 0$, and $\nu'_{a_l} = 1$. Consequently, $\mu_{a_l} = a_l$ and $\mu_{a_k} = a_k$ (where $A = \{a_l, a_k\}$). By $\nu_{a_l} = 0$, $a_l P_{a_l}(a_l, p_{a_l})$. This together with $\nu'_{a_l} = 1$ implies $\mu'_{a_l} = a_k$ and $\mu'_{a_k} = a_l$ as $|A| = 2$. Note next that $x_{a_l} R_{a_l} x_{a_k}$ and $x_{a_k} R_{a_k} x_{a_l}$ as x is an equilibrium state. Because $\mu_{a_j} = a_j$ for $j = 1, 2$, it must be the case that $p_{\mu_j} \geq p'_{\mu_j}$ for $j = 1, 2$ otherwise x' cannot be an equilibrium state. Now, $p_{\mu_j} > p'_{\mu_j}$ for some j contradicts that p is a minimum equilibrium price vector in Π_R . Hence, $p_{\mu_j} = p'_{\mu_j}$ for $j = 1, 2$, and, consequently, $x_{a_l} I_{a_l} x_{a_k}$ and $x_{a_k} I_{a_k} x_{a_l}$ as x' is an equilibrium state. But then $x'' = (\mu'', \nu'', p'') = (\mu', \nu', p)$ is an equilibrium state at the minimum equilibrium price vector p with $|\nu''| > |\nu|$ contradicting that the cardinality $|\nu|$ is maximal at state x . Hence, state x cannot be an equilibrium.

The proof that f is a maximum trade mechanism on the domain \mathcal{R} only if $1 \leq |A| \leq 2$ follows directly from Example 1. To see this, note that in order to prove the result, it suffices to find a profile $R \in \mathcal{R}$ for an arbitrary $|A| \geq 3$ where the mechanism can be manipulated by some agent. Consider now Example 1, and suppose that we add an arbitrary but finite number of agents to the set $A = \{1, 2, 3\}$, and that $v_{j0} = 0$ and $v_{jk} = -2$ for $j = 4, \dots, n$, and all $k \in A^*$ where $A^* = \{1, \dots, 3, 4, \dots, n\}$. Suppose, in addition, that $v_{jk} = -2$ for $j = 1, 2, 3$ and $k = 4, \dots, n$. The added agents will always rent the house they currently live in and will not affect the outcome of the mechanism f for agents 1, 2, and 3. Because Example 1 demonstrates that a minimum price mechanism need not make an identical selection as a maximum trade mechanism when $|A| = 3$, the result will then carry over to the case when $|A| \geq 3$. \square

Proposition 4. A maximum trade mechanism f is manipulable if $|A| > 3$.

Proof. We will prove the proposition by identifying a profile $R \in \mathcal{R}$ where some agent can manipulate the outcome of an arbitrary maximum trade mechanism f when $|A| > 3$. Using the same arguments as in the proof of Proposition 2, it is sufficient to demonstrate the

result for $|A| = 4$. As in Example 2, let preferences be represented by quasi-linear utility functions where:

$$V = (v_{ah}) = \begin{pmatrix} 0 & 0 & -2 & 0 & -2 \\ 0 & -2 & 0 & 0 & -2 \\ 0 & 2 & -2 & -2 & 1 \\ 0 & -2 & 2 & -2 & 1 \end{pmatrix}.$$

In the remaining part of the proof, we let the profile $R \in \mathcal{R}$ denote the preferences that are represented by the above values. Note first that $x = (\mu, \nu, p)$ is an equilibrium state for $\mu = (1, 3, 4, 2)$, $\nu = (1, 1, 1, 1)$, and $p = (1, 0, 0, 0)$. Note, in particular, that $|\nu| = 4$. This also means that any selection $x' = (\mu', \nu', p')$ of a maximum trade mechanism f at profile R must have the property that $|\nu'| = 4$. But then it must also be the case that $\mu'_2 = 3$ and $p'_3 = 0$ at any selection x' of a maximum trade mechanism. This follows because if $\mu'_2 \neq 3$, then it must be the case that $\mu'_2 = 2$ and $\nu'_2 = 0$ otherwise agent 2 will envy the agent a with $\mu'_a = 2$ at any feasible price vector. This contradicts that x' is an equilibrium. But $\mu'_2 = 2$ and $\nu'_2 = 0$ contradicts that x' is selected by a maximum trade mechanism at profile R as this implies that $|\nu'| < 4$. Hence, $\mu'_2 = 3$ and $p'_3 = 0$. By using similar arguments, it is clear that $\mu'_1 = 1$ and $\nu'_1 = 1$. As a consequence, $\mu'_3 = 4$, $\nu'_3 = 1$, $p'_1 - p'_4 \geq 1$, and $p'_4 = 0 \leq p'_4 \leq 1$. But if $\mu'_3 = 4$ and $0 \leq p'_4 \leq 1$ at any selection $x' = (\mu', \nu', p')$ of a maximum trade mechanism f at profile R , the maximal utility that agent 3 can obtain at profile R equals $v_{34} - 0 = 1$.

Consider next the profile $\tilde{R} \in \mathcal{R}$ where all agents except agent 3 have the same values as in profile R , and:

$$\tilde{v}_{3h} = (0, 0, -2, -2, -2),$$

i.e., agent 3 misrepresents his values. Note first that $|\nu''| \leq 3$ for all $x'' \in \mathcal{E}_{\tilde{R}}$. To see this, suppose that $|\nu''| = 4$ for some $x'' \in \mathcal{E}_{\tilde{R}}$. By using the similar arguments as in the above, it must then be the case that $\mu''_2 = 3$, $\mu''_3 = 1$, and $p''_1 = p''_3 = 0$. But then agent 1 must be assigned house 2 or 4 and will, consequently, envy agents 2 and 3 at any feasible price vector. This contradicts that x'' is an equilibrium. Hence, $|\nu''| \leq 3$ for all $x'' \in \mathcal{E}_{\tilde{R}}$.

We next remark that $|\nu''| = 3$ for any maximum trade mechanism f at profile \tilde{R} . This follows directly as $x'' = (\mu'', \nu'', p'')$ is an equilibrium state at profile \tilde{R} for $\mu'' = (3, 2, 1, 4)$, $\nu'' = (1, 0, 1, 1)$, and $p'' = (0, 1, 0, 0)$.

The final part of the proof demonstrates that $\mu''_3 = 1$ and $p''_3 = 0$ for any selection x'' made by a maximum trade mechanism f at profile \tilde{R} . This completes the proof of the proposition as the utility of agent 3 when misrepresenting and when reporting truthfully then equals $u_{31}(p'') = 2 - 0 = 2$ and (at most) $v_{34} - 0 = 1$, respectively. Suppose first that $\mu''_3 \neq 1$. Then it must be the case that $\mu''_3 = 3$ and $v''_3 = 0$ because if this is not the case, then agent 3 will prefer continuing renting, i.e. $3\tilde{P}_3 x''_3$, at any feasible price vector, contradicting individual rationality of x'' . But if $\mu''_3 = 3$ and $v''_3 = 0$, it must be the case that $\mu''_2 = 2$ and $\nu''_2 = 0$ because if this is not the case, then agent 2 will envy the agent a with $\mu''_a = 2$ at any feasible price vector contradicting that x'' is an equilibrium. But if $\nu''_2 = \nu''_3 = 0$, then $|\nu''| < 3$ which contradicts that x'' is selected by a maximum trade mechanism f . Hence, $\mu''_3 = 1$. But if $\mu''_3 = 1$, then it must be the case that $p''_3 = 0$ because

if this not is the case, then agent 3 will envy the agent a with $\mu''_a = 3$ at any feasible price vector contradicting that x'' is an equilibrium. Hence, $\mu''_3 = 1$ and $p''_3 = 0$ as desired. \square

To prove the subsequent results, some consequences of the domain restriction (stated as lemmas) are derived for which some additional concepts are needed.

Let $q > 1$ and $a_j \in A$ for $1 \leq j \leq q$. Given two assignments μ and μ' , a *trading cycle* from μ to μ' is a sequence $G = (a_1, \dots, a_q)$ of distinct agents such that $\mu'_{a_j} = \mu_{a_{j+1}}$, for $1 \leq j < q$ and $\mu'_{a_q} = \mu_{a_1}$. For simplicity, we will use the same notation for the sequence G and the corresponding set $G = \{a_1, \dots, a_q\}$. Note that the complete trade from μ to μ' can be decomposed uniquely into a number of trading cycles.

Let $R, R' \in \tilde{\mathcal{R}}$ be two profiles such that $R' = (R'_C, R_{-C})$ for some $C \subseteq A$ (where possibly $C = \emptyset$ and $R = R'$), and consider two equilibrium states $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ where $x'_a P_a x_a$ for all $a \in C$. Let H_0, H_1, H_2 , and H_3 be defined as:

$$\begin{aligned} H_0 &= \{a \in A : \mu'_a = \mu_a = a\}, \\ H_1 &= \{h \in H : p'_h < p_h\} \setminus H_0, \\ H_2 &= \{h \in H : p'_h = p_h\} \setminus H_0, \\ H_3 &= \{h \in H : p'_h > p_h\} \setminus H_0. \end{aligned}$$

Lemma 1. Let R and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{\mathcal{R}}$, and $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states. Let $G = (a_1, \dots, a_q)$ be a trading cycle from μ to μ' , and $(\mu_{a_1}, \dots, \mu_{a_q})$ the corresponding set of houses.

(i) If $a_k \in G$ and $x'_{a_k} P_{a_k} x_{a_k}$, then $\mu'_{a_k} \in H_1$.

(ii) If $\mu_a \in H_1$, $\mu'_a \in H_2 \cup H_3$ and $a \notin C$, then $x_a = a$, $\mu'_a \in H_2$, and $x_a I_a x'_a$.

Proof. Part (i) of the lemma is proved by contradiction. Assume that $a_k \in G$ and $x'_{a_k} P_{a_k} x_{a_k}$, but that $\mu'_{a_k} \notin H_1$. Note first that $\mu'_{a_k} \neq a_k$. To see this, suppose that $\mu'_{a_k} = a_k$ or, equivalently, that $x'_{a_k} = a_k$. Because x is an equilibrium state, it must be the case that $x_{a_k} R_{a_k} a_k$. But then $x_{a_k} R_{a_k} x'_{a_k}$, which contradicts the assumption that $x'_{a_k} P_{a_k} x_{a_k}$. Hence, $\mu'_{a_k} \neq a_k$ and $\nu'_{a_k} = 1$. Note next that $x_{a_k} R_{a_k} (\mu'_{a_k}, p)$ and $(\mu'_{a_k}, p) R_{a_k} (\mu'_{a_k}, p')$ because x is an equilibrium state and $\mu'_{a_k} \notin H_1$, respectively. But then $x_{a_k} R_{a_k} (\mu'_{a_k}, p')$. Now, $x'_{a_k} = (\mu'_{a_k}, p')$ because $\mu'_{a_k} \neq a_k$. Consequently, $x_{a_k} R_{a_k} x'_{a_k}$ which, again, contradicts the assumption that $x'_{a_k} P_{a_k} x_{a_k}$. Hence, $\mu'_{a_k} \in H_1$.

To prove Part (ii) of the lemma, note that because $\mu_a \in H_1$ and $\mu'_a \in H_2 \cup H_3$, by assumption, it follows that $\mu_a \neq \mu'_a$ and that agent a must belong to some trading cycle from μ to μ' . Since $a \notin C$ we have $R'_a = R_a$. Thus, $x_a R_a (\mu'_a, p) R_a x'_a$ as $\mu'_a \in H_2 \cup H_3$ and $x'_a = (\mu'_a, p')$. Suppose that $\mu'_a \in H_3$. If $x_a = a$, then $a R_a (\mu'_a, p) P_a x'_a$, which contradicts that x' is individually rational. If $x_a \neq a$, then $(\mu_a, p') P_a (\mu_a, p) R_a x'_a$, which is a contradiction to $x' \in \mathcal{E}_{R'}$. Hence, $\mu'_a \notin H_3$, i.e., $\mu'_a \in H_2$. It now follows directly that if $x_a = a$ then $x_a I_a x'_a$, and if $x_a \neq a$ then $(\mu_a, p') P_a x'_a$, which is a contradiction to $x' \in \mathcal{E}_{R'}$. Hence, $x_a = a$, $\mu'_a \in H_2$, and $x_a I_a x'_a$. \square

Lemma 2. Let R and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{\mathcal{R}}$, and $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let also $G = (a_1, \dots, a_q)$ be a trading cycle from μ to μ' , and $(\mu_{a_1}, \dots, \mu_{a_q})$ the corresponding set of houses. Then $\mu_{a_k} \in H_1$ for some $a_k \in G$ implies that $\mu_{a_j} \notin H_3$ for all $a_j \in G$.

Proof. For notational simplicity, let $h_j = \mu_{a_j}$ for all $1 \leq j \leq q$. To obtain a contradiction, suppose that $h_k \in H_1$ but $h_l \in H_3$ where, without loss of generality, $k < l$ and $h_j \in H_2$ for all $k < j < l$. A first observation is that $a_j \notin C$ for $k \leq j < l$. This follows directly from Lemma 1(i) as $\mu'_{a_j} \in H_2 \cup H_3$ for all $k \leq j < l$ by construction. Hence, $R'_{a_j} = R_{a_j}$ for all $k \leq j < l$. We will consider the cases when $k + 1 = l$ and $k + 1 < l$. By Lemma 1(ii), $x_{a_k} = a_k = h_k$ and $h_{k+1} \in H_2$, implying that $k + 1 = l$ is impossible.

A second observation is that $x'_{a_j} I_{a_j} x_{a_j}$ for $k \leq j < l$. For $j = k$ this follows from Lemma 1(ii). For all $k < j < l - 1$, it follows from the facts that $h_j, h_{j+1} \in H_2$, $R'_{a_j} = R_{a_j}$, and that x and x' are equilibrium states. For $j = l - 1$, we have $a_{l-1} \notin C$, $\mu_{a_{l-1}} \in H_2$ and $\mu'_{a_{l-1}} \in H_3$. Applying Lemma 1(ii) to R' now yields $x'_{a_{l-1}} = a_{l-1}$ and $(h_{l-1}, p) I'_{a_{l-1}} a_{l-1}$. Now, because $k + 1 < l$ and both $x_{a_k} = a_k$ and $x'_{a_{l-1}} = a_{l-1}$, there exist l' and l'' where $k \leq l' < l'' \leq l - 1$ such that for all $l' < j < l''$ we have $x_j \neq a_j \neq x'_j$ and both $x_{a_{l'}} = a_{l'}$ and $x'_{a_{l''}} = a_{l''}$. But then:

- (a) $x'_{a_{l'}} = (h_{l'+1}, p)$, $a_{l'} I_{a_{l'}} (h_{l'+1}, p)$ and $R'_{a_{l'}} = R_{a_{l'}}$,
- (b) $(h_j, p) I_{a_j} (h_{j+1}, p)$ and $R'_{a_j} = R_{a_j}$ for all $l' < j < l''$,
- (c) $(h_{l''}, p) I'_{a_{l''}} a_{l''}$ (where $R'_{a_{l''}} = R_{a_{l''}}$ if $l'' < l$).

Now by (a)–(c), houses $a_{l'}$ and $a_{l''}$ are connected by indifference at profile R' which contradicts that the profile R' belongs to $\tilde{\mathcal{R}}$. \square

An immediate consequence of Lemma 2 is that the set of trading cycles from μ to μ' can be partitioned into two disjoint groups as explained in the following definition.

Definition 5. Let R and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{\mathcal{R}}$, and $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let $A^+ \subseteq A$ be such that $a \in A^+$ precisely when there is a trading cycle G from μ to μ' such that $a \in G$ and $\mu_{a'} \in H_1$ for some $a' \in G$. Let $A^- = A \setminus A^+$.

The notations A^+ and A^- are chosen because all agents in A^+ are weakly better off at the equilibrium state x' than at the equilibrium state x , while no agent in A^- is strictly better off at the equilibrium state x' than at the equilibrium state x . This is demonstrated in the next lemma.

Lemma 3. Let R and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{\mathcal{R}}$, and $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let $G \subseteq A$ be a trading cycle from μ to μ' . If $x'_a P_a x_a$ for some agent $a \in G$, then $G \subseteq A^+$ and $x'_a R_{\hat{a}} x_{\hat{a}}$ for all $\hat{a} \in G$.

Proof. Note first that Lemma 1(i) implies that $\mu'_a \in H_1$ since $x'_a P_a x_a$. Hence, $G \subseteq A^+$ and $\mu_{\hat{a}} \in H_1 \cup H_2$ for all $\hat{a} \in G$ by Lemma 2 and by the fact that G is a trading cycle from μ to μ' . Note next that either $x'_{\hat{a}} P_{\hat{a}} x_{\hat{a}}$ or $x_{\hat{a}} R_{\hat{a}} x'_{\hat{a}}$ for all $\hat{a} \in G$. In the latter case, $\hat{a} \notin C$ and $R'_{\hat{a}} = R_{\hat{a}}$. But then from $\mu_{\hat{a}} \in H_1 \cup H_2$ we obtain $x'_{\hat{a}} R_{\hat{a}}(\mu_{\hat{a}}, p') R_{\hat{a}}(\mu_{\hat{a}}, p)$ because x' is an equilibrium state. If $x_{\hat{a}} = (\mu_{\hat{a}}, p)$, then $x'_{\hat{a}} R_{\hat{a}} x_{\hat{a}}$. Otherwise, $x_{\hat{a}} = \hat{a}$ from the fact that x' is an equilibrium state, $x'_{\hat{a}} R_{\hat{a}} \hat{a}$. Hence, $x'_{\hat{a}} R_{\hat{a}} x_{\hat{a}}$ for all $\hat{a} \in G$. \square

Theorem 1. There is a unique minimum equilibrium price vector $p^* \in \Pi_R$ for each profile $R \in \tilde{\mathcal{R}}$.

Proof. Let $R \in \tilde{\mathcal{R}}$, and let $x' = (\mu', \nu', p') \in \mathcal{E}_R$ and $x'' = (\mu'', \nu'', p'') \in \mathcal{E}_R$ be two equilibrium states. We will demonstrate that $x = (\mu, \nu, p) \in \mathcal{E}_R$ for some assignments μ and ν if $p_h = \min\{p'_h, p''_h\}$ for each $h \in H$. The proof of the theorem then follows directly since the set Π_R is closed and bounded from below.

Consider the equilibrium states $x' = (\mu', \nu', p')$ and $x'' = (\mu'', \nu'', p'')$, and the trading cycles from μ' to μ'' . Let the sets A^+ and A^- be defined as in Definition 5 with the restriction $R = R'$, and define $p_h = \min\{p'_h, p''_h\}$ for each $h \in H$. Let:

$$\mu_a = \begin{cases} \mu''_a & \text{if } a \in A^+, \\ \mu'_a & \text{if } a \in A^-. \end{cases}$$

Note that the assignment μ defined in this way becomes bijective because no agent $a \in A^+$ belongs to a trading cycle containing an agent in A^- by Lemma 2. Let now $x_a = x''_a$ and $\nu_a = \nu''_a$ for all $a \in A^+$, and $x_a = x'_a$ and $\nu_a = \nu'_a$ for all $a \in A^-$. To prove the theorem, we need to demonstrate that $x = (\mu, \nu, p)$ is an equilibrium state. Now, the following is true for any $a \in A^+$ (recall that $R = R'$):

- (a) $x''_a R_a a$ because of individual rationality,
- (b) $x''_a R_a(h, p'')$ for all $h \in H$ as x'' is an equilibrium state,
- (c) $x''_a R_a x'_a$ by Lemma 3 since $a \in A^+$,
- (d) $x'_a R_a(h, p')$ for all $h \in H$ as x' is an equilibrium state.

From (a)–(d) in the above and the construction that $x_a = x''_a$ and $\nu_a = \nu''_a$ for all $a \in A^+$, it now follows that $x_a R_a a$ and $x_a R_a(h, p)$ for all $a \in A^+$ and all $h \in H$. Symmetric arguments now give that $x_a R_a a$ and $x_a R_a(h, p)$ for all $a \in A^-$ and all $h \in H$. Hence, $x = (\mu, \nu, p)$ is an equilibrium state if the assignment ν implies maximal trade (see Definition 1). If this condition is not satisfied, it only remains to change the assignment ν so trade becomes maximal. Hence, $x = (\mu, \nu, p)$ is an equilibrium state. \square

Proposition 5. Let f be a minimum price mechanism. Then f is a maximum trade mechanism on the domain $\tilde{\mathcal{R}}$.

Proof. Let $R \in \tilde{\mathcal{R}}$. To obtain a contradiction, suppose that f is a minimum price mechanism but not a maximum trade mechanism. Let also the state $x' = (\mu', \nu', p') \in \mathcal{E}_R$ be selected by the minimum price mechanism, i.e., p' is the unique minimum equilibrium price vector in Π_R by Theorem 1. From Theorem 1 and the assumption that f is not a maximum trade mechanism, it then follows that there is a state $x = (\mu, \nu, p) \in \mathcal{E}_R$ where $p' \neq p$, $p'_h \leq p_h$ for all $h \in H$, and $|\nu'| < |\nu|$.

Recall next that the complete trade from μ to μ' can be decomposed uniquely into a number of trading cycles. Because $|\nu'| < |\nu|$, by assumption, this means that there must be a trading cycle $G = (a_1, \dots, a_q)$ from μ to μ' where $\sum_{a \in G} \nu_a > \sum_{a \in G} \nu'_a$, and, consequently, an agent $a_l \in G$ with $\mu'_{a_l} = a_l$ and $\nu'_{a_l} = 0$. Note also that $\mu_{a_j} \in H_1 \cup H_2$ for all $j \in G$ as $p'_h \leq p_h$ for all $h \in H$.

Let now agent $a_l \in G$ be chosen as above, and let $a_j \in G$ for all $1 \leq j \leq q$. We will next demonstrate that $\mu_{a_l} \in H_2$ and $a_l I_{a_l}(\mu_{a_l}, p)$. As $\mu_{a_l} \in H_1 \cup H_2$, it suffices to show that $\mu_{a_l} \notin H_1$ to prove the first part of the statement. To obtain a contradiction, suppose that $\mu_{a_l} \in H_1$. Because $\mu_{a_l} \in H_1$, $\mu_{a_l} \neq a_l$, and $(\mu, \nu, p) \in \mathcal{E}_R$, it follows that:

$$(\mu_{a_l}, p') P_{a_l}(\mu_{a_l}, p) R_{a_l} a_l.$$

But then state (μ', ν', p') cannot belong to \mathcal{E}_R since $x'_{a_l} = a_l$. Hence, $\mu_{a_l} \in H_2$. But if $\mu_{a_l} \in H_2$ and $x'_{a_l} = a_l$, it is immediately clear that $x'_{a_l} I_{a_l} x_{a_l}$ as both x and x' belong to \mathcal{E}_R . Note that the latter condition may also be written as $a_l I_{a_l}(\mu_{a_l}, p)$ since $x'_{a_l} = a_l$.

Let again agent $a_l \in G$ be defined as in the above. Given the above findings, we next remark that either (i) $\mu_{a_j} \in H_2$ for all $k < j \leq l$ and $\mu_{a_k} \in H_1$ for some $a_k \in G$, or (ii) $\mu_{a_j} \in H_2$ for all $a_j \in G$. We will demonstrate that both these cases lead to the desired contradiction.

Case (i). Suppose first that $x_{a_k} \neq a_k$. Because $(\mu, \nu, p) \in \mathcal{E}_R$, $\mu_{a_k} \in H_1$, and $\mu'_{a_k} = \mu_{a_{k+1}} \in H_2$ it follows that:

$$(\mu_{a_k}, p') P_{a_k}(\mu_{a_k}, p) R_{a_k} x'_{a_k},$$

which contradicts that $(\mu', \nu', p') \in \mathcal{E}_R$. Hence, $x_{a_k} \neq a_k$ cannot be the case. Suppose instead that $x_{a_k} = a_k$, and note that $x_{a_j} I_{a_j} x'_{a_j}$ for all $k \leq j \leq l-1$. This follows as both x and x' belong to \mathcal{E}_R and $\mu_{a_j} \in H_2$ for all $k < j \leq l$. But then houses a_k and a_l are connected by indifference at prices p as $a_k I_{a_k}(a_{k+1}, p)$ and $a_l I_{a_l}(\mu_{a_l}, p)$. Hence, $x_{a_k} = a_k$ cannot be the case.

Case (ii). If $\mu_{a_j} \in H_2$ for all $a_j \in G$, it follows, by the same arguments as in Case (i), that $x_{a_j} I_{a_j} x'_{a_j}$ for all $a_j \in G$. Now, if $\mu_{a_{l'}} = a_{l'}$ for some $a_{l'} \in G$ and $\mu_{a_j} \neq a_j$ for all $l' < j \leq l$, houses $a_{l'}$ and a_l are connected by indifference at prices p as $a_{l'} I_{a_{l'}}(a_{l'+1}, p)$ and $a_l I_{a_l}(\mu_{a_l}, p)$. On the other hand, if $\mu_{a_{l'}} \neq a_{l'}$ for all $a_{l'} \in G$, then the trade at state x' can be increased since $\mu'_{a_l} = a_l$ and $\nu'_{a_l} = 0$, which contradicts that $x' \in \mathcal{E}_R$. \square

For any assignment μ , let $|\mu| = |\{a \in A : \mu_a \neq a\}|$ denote the number of agents who do not stay in their house (or who do not exercise their outside option). For convenience, we will call $|\mu|$ the cardinality of μ . The following lemma shows that for profiles in $\tilde{\mathcal{R}}$, it suffices to maximize the cardinality of $|\mu|$ instead of the cardinality of $|\nu|$.

Lemma 4. Let $R \in \tilde{\mathcal{R}}$ and $x = (\mu, \nu, p)$ be a state such that for all $a \in A$: (i) $x_a R_a a$ and (ii) $x_a R_a(h, p)$ for all $h \in H$. If the cardinality of $|\mu|$ is maximal among all states satisfying (i) and (ii) with price vector p , then x is an equilibrium state, i.e., $|\nu|$ is maximal among all states satisfying (i) and (ii) with price vector p .

Proof. To obtain a contradiction, suppose that $|\nu|$ is not maximal among all states satisfying (i) and (ii) with price vector p . Then there exists an equilibrium state $x' = (\mu', \nu', p')$ with $p' = p$ and $|\nu'| > |\nu|$. Then for some $a \in A$, $\nu'_a = 1$ and $\nu_a = 0$. Thus, $\mu_a = a$ and $x_a = a$. If $\mu'_a = a$, then $(a, p_a) I_a a$ which contradicts our assumption $\nu_a = 0$ only if $a P_a(a, p_a)$. Thus, $\mu'_a \neq a$ and a belongs to some trading cycle $G = (a_1, \dots, a_q)$ from μ to μ' where $a_1 = a$ and $\mu'_{a_j} = \mu_{a_{j+1}}$. Note that $a I_a(\mu'_a, p)$. If for all $1 \leq j \leq q$, $\mu'_{a_j} \neq a_j$, then (μ, ν, p) could not have maximized $|\mu|$ (simply define $x'' = (\mu'', \nu'', p'')$ by $p'' = p$, $x''_{a''} = x_{a''}$ for all $a'' \in A \setminus G$, and $x''_{a''} = x'_{a''}$ for all $a'' \in G$; then $|\mu''| > |\mu|$), which is a contradiction. Thus, choose $1 < l \leq q$ minimal such that $\mu'_{a_l} = a_l$. Note that $\mu_{a_l} \neq a_l$. But then $a_1 I_{a_1}(\mu'_{a_1}, p)$, $(\mu_{a_j}, p) I_{a_j}(\mu_{a_{j+1}}, p)$ for all $1 < j < l$, and $(\mu_{a_l}, p) I_{a_l} a_l$, which contradicts the fact that R is not connected by indifference. \square

Note that for any two equilibrium states $x = (\mu, \nu, p)$ and $x' = (\mu', \nu', p')$ with $p = p'$, it holds that:

$$x_a I_a x'_a \text{ for all } a \in A. \quad (1)$$

Condition (1) together with Lemma 4 have the important consequence that without loss of generality, for profiles $R \in \tilde{\mathcal{R}}$ below in Lemma 5 we focus on equilibrium states $x = (\mu, \nu, p)$ where the cardinality of $|\mu|$ is maximized. Lemma 5 is the key in the proofs of Theorems 2 and 3.

Lemma 5. Let $R, R' \in \tilde{\mathcal{R}}$ be two profiles such that $R' = (R'_C, R_{-C})$ for some $C \subset A$. Let also $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states such that $x'_a P_a x_a$ for all $a \in C$ and $|\mu|$ is maximized among all equilibrium states with price vector p . If $H_1 \neq \emptyset$, then there is a subset $S \subseteq H_1$ such that $A_S = \emptyset$ where:

$$A_S = \{a \in A : \mu_a \notin S \text{ and } x_a I_a(h, p) \text{ for some } h \in S \setminus \{a\}\}.$$

Proof. Without loss of generality, we may assume that $C = \{a \in A : x'_a P_a x_a\}$ as $R'_a = R_a$ is an allowed report for all agents $a \in C$. To obtain a contradiction, suppose that $A_S \neq \emptyset$ for each $S \subseteq H_1$. Then $A_{H_1} \neq \emptyset$ and there is an agent $a_0 \in A$ with $\mu_{a_0} \notin H_1$, $x_{a_0} I_{a_0}(h, p)$ for some $h \in H_1$, and $h \neq a_0$. Now, $x_{a_0} R_{a_0} a_0$ by individual rationality and $(h, p') P_{a_0}(h, p)$ as preferences are strictly monotonic, $h \neq a_0$ and $p'_h < p_h$. Hence, $(h, p') P_{a_0}(h, p) I_{a_0} x_{a_0} R_{a_0} a_0$ and, obviously, $a_0 \in C$ and $\mu'_{a_0} \in H_1$.

Note next that agent a_0 belongs to some trading cycle $G = (a_t, a_{t-1}, \dots, a_1, a_0)$ from μ to μ' . Since $\mu'_{a_0} \in H_1$, it follows from Lemma 2 that $\mu_{a_j} \in H_1 \cup H_2$ for all $0 \leq j \leq t$, and $\mu_{a_0} \in H_2$ because $\mu_{a_0} \notin H_1$. Let $0 < l \leq t$ be minimal such that $\mu_{a_l} \in H_1$, and note that such an index exists because $\mu'_{a_0} = \mu_{a_t} \in H_1$. But then $\mu'_{a_l} \in H_2$ and, consequently, $a_l \notin C$ by Lemma 1(i). Because $\mu_{a_l} \in H_1$ and $\mu'_{a_l} \in H_2$, Lemma 1(ii) yields $x_{a_l} = a_l$

and $a_l I_{a_l}(\mu_{a_{l-1}}, p)$. Furthermore, for $0 < j < l$, we have $\mu'_{a_j} \in H_2$, $a_j \notin C$, $x_{a_j} I_{a_j} x'_{a_j}$ and $x'_{a_j} \neq a_j$ because R belongs to $\tilde{\mathcal{R}}$.

It is next established that $x_{a_j} P_{a_j}(a_l, p)$ for all $0 \leq j < l$. To obtain a contradiction, suppose that $x_{a_0} I_{a_0}(a_l, p)$. Then it is possible to define an assignment μ'' such that $\mu''_a = \mu_a$ for $a \in A \setminus \{a_0, a_1, \dots, a_l\}$, $\mu''_{a_j} = \mu_{a_{j-1}}$ for $0 < j \leq l$, and $\mu''_{a_0} = a_l$. Further, let $\nu''_a = \nu_a$ for $a \in A \setminus \{a_0, a_1, \dots, a_l\}$ and $\nu''_a = 1$ for $a \in \{a_0, a_1, \dots, a_l\}$. Note that $\nu''_a = 1$ for $a \in \{a_0, a_1, \dots, a_l\}$ is a utility maximizing choice for agent a since for $a = a_j$, $x_a R_a a$, $x_a R_a(h, p)$ for all $h \in H$, $x_{a_j} I_{a_j}(\mu_{a_{j-1}}, p)$, and both $x'_{a_j} \neq a_j$ and $\nu'_{a_j} = 1$. Then the triple (μ'', ν'', p) satisfies the requirements of Definition 1. However, by comparing $x'' = (\mu'', \nu'', p)$ and $x = (\mu, \nu, p)$, we see that $|\mu''| > |\mu|$, i.e. $\mu_{a_l} = a_l$ while $\mu''_{a_l} \neq a_l$ (and $\mu''_{a_j} = \mu'_{a_j} \neq a_j$ for $0 \leq j < l$). This contradicts that x is an equilibrium state where the cardinality of μ is maximized. Similar arguments can be used to derive a contradiction if $x_{a_j} I_{a_j}(a_l, p)$ for some $0 < j < l$. Hence, $x_{a_j} P_{a_j}(a_l, p)$ for all $0 \leq j < l$.

Recall next that $A_S \neq \emptyset$ for each $S \subseteq H_1$ by assumption. Then because $a_l \in H_1$ and $x_{a_l} = a_l$, it follows that $A_{\{a_l\}} \neq \emptyset$. Let $\hat{a}_0 \in A_{\{a_l\}}$, and note that $\hat{a}_0 \notin \{a_0, a_1, \dots, a_l\}$ as $x_{a_j} P_{a_j}(a_l, p)$ for all $0 \leq j < l$ by the above conclusion. Consider next the following two cases:

- (I) Suppose that $\mu_{\hat{a}_0} \notin H_1$. Then, again, \hat{a}_0 belongs to some trading cycle $\hat{G} = (\hat{a}_t, \hat{a}_{t-1}, \dots, \hat{a}_1, \hat{a}_0)$ from μ to μ' . Again $\hat{a}_0 \in C$ and $\mu_{\hat{a}_0} \in H_2$, and there exists $0 < k \leq t$ such that $\mu_{\hat{a}_k} \in H_1$ and both $\mu_{\hat{a}_j} \in H_2$ and $x'_{\hat{a}_j} \neq \hat{a}_j$ for all $0 < j < k$. Again $x_{\hat{a}_k} = \hat{a}_k$ and $\hat{a}_k I_{\hat{a}_k}(\mu'_{\hat{a}_k}, p)$. Because G and \hat{G} are trading cycles, we have $\hat{a}_j \notin \{a_0, a_1, \dots, a_l\}$ for all $0 \leq j \leq k$. Now if (a) $\hat{a}_k = \mu'_{a_0}$ or (b) for some $0 \leq j < k$, $x_{\hat{a}_j} I_{\hat{a}_j}(\hat{a}_k, p)$ or (c) for some $0 \leq j \leq l$ and $0 \leq j' \leq k$, we have $x_{a_j} I_{a_j}(\mu_{\hat{a}_{j'}}, p)$, then, in all cases (a), (b) and (c), we can construct similarly as above from x an equilibrium state x'' where the cardinality of μ'' is larger compared to the cardinality of μ , which is a contradiction. Thus, suppose that (a), (b) and (c) are not true, in particular, $x_{a_j} P_{a_j}(\hat{a}_k, p)$ for all $0 \leq j \leq l$, and $x_{\hat{a}_j} P_{\hat{a}_j}(\hat{a}_k, p)$ for all $0 \leq j < k$. By $\hat{a}_k \in H_1$ and the above assumption that $A_S \neq \emptyset$ for each $S \subseteq H_1$, it then follows that $A_{\{\hat{a}_k\}} \neq \emptyset$. Let $a'_0 \in A_{\{\hat{a}_k\}}$. Now we have $a'_0 \notin \{\hat{a}_0, \hat{a}_1, \dots, \hat{a}_k\} \cup \{a_0, a_1, \dots, a_l\}$.
- (II) Suppose that $\mu_{\hat{a}_0} \in H_1$. Now, if $x_{a_j} I_{a_j}(\mu_{\hat{a}_0}, p)$ for some $0 \leq j \leq l$, it is possible to construct, in a similar fashion as in the above, an equilibrium state x'' where the cardinality of μ'' trade is larger compared to the cardinality of μ , which is a contradiction. Thus, $x_{a_j} P_{a_j}(\mu_{\hat{a}_0}, p)$ for all $0 \leq j \leq l$. By $\mu_{\hat{a}_0} \in H_1$ and the above assumption that $A_S \neq \emptyset$ for each $S \subseteq H_1$, it then follows that $A_{\{\mu_{\hat{a}_0}\}} \neq \emptyset$. Let $a'_0 \in A_{\{\mu_{\hat{a}_0}\}}$. Since $x_{a_j} P_{a_j}(\mu_{\hat{a}_0}, p)$ for all $0 \leq j \leq l$, we have $a'_0 \notin \{\hat{a}_0, a_0, a_1, \dots, a_l\}$.

We next observe that if $\mu_{a'_0} \notin H_1$, then as in (I) we can find another sequence a'_0, a'_1, \dots, a'_o where $x_{a'_o} = a_o \in H_1$, $\mu_{a'_j} \in H_2$, $\mu_{a'_j} = \mu_{a'_{j-1}}$, $x'_{a'_j} \neq a'_j$ for $0 < j < o$, and $\mu'_{a'_o} = \mu_{a'_{o-1}}$; and otherwise as in (II) $\mu_{a'_0} \in H_1$. Then either we can use similar arguments as above to construct another equilibrium state x'' from x where the cardinality of μ'' trade is larger

compared to the cardinality of μ or otherwise we continue to construct another (disjoint) sequence as in (I) or (II) (where in (II) the sequence consists of one agent) which eventually leads to a contradiction because in any such sequence there is a new agent $a \in A$, who does not belong to any of the previously identified sequences, with $\mu_a \in H_1$. The contradiction then follows as H_1 is finite. \square

Theorem 2. Let f be a minimum price mechanism. Then f is group non-manipulable on the domain $\tilde{\mathcal{R}}$.

Proof. To obtain a contradiction, first suppose that some nonempty group $C \subseteq A$ can manipulate the minimum price mechanism f at a profile $R \in \tilde{\mathcal{R}}$ by reporting preferences $R' = (R'_C, R_{-C}) \in \tilde{\mathcal{R}}$. More precisely, let $x = (\mu, \nu, p) \in \mathcal{E}_R$ and $x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be two equilibrium states such that $x'_a P_a x_a$ for all $a \in C$. By condition (1) and as in the proof of Lemma 5, we will, without loss of generality, assume that $C = \{a \in A : x'_a P_a x_a\}$ and the cardinality of μ is maximized among all equilibrium states with price vector p . Then $\mu'_a \in H_1$ for all $a \in C$ by Lemma 1(i) as $x'_a P_a x_a$ for all $a \in C$ and $C \neq \emptyset$. Hence, $H_1 \neq \emptyset$ and $p_h > p'_h \geq \underline{p}_h$ for all $h \in H_1$. Let now:

$$A_S = \{a \in A : \mu_a \notin S \text{ and } x_a I_a(h, p) \text{ for some } h \in S \setminus \{a\}\}.$$

From Lemma 5, it follows that there exists a nonempty set $S \subseteq H_1$ such that $A_S = \emptyset$. But then it is possible to decrease p_h for all $h \in S$ (since $p_h > p'_h \geq \underline{p}_h$ for all $h \in S$, and if $h = a \in S$ and $\mu_a \notin S$, then agent a is not affected by the decrease because $x_a R_a a$ and $p_a > \underline{p}_a$) and obtain a new equilibrium state by the Perturbation Lemma in Alkan et al. (1991) which contradicts that state x is selected by f at profile R . Hence, f is group non-manipulable at R .

Second suppose that the nonempty group $C \subseteq A$ can manipulate the minimum price mechanism f at $R \in \tilde{\mathcal{R}}$ by reporting preferences $R' = (R'_C, R_{-C}) \in \mathcal{R}$. More precisely, let $f(R) = x = (\mu, \nu, p) \in \mathcal{E}_R$ and $f(R') = x' = (\mu', \nu', p') \in \mathcal{E}_{R'}$ be such that $x'_a P_a x_a$ for all $a \in C$. Because $x_a R_a a$ for all $a \in C$, we have $\mu'_a \neq a$ for all $a \in C$. Let $R'' = (R''_C, R_{-C})$ be such that for all $a \in C$, R''_a is quasi-linear with (i) $v_{a\mu'_a} = p'_{\mu'_a} - \epsilon^a$ where $0 < \epsilon^a < 1$, (ii) $v_{a0} = v_{aa} = -1$ and (iii) $v_{ah} = p'_h - k_h^a$ for all $h \in H \setminus \{\mu'_a, a\}$ where $k_h^a > -(p'_h - \underline{p}_h) - 1$. Below we show that the numbers ϵ^a and k_h^a can be chosen such that $R'' \in \tilde{\mathcal{R}}$.

If $|C| = 1$, say $C = \{1\}$, and $R'' \notin \tilde{\mathcal{R}}$, then there exist two houses h and h' in H and $p \in \Omega$ such that there are a sequence of distinct agents (a_1, \dots, a_q) of agents and a sequence of distinct houses (h_1, \dots, h_{q+1}) such that $h = h_1 = a_1$, $h' = h_{q+1} = a_q$, $a_1 I''_{a_1}(h_2, p)$, $a_q I''_{a_q}(h_q, p)$ and $(h_j, p) I''_{a_j}(h_{j+1}, p)$ for all $2 \leq j \leq q-1$. Because $R \in \tilde{\mathcal{R}}$, we must have $1 \in \{a_1, \dots, a_q\}$, say $a_l = 1$ with $l \in \{1, \dots, q\}$, say $1 < l$ (or $l < q$). Since $1 < l \leq q$, then the indifference $a_1 I''_{a_1}(h_2, p)$ and $(h_j, p) I''_{a_j}(h_{j+1}, p)$ determine uniquely the price p_{h_l} (and similarly, if $l < q$, the price $p_{h_{l+1}}$ is uniquely determined). Then we may distort (arbitrarily small) the parameter $k_{h_l}^1$ and such that $(h_l, p) P''_1(h_{l+1}, p)$ (if $l < q$) or $(h_l, p) P''_1 1$ (if $l = q$). In general, for any $2 \leq l \leq n-1$, let $\vec{h} = (h_1, \dots, h_l)$ be a sequence of l distinct houses and $\vec{a} = (a_1, \dots, a_{l-1})$ be a sequence $l-1$ distinct agents in $A \setminus \{1\}$ such that $h = h_1 = a_1$ and $h_j \neq a_j \neq h_{j+1}$ for $2 \leq j \leq l-1$. We say that (\vec{h}, \vec{a})

ends in h_l . Denote all these sequences (\vec{h}, \vec{a}) by \mathcal{S} . The empty sequence \emptyset corresponds to 1 and belongs by convention to \mathcal{S} . Now if for some $p \in \Omega$ we have $a_1 I''_{a_1}(h_2, p)$ and $(h_j, p) I''_{a_j}(h_{j+1}, p)$ for all $2 \leq j \leq l-1$, then the price of h_l is uniquely determined and we denote it by $p_{h_l}^{\vec{h}, \vec{a}}$. Note that the set \mathcal{S} is finite, and we may choose an order of the houses $H \setminus \{1\}$ starting with μ'_1 , say $\mu'_1 = h_1, h_2, \dots, h_{n-1}$ such that (i) we increase ϵ^1 by an amount $\delta(h_1) > 0$ in order to break indifferences but not reversing any strict preferences: choose $\delta(h_1) > 0$ such that for $v_{1\mu'_1} = p'_{\mu'_1} - \epsilon^1 + \delta(\mu'_1)$ we have for any $h \in H \setminus \{h_1\}$ and any sequences (\vec{h}, \vec{a}) ending in h_1 and (\vec{h}', \vec{a}') ending in h : (a) (for $h = 1$) $p'_{h_1} - \epsilon^1 - p_{h_1}^{\vec{h}, \vec{a}} < -1$ iff $p'_{h_1} - \epsilon^1 + \delta(h_1) - p_{h_1}^{\vec{h}, \vec{a}} < -1$ (b) (for $h \neq 1$) $p'_{h_1} - \epsilon^1 - p_{h_1}^{\vec{h}, \vec{a}} < p'_h - k_h^1 - p_h^{\vec{h}', \vec{a}'}$ iff $p'_{h_1} - \epsilon^1 + \delta(h_1) - p_{h_1}^{\vec{h}, \vec{a}} < p'_h - k_h^1 - p_h^{\vec{h}', \vec{a}'}$, i.e. all indifferences are broken favor of h_1 ; and (ii) for $2 \leq j \leq n-1$ we increase $k_{h_j}^1$ by an amount $\delta(h_{j-1}) > \delta(h_j) > 0$ in order to break indifferences but not reversing any strict preferences: for $v_{1h_j} = p'_{h_j} - k_{h_j}^1 + \delta(h_j)$ we have for any $h \in H \setminus \{h_1, \dots, h_j\}$ and any sequences (\vec{h}, \vec{a}) ending in h_j and (\vec{h}', \vec{a}') ending in h we have (a) (for $h = 1$) $p'_{h_j} - k_{h_j}^1 - p_{h_j}^{\vec{h}, \vec{a}} < -1$ iff $p'_{h_j} - k_{h_j}^1 + \delta(h_j) - p_{h_j}^{\vec{h}, \vec{a}} < -1$ (b) (for $h \neq 1$) $p'_{h_j} - k_{h_j}^1 - p_{h_j}^{\vec{h}, \vec{a}} < p'_h - k_h^1 - p_h^{\vec{h}', \vec{a}'}$ iff $p'_{h_j} - k_{h_j}^1 + \delta(h_j) - p_{h_j}^{\vec{h}, \vec{a}} < p'_h - k_h^1 - p_h^{\vec{h}', \vec{a}'}$, i.e. all remaining indifferences are broken favor of h_j . Thus, we may without loss of generality assume that $R'' \in \tilde{\mathcal{R}}$ when $|C| = 1$. If $|C| \geq 2$, then we may use the above argument to replace the preferences of the agents in C in R one at a time by quasi-linear preferences: for example, if $C = \{1, 2\}$, then first we replace R_1 by R'_1 and obtain $(R'_1, R_{-1}) \in \tilde{\mathcal{R}}$ and then we obtain $(R'_1, R'_2, R_{-1,2}) \in \tilde{\mathcal{R}}$.

Note that $x' \in \mathcal{E}_{R''}$ because $x' \in \mathcal{E}_{R'}$. Since $R'' \in \tilde{\mathcal{R}}$, there is a unique minimum equilibrium price vector $p^* \in \Pi_{R''}$. Let $x^* = (\mu^*, \nu^*, p^*)$ be minimum price equilibrium for R'' . By $x' \in \mathcal{E}_{R''}$ and Theorem 1, $p^* \leq p'$. Thus, by Proposition 1 (iii), for all $a \in C$, $x_a^* R''_a x'_a$ and by construction of R''_a , $\mu_a^* = \mu'_a$. Hence, for all $a \in C$, by $p'_{\mu_a^*} \geq p_{\mu_a^*}^*$ and $x'_a P_a x_a$, we have $x_a^* P_a x_a$. Therefore, the nonempty group $C \subseteq A$ can manipulate the minimum price mechanism f at $R \in \tilde{\mathcal{R}}$ by reporting preferences $R'' = (R''_C, R_{-C}) \in \tilde{\mathcal{R}}$, which is a contradiction to the first part of the proof. \square

Proposition 3. A minimum price mechanism f is non-manipulable on the domain \mathcal{R} if and only if $1 \leq |A| \leq 3$.

Proof. We first prove that a minimum price mechanism is non-manipulable if $1 \leq |A| \leq 3$. We will, however, only prove the result for $|A| = 3$ as the proof for $1 \leq |A| \leq 2$ is a special case of the proof for $|A| = 3$. To obtain a contradiction, suppose that $|A| = 3$ and that some nonempty group $C \subseteq A$ can manipulate a minimum price mechanism f at a profile $R \in \tilde{\mathcal{R}}$ by reporting preferences $R' = (R'_C, R_{-C}) \in \tilde{\mathcal{R}}$. Let also $x = (\mu, \nu, p)$ and $x' = (\mu', \nu', p')$ represent the selections of f at profiles R and R' , respectively. As in the proof of Lemma 5, we will, without loss of generality, assume that $C = \{a \in A : x'_a P_a x_a\}$. Note also that $\mu'_a \in H_1$ for all $a \in C$ by Lemma 1(i) as $x'_a P_a x_a$ for all $a \in C$.

We need to demonstrate that $|H_1| = 3$. The conclusion then follows by the Perturbation Lemma in Alkan et al. (1991) in the same fashion as in the proof of Theorem 2. Note first

that $|H_1| \geq 1$, by the above observation, as $C \neq \emptyset$. To obtain a contradiction to $|H_1| = 3$, suppose that $|H_1| = 2$. This also means that there is an agent $a \in A$ with $\mu'_a \notin H_1$. Suppose next, without loss of generality, that $a = 3$, and note that $3 \notin C$. If $\mu_3 = \mu'_3$, then it must be the case that $x_3 P_3(\mu_j, p)$ for $j = 1, 2$, otherwise x' cannot be an equilibrium since $\mu_j \in H_1$ for $j = 1, 2$. But in this case, is possible to decrease the prices of houses $\mu_1 \in H_1$ and $\mu_2 \in H_1$ at state x , by the Perturbation Lemma in Alkan et al. (1991), and obtain a new equilibrium which contradicts that p is a minimum equilibrium price vector at profile R . Hence, $\mu_3 \neq \mu'_3$, and consequently, $\mu_3 \in H_1$ as $\mu'_3 \notin H_1$ and $|H_2| = 2$. But this also means that $\mu_3 = 3$, $\mu'_3 \in H_2$, and $x_3 I_3(\mu'_3, p)$, otherwise, state x' cannot be an equilibrium as $\mu_3 \in H_1$.

Suppose next, without loss of generality, that $\mu'_3 = 2$, i.e., that $1, 3 \in H_1$. Hence, $x_3 I_3(2, p)$, by the above conclusion, and $x_3 P_3(1, p)$ otherwise x' cannot be an equilibrium at profile R as $1 \in H_1$. But then it must be the case that $x_j I_j(1, p_1)$ for the agent $j \neq 3$ with $\mu_j \neq 1$ otherwise it is possible to decrease $p_1 \in H_1$ and obtain a new equilibrium which contradicts that p is a minimum equilibrium price vector at profile R . By identical arguments, it must also be the case that $x_k I_k(3, p_3)$ some agent $k \neq 3$. If $\mu_k = 2$, agents k and 3 can swap houses at allocation x and a new equilibrium is obtained, but it is also possible to decrease the prices of all houses but 2 and obtain a new equilibrium, by the Perturbation Lemma in Alkan et al. (1991), which contradicts that p is a minimum equilibrium price vector at profile R . If $\mu_k = 1$, a new equilibrium can be obtained at prices p and for $\mu''_j = 1$, $\mu''_k = 3$, and $\mu''_3 = 2$, and it is again possible to decrease the prices of all houses but 2 to obtain a new equilibrium at profile R . Hence, in both cases it is possible to obtain a contradiction to $|A| = 2$. Hence, $|H_1| = 3$ as desired.

We next prove that a minimum price mechanism f is non-manipulable on the domain \mathcal{R} *only if* $1 \leq |A| \leq 3$, i.e., that f can be manipulated by some agent, at some profile $R \in \mathcal{R}$, whenever $|A| > 3$. The proof is based on Example 2, and by using the same arguments as in the proof of Proposition 2, it is sufficient to demonstrate the result for $|A| = 3$.

Consider now the state $\hat{x} = (\hat{\mu}, \hat{\nu}, \hat{p}) \in \mathcal{E}_R$ and suppose that \hat{p} is a minimum equilibrium price vector. We first demonstrate that $\hat{p} = p' = (1, 0, 0, 0)$ or $\hat{p} = p'' = (0, 1, 0, 0)$. From Example 2, it is clear that either $\hat{p}_1 < 1$ or $\hat{p}_2 < 1$ as \hat{p} is a minimum equilibrium price vector. Suppose that $\hat{p}_1 < 1$. Then $\hat{\mu}_3 = 1$ by envy-freeness as \hat{x} is an equilibrium. Consequently, $\hat{\mu}_1 = 3$, $\hat{p}_3 = 0$ by individual rationality for agent 1, and it then follows that $\hat{\mu}_2 = 2$ by individual rationality for agent 2. But then it must be the case that $\hat{\mu}_4 = 4$ and $\hat{p}_2 \geq 1$ because otherwise agent 4 will envy agent 2 at state \hat{x} . Hence, $\hat{p} \geq p''$. But then $\hat{p} = p''$, by definition, as \hat{p} is a minimum equilibrium price vector by assumption. Analogous arguments can be used to show that $\hat{p} = p'$ if $\hat{p}_2 < 1$.

Let now f be minimum price mechanism on domain \mathcal{R} . Then f chooses either p' or p'' . If f chooses p' , then agent 3's utility is equal to $v_{34} - p'_4 = 1$. Let R' denote the profile of quasi-linear preferences where the entry v_{32} in the matrix V from Example 2 is replaced by $v'_{32} = 2$. Obviously, $x' \notin \mathcal{E}_{R'}$ because $(2, p'_2) P'_3 x'_3$. On the other hand, it is easy to check that $x'' \in \mathcal{E}_{R'}$. Also, p'' is the unique minimum equilibrium price vector at profile R' . To see this, suppose that $\hat{x} = (\hat{\mu}, \hat{\nu}, \hat{p}) \in \mathcal{E}_{R'}$ and that $\hat{p} \neq p''$ is a minimum equilibrium price vector at profile R' . Then $\hat{p}_2 < 1$, which implies that $\hat{\mu}_4 = 2$ and $\hat{\mu}_3 = 1$. But then

individual rationality cannot be satisfied for both agents 1 and 2 at state x'' . Thus, p'' must be chosen by f for profile R' . Then, by individual rationality for agents 1 and 2, it follows that agent 3 must receive house 1. Because $R' = (R'_3, R_{-3})$ and agent 3's utility from $(1, p'')$ under R_3 is equal to $v_{31} - p''_1 = 2 > 1$, agent 3 can profitably manipulate f at R .

If f chooses p'' , it can be shown, by identical arguments as in the above, that agent 4 can manipulate the mechanism by replacing the entry v_{41} in the matrix V from Example 2 is by $v'_{41} = 2$. \square

Proposition 6. For any given profile $R \in \mathcal{R}$, a Dutch price sequence $(p^t)_{t=1}^T$ contains a finite number of price vectors, i.e., $T < \infty$.

Proof. To obtain a contradiction, suppose that $(p^t)_{t=1}^T$ is a Dutch price sequence with $T = \infty$. Let also $(x^t)_{t=1}^T$ be a supporting sequence of equilibrium states, where $x^t = (\mu^t, \nu^t, p^t)$. Because there is only a finite number of assignments, there is an infinite subsequence $(t_j)_{j=1}^\infty$ of steps such that for some finite number j' and some assignments μ' and ν' , it holds that $\mu^{t_j} = \mu'$ and $\nu^{t_j} = \nu'$ for all $j \geq j'$. Let also $p^t \rightarrow p^e$ as $t \rightarrow \infty$. We will demonstrate that $p^e \in \Pi^{\nu'}$, which contradicts that $T = \infty$ since $\nu^{t_j} = \nu'$.

Consider now the states $x^{t_j} = (\mu^{t_j}, \nu^{t_j}, p^{t_j}) = (\mu', \nu', p^{t_j})$ for $j \geq j'$. Because $x^{t_j} \in \mathcal{E}$, the following holds for all $a \in A$:

$$x_a^{t_j} R_a a \text{ and } x_a^{t_j} R_a(h, p^{t_j}) \text{ for all } j \geq j' \text{ and for all } h \in H.$$

Moreover, by continuity, the following holds for all $a \in A$:

$$(\mu', \nu', p^e) R_a a \text{ and } (\mu', \nu', p^e) R_a(h, p^e) \text{ for all } h \in H.$$

But then $p^e \in \Pi^{\nu'}$. \square

Theorem 3. For any given profile $R \in \tilde{\mathcal{R}}$, a Dutch price sequence $(p^t)_{t=1}^T$ is unique and p^T is the unique minimum equilibrium price vector in Π_R .

Proof. We first prove that a Dutch price sequence is unique. To do this, consider the profile $R \in \tilde{\mathcal{R}}$, and let $x' = (\mu', \nu', p) \in \mathcal{E}_R$ and $x'' = (\mu'', \nu'', p) \in \mathcal{E}_R$ be two equilibrium states with a common price vector p (recall from Theorem 1 that the minimum price vector is unique on the domain $\tilde{\mathcal{R}}$). We need to demonstrate that $\nu' = \nu''$ to complete the proof. To see this, consider the correspondence ξ from the definition of a Dutch price sequence. If $\nu' = \nu''$, then there is just *one* state in $\xi(p)$. This also means that, for a given profile $R \in \tilde{\mathcal{R}}$, there is just one Dutch price sequence.

Suppose now that $\nu' \neq \nu''$, and consider a trading cycle $G = (a_1, \dots, a_q)$ from μ' to μ'' such that $\nu'_{a_1} = 0$. Such a trading cycle exists since $\nu' \neq \nu''$ by assumption. For each agent a_j in the trading cycle G , let $r_j = (\nu'_{a_j}, \nu''_{a_j})$. Then r_j is equal to $(0, 1)$, $(1, 1)$ or $(0, 1)$, but not $(0, 0)$ as agent a_j is included in G . Moreover, $r_1 = (0, 1)$ by assumption. Consider then the sequences $(r_l, r_{l+1}, \dots, r_{q'})$ for $1 \leq l \leq q' \leq q$. Assume first that $r_j \neq (1, 0)$ for all j , i.e., that $r_j = (0, 1)$ or $r_j = (1, 1)$ for all j . This means that trade is not maximal at the

state x' . Note also that $x'_{a_j} I_{a_j} x''_{a_j}$ for all j since prices are the same at the two states x' and x'' . Hence, there must be a q' such that $r_{q'} = (1, 0)$. But since $r_1 = (0, 1)$, there must also be an l such that $r_l = (0, 1)$ and $r_j = (1, 1)$ for $l < j < q'$. In that case, houses μ'_{a_l} and $\mu''_{a_{q'}}$ are connected by indifference, contradicting our assumption that $R \in \tilde{\mathcal{R}}$. Hence, $\nu' = \nu''$ must be the case, and, consequently, a Dutch price sequence is unique.

We next prove that p^T is the unique minimum equilibrium price vector p^* in Π_R . From the first part of this theorem and Proposition 6 we know that a Dutch price sequence is unique and finite. Suppose that it converges to the price vector p^e , but that $p^* \leq p^e$ and $p^* \neq p^e$. Let $x^e = (\mu^e, \nu^e, p^e)$ and $x^* = (\mu^*, \nu^*, p^*)$ be two corresponding equilibrium states in \mathcal{E}_R . Then $H_1 \neq \emptyset$, so $A(S) = \emptyset$ for some $S \subset H_1$ by Lemma 5. Note also that $S \neq \emptyset$ and $p^e \in \Pi^e$. Consider now the set of houses S and the set of agents \hat{A} that are assigned houses in S , i.e., $\hat{A} = \{a \in A : \mu_a \in S\}$. Let also $A^0 = \{A \in \hat{A} : \nu^e = 0\}$ and $A^1 = \{A \in \hat{A} : \nu^e = 1\}$.

Since $p^* \leq p^e$ and $p^* \neq p^e$, for each $\varepsilon > 0$, there is a state $x^\varepsilon = (\mu^\varepsilon, \nu^\varepsilon, p^\varepsilon) \in \mathcal{E}_R$ such that $p^m \leq p^\varepsilon \leq p^e$, $p^m \neq p^\varepsilon \neq p^e$. Moreover, since $A(S) = \emptyset$, $x_a^\varepsilon = x_a^e$ can be chosen for $a \in A \setminus \hat{A}$ for “sufficiently small” ε . Since there is only a finite number of assignments ν , there is an infinite and increasing sequence $(\varepsilon_j)_{j=1}^\infty$ such that $\nu^{\varepsilon_j} = \nu^*$ is constant for all j , and $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$. Hence, $x^{\varepsilon_j} = (\mu^{\varepsilon_j}, \nu^*, p^{\varepsilon_j})$. But then $x^* = (\mu^*, \nu^*, p^e) \in \xi(x^{T-1})$ for some μ^* by continuity of preferences. But then there is a $p \in \Pi^{\nu^*}$ such that $p \leq p^e$, $p \neq p^e$. This means that the Dutch price sequence cannot stop at T , which contradicts our assumptions. Hence, $p^e = p^*$ must be the case. \square

Proposition 7 For any profile $R \in \mathcal{R}$, the minimum price mechanism generates a weakly higher revenue to the public authority compared to the current U.K. system.

Proof. The current U.K. system can, for any profile $R \in \mathcal{R}$, be represented by a state $x = (\mu, \nu, p)$ where $p_h = \infty$ for all $h \in H$ as the interpretation of this situation, due to the assumption that each house in H is bounded desirable, is that each tenant $a \in A$ is given a take-it-or-leave-it offer either to buy house $h = a$ at price $p_h = \underline{p}_h$, or to continue renting house $h = a$.

Suppose now that the state $x' = (\mu', \nu', p')$ is selected by the minimum price mechanism at profile $R \in \mathcal{R}$. Note first that if $\nu_a = 1$, then $\nu'_a = 1$. This follows trivially because if $\nu_a = 1$ but $\nu'_a = 0$, then $(a, \underline{p}_a) R_a a$ and $a P_a(a, \underline{p}_a)$, respectively, which is a logical contradiction. This also means that all agents $a \in A$ that buy a house at state x also buy a house at state x' . Furthermore, all agents that buy a house at state x belong to one of the following two sets:

$$\begin{aligned} S &= \{a \in A : \mu_a = \mu'_a \text{ and } \nu_a = 1\}, \\ T &= \{a \in A : \mu_a \neq \mu'_a \text{ and } \nu_a = 1\}. \end{aligned}$$

It is clear that the revenue for the public authority from the sales to the agents in S is identical at states x' and x since $\mu_a = a = \mu'_a$ in this case. Because each agent in T must be involved in a trading cycle from μ to μ' and $p' \geq \underline{p}$, it is clear that the revenue for the public authority from the sales to the agents in T is weakly higher at state x' than at state

x . This proves the statement. Note that there may be agents $a \in T$ with $\nu_a = 0$ but $\nu'_a = 1$ which further increases the revenue for the public authority by switching from state x to state x' . \square

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