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Department of Economics
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Strategy-Proof House Allocation with Price Restrictions

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Strategy-Proof House Allocation with Price Restrictions*

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Abstract

This paper considers a house allocation problem with no initial ownership and where prices are bounded from below and above by exogenously given price restrictions. This type of housing market contains, e.g., the “assignment market” and the “student placement problem” as special cases. A mechanism called the minimal RPE mechanism is defined, and two main results are obtained. First, it is demonstrated that the mechanism is manipulable at some profile in the full preference domain \mathcal{R} . Second, it is proved that there is a subset $\tilde{\mathcal{R}} \subset \mathcal{R}$ of the full domain, containing almost all profiles in \mathcal{R} , such that the minimal RPE mechanism is strategy-proof in that subset.

JEL Classification: C78; D71.

Keywords: house allocation; matching; strategy-proofness; domain restrictions.

1 Introduction

We consider a house allocation problem with no initial ownership where prices are bounded from below and above by exogenously given price restrictions. Because the houses are indivisible objects, the upper and lower price restrictions are allowed to coincide, and the upper and lower price restrictions can take any real number, this problem contains, e.g., the models by [Balinski and Sönmez \(1999\)](#), [Demange and Gale \(1985\)](#), [Hylland and Zeckhauser \(1979\)](#), [Leonard \(1983\)](#), and [Shapley and Shubik \(1972\)](#), as special cases. Our main objective is to define a strategy-proof allocation mechanism for this type of housing

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market.¹ One difficulty in achieving this task is that, due to the exogenously given price constraints, a price equilibrium need not exist for all preference profiles in the full preference domain (i.e., all profiles where preferences are rational, monotonic, and continuous). Consequently, previous results, from, e.g., [Demange and Gale \(1985\)](#) and [Leonard \(1983\)](#), cannot be used directly to construct a strategy-proof allocation mechanism. However, as is demonstrated in this paper, the basic idea in those papers, i.e., that a minimal price vector can be used as a main ingredient in a strategy-proof allocation mechanism, can also be used in non-equilibrium situations.

Since the price restrictions exclude equilibrium for certain preference profiles, the concept of a rationing price equilibrium (RPE, henceforth) is introduced. This concept is partly characterized by a priority-order that is used to determine the allocation when no price equilibrium exists, and partly characterized by a price condition stating that prices on “overdemanded” houses must equal the upper price bound. The investigated allocation mechanism then selects an RPE with a price vector that is minimal in the set of RPE price vectors. This mechanism is called a minimal RPE mechanism.

The first main insight of this paper is negative. More explicitly, it is shown that a minimal RPE mechanism is manipulable at some profile in the full preference domain \mathcal{R} . This negative finding provides a rationale for restricting the preference domain, and it also adds another example to the long line of examples that demonstrate that allocation mechanisms or social choice rules often need to be defined on restricted domains (e.g., single-peaked domains, or domains where the preferences satisfy the single-crossing condition or the intermediateness condition, etc.) to avoid impossibility results.²

The second main result of this paper is positive and demonstrates that there is a subset $\tilde{\mathcal{R}} \subset \mathcal{R}$ of the full domain, containing, in a mathematical meaning, almost all profiles in \mathcal{R} , such that the minimal RPE mechanism is strategy-proof in that subset. This result extends a partial result obtained for the minimal RPE mechanism in [Andersson and Svensson \(2014\)](#). In that paper, the mechanism was defined only on the subset $\tilde{\mathcal{R}}$, and it was shown that it is impossible for any group of agents to manipulate the mechanism if they report preferences restricted to profiles in $\tilde{\mathcal{R}}$. Hence, the result did not reveal if it is possible to manipulate the mechanism with a profile in $\mathcal{R} \setminus \tilde{\mathcal{R}}$. In this meaning, strategy-proofness was not proved. This in combination with the fact that the domain \mathcal{R} is not a hyperrectangle is problematic as the reports of the truthful agents *and* the misrepresenting agents jointly determine whether or not the “new” preference profile belongs to the restricted domain $\tilde{\mathcal{R}}$, meaning that it is not possible for a group of agents to know that it is impossible to manipulate until they know the reported preferences of the truthful agents. As the main result of this paper demonstrates that the minimal RPE mechanism in fact is strategy-proof, the agents need not to have any information about other agents’ preferences, or, put differently, the “new” profile is always allowed to belong to the rectangular domain \mathcal{R} .

The remaining part of this paper is organized as follows. Section 2 describes the basic

¹Some recent papers have considered a housing market with price restrictions, see, e.g., [Andersson et al. \(2015\)](#), [Herings \(2015\)](#), [Talman and Yang \(2008\)](#) or [Zhu and Zhang \(2011\)](#), but none of these papers address the issue of strategy-proofness.

²See, e.g., [Barberà et al. \(2013\)](#) for a nice overview of domain restrictions and their consequences.

economy and introduces a number of definitions and remarks. All results of this paper are stated and proved in Section 3. The Appendix contains a measure on sets of profiles such that $\tilde{\mathcal{R}} = \mathcal{R}$ a.e.

2 The Model and Basic Definitions

Let $H = \{1, \dots, m\}$ and $A = \{1, \dots, n\}$ denote the finite sets of *houses* and *agents*, respectively. Agents wish to buy at most one house and have an option not to buy any house at all. This outside option is formally represented by a *null house*, denoted by 0. These houses are available in an unlimited number of copies.

An *assignment* is a mapping $\mu : A \rightarrow H \cup \{0\}$ such that $\mu_a = \mu_{a'}$ for $a \neq a'$ only if $\mu_a = 0$. Hence, two distinct agents can not be assigned the same house in H . Denote by μ_0 the set of houses that is not assigned to any agent at assignment μ . The null house is always included in this set as its supply is unlimited. Hence:

$$\mu_0 = \{h \in H : \mu_a \neq h \text{ for all } a \in A\} \cup \{0\}.$$

Let $p \in \mathbb{R}^{m+1}$ be a *price vector*. A coordinate in p is denoted by p_h and represents the price of house $h \in H \cup \{0\}$. The price of the null house is, without loss of generality, always assumed to equal zero, i.e., $p_0 = 0$. Price vectors are assumed to be restricted by exogenously given lower and upper bounds denoted by $\underline{p} \in \mathbb{R}^{m+1}$ and $\bar{p} \in \mathbb{R}^{m+1}$, respectively, where $\underline{p} \leq \bar{p}$. As $p_0 = 0$, by assumption, it is clear that $\underline{p}_0 = \bar{p}_0 = 0$. The *price space* is given by:

$$\Omega = \{p \in \mathbb{R}^{m+1} : \underline{p}_h \leq p_h \leq \bar{p}_h \text{ for } h \in H \cup \{0\}\}.$$

Each agent $a \in A$ has *preferences* on pairs of houses and prices.³ Denote by R_a agent a 's preference relation on the set of houses and prices $(H \cup \{0\}) \times \mathbb{R}$. The corresponding strict and indifference relations are denoted by P_a and I_a , respectively. Preferences are assumed to be *rational* and *strictly monotonic* for all agents $a \in A$, i.e., R_a is a complete and transitive binary relation on $(H \cup \{0\}) \times \mathbb{R}$ and $(h, p'_h) P_a (h, p_h)$ if $p'_h < p_h$. It is also, for all agents $a \in A$, assumed that $(0, 0) P_a (h, p_h)$ for p_h "sufficiently large". Preferences are further assumed to be *continuous*, i.e., the sets $\{p_h \in \mathbb{R} : (h, p_h) R_a (h', p'_{h'})\}$ and $\{p_h \in \mathbb{R} : (h', p'_{h'}) R_a (h, p_h)\}$ are closed for each $a \in A$ and all $h, h' \in H \cup \{0\}$ and all $p'_{h'} \in \mathbb{R}$. All preference relations R_a satisfying the above properties for agent $a \in A$ are gathered in the set \mathcal{R}_a . A (preference) *profile* is a list $R = (R_1, \dots, R_n)$ of the agents' preferences. This list belongs to the set $\mathcal{R} = \mathcal{R}_1 \times \dots \times \mathcal{R}_n$. Finally, we adopt the notational convention of writing a profile $R \in \mathcal{R}$ as $R = (R_C, R_{-C})$ for $C \subset A$.

There is also a *priority-order* π , i.e., a bijection $\pi : A \rightarrow A$, where the highest ranked agent a has $\pi_a = 1$, the second highest ranked agent a' , $\pi_{a'} = 2$, and so on.

The following definitions have already been extensively discussed in [Andersson and Svensson \(2014\)](#) so they are stated here without any further discussion.

³For convenience, we will often let $(h, p) \equiv (h, p_h)$, i.e., by (h, p) we mean house h at price p_h in the price vector p .

Definition 1. A *state* is a pair $x = (\mu, p)$ where μ is an assignment and $p \in \Omega$ is a price vector. A state can also be written as $x = (x_1, \dots, x_n)$ where $x_a = (\mu_a, p_a)$ for $a \in A$.

Definition 2. For a given profile $R \in \mathcal{R}$, a state $x = (\mu, p)$ is *priority respecting* if for each $a, a' \in A$, (i) $x_{a'} P_a x_a$ only if $\pi_{a'} < \pi_a$, and (ii) $x_a R_a(h, p)$ if $h \in \mu_0$.

Definition 3. For a given profile $R \in \mathcal{R}$, a state $x = (\mu, p)$ and the assignment μ are *constrained efficient* if x is priority respecting, and there is no priority respecting state $x' = (\mu', p)$ that Pareto dominates x .

Definition 4. For a given profile $R \in \mathcal{R}$, a state $x = (\mu, p)$ is a *rationing price equilibrium* (RPE, henceforth) if (i) $p_h = \underline{p}_h$ for all $h \in \mu_0$, (ii) for all $a, a' \in A$, $p_{\mu_{a'}} = \bar{p}_{\mu_{a'}}$ if $x_{a'} P_a x_a$, and (iii) x is constrained efficient.

For a given profile $R \in \mathcal{R}$, the set of RPE states is denoted by Σ_R . Let also $\Sigma = \cup_{R \in \mathcal{R}} \Sigma_R$. Moreover, for a given profile $R \in \mathcal{R}$, a price vector p is an RPE price vector if there is an assignment μ such that the state (μ, p) is a rationing price equilibrium. The set of RPE price vectors at profile R is denoted by Π_R .

Definition 5. For a given profile $R \in \mathcal{R}$, a vector p^m is a *minimal RPE price vector* if $p^m \in \Pi_R$ and for $p \in \Pi_R$, $p \leq p^m$ only if $p = p^m$.

Definition 6. Let $\mathcal{R}^* \subset \mathcal{R}$. An (allocation) *mechanism* with domain \mathcal{R}^* is a function $f : \mathcal{R}^* \rightarrow \Sigma$.

Definition 7. A *minimal RPE mechanism* is an allocation mechanism f with domain $\mathcal{R}^* \subset \mathcal{R}$ where $f(R) = (\mu, p)$ is any selection of RPE states such that p is a minimal RPE price vector in Π_R .

Definition 8. A mechanism f is *manipulable* at a profile $R \in \mathcal{R}$ by a nonempty group of agents $C \subset A$ if there is a profile $R' = (R'_C, R_{-C})$, and two states $f(R) = x = (\mu, p)$ and $f(R'_C, R_{-C}) = x' = (\mu', p')$, such that $x'_a P_a x_a$ for all $a \in C$. If the mechanism f is not manipulable by any group $C \subset A$ at profile R , it is *strategy-proof* at profile R . Given $\mathcal{R}^* \subset \mathcal{R}$, the mechanism f is *group strategy-proof on the domain \mathcal{R}^** if for any profile $R \in \mathcal{R}^*$, f is *group strategy-proof* at profile R .

We end this section with a few remarks related to equilibrium notions, efficiency, and strategy-proofness. A first remark is that in the definition of an RPE, the priority-order is used only to define the assignment of agents to houses in non-price equilibrium situations, i.e., in situations where prices alone cannot define the assignment and, hence, where there in principal only are indivisible items. If the priority-order is relaxed, the concept of constrained efficiency can be replaced with efficiency in the definition of an RPE, and the existence of an “equilibrium” is still guaranteed for all profiles in \mathcal{R} . However, it is well-known that a priority structure often is a consequence of efficiency and strategy-proofness, e.g., in models with only indivisible items ([Balinski and Sönmez, 1999](#); [Larsson](#)

and Svensson, 2005; Svensson, 1999; Pápai, 2000), meaning that efficiency and strategy-proofness are incompatible unless some kind of priority structure is imposed. Hence, if the priority-order is relaxed in the considered model, it is not unlikely that there will be a tradeoff between efficiency and strategy-proofness in the sense that if efficiency is imposed, then strategy-proofness is lost.

A second remark is that there are alternative equilibrium notions for house allocation problems with price restrictions. In particular, Talman and Yang (2008) analyze such problem using the notion of a *constrained Walrasian equilibrium* (CWE, henceforth). This notion is weaker than RPE⁴, and its rationing system is endogenous. Because there is no efficiency requirement in a CWE, there are profiles in \mathcal{R} where it is possible to make a Pareto improving reallocation of the houses among the agents and at the same time respect all requirements of a CWE. Note that the latter may be true even if an arbitrary priority-order is attached to the concept. This can never occur at a RPE since it is constrained efficient by definition.⁵

A final remark is related to the core. More specifically, Andersson et al. (2015, Theorem 3.4) demonstrate that the ascending auction in Talman and Yang (2008) always finds a *core allocation*, as defined in Andersson et al. (2015, Definition 2.1), in a finite number of steps. By using almost identical arguments as in their proof, it follows immediately that also a minimal RPE mechanism always selects a core allocation.

3 Results

The first main insight of the paper is that the minimal RPE mechanism is manipulable on the full preference domain. This is demonstrated in the following example.

Example 1. Let $A = \{1, 2, 3, 4, 5\}$, $H = \{1, 2, 3, 4, 5\}$, $\pi_a = a$ for all $a \in A$, and $\underline{p}_h = 0$ and $\bar{p}_h = 1$ for all $h \in H$. Suppose further, that preferences over bundles (h, p) , for each agent $a \in A$, are represented by a quasi-linear utility function $u_{ah}(p) = v_{ah} - p_h$ for $h \in H$ where:

$$V = (v_{ah}) = \begin{pmatrix} 10 & 10 & 0 & 0 & 0 \\ 10 & 0 & 0 & 5 & 0 \\ 0 & 10 & 0 & 0 & 5 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{pmatrix},$$

and $u_{a0}(p) = -10$. The above quasi-linear preferences are gathered in the profile $R \in \mathcal{R}$. We next remark that there are *exactly* two minimal RPE price vectors, p and p' , at profile R , where $p_1 = p_2 = p_5 = 1$, $p_3 = p_4 = 0$, $p'_1 = p'_2 = p'_4 = 1$, and $p'_3 = p'_5 = 0$. The vector p is obtained *only* at the assignment $\mu = (2, 1, 5, 4, 3)$, and the vector p' is obtained *only* at

⁴The formal proofs of the claims and properties stated in the remaining part of this section are available from the authors upon request.

⁵See Andersson and Svensson (2014) for further discussions.

the assignment $\mu' = (1, 4, 2, 3, 5)$. Hence, the mechanism f selects either (μ, p) or (μ', p') at profile R .

If the mechanism selects state (μ', p') , agent 4 can manipulate the mechanism by reporting \hat{R}_4 . At this report, preferences are represented by a quasi-linear utility function with the following values: $\hat{v}_{4j} = v_{4j}$ for $j \in \{1, 2, 3, 4\}$ and $\hat{v}_{45} = 5$. It is easy to check that the unique selection of f at profile $\hat{R} = (\hat{R}_4, R_{-4})$ is state (μ, p) . Because $(\mu_4, p)P_4(\mu'_4, p')$, agent 4 can successfully manipulate the mechanism. Similarly, if the mechanism selects state (μ, p) , agent 5 can manipulate the mechanism by reporting \hat{R}_5 . At this report, preferences are represented by a quasi-linear utility function with the following values: $\hat{v}_{5j} = v_{5j}$ for $j \in \{1, 2, 3, 5\}$ and $\hat{v}_{54} = 5$. \square

The consequence of the above example is that a minimal RPE mechanism can only be strategy-proof on a restricted domain. For this purpose, we will consider the domain $\tilde{\mathcal{R}} \subset \mathcal{R}$ that contains all profiles where no two houses are “connected by indifference” at any price vector in Ω .⁶ We will later (Theorem 1) demonstrate that the minimal RPE mechanism indeed is group strategy-proof on this domain.

Definition 9. For a given profile $R \in \mathcal{R}$, two houses, h' and h'' in $H \cup \{0\}$, are *connected by indifference* if there is a price vector $p \in \Omega$, a sequence of distinct agents (a_1, \dots, a_t) , and a sequence of distinct houses (h_1, \dots, h_{t+1}) such that:

- (i) $h' = h_1$ and $h'' = h_{t+1}$,
- (ii) $p_{h'} = \bar{p}_{h'}$ and $p_{h''} = \bar{p}_{h''}$,
- (iii) $(h_j, p)I_{a_j}(h_{j+1}, p)$ for all $1 \leq j \leq t$.

We first remark that houses 1 and 2 are connected by indifference at profile R in Example 1 as $(1, \bar{p}_1)I_1(2, \bar{p}_2)$. Consequently, $R \notin \tilde{\mathcal{R}}$ but $R \in \mathcal{R}$. We next remark that the set $\tilde{\mathcal{R}}$, in a mathematical meaning, contains almost all profiles in \mathcal{R} . In the Appendix, a measure on sets of profiles is defined such that $\tilde{\mathcal{R}} = \mathcal{R}$ a.e., but it can intuitively be understood in the following way. Let $R \in \mathcal{R}$ be any profile and h' and h'' be any two distinct houses in $H \cup \{0\}$. Further, let $(a_j)_{j=1}^t$ and $(h_j)_{j=1}^{t+1}$ be any two sequences of distinct agents and houses, respectively, such that $h' = h_1$ and $h'' = h_{t+1}$. Then, because of monotonicity and continuity of preferences, there is a unique sequence $(p_{h_j})_{j=1}^t$ of prices such that $(h_j, p_{h_j})I_{a_j}(h_{j+1}, p_{h_{j+1}})$ for all $j < t$. But Definition 9 also requires that $(h_t, p_{h_t})I_{a_t}(h_{t+1}, p_{h_{t+1}})$. This will occur with probability zero in most cases, e.g., if preferences are quasi-linear and represented by utility functions of type $u_{ah}(p) = v_{ah} - p_h$ for $v_{ah} \in \mathbb{R}$, $a \in A$ and $h \in H \cup \{0\}$, and the various values v_{ah} are randomly drawn from a bounded interval in \mathbb{R} . In this meaning, a limitation to profiles in $\tilde{\mathcal{R}}$ excludes very few profiles in \mathcal{R} .

⁶Note that Definition 9, below, is weaker than the one originally used by [Andersson and Svensson \(2014, Definition 5\)](#). It is, however, straightforward to check that their main non-manipulability result (restated as Lemma 1 in this paper) also holds for the weaker definition.

Theorem 1. Let f be a minimal RPE mechanism with domain \mathcal{R} . Then f is group strategy-proof on the domain $\tilde{\mathcal{R}}$.

Before proving Theorem 1, we note that the definition of group strategy-proofness assumes that it is not possible for a deviating group of agents to transfer money among themselves after the outcome of the rule has been realized. If such side payments would be allowed, it would be possible for the group A to gain by strategic misrepresentation at any profile in $\tilde{\mathcal{R}}$ where the minimal RPE mechanism selects a state $x = (\mu, p)$ with $p \neq \underline{p}$.⁷ Similar findings have previously been reported by, e.g., Demange and Gale (1985, p.875–876) and Schummer (2000a, p.307), for generalizations of the second-price auction. We also remark that another type of manipulation may occur when a group of agents bribe some other group of agents to misrepresent their preferences, and if this misrepresentation results in a net gain for all agents involved in the bribing situation. If such situation is impossible, the mechanism is said to be *bribe proof* (Massó and Neme, 2007; Schummer, 2000a,b). Also for this type of manipulation, negative results are available under general circumstances. For example, Schummer (2000b) proves that the only mechanisms that satisfy the bribe proofness axiom are essentially constant mechanisms.

To prove Theorem 1, we will make use of a number of lemmas. As Lemmas 1 and 2 are equivalent to Theorem 3 and Lemma 5 in Andersson and Svensson (2014), respectively, they are stated without a proof.

Lemma 1. Let f be a minimal RPE mechanism with domain $\tilde{\mathcal{R}}$. Then there is no profile $R \in \tilde{\mathcal{R}}$ and no group $C \subset A$ of agents such that the group C can manipulate f by reporting a profile $(R'_C, R_{-C}) \in \tilde{\mathcal{R}}$.

Theorem 1 is a substantial extension of the Lemma 1. The weakness in the latter result is that if a group C of agents have to decide if it is not possible to manipulate at a given profile R , they are restricted to choose preferences R'_C such that no two houses are connected by indifference at the profile (R'_C, R_{-C}) , i.e., such that the profile (R'_C, R_{-C}) belongs to $\tilde{\mathcal{R}}$. To do that, the group must first know the preferences of the group $A \setminus C$ as the domain $\tilde{\mathcal{R}}$ is not a hyperrectangle. Strategy-proofness, as in Theorem 1, on the other hand, requires non-manipulability even if the reported preferences (R'_C, R_{-C}) belongs to $\mathcal{R} \setminus \tilde{\mathcal{R}}$.

The following lemma states that the “trade” between two states $x, x' \in \Sigma_R$, where $R \in \tilde{\mathcal{R}}$, can be decomposed uniquely into a number of trading cycles such that all agents in a cycle are weakly better off at x' than at x with strict preference for some agent, or all agents in a cycle are weakly better off at x than at x' . More precisely, a *trading cycle* from

⁷To see this, suppose that $f(R) = x = (\mu, p)$ with $p_h > \underline{p}_h$ for some $h \in H$, and $f(R'_C, R_{-C}) = x' = (\mu', p')$. Assume further that $C = A$ and that R'_a is represented by a quasi-linear utility function $u_{ah}(p) = v_{ah} - p_h$ where $v_{a\mu_a} = \bar{p}_{\mu_a}$ and $v_{ah} = \underline{p}_h$ for each $a \in A$ and for any $h \neq \mu_a$. Then, \underline{p} is the unique minimal RPE price vector at profile R' by construction of the preferences, and, consequently, $f(R') = (\mu', p') = (\mu, \underline{p})$. Because $p_h > \underline{p}_h$ for some $h \in H$, the set $A' = \{a \in A : p_{\mu_a} > \underline{p}_{\mu_a}\}$ is nonempty. Since $f(R') = (\mu', p') = (\mu, \underline{p})$, it then follows that $x'_a P_a x_a$ for all $a \in A'$ and $x_a I_a x'_a$ for all $a \in A \setminus A'$ (the set $A \setminus A'$ is nonempty by Theorem 1). By monotonicity of the preferences, it is now clear that the agents in A' can transfer some “small” amount of money to the agents in $A \setminus A'$ after the outcome of the mechanism has been realized in such a way that all agents in A gain strictly by the misrepresentation.

μ to μ' is a sequence $(a_j)_{j=1}^t$ of distinct agents such $\mu'_{a_j} = \mu_{a_{j+1}}$ for $j < t$, where $\mu'_{a_t} = \mu_{a_1}$ if the cycle is *closed*, and where $\mu'_{a_t} \in \mu_0$ and $\mu_{a_1} \in \mu'_0$ if the cycle is *open*.

Lemma 2. Let $R \in \tilde{\mathcal{R}}$ and $x, x' \in \Sigma_R$. Then there are unique partitions $H = H_1 \cup H_2$ and $A = A_1 \cup A_2$ of the sets of houses and agents, respectively, such that $a \in A_1$ if and only if a belongs to a trading cycle where all agents are weakly better off at x' than at x and some agent in the trading cycle is strictly better off. Moreover, $A_2 = A \setminus A_1$, $H_1 = \{h \in H : h = \mu'_a \text{ for some } a \in A_1\}$, and $H_2 = H \setminus H_1$.

Lemma 3. Let $R \in \tilde{\mathcal{R}}$ and $x, x' \in \Sigma_R$, where $x = (\mu, p)$, $x' = (\mu', p')$, $p' \leq p$, and $p' \neq p$. Then, $x_a I_a x'_a$ for all $a \in A_2$.

Proof. To obtain a contradiction, suppose that $x_a P_a x'_a$ for some $a \in A_2$ (we need not consider any other case by the construction of A_2). Define next the assignment μ'' as: $\mu''_a = \mu'_a$ if $a \in A_1$ and $\mu''_a = \mu_a$ if $a \in A_2$. Then, the assignment μ'' defines a Pareto improvement of μ' at the state x' since $p'_h = p_h$ for all h with $h = \mu_a$ and $a \in A_2$. This contradicts that x' being an RPE state if, in addition, the state $x'' = (\mu'', p')$ is priority respecting, i.e., if (i) the priorities are respected at state x'' and (ii) $x''_a R_a(h, p)$ for all $a \in A$ if $h \in \mu''_0$. Cases (i) and (ii) are proved separately as follows:

- (i) Suppose that $x''_{a'} P_a x''_a$ for some $a, a' \in A$. We will demonstrate that the priorities are respected at state x'' in all possible cases, i.e., that $\pi_{a'} < \pi_a$ in all possible cases. If $a, a' \in A_1$ or if $a, a' \in A_2$, then $\pi_{a'} < \pi_a$ since $x, x' \in \Sigma_R$. If $a \in A_1$ and $a' \in A_2$, then $x''_a = x'_a$ and $x''_{a'} = x_{a'}$, so $x''_{a'} P_a x''_a$ imply that $x_{a'} P_a x'_a R_a x_a$ and, hence, $\pi_{a'} < \pi_a$ since $x \in \Sigma_R$. Finally, if $a' \in A_1$ and $a \in A_2$ then $x''_a = x_a$, $x''_{a'} = x'_{a'}$, and $x''_{a'} P_a x''_a$ imply that $x'_{a'} P_a x_a R_a x'_a$ and, hence, $\pi_{a'} < \pi_a$ since $x' \in \Sigma_R$.
- (ii) Consider μ''_0 and let $G = (a_j)_{j=1}^t$ be a trading cycle $\mu \rightarrow \mu'$. Suppose first that G is closed. Then, $\mu_{a_j} \notin \mu''_0$ for all j . Hence, we need only consider open trading cycles. Suppose therefore that G is open, i.e., that $\mu_{a_1} \in \mu'_0$ and $\mu'_{a_t} \in \mu_0$. Then if $G \subset A_1$, $h = \mu_{a_1} \in \mu'_0 \cap \mu''_0$ and, hence, $x'_a R_a(h, p')$ for all $a \in A$. Thus, $x''_a R_a(h, p')$ for all $a \in A$. If $G \subset A_2$ then $h = \mu'_{a_t} \in \mu_0 \cap \mu''_0$ and, hence, $x_a R_a(h, p)$ for all $a \in A$, and then $x''_a R_a(h, p')$ for all $a \in A$, since $p_h = p'_h$ when $h = \mu'_{a_t}$ and $a_t \in A_2$.

Hence, the state x' can not be an RPE state which contradicts our assumptions, and, consequently, $x_a I_a x'_a$ for all $a \in A_2$. \square

Let now $G = (a_1, \dots, a_t)$ and $S = (h_1, \dots, h_{t+1})$ be any two sequences of distinct agents and houses, respectively, and let a_l , $1 \leq l \leq t$, be any agent in the sequence G . Define next T to be the set of all such triples (G, S, a_l) .

Lemma 4. Let $R \in \mathcal{R}$. Then:

- (i) T is a finite set,

- (ii) For any triple (G, S, a_l) , there are unique prices p_{h_j} , for $1 \leq j \leq t+1$, such that $p_{h_1} = \bar{p}_{h_1}, p_{h_{t+1}} = \bar{p}_{h_{t+1}}$, and $(h_j, p_{h_j})I_{a_j}(h_{j+1}, p_{h_{j+1}})$ for all $j \neq l$,
- (iii) $R \in \tilde{\mathcal{R}}$ if and only if there is no triple $(G, S, a_l) \in T$ such that $(h_l, p_{h_l})I_{a_l}(h_{l+1}, p_{h_{l+1}})$.

Proof. Case (i) follows since A and H are finite sets, case (ii) follows from the continuity and monotonicity properties of the preferences, while case (iii) is a direct consequence of Definition 9. \square

In all of the remaining proofs, a specific preference representation will play an important role. This preference representation is next formally defined.

Definition 10. Let $a' \in A, h' \in H$, and $\delta = 1 + \max\{\bar{p}_h : h \in H\} - \min\{p_h : h \in H\}$. Then, preferences $R_{a'h'} \in \mathcal{R}_{a'}$ are represented by a quasi-linear utility function $u_{a'h}(p) = v_{a'h} - p_h$ on the set of houses and prices $(H \cup \{0\}) \times \mathbb{R}$ where the values $v_{a'h}$ are given by:

$$\begin{aligned} v_{a'h} &= (1 - h)\delta \text{ if } h \neq h', \\ v_{a'h'} &= 2\delta. \end{aligned}$$

Lemma 5. Let $R \in \tilde{\mathcal{R}}$, and consider the sets of distinct agents and distinct houses, $C = \{\hat{a}_1, \dots, \hat{a}_k\} \subset A$ and $\{\hat{h}_1, \dots, \hat{h}_k\} \subset H \cup \{0\}$, respectively. Let further $R'_{a_j} = R_{a_j h_j}$ for all $a_j \in C$. Then $(R'_C, R_{-C}) \in \tilde{\mathcal{R}}$.

Proof. To obtain a contradiction, suppose that $(R'_C, R_{-C}) \notin \tilde{\mathcal{R}}$. Then, by Lemma 4, there is a triple $(G, S, a_l) \in T$ and unique prices p_{h_j} , for all $h_j \in S$, such that $p_{h_1} = \bar{p}_{h_1}, p_{h_{t+1}} = \bar{p}_{h_{t+1}}$, and:

$$(h_j, p_{h_j})I_{a_j}(h_{j+1}, p_{h_{j+1}}) \text{ for all } 1 \leq j \leq t. \quad (1)$$

Since $R \in \tilde{\mathcal{R}}$, by assumption, there is an agent $\hat{a}_j \in C \cap G$. Let now $\hat{a}_j = a_l$. Because $a_l \in C$, it follows by the construction of the preferences that $(h_l, p_{h_l})I_{a_l}(h_{l+1}, p_{h_{l+1}})$ if $v_{a_l h_l} - p_{a_l} = v_{a_l h_{l+1}} - p_{a_{l+1}}$, i.e., if $v_{a_l h_l} - v_{a_l h_{l+1}} = p_{a_l} - p_{a_{l+1}}$. Hence, if $|v_{a_l h_l} - v_{a_l h_{l+1}}| \neq |p_{a_l} - p_{a_{l+1}}|$, then also $v_{a_l h_l} - v_{a_l h_{l+1}} \neq p_{a_l} - p_{a_{l+1}}$. Because $|p_{a_l} - p_{a_{l+1}}| < \delta$, by construction, we will therefore obtain a contradiction to the assumption that $(R'_C, R_{-C}) \notin \tilde{\mathcal{R}}$ if $|v_{a_l h_l} - v_{a_l h_{l+1}}| \geq \delta$. But this condition must hold by the construction of the preferences, and the facts that $h_l \neq h_{l+1}, h_l \in H \cup \{0\}$, and $h_{l+1} \in H \cup \{0\}$ as:

$$\begin{aligned} |v_{a_l h_l} - v_{a_l h_{l+1}}| &= |(1 + h_{l+1})\delta| \text{ if } h_l = \hat{h}_j, \\ |v_{a_l h_l} - v_{a_l h_{l+1}}| &= |-(1 + h_l)\delta| \text{ if } h_{l+1} = \hat{h}_j, \\ |v_{a_l h_l} - v_{a_l h_{l+1}}| &= |(h_{l+1} - h_l)\delta| \text{ if } h_l, h_{l+1} \neq \hat{h}_j. \end{aligned}$$

Hence, $(h_l, p_{h_l})I_{a_l}(h_{l+1}, p_{h_{l+1}})$ can never be the case, and, therefore, houses h_l and h_{l+1} cannot be connected by indifference, contradicting the assumption that $(R'_C, R_{-C}) \notin \tilde{\mathcal{R}}$. Thus, $(R'_C, R_{-C}) \in \tilde{\mathcal{R}}$. \square

Lemma 6. Consider state $(\mu, p) \in \Sigma_{R'}$ where $R' = (R'_C, R_{-C}) \in \tilde{\mathcal{R}}$, $R'_{a_j} = R_{a_j h_j}$, and $\mu_{a_j} = h_j$ for $a_j \in C$. Then, $x_{a_j} P_{a_j} x_k$ for all $a_j \in C$ and all $k \in A \setminus \{a_j\}$.

Proof. This follows directly from the construction of $R_{a_j h_j}$ and the assumption that $\mu_{a_j} = h_j$, i.e., the condition $x_{a_j} P_{a_j} x_k$ for some arbitrary $k \in A \setminus \{a_j\}$ is equivalent to:

$$(1 + h_k)\delta > p_{h_j} - p_{h_k}.$$

But this condition always holds as $h_k \in H \cup \{0\}$, i.e., the value of the left hand side of the condition is always weakly greater than δ , whereas the left hand side always is strictly smaller than δ by the construction of δ . \square

Proof of Theorem 2. Let $R \in \tilde{\mathcal{R}}$ and $f(R) = x = (\mu, p)$. To obtain a contradiction, suppose that the agents in $C \subset A$ can manipulate f at profile $R \in \tilde{\mathcal{R}}$. Then there is a profile $R^1 = (R^1_C, R_{-C})$ and a state $f(R^1) = x^1 = (\mu^1, p^1)$ such that $x^1_a P_a x_a$ for all $a \in C$ by Definition 8. Define now $R^2 = (R^2_C, R_{-C})$ to be the profile where $R^2_a = R_{a\mu^1_a}$ for all $a \in C$. Because $R \in \tilde{\mathcal{R}}$, it then follows from Lemma 5 that $R^2 \in \tilde{\mathcal{R}}$.

The first step in the proof is to demonstrate that there is a state $x^2 \in \Sigma_{R^2}$ such that $x^2 = (\mu^2, p^1)$ for some assignment μ^2 . For this purpose, consider state $x^1 = (\mu^1, p^1)$ at the profile R^2 . We note the following:

- (i) $p^1_h = \underline{p}_h$ for all $h \in \mu^1_0$ as $x^1 \in \Sigma_{R^1}$.
- (ii) $x^1_a R^2_a(h, p)$ if $h \in \mu^1_0$. For $a \notin C$, this follows since $x^1 \in \Sigma_{R^1}$ and $R^2_a = R^1_a$. For $a \in C$, this follows from Lemma 6 as agent a strictly prefers bundle x^2_a over any other bundle at preferences R^2_a .
- (iii) For all $a, a' \in A$, $p^1_{\mu^1_{a'}} = \bar{p}_{\mu^1_{a'}}$ if $x^1_{a'} P^2_a x^1_a$. For $a \notin C$, this follows because $x^1 \in \Sigma_{R^1}$ and $R^2_a = R^1_a$. For $a \in C$, the conclusion again follows directly from Lemma 6.
- (iv) $x^1_{a'} P^2_a x^1_a$ only if $\pi_{a'} < \pi_a$. For $a \notin C$, this follows as $x^1_{a'} P^2_a x^1_a$ for some $a, a' \in A$ implies that $\pi_{a'} < \pi_a$ since $x^1 \in \Sigma_{R^1}$ and $R^2_a = R^1_a$. For $a \in C$, the conclusion again follows directly from Lemma 6.

From Definition 7 and points (i)–(iv), it then follows that if μ^1 is constrained efficient at profile R^2 , then we can choose $x^2 = x^1$ to prove that there is a state $x^2 \in \Sigma_{R^2}$. On the other hand, if μ^1 is not constrained efficient, then there is a constrained efficient assignment μ^2 , at profile R^2 and the price vector p^1 , which is a Pareto improvement of μ^1 . Then we can set $x^2 = (\mu^2, p^1)$. In this case, we remark that μ^2 can be chosen so that points (i)–(iii) above are satisfied since μ^2 is a Pareto improvement. Point (iv) is satisfied by definition since μ^2 has to be priority respecting. Hence, there is a state $x^2 \in \Sigma_{R^2}$.

We next note that by the construction of R^2_a and the fact that $x^2 \in \Sigma_{R^2}$, it follows directly that $x^2_a = x^1_a$ for all $a \in C$. Let now $f(R^2) = x^3 = (\mu^3, p^3)$. Then $p^3 \leq p^2$, since $x^2, x^3 \in \Sigma_{R^2}$ and p^3 is the unique minimal RPE vector. The uniqueness property follows from the fact that $R^2 \in \tilde{\mathcal{R}}$ and Theorem 2 in [Andersson and Svensson \(2014\)](#).

We next show that $x_a^3 R_a x_a^2$ for all $a \in A$. Consider now the profile R^2 , the states x^2 and x^3 , and the trade $\mu^2 \rightarrow \mu^3$. Let $H = H_1 \cup H_2$ and $A = A_1 \cup A_2$ be the partitions as defined in Lemma 2. If $p_h^3 < p_h^2$ and $\mu_a^2 = h$, then $x_a^3 P_a^2 x_a^2$ and, hence, $a \in A_1$ and $h \in H_1$. Then also $p_h^3 = p_h^2$ for all $h \in H_2$. Moreover, by Lemma 3, $x_a^3 I_a^2 x_a^2$ for all $a \in A_2$. Hence, $x_a^3 R_a^2 x_a^2$ for all $a \in A$.

Note next that $x_a^3 P_a x_a$ for all $a \in C$. This follows as $x_a^3 R_a x_a^2$, $x_a^1 P_a x_a$, and $x_a^2 = x_a^1$ for all $a \in C$. But because $f(R^2) = x^3$, the group $C \subset A$ can manipulate the mechanism f at profile $R \in \tilde{\mathcal{R}}$ by reporting profile $(R_C^2, R_{-C}) \in \tilde{\mathcal{R}}$. But that is a contradiction to Lemma 1. Hence, the minimal RPE mechanism with domain \mathcal{R} is group strategy-proof on the domain $\tilde{\mathcal{R}}$. \square

Appendix: A Measure where $\tilde{\mathcal{R}} = \mathcal{R}$ a.e.

This Appendix provides a measure on subsets of profiles in \mathcal{R} such that the subset of profiles where two houses are connected by indifference has measure zero.

Let $h \in H$ and let D_h be the set of continuous and strictly decreasing functions $f : [\underline{p}_h, \bar{p}_h] \rightarrow [0, 1]$. Then each preference R_a , for each agent $a \in A$, can be represented by a list $u_a = (u_{ah})_1^n$ of utility functions $u_{ah} \in D_{ah}$, where $D_{ah} = D_h$, and each such list represents a preference R_a . Hence, $u_a \in \times_h D_{ah}$. Moreover, a preference profile $R = (R_1, \dots, R_n)$ can be represented by an element in $U = \times_{ah} D_{ah}$.

For each $h \in H$, let \hat{m}^h be the Lebesgue measure defined for measurable subsets of the interval $[\underline{p}_h, \bar{p}_h]$. Let also, for a subset $D' \subset D_h$:

$$m^h D' = \hat{m}^h \{y : y = f(\bar{p}_h) \text{ for some } f \in D'\}.$$

Note that by this definition of the measure of a subset $D' \subset D_h$ of utility functions, we have that $m^h D' = 0$ if for some $y \in [0, 1]$ and for all $u_{ah} \in D'_{ah}$, $u_{ah}(\bar{p}_h) = y$. This means that if utility functions u_{ah} are drawn randomly, with equal probability, from the set D_{ah} , the probability that $u_{ah} \in D'_{ah}$ is zero. Finally, denote the product measure on U by $m = \times_{ah} m^h$.

Proposition 1. $\tilde{\mathcal{R}} = \mathcal{R}$ a.e.

Proof. Let (a_1, \dots, a_t) be a sequence of distinct agents and (h_1, \dots, h_{t+1}) a sequence of distinct houses, and let $S \subset \times_{ah} D_{ah}$ be the subset of profiles where two distinct houses h' and h'' are connected by indifference for those two sequences, i.e.:

$$S = \{u \in U : \text{there is } p \in \Omega \text{ such that all conditions of Definition 9 are satisfied}\}.$$

To complete the proof, we need to demonstrate that $mS = 0$. To achieve this task, decompose U according to $U = U' \times U''$, where $U'' = D_{a_t h_{t+1}}$. Let $u' \in U'$ and:

$$g(u') = m_{a_t h_{t+1}} \{u_{a_t h_{t+1}} : (u', u_{a_t h_{t+1}}) \in S\}.$$

Then:

$$mS = \int_{u' \in U'} g dm',$$

where $m' = \times_{ah \neq a_t h_{t+1}} m^h$ (see, e.g., Royden, 1968, Lemma 16, p.266). However, $g(u') = 0$. To see this, note that p_{h_j} , for $j \leq t$, are uniquely determined by monotonicity of preferences. This means that the functions $u_{a_t h_{t+1}}$ with $(u', u_{a_t h_{t+1}}) \in S$ have to satisfy $u_{a_t h_{t+1}}(\bar{p}_{h_{t+1}}) = u'_{a_t h_t}(p_{h_t})$. But then $g(u') = 0$.

But now it follows that the set of profiles where two houses are connected by indifference has measure zero, since there is only a finite number of sequences of distinct agents (a_1, \dots, a_t) and distinct houses (h_1, \dots, h_{t+1}) . \square

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