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Department of Economics
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On Axioms Underlying Use of Reserve Price

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ON AXIOMS UNDERLYING USE OF RESERVE PRICE

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ABSTRACT. This paper establishes reserve price as an ethical necessity contrary to its popular interpretation as an instrument of revenue generation. It provides an axiomatic justification to reserve pricing at Vickrey auction in single as well as multiple objects settings. It also provides a multi-object axiomatization of another popular class reserve price mechanisms called *maxmed* mechanisms introduced by Sprumont [JET,2013]. In general, a topological interpretation of reserve price is provided as the *infimum* of the set of non-negative real numbers satisfying the following property: if all agents bid the same number from this set, then at least one object is allocated.

For the single object setting, (i) it is shown that any anonymous strategy-proof mechanism that satisfies non-bossiness (in decision) must have an allocation rule same as that of a Vickrey auction with reserve price (VARP) and (ii) an axiomatization is provided for the class of VARP mechanisms. The same results are shown to hold in multiple objects context under an additional ethical axiom *minimal impartiality* (which requires that either all objects or no object be allocated at any profile where all agents report the same value) and a technical regularity condition. These results are further extended to provide a characterization of *maxmed* mechanisms in the multiple object setting.

JEL classification: C72; C78; D71; D63

Keywords: Anonymous, non-bossy, strategy-proof mechanism, *maxmed* mechanisms

1. INTRODUCTION

It is well known that reserve pricing at auctions is an important method of ensuring that the seller revenue is not too low in settings where competition among bidders is low and bidders are asymmetric (Ausubel and Cramton [3]). Vickrey auctions, on other hand, ensure that the objects are allocated efficiently and that agents have no incentive to misreport irrespective of what other agents are reporting. Therefore, Vickrey auction with reserve price [VARP]¹ is a useful mechanism that achieves both objectives of efficient allocation of objects and avoidance of low seller revenues. Hence, VARP has been widely observed in practical applications.² However, there is a dearth of literature on normative justifications

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¹Vickrey auction with reserve price is a mechanism with a special allocation rule where objects are allocated only to agents whose bids are not less than the reserve price. Further, winners of object pay the maximum of the reserve price and the greatest losing bid as price and non-winners pay nothing.

²As pointed out by Lucking-Reiley [6], VARP was found to be held as early as 1897 by pioneering stamp dealer William P. Brown of New York.

of VARP. This is in contrast to a large literature characterizing Vickrey auction without reserve prices.³ The first half of this paper, attempts to fill this gap by providing axiomatic justification to reserve pricing at Vickrey auctions, both in single and multiple objects setting. The second half of the paper discusses another popular class of mechanisms involving reserve price called *maxmed* mechanisms introduced by Sprumont [17].⁴ In particular, it generalizes the maxmed mechanisms to the multiple identical objects setting and provides an axiomatization.

We consider the problem of allocating homogeneous indivisible objects with monetary transfers and unit demand, under the condition that the planner may decide not to allocate some or all of the objects. This problem has several practical applications where the objects may be houses, jobs, locations etc. The agents are assumed to have quasi-linear preferences over the object and money. In particular, each agent has a non-negative valuation for the object which is private information, and has utility as a linear function of money. The planner, therefore, needs to design a mechanism to ensure truthful revelation of valuations. A mechanism in this context, is a pair consisting of an assignment function which determines which agent gets an object, and a vector of monetary transfers. Note that our setting allows for no objects to be allocated to any agent.

We completely characterize the class of ethical mechanisms immune to strategic manipulation (by a single agent) and show that each member of this class must have an allocation rule same as that of VARP mechanisms. We describe ethical mechanisms by invoking three notions of fairness: *anonymity in welfare* used by Ashlagi and Serizawa [1], Hashimoto and Saitoh [5], *no-envy* introduced by Foley [4], Varian [19] and *non-bossiness* used by Satterthwaite and Sonnenschein [16], Svensson [18]. A mechanism is said to satisfy anonymity in welfare if utility levels of any two agents get interchanged, when their valuations are interchanged with all other agents' valuations remaining unchanged. A mechanism is said to satisfy no-envy, if each agent prefers his/her bundle of object and money, to that of others. We use a weaker version of non-bossiness (also used by Mishra and Quadir [7]), that is, *non-bossiness in decision* which requires that no agent can change his/her valuation in such a way that the allocation decision of that agent does not change, but the allocation decision of some other agent changes.⁵ Further, we describe mechanisms immune to manipulation by invoking the concept of strategy-proofness. A mechanism is said to be strategy-proof if truth-telling is a weakly dominant strategy for all agents in the direct revelation game induced by it.

In the single object case, we show that a strategy-proof mechanism satisfies anonymity and non-bossiness only if it has an allocation rule same as that of a VARP mechanism. Then we completely characterize the class of mechanisms that satisfy anonymity, strategy-proofness and non-bossiness. Any mechanism in this class, satisfies a mild *zero-utility* condition (requiring that any agent with zero valuation for the object should get zero utility by participating in the mechanism), if and only if it is a VARP mechanism.

³Moulin [8], Ashlagi and Serizawa [1], Sakai [13], [15], Saitoh and Serizawa [12], Mukherjee [9] etc.

⁴These mechanisms, too, have the same allocation rule as VARP mechanism but have a different transfer function.

⁵This weaker axiom imposes no restriction on transfers.

However, these characterizations fail to hold in the multiple homogeneous objects case straightaway. That is because, with multiple objects, any subset of objects may be withheld by the planner leading to a proliferation of the number of possible decisions at any valuation profile. Therefore, we introduce a regularity condition (applicable only at some valuation profiles) and an additional ethical axiom *minimal impartiality* (which requires that either all or no object be allocated at any profile where all agents have bid the same value). We show that any *regular* mechanism satisfies minimal impartiality along with anonymity, non-bossiness, strategy-proofness and zero-utility, if and only if it is a VARP mechanism of a special kind.

Finally, we characterize the class of regular, minimally impartial, anonymous, non-bossy, strategy-proof mechanisms satisfying no-envy. Further, within this class of mechanisms, we identify the Pareto undominated, individually rational and feasible mechanisms. We then show that these mechanisms are the general version of the maxmed mechanisms of Sprumont [17].

Perhaps the most popular paper on reserve pricing at auctions is Myerson [10]. Myerson [10], in an independent private value setting, identifies a particular VARP mechanism as the revenue maximizing mechanism under the assumptions of: (i) symmetric bidders, (ii) distribution of valuations satisfying a regularity condition and (iii) the planner knowing this distribution with certainty. Further, Myerson [10] obtains a revenue maximizing mechanism involving different reserve prices for different agents, if assumption (i) is violated. In contrast, the present paper makes no such assumption and shows that, in an independent private value setting, any ethical (anonymous and non-bossy) and strategy-proof mechanism must have a non-negative reserve price (same across all agents) and must allocate objects as in a VARP mechanism.

The other papers relevant to the present analysis are Athanasiou [2], Sakai [14] and Sprumont [17]. Athanasiou [2] characterizes the set of Pareto undominated mechanisms that satisfy individual rationality in the class of anonymous and strategy-proof mechanisms. Sprumont [17] too investigates Pareto undominated mechanisms satisfying individual rationality but restricts attention to the class of feasible mechanisms that are anonymous, envy-free and strategy-proof mechanisms. For the two agent case, results of both these papers coincide to a special class of mechanisms with VARP allocation rule. For the general n agent case, Sprumont [17] obtains a family of single parameter maxmed mechanisms with VARP allocation rule. This paper characterizes a *larger* class of mechanisms with VARP allocation rule (that contain the maxmed mechanisms) using the three axioms: anonymity, non-bossiness and strategy-proofness. Further, it completely characterizes the class of VARP mechanisms. Also, unlike these papers, the present paper extends the characterizations to a multiple homogeneous object setting (which is a complicated exercise since the multiple objects setting allows a strict subset of objects to be not allocated at some profiles). Finally, this paper generalizes maxmed mechanisms to the multiple homogeneous objects setting and provides an axiomatization.

Sakai [14], too, restricts attention to the single object allocation problem and provides axiomatizations of VARP mechanisms using weak versions of efficiency, individual rationality and equal treatment of equals. In contrast the present paper does not use any efficiency axiom in any of the characterizations. That is, all the efficiency aspects embedded in VARP arise in our setting as an implication of strategic and fairness axioms, both in the single as well as multiple objects setting.

Section 2 states the model and definitions. Section 3 contains the main results with first subsection focusing on the single object case and the second subsection stating the multiple objects case. Section 5 is the appendix containing proofs.

2. MODEL

We consider a situation where m homogeneous objects need to be allocated to agents in $N = \{1, 2, \dots, n\}$ with unit demand and the restriction $1 \leq m \leq n - 1$. Each agent $i \in N$ has an independent private valuation $v_i \in \mathbb{R}_+$. A mechanism is a tuple (d, τ) such that at any reported profile of valuations $b \in \mathbb{R}_+^N$, each agent i is allocated a transfer $\tau_i(b) \in \mathbb{R}$ and a decision $d_i(b) \in \{0, 1\}$. $d_i(b) = 1$ implies that agent i gets an object, while $d_i(b) = 0$ stands for i not getting the object. For any reported valuation profile $b \in \mathbb{R}_+^N$, define $W(b) := \{i \in N \mid d_i(b) = 1\}$ to be the set of agents that are allocated an object. Note that at any reported profile of valuations $b \in \mathbb{R}_+^N$, $\sum_i^n d_i(b) \leq m$. That is, all objects need not get allocated at all reported profiles.

The utility to agent i with a true valuation of v_i at any reported profile $b \in \mathbb{R}_+^N$, from the mechanism (d, τ) is given by $u((d_i(b), \tau_i(b)); v_i) = v_i d_i(b) + \tau_i(b)$. Let $\forall i \in N, \forall b \in \mathbb{R}_+^N$, $b_{-i} := (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$. Define $M(x) := \max\{x_t\}$ for all $x = (x_t)_{t=1}^p, \forall p \in \mathbb{N}$. That is, $M(x)$ is the largest element of any vector x in Euclidean space of any dimension. Also define $b(r)$ to be the r th ranked valuation in a non-increasing arrangement of coordinates of any $b \in \mathbb{R}_+^N$ for all $r = 1, 2, \dots, n$. In case of ties, w.l.o.g. we use the tie-breaking rule $1 \succ \dots \succ n$. Therefore $b(1) = M(b)$ for all $b \in \mathbb{R}_+^N$.⁶ Define $B_0 := \{v \in \mathbb{R}_+^N : W(v) = \emptyset\}$ to be the set of profiles at which no object is allocated. Finally, define for any $x \geq 0$, $\bar{x}^t := (x, x, \dots, x) \in \mathbb{R}_+^t$ for all $t = 1, 2, \dots, n$. Therefore, $\bar{x}^n = (x, x, \dots, x) \in \mathbb{R}_+^n$ and $\bar{x}^{n-1} = (x, x, \dots, x) \in \mathbb{R}_+^{n-1}$.

The following definition states the Vickrey auction with reserve price $r \geq 0$ as a mechanism with an allocation rule that does not allocate the object to any agent who has bid less than r .

Definition 1. Any mechanism (d^r, τ^r) is said to be a *Vickrey auction with reserve price $r \geq 0$* (VARP) if for all $i \in N$ and all $v \in \mathbb{R}_+^N$,

- $v_i < \max\{v_{-i}(m), r\} \implies d_i^r(v) = 0$
- $\tau_i^r(v) = \begin{cases} 0 & \text{if } d_i^r(v) = 0 \\ -\max\{v_{-i}(m), r\} & \text{if } d_i^r(v) = 1 \end{cases}$

Define Γ to be the class of Vickrey auctions with reserve prices $\{d^r, \tau^r\}_{r \geq 0}$.

The popular strategic axiom in independent private values setting, strategy-proofness, eliminates the incentive to misreport valuation for each agent. It is defined as follows.

Definition 2. A mechanism (d, τ) is *strategy-proof* (SP) if $\forall i \in N, \forall v_i, v'_i \in \mathbb{R}_+, \forall v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$,

$$u(d_i(v_i, v_{-i}), \tau_i(v_i, v_{-i}); v_i) \geq u(d_i(v'_i, v_{-i}), \tau_i(v'_i, v_{-i}); v_i)$$

⁶For any $i \neq j, i \succ j$ means that the tie is broken in favour of agent i . That is, if $M(b) = b_3 = b_7$ and $3 \succ 7$, then $b(1) = b_3$.

Therefore, an SP mechanism guarantees that revealing the true valuation is a weakly dominant strategy for each agent.

The following three definitions pertain to three popular notions of fairness. The first states the fairness concept of anonymity in welfare which requires the allocation to each agent, be independent of his/her identity. The second states the notion of non-bossiness in decision which requires that if an agent cannot change his/her own decision (of getting an object or not) by changing his/her valuation, then he/she should not be able to change the decision of any other agent. The third states the notion of no-envy which requires that every agent prefers his/her own allocation (of decision and transfer from the mechanism) to that of any other agent.

Definition 3. A mechanism (d, τ) satisfies *anonymity in welfare* (AN) if for all $i \neq j \in N$ and all $v \in \mathbb{R}_+^N$,

$$u(d_i(v_i, v_j, v_{-i-j}), \tau_i(v_i, v_j, v_{-i-j}); v_i) = u(d_j(v_j, v_i, v_{-i-j}), \tau_j(v_j, v_i, v_{-i-j}); v_j)$$

Definition 4. A mechanism (d, τ) satisfies *non-bossiness in decision* (NBD) if for all $i \in N$, all $v \in \mathbb{R}_+^N$ and all $v'_i \in \mathbb{R}_+$,

$$d_i(v) = d_i(v'_i, v_{-i}) \implies d_j(v) = d_j(v'_i, v_{-i}), \forall j \neq i$$

Definition 5. A mechanism (d, τ) satisfies *no-envy* (NE) if for all $i \neq j \in N$, all $v \in \mathbb{R}_+^N$,

$$u(d_i(v), \tau_i(v); v_i) \geq u(d_j(v), \tau_j(v); v_j)$$

The following axiom requires that all agents get a non-negative utility at all possible profiles so as to ensure voluntary participation in the mechanism.

Definition 6. A mechanism (d, τ) satisfies *individual rationality* (IR) if for all $i \in N$, all $v \in \mathbb{R}_+^N$,

$$v_i d_i(v) + \tau_i(v) \geq 0$$

The following axiom implies the fairness perception that if an agent has zero valuation for the object, then the agent must not get a positive or negative utility by merely participating in the mechanism.

Definition 7. A mechanism (d, τ) satisfies *zero-utility* if for all $i \in N$ and all $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$,

$$u(d_i(0, v_{-i}), \tau_i(0, v_{-i}); 0) = 0$$

The following axiom of feasibility requires that the sum of transfers not exceed zero for any profile of valuations and thus, ensures that implementing fair mechanisms do not entail wastage of resources.

Definition 8. A mechanism (d, τ) satisfies *feasibility* (F) if for all $v \in \mathbb{R}_+^N$,

$$\sum_{i \in N} \tau_i(v) \leq 0$$

Finally, for any given class of mechanisms \mathcal{F} , define a transitive preference relation on the mechanisms in \mathcal{F} in the following manner. For any two mechanisms $(d, \tau), (d', \tau') \in \mathcal{F}$, we say that $(d, \tau) \succeq (d', \tau')$ if for all $i \in N$ and all $v \in \mathbb{R}_+^N$, $u(d_i(v), \tau_i(v); v_i) \geq u(d'_i(v), \tau'_i(v); v_i)$. If in addition, the inequality is strict for some i and some v , then we

write that $(d, \tau) \succ (d', \tau')$ and say that (d, τ) *Pareto dominates* (d', τ') . If all the inequalities are equalities, we write that $(d, \tau) \sim (d', \tau')$ and say that (d, τ) is *Pareto equivalent* to (d', τ') . Further, we call the class of mechanisms in \mathcal{F} that are not dominated by any other mechanism in \mathcal{F} , as the set of *Pareto undominated* mechanisms in \mathcal{F} .

3. MAIN RESULTS

3.1. Single object. In this section we study the single object case. The following theorem states that any mechanism satisfying AN, SP and NBD must have an allocation rule same as that of a VARP mechanism. That is, any such mechanism must have an associated reserve price (which is shown to be equal to the η of Lemma 3 in subsection 5.1 of Appendix).

Theorem 1. A mechanism (d, τ) satisfies properties AN, SP and NBD only if $\exists r \geq 0$ such that for all $i \in N$ and all $v \in \mathbb{R}_+^N$,

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(1), r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(1), r\} \end{cases}$$

Proof: See Appendix. □

Define \mathcal{M} to be the class of mechanisms (d, τ) that satisfy the following property. There exists an $r \geq 0$ such that $\forall i \in N$ and $\forall v \in \mathbb{R}_+^n$,

$$\begin{aligned} \bullet \quad d_i(v) &= \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(1), r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(1), r\} \end{cases} \\ \bullet \quad \tau_i(v) &= \begin{cases} K(v_{-i}) - \max\{v_{-i}(1), r\} & \text{if } d_i(v) = 1 \\ K(v_{-i}) & \text{if } d_i(v) = 0 \end{cases} \end{aligned}$$

where $K : \mathbb{R}_+^{n-1} \mapsto \mathbb{R}$ is a *symmetric* function. Note that Γ (the set of all VARP mechanisms) is the set of mechanisms in \mathcal{M} that have a special $K(\cdot)$ function where $K(z) = 0$ for all $z \in \mathbb{R}_+^{n-1}$.

The following corollary completely characterizes the class of mechanisms that satisfy AN, SP and NBD.

Corollary 1. Any mechanism (d, τ) satisfies AN, SP and NBD *if and only if* $(d, \tau) \in \mathcal{M}$.

Proof: The sufficiency is easy to check. The necessity follows from Proposition 4 (in subsection 5.1 of Appendix) and Theorem 1. □

Remark 1. The Corollary 1 characterizes a large class of mechanisms with VARP allocation rule. The *maxmed* mechanisms characterized by Sprumont [17] can be obtained by setting

$$K(v_{-i}) = \text{med} \left\{ 0, v_{-i}(1) - r, \frac{r}{n-1} \right\}, \forall v \in \mathbb{R}_+^n, \forall i \in N$$

in the Corollary 1. Therefore, the *maxmed* mechanisms belong to this class.

The following corollary provides a complete characterization of the VARP mechanisms in the single object setting.

Corollary 2. A mechanism (d, τ) satisfies AN, SP, NBD, and zero-utility if and only if $(d, \tau) \in \Gamma$.

Proof: The proof of sufficiency is easy to check. To see the necessity, fix any $i \in N$ and any $v_{-i} \in \mathbb{R}_+^{n-1}$. Consider the profile $(0, v_{-i})$. From the Theorem 1 and zero-utility condition, it follows that $u_i(d_i(0, v_{-i}), \tau_i(0, v_{-i}); 0) = K(v_{-i}) = 0$. Hence, the result follows. \square

Remark 2. Sakai [14] provides a characterization of the class of VARP allocation rules using the axioms of *weak efficiency* and *non-imposition* in the single object setting.⁷ However, our characterizations do not use any efficiency axioms. In fact, both the reserve price as well as the efficiency properties embedded in the VARP mechanisms, arise in this paper, as implications of the fairness axioms AN, NBD and the strategic axiom SP.

3.2. Multiple objects. In this section we study the case where the number of objects m can take any integer value from 2 to $n - 1$. Ideally, the results in single object case should translate directly to the multiple homogeneous objects (with unit demand) setting. However, that is not the case. The reason for this are the two complications arising out of the multiple objects setting. First, at any profile, any one agent getting an object no longer implies that other agents do not get any object. Second, there may exist profiles where only a subset of objects are allocated.

To accommodate for these two difficulties, we invoke a regularity condition and an ethical axiom named “minimal impartiality”.

Definition 9. A mechanism (d, τ) is said to be *regular* if

- (a) For any sequence $\{y_k\} \subset \mathbb{R}_+$ such that $\{y_k\} \rightarrow y$; if for all $k, \bar{y}_k^n \notin B_0$ then

$$\bar{y}^n \notin B_0$$

- (b) For any $i \in N$ and any $v \in \mathbb{R}_+^n$, if $v_i = \inf\{x \geq 0 : \bar{x}^n \notin B_0\}$ and $d_i(v) = 1$, then $\forall j \neq i \in N$,

$$d_j(y, v_{-i}) = d_j(v), \forall y \in [0, v_i)$$

Therefore, a mechanism is regular if the associated allocation rule satisfies the technical conditions (a) and (b). Condition (a) is a mild continuity property which requires that objects be allocated at any profile \bar{y}^n , if there exists a sequence of profiles converging to \bar{y}^n such that at each member profile of this sequence: (i) all agents bid the same value and (ii) at least one object is allocated. Condition (b) requires the allocation rule to be well-behaved at profiles where some agent i has bid the value $\inf\{x \geq 0 : \bar{x}^n \notin B_0\}$ and gets an object. The allocation decision of all agents other than i at such profiles, should not change in response to any unilateral reduction in i 's report.

Definition 10. A mechanism (d, τ) satisfies *minimal impartiality* (MI) if for all $x \geq 0$,

$$\bar{x}^n \notin B_0 \implies |W(\bar{x}^n)| = m$$

⁷Non-imposition requires that for any $i \in N$ and any $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$, $\tau_i(0, v_{-i}) = 0$. That is, any mechanism satisfying non-imposition would give zero transfers to any agent bidding 0 irrespective of what other agents are bidding. It can easily be seen that Corollary 2 continues to hold if zero-utility condition is replaced by non-imposition in its statement. That is because non-imposition is a stronger condition than zero-utility.

The MI axiom is an ethical axiom which imposes impartial behaviour on the allocation rules at profiles where all agents bid the same value. It requires that either all objects be allocated or no object be allocated at such profiles. We assume w. l. o. g. that at any such profile, if objects are allocated then the agents in $\{1, 2, \dots, m\}$ (the set of top m agents according to the assumed tie-breaking rule) get an object each. Note that this axiom is trivially satisfied in the single object context.

Before presenting the main theorem in this subsection, we prove the following lemma which states that for any regular mechanism, if there exists a bid profile such that (i) all agents bid the same value and (ii) at least one object is allocated: then some subset of objects must be allocated at the profile where each agent bids the value $\inf\{x \geq 0 : \bar{x}^n \notin B_0\}$.

Lemma 1. If a mechanism (d, τ) is regular and $\{x \geq 0 : \bar{x}^n \notin B_0\} \neq \emptyset$ then

$$\bar{\eta}^n \notin B_0$$

where $\eta = \inf\{x \geq 0 : \bar{x}^n \notin B_0\}$.

Proof: By definition of η , if $\{x \geq 0 : \bar{x}^n \notin B_0\} \neq \emptyset$, then $\eta \in [0, \infty)$. Therefore, there exists a sequence $\{y_k\}$ converging to η such that $y_k > \eta$ for all k . Therefore, by Lemma 3, $\bar{y}_k^n \notin B_0, \forall k$. The result follows from the continuity condition (a) in Definition 9. \square

The main theorem in this section states that any regular mechanism satisfying MI, AN, SP and NBD must have an allocation rule same as that of VARP mechanism for m homogeneous objects. That is, any such mechanism allocates a object to any agent who has bid greater than the reserve price as well as the $m + 1$ th highest bid.

Theorem 2. A regular mechanism (d, τ) satisfies MI, AN, SP and NBD only if $\exists r \geq 0$ such that $\forall i \in N, \forall v \in \mathbb{R}_+^n$,

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(m), r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(m), r\} \end{cases}$$

Proof: See Appendix. \square

Define T^k to be the top $k \in N$ agents according to the tie-breaking rule (which is essentially a linear order on N). Given our assumed tie-breaking rule $1 \succ 2 \succ \dots \succ n$, $T^k = \{1, 2, \dots, k\}$. We now define a subclass of VARP mechanisms that respects its tie-breaking rules in a particular way.

Definition 11. Γ' is the class of mechanism $(d^r, \tau^r) \in \Gamma$ such that

- For all $x \geq r, d_i(\bar{x}^n) = 1$ for all $i \in T^m$.
- For any $v \in \mathbb{R}_+^n$ such that $\exists i \in N$ with $v_i = r$,

$$d_i(v) = 1 \implies i \in T^m$$

Γ' is the class of VARP mechanisms that satisfy two conditions. The first condition requires that all objects are allocated to the top m agents according to the tie-breaking rule, at any profile where (i) all agents have bid the same value and (ii) this value is not less than the reserve price. The second condition requires that any agent bidding the reserve price should get an object *only if* he/she is one of the top m agents according to the tie-breaking rule. Therefore, if there are 3 agents and 2 objects to be allocated along with a tie-breaking

rule $1 \succ 2 \succ 3$: a VARP mechanism in Γ' with reserve price r should have $d_3(r, x, r) = 0$ for all $x \geq 0$. Further, any mechanism in Γ' is Pareto equivalent to any VARP mechanism with same *reserve price* r belonging to $\Gamma \setminus \Gamma'$.

The following corollary characterizes the class Γ' in a manner analogous to Corollary 2 and shows that Γ' is the class of regular mechanisms that satisfy AN, SP, NBD, MI and zero-utility.

Corollary 3. A regular mechanism (d, τ) satisfies AN, SP, NBD, MI and zero-utility if and only if $(d, \tau) \in \Gamma'$.

Proof: The proof of sufficiency is easy to check. The proof of necessity follows in a manner identical to that in Corollary 2. \square

Remark 3. The independence of AN, SP, NBD and IR axioms can easily be seen. To check for the independence of the restrictions embodied in the regularity restriction and MI axiom, consider a simple setting where $m = 2$ and $n = 3$. Consider a VARP mechanism⁸ with reserve price r such that $d(r, r, r) = (1, 1, 0)$, $d(r - \epsilon, r, r) = (0, 1, 1)$, $d(r, r - \epsilon, r) = (1, 0, 1)$ and, $d(r - \epsilon, r - \epsilon, r) = (0, 0, 1)$ for any $\epsilon \in (0, r)$. It can be checked that this mechanism satisfies MI and condition (a) of regularity but violates condition (b) of regularity. Similarly a VARP mechanism with reserve price r that satisfies MI, condition (b) of regularity but not condition (a) of regularity if $d(r, r, r) = (0, 0, 0)$. Finally, consider a VARP mechanism with reserve price r with tie-breaking rule $1 \succ 2 \succ 3$ such that $d(x, x, x) = (1, 0, 0)$ for all $x \geq r$ and $d(r - \epsilon, r, r) = (0, 1, 0)$, $d(r, r - \epsilon, r) = (1, 0, 0)$ and, $d(r - \epsilon, r - \epsilon, r) = (0, 0, 0)$ for any $\epsilon \in (0, r)$. This mechanism satisfies both conditions of regularity but does not satisfy MI.

We now investigate the mechanisms in Γ' that satisfy no-envy.

Theorem 3. If a regular mechanism (d, τ) satisfies NE, AN, SP, NBD and MI, then there exists $r \geq 0$ such that for all $i \in N$ and all $v_{-i} \in \mathbb{R}_+^{n-1}$,

- (A): If $v_{-i}(m) \geq r$, then $K(v_{-i}) = f(v_{-i}(m))$ where $f() : [r, \infty) \mapsto \mathbb{R}$ is an arbitrary continuous and non-decreasing real valued function with a slope not exceeding 1 at all points in $[r, \infty)$.
- (B): If $v_{-i}(m) < r$, then $K(v_{-i}) = C$ where C is an arbitrary real constant.

Proof:

Proof of (A): From Theorem 2 it follows that $\exists r \geq 0$ such that $\forall z \in \mathbb{R}_+^{n-1}$, $T(z) = \max\{z(m), r\}$. Fix an arbitrary $z \in \mathbb{R}_+^{n-1}$. W. l. o. g. suppose that $z_k = z(k)$ for all $k = 1, \dots, n-1$. To establish that $K(z) = f(z_m)$ (that is, $K(z)$ is a function of z_m only), we need to show that (i) for all $k = 1, \dots, m-1$, $K(x_k, z_{-k}) = K(z)$, $\forall x_k \geq z_m$ and (ii) for all $k = m+1, \dots, n-1$, $K(x_k, z_{-k}) = K(z)$, $\forall x_k < z_m$.

Suppose (i) does not hold. That is, there exists a $k \in \{1, \dots, m-1\}$ and an $x'_k \geq z_m$ such that $K(x'_k, z_{-k}) \neq K(z)$. Consider the profile v such that $v_1 = x'_k$ and $v_{-1} = z$. Therefore, for all $t = 1, \dots, n-1$, $v_{t+1} = z_t$ and so, by Theorem 2, $d_1(v) = d_{k+1}(v) = 1$. By NE,

⁸Note that any VARP mechanism satisfies AN, SP, NBD and IR.

it must be that $\tau_1(v) = \tau_{k+1}(v)$ ⁹, which implies that $-z_m + K(z) = -z_m + K(x'_k, z_{-k})$ and hence, contradiction. Now, suppose that (ii) does not hold. That is, there exists a $k \in \{m+1, \dots, n-1\}$ and an $x''_k < z_m$ such that $K(x''_k, z_{-k}) \neq K(z)$. As before, construct a profile w such that $w_1 = x''_k$ and $w_{-1} = z$. Then, by Theorem 2, $d_1(w) = d_{k+1}(w) = 0$ implying that $\tau_1(w) = \tau_{k+1}(w) \implies K(z) = K(x''_k, z_{-k})$. Hence, contradiction.

Finally consider an arbitrary profile \bar{v} such that $\bar{v}(m+1) \geq r$. Suppose w. l. o. g. that $\bar{v}_k = \bar{v}(k)$ for all $k = 1, \dots, n$. By Theorem 2, $d_m(\bar{v}) = 1, d_{m+1}(\bar{v}) = 0$. NE for the pair of agents $\{m, m+1\}$ implies that $0 \leq K(\bar{v}_{-\{m+1\}}) - K(\bar{v}_{-m}) \leq \bar{v}_m - \bar{v}_{m+1}$. From the discussion above, it follows that $0 \leq f(\bar{v}_m) - f(\bar{v}_{m+1}) \leq \bar{v}_m - \bar{v}_{m+1}$. Since, the profile \bar{v} was arbitrarily chosen, we have established that for all $x \geq y \geq r$, $f(x) \geq f(y)$. That is $f(\cdot)$ is a non-decreasing function with a slope less than 1 over the interval $[r, \infty)$. To establish continuity of $f(\cdot)$, for any $x \geq r$, consider any sequence $\{y_x^t\} \subset [r, \infty)$ such that $\lim_{t \rightarrow \infty} y_x^t = x$. Then, for all t , it must be that $0 \leq |f(x) - f(y_x^t)| \leq |x - y_x^t|$ implying that $\lim_{t \rightarrow \infty} f(y_x^t) = f(x)$. Since, $x \geq r$ was arbitrarily chosen, continuity of $f(\cdot)$ follows. \square

Proof of (B): As before, note that Theorem 2 implies that $\exists r \geq 0$ such that $\forall z \in \mathbb{R}_+^{n-1}$, $T(z) = \max\{z(m), r\}$. Define for any $z \in \mathbb{R}_+^{n-1}$, $k^z := |\{z_i : z_i \geq r\}|$. Also define for all $t = 0, \dots, n-1$, $S^t := \{z \in \mathbb{R}_+^{n-1} : k^z = t\}$. Therefore, the set of all $z \in \mathbb{R}_+^{n-1}$ such that $z(m) < r$ is $S^* := \cup_{t=0}^{m-1} S^t$. In the following three steps, we show that for all $z \in S^*$, $K(z) = C$ where C is an arbitrary real constant.

STEP 1: For all $z \in S^0$, $K(z) = C$ where C is an arbitrary real constant.

Proof of Step: Suppose there exists $z, z' \in S^0$ such that $K(z) \neq K(z')$. Construct a sequence of profiles $\{v^t\}_{t=1}^{n-1}$ such that $v_1^1 = z'_1, v_{-1}^1 = z$, and for all $2 \leq t \leq n-1$, $v_i^t = z'_i$ with $v_{-t}^t = v_{-t}^{t-1}$. Since for all t , $v^t \in [0, r]^n$, by Theorem 2, $d_i(v^t) = 0$ for all $i \in N$. Therefore, by NE, for all $i \neq j \in N$, $\tau_i(v^t) = \tau_j(v^t) \implies K(v_{-i}^t) = K(v_{-j}^t)$. Therefore, $K(v_{-1}^1) = K(v_{-2}^2) = \dots = K(v_{-\{n-1\}}^{n-1})$. By definition $v_{-1}^1 = z$ and $v_{-\{n-1\}}^{n-1} = z'$. Thus, we arrive at a contradiction.

STEP 2: For any $t = 1, \dots, m-1$ and any $z \in S^t$, $K(z) = C^t$ where C^t is an arbitrary real constant.

Proof of Step: Fix a $t = 1, \dots, m-1$ and consider any $z \in S^t$. We first prove that (i) for all $k = 1, \dots, t$, $K(x_k, z_{-k}) = K(z)$, $\forall x_k \geq z_k$; and (ii) for all $k = t+1, \dots, n-1$, $K(x_k, z_{-k}) = K(z)$, $\forall x_k < z_k$. Suppose that (i) is not true. Then there exists a $k \in \{1, \dots, t\}$ such that $K(x_k, z_{-k}) \neq K(z)$ for some $x_k \geq z_k$. Consider the profile v such that $v_1 = x_k, v_2 = z_k$ and $v_{-1-2} = z_{-k}$. As before, from Theorem 2 and NE, it follows that $\tau_1(v) = \tau_2(v) \implies K(z) = K(x_k, z_{-k})$ and hence, contradiction. Similarly, suppose that (ii) is not true and so, there exists $l \in \{t+1, \dots, n-1\}$ such that $K(x_l, z_{-l}) \neq K(z)$ for some $x_l < z_l$. As before, construct a profile v' such that $v'_1 = x_l, v'_2 = z_l$ and $v'_{-1-2} = z_{-l}$. From Theorem 2 and NE, it follows that $\tau_1(v') = \tau_2(v') \implies K(z) = K(x_l, z_{-l})$ and hence, contradiction. From the symmetry of the $K(\cdot)$ functions (implied by AN), the result follows.

⁹It is well known that in a quasilinear setting NE implies that any two agents receiving the same decision also get the same transfer. This is because, for any profile v and any $i \neq j \in N$, NE implies that $v_i(d_i(v) - d_j(v)) \geq t_j(v) - t_i(v) \geq v_j(d_i(v) - d_j(v))$.

STEP 3: For all $t = 1, \dots, m-1$, $C^t = C$.

Proof of Step: We accomplish this proof by induction. We first show that $C^1 = C$. To see this, consider any profile v such that $v_1 \geq r > v_2 \geq \dots \geq v_n$. Therefore, $v_{-1} \in S^0$ and for all $j \neq 1$, $v_{-j} \in S^1$. Therefore, by the earlier steps, $K(v_{-1}) = C$ and $K(v_{-j}) = C^1$ for all $j \neq 1$. Further, by Theorem 2, $d_1(v) = 1, d_2(v) = 0$. Therefore, NE implies that $v_2 - r \leq C^1 - C \leq v_1 - r$. Since v_1 and v_2 could have been chosen arbitrarily close to r satisfying $v_1 > r > v_2$, it must be that $C^1 = C$. Now, suppose that $C^t = C$ for some t such that $0 \leq t \leq m-2$. We shall then show that $C^{t+1} = C$. To see this, consider the profile v such that $v_1 \geq \dots \geq v_{t+1} > r > v_{t+2} \geq \dots \geq v_n$. By the earlier steps, $v_{-\{t+1\}} \in S^t \implies K(v_{-\{t+1\}}) = C^t = C$ and $v_{-\{t+2\}} \in S^{t+1} \implies K(v_{-\{t+2\}}) = C^{t+1}$. Also, by Theorem 2, $d_{t+1}(v) = 1, d_{t+2}(v) = 0$. Then, by NE, $v_{t+2} - r \leq C^{t+1} - C \leq v_{t+1} - r$. Since, v_{t+1} and v_{t+2} can be chosen arbitrarily close to r satisfying $v_{t+2} < r < v_{t+1}$, it must be that $C^t = C$. \square

The following proposition states the additional restrictions imposed by the axioms IR and F, on the $K(\cdot)$ function specified by Theorem 3 above.

Proposition 1. If a regular mechanism (d, τ) satisfies NE, AN, SP, NBD, MI, IR and F, then the following statements hold true.

(A): If $r > 0$, then for all $i \in N$ and all $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$,

$$K(v_{-i}) = \begin{cases} f(v_{-i}(m)) & \text{if } v_{-i}(m) \geq r \\ 0 & \text{otherwise} \end{cases}$$

where $f(\cdot) : [r, \infty) \mapsto \mathbb{R}$ is a continuous, non-decreasing function with slope not exceeding 1 at all points in $[r, \infty)$ such that $f(x) \in [0, \frac{mr}{n-m}]$ for all $x \geq r$.

(B): If $r = 0$, then for all $i \in N$ and all $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$,

$$K(v_{-i}) = 0$$

(C): For all $r \geq 0$, $f(r) = 0$.

Proof:

Proof of (A): To prove the statement, we need to show that IR and F imply the following two restrictions on the results in Theorem 3: (i) $C = 0$ and (ii) $f(x) \in [0, \frac{mr}{n-1}]$ for all $x \geq r > 0$. To see (i), consider a profile $v \in [0, r]^n$. Theorem 2 implies that $d_i(v) = 0$ and so $u(d_i(v), \tau_i(v); v_i) = K(v_{-i}) = C$ (since by construction, $v_{-i}(m) < r$) for all $i \in N$. Therefore, IR implies that $C \geq 0$ for all $i \in N$ and F implies that $\sum_{i \in N} K(v_{-i}) = nC \leq 0$. Thus, it must be that $C = 0$. To prove (ii), fix any $x \geq r$ and consider the profile v^x such that $v_1^x \geq \dots \geq v_m^x \geq r > v_{m+1}^x \geq \dots \geq v_n^x$ and $v_m^x = x$. Theorem 3 implies that for all $i \in \{1, \dots, m\}$, $K(v_{-i}) = C = 0$ and for all $i \in \{m+1, \dots, n\}$, $K(v_{-i}) = f(x)$. Further, by Theorem 2 and IR, for all $i \in \{m+1, \dots, n\}$, $u(d_i(v), \tau_i(v); v_i) = K(v_{-i}) = f(x) \geq 0$. F implies that $(n-m)f(x) + m(0-r) \leq 0 \implies f(x) \leq \frac{mr}{n-m}$. \square

Proof of (B): By Theorem 3, if $r = 0$ then for all $v \in \mathbb{R}_+^N$ and all $i \in N$, $K(v_{-i}) = f(v_{-i}(m))$. Suppose there exists $y > 0$ such that $f(y) > 0$. Consider profile v where $v_k = v(k)$ for all $k = 1, \dots, n$, $v_m = y$ and $v_{m+1} = \epsilon \in (0, \min \{y, \frac{(n-m)f(y)}{m}\})$. Therefore, by Theorem 2,

$d_i(v) = 1$ for all $i = 1, \dots, m$ and $d_i(v) = 0$ for all $i = m + 1, \dots, n$. Note that, $f(\epsilon) \geq 0$ (or else IR is violated at the profile $\bar{\epsilon}^n$), and so by the particular choice of ϵ , F is violated at profile v . Further, it follows trivially from IR and F that $f(0) = 0$ when $r = 0$.¹⁰ Therefore, it must be that $f(y) = 0, \forall y \geq 0$. Thus, the result follows. \square

Proof of (C): The result trivially follows if $r = 0$. If $r > 0$ and $f(r) > 0$ then consider a profile v such that for all $k = 1, \dots, n$, $v_k = v(k)$, $v_m = r$ and $v_{m+1} = r - \epsilon$ where $\epsilon \in (0, f(r))$. Note that by statement (A), $K(v_{-m}) = 0$ since $v_{-m}(m) = v_{m+1} < r$. Further, by Theorem 2, $d_m(v) = 1$ and $d_{m+1}(v) = 0$. Therefore, $u(d_m(v), \tau_m(v); v_m) = \epsilon < f(r) = K(v_{-\{m+1\}}) = u(d_{m+1}(v), \tau_{m+1}(v); v_{m+1})$ which violates NE and hence, contradiction. Therefore, as argued above, by IR, $f(r) = 0$. \square

Define the class of mechanisms $\mathcal{G} = \{(d^r, \tau^r)\}_{r \geq 0}$ such that,

- (1) $T(v_{-i}) = \max\{v_{-i}(m), r\}$ for all $i \in N$ and all $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$.
- (2) $K(v_{-i}) = \begin{cases} 0 & \text{if } v_{-i}(m) \leq r \\ v_{-i}(m) - r & \text{if } v_{-i}(m) \in (r, \frac{nr}{n-m}) \\ \frac{nr}{n-m} & \text{if } v_{-i}(m) \geq \frac{nr}{n-m} \end{cases}$ for all $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$.
- (3) For any $v \in \mathbb{R}^N$ such that $\exists i \in N$ with $v_i = r$,

$$d_i(v) = 1 \implies i \in \mathbf{T}^m$$

Using the definition above we state the following proposition.

Proposition 2. A mechanism (d, τ) is Pareto undominated in the class of regular mechanisms satisfying NE, AN, SP, NBD, MI, IR and F if and only if,

$$(d, \tau) \in \mathcal{G}$$

Proof: The proof of sufficiency is easy to check. The proof of necessity is as follows. From Proposition 2 and Theorems 2 and 3, it follows that any regular mechanism satisfying NE, AN, SP, NBD, MI, IR and F; must satisfy conditions 1 and 3 in definition of \mathcal{G} and have $K(v_{-i}) = 0$ as long as $v_{-i}(m) \leq r$ for all i and all v . Consider a regular mechanism (d', τ') satisfying NE, AN, SP, NBD, MI, IR and F, such that there exists a $z \in \mathbb{R}_+^{n-1}$ with $z(m) \in (r, \frac{nr}{n-m})$ and $K(z) < z(m) - r$. Define the set $P_z := \{v \in \mathbb{R}_+^n : \exists i \in N \text{ such that } v_{-i} = z\}$. Also define, for all $v \in P_z$, the set $a_z^v := \{i \in N : v_{-i} = z\}$. Therefore, by Result 1, P_z is the set of all possible profiles v such that all agents i in a_z^v are assigned a transfer

$$\tau'_i(v) = \begin{cases} K(z) & \text{if } d'_i(v) = 0 \\ K(z) - \max\{z(m), r\} & \text{otherwise} \end{cases}$$

by the mechanism (d', τ') .

Construct another regular mechanism (d'', τ'') satisfying NE, AN, SP, NBD, MI, IR and F such that **(a)** $(d''_i(v), \tau''_i(v)) = (d'_i(v), \tau'_i(v))$ for all $i \in N$ and all $v \in \mathbb{R}^n \setminus P_z$, **(b)** $d''_i(v) = d'_i(v)$ for all $i \in N$ and all $v \in P_z$, **(c)** $\tau''_i(v) = \tau'_i(v)$ for all $i \notin a_z^v$ and all $v \in P_z$, and **(d)** for all $i \in a_z^v$ and all $v \in P_z$,

$$\tau''_i(v) = \begin{cases} z(m) - r & \text{if } d''_i(v) = 0 \\ z(m) - r - \max\{z(m), r\} & \text{otherwise} \end{cases}$$

¹⁰By considering the profile $\bar{0}^n$.

Since by supposition, $K(z) < z(m) - r$, it can easily be seen that (d'', τ'') Pareto dominates (d', τ') . This implies that any Pareto undominated mechanism in the aforementioned class, must have $K(z) = z(m) - r$ for all $z \in \mathbb{R}^{n-1}$ with $z(m) \in (r, \frac{nr}{n-m})$. Similarly, we can argue that for all $z \in \mathbb{R}^{n-1}$ with $z(m) \geq \frac{nr}{n-m}$, any Pareto undominated mechanism in the aforementioned class must have $K(z) = \frac{mr}{n-m}$. Hence, the result follows. \square

Remark 4. It can easily be seen from Theorem 3 and Proposition 2 that \mathcal{G} is also the class of Pareto undominated mechanisms that satisfy IR and F; in the class of regular mechanisms that satisfy NE, AN, SP, NBD and MI (described by Theorem 3). Further, in definition of \mathcal{G} , the condition 2 can be written as

$$K(v_{-i}) = \text{med} \left\{ 0, v_{-i}(m) - r, \frac{mr}{n-m} \right\}, \forall i \in N, \forall v \in \mathbb{R}^N$$

where $\text{med}\{x, y, z\}$ is defined as the median of any three real numbers x, y, z . Therefore, when $m = 1$, every mechanism in \mathcal{G} is Pareto equivalent to the *maxmed* mechanism of Sprumont [17] having the same reserve price. Thus, Proposition 2, generalizes the maxmed mechanisms on Sprumont [17] to the multiple objects setting.¹¹

4. CONCLUSION

This paper provides an justification to reserve pricing at auctions using normative and strategic axioms unrelated to revenue considerations. In particular, it provides a topological interpretation of a reserve price as the *infimum* of the set of non-negative real numbers satisfying the following property: if all agents bid the same number from this set, then at least one object is allocated. Further, the paper shows how certain degree of efficiency is necessary in mechanisms that satisfy certain ethical and strategic axioms. The two main contributions of this paper are (i) axiomatization of VARP mechanisms in single as well as multiple objects settings and (ii) generalization and axiomatization of *maxmed* mechanisms of Sprumont [17] in the multiple homogeneous objects setting. Whether these results continue to hold in a multiple heterogeneous objects setting would be an interesting question for future research.

5. APPENDIX

5.1. Preliminary Results. Recall that for all $v \in \mathbb{R}_+^N$, $W(v)$ is the set of agents who get an object at the profile v . Since there are m objects to be allocated and not all objects are allocated at all profiles, $|W(v)| \leq m, \forall v \in \mathbb{R}_+^N$.

The following result establishes that the decision rule implicit in any strategy-proof mechanism must be non-decreasing in nature. In particular, for any profile of valuations v_{-i} there must exist a threshold price $T_i(v_{-i})$ such that agent i gets an object if v_i *strictly* exceeds $T_i(v_{-i})$ and fails to get an object if v_i is *strictly* less than $T_i(v_{-i})$. Further, the transfer of agent i when getting the object, must exceed that when not getting the object, by $T_i(v_{-i})$.

¹¹However, when $m = 1$, \mathcal{G} is actually a subset of the class of maxmed mechanisms characterized in Sprumont [17]. This is because, we need to impose an additional restriction (to account for the additional complications of multiple objects setting) which says that no agent *other than the pre-fixed agent in whose favour ties are broken*, should get the object by reporting the reserve price.

Result 1. Any mechanism (d, τ) satisfies SP only if $\forall i \in N$ and $\forall v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$, there exist real valued functions $K_i : \mathbb{R}_+^{N \setminus \{i\}} \mapsto \mathbb{R}$ and $T_i : \mathbb{R}_+^{N \setminus \{i\}} \mapsto \mathbb{R} \cup \{+\infty\}$ such that

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > T_i(v_{-i}) \\ 0 & \text{if } v_i < T_i(v_{-i}) \end{cases} \quad \text{and} \quad \tau_i(v) = \begin{cases} K_i(v_{-i}) - T_i(v_{-i}) & \text{if } d_i(v) = 1 \\ K_i(v_{-i}) & \text{if } d_i(v) = 0 \end{cases}$$

Proof: The results follow from Proposition 9.27 in Nisan [11] and Lemma 1 in Mukherjee [9]. \square

Remark 5. Note that Result 1 allows for arbitrary tie breaking in case there exists a valuation profile $v \in \mathbb{R}_+^N$ such that $v_i = T_i(v_{-i})$ for any $i \in N$. Thus, Result 1 establishes that for any valuation profile v , $d_i(v) = 1$ implies that $v_i \geq T_i(v_{-i})$ and $d_i(v) = 0$ implies that $v_i \leq T_i(v_{-i})$. Further, if the $T(\cdot)$ were independent of agent labels, then at any profile v with $d_i(v) = 1, d_j(v) = 0$ and $v_i = v_j$, $T(v_{-i}) = T(v_{-j}) = v_i$ for all $i \neq j \in N$.

The following proposition establishes that the threshold functions corresponding to any anonymous and strategy-proof functions must be independent of respective agent labels.

Proposition 3. Any mechanism (d, τ) that satisfies AN and SP must satisfy the following properties

- (1) $T_i(z) = T(z)$ for all $z \in \mathbb{R}_+^{n-1}$ and all $i \in N$.
- (2) $K_i(z) = K(z)$ for all $z \in \mathbb{R}_+^{n-1}$ and all $i \in N$.

Proof: Suppose there exists some $z \in \mathbb{R}_+^{n-1}$ such that $T_1(z) \neq T_2(z)$. W. l. o. g. suppose that $T_1(z) > T_2(z)$. Construct the profile v such that $v_{-1} = z$ and $v_1 \in (T_2(z), T_1(z))$. Then, from Result 1 it follows that $d_1(v) = 0$ since $v_1 < T_1(v_{-1}) = T_1(z)$ by construction. Now, consider the profile $v' = (v'_1, v'_2, v_{-1-2})$ where $v'_1 = v_2$ and $v'_2 = v_1$. Note that $v'_{-2} = v_{-1} = z$. Therefore, $d_2(v') = 1$ since $v'_2 = v_1 > T_2(z) = T_2(v'_{-2})$. Further, AN requires that $u(d_1(v), \tau_1(v); v_1) = u(d_2(v'), \tau_2(v'); v'_2)$. Then, from Result 1 it follows that $v_1 + K_2(z) - T_2(z) = K_1(z)$ for all $v_1 \in (T_2(z), T_1(z))$. This leads to contradiction. Thus, arguing in this manner we can show that $T_i(z) = T_{i+1}(z)$ for all $i = 1, \dots, n-1$ and all $z \in \mathbb{R}_+^{n-1}$. Therefore, statement (1) follows.

To prove statement 2, we fix any $x \in \mathbb{R}_+^{n-1}$ and show that $K_1(x) = K_2(x)$. Consider the profile v such that $v_1 = v_2$ and $v_{-1} = v_{-2} = x$. If $d_1(v) = d_2(v) = 0$ then by AN, $u(d_1(v), \tau_1(v); v_1) = u(d_2(v), \tau_2(v); v_2)$ implying $K_1(x) = K_2(x)$. If $d_1(v) = 1, d_2(v) = 0$ then AN, Remark 5 and statement (1), imply that $v_1 + K_1(x) - T(x) = K_2(x)$ and $T(x) = v_1$ leading to the conclusion $K_1(x) = K_2(x)$. Similarly, if $d_1(v) = 0, d_2(v) = 1$, we can show that $K_1(x) = K_2(x)$. Finally, if $d_1(v) = d_2(v) = 1$, then by AN and statement (1), $v_1 + K_1(x) - T(x) = v_2 + K_2(x) - T(x)$ implying $K_1(x) = K_2(x)$. Thus, arguing in this manner we can show that $K_i(x) = K_{i+1}(x)$ for all $i = 1, \dots, n-1$ and all $x \in \mathbb{R}_+^{n-1}$. Therefore, statement (2) follows. \square

It is well known that the notion of AN defined in Definition 3 is the same as that in the following definition.

Definition 12. A mechanism (d, τ) satisfies AN if for all $i \in N$, all $v \in \mathbb{R}_+^n$ and all bijections $\pi : N \mapsto N$,

$$u(d_i(v), \tau_i(v); v_i) = u(d_{\pi i}(\pi v), \tau_{\pi i}(\pi v); \pi v_{\pi i})$$

where $\pi v := \left(v_{\pi^{-1}(k)} \right)_{k=1}^n$.

We use this definition of AN to state the following proposition.

Proposition 4. For any mechanism (d, τ) that satisfies AN and SP, the $K(\cdot)$ and the $T(\cdot)$ functions must be symmetric.¹²

Proof: Suppose not. That is, there exists a $z \in \mathbb{R}_+^{n-1}$, an $i \in N$ and a bijection $\pi : N \setminus \{i\} \mapsto N \setminus \{i\}$ such that $T(z) \neq T(\pi z)$. W. l. o. g. assume that $T(z) < T(\pi z)$ and fix any $x \in (T(z), T(\pi z))$. Consider a bijection $\pi' : N \mapsto N$ such that $\pi' i = i$ and $\pi' j = \pi j$ for all $j \neq i$ and the profile v such that $v_i = x$ and $v_{-i} = z$. By Result 1, $d_i(v) = 1$ and $d_i(\pi' v) = 0$ because $(\pi' v)_{-i} = \pi z$. By AN, $u(d_i(v), \tau_i(v); x) = u(d_i(\pi' v), \tau_i(\pi' v); x)$ implying that $x + K(z) - T(z) = K(\pi z)$. Since x was chosen arbitrarily from the interval $(T(z), T(\pi z))$, we get a contradiction.

To show the $K(\cdot)$ function to be symmetric, consider any $i \in N$, any profile $v \in \mathbb{R}_+^n$ and any bijection $\hat{\pi} : N \mapsto N$ such that $\hat{\pi} i = i$. Since $T(\cdot)$ function has already shown to be symmetric above, $T(v_{-i}) = T((\hat{\pi} v)_{-i})$ and so, $d_i(v) = d_i(\hat{\pi} v)$. By AN, $u(d_i(v), \tau_i(v); v_i) = u(d_i(\hat{\pi} v), \tau_i(\hat{\pi} v); v_i)$, which implies that $K(v_{-i}) = K((\hat{\pi} v)_{-i})$. Hence, the result follows. \square

The following lemma states that any mechanism satisfying AN, SP and NBD must also satisfy a *weaker* form of efficiency which requires that whenever objects are allocated, the allocation must be efficient.

Lemma 2. A mechanism (d, τ) satisfies AN, SP and NBD only if $\forall v \in \mathbb{R}_+^n, i \neq j \in N$,

$$d_i(v) = 1 \text{ and } d_j(v) = 0 \implies v_i \geq v_j$$

Proof: Fix any profile $v \in \mathbb{R}_+^n$ and suppose w. l. o. g. that $d_1(v) = 0, d_2(v) = 1, v_1 > v_2$. Consider the profile $\tilde{v} := (v_1, v_{-2})$. By SP, $d_2(v) = 1 \implies d_2(\tilde{v}) = 1$, and so, by NBD, $d_1(v) = 0 \implies d_1(\tilde{v}) = 0$. Therefore, by Remark 5, $T(v_{-2}) = v_1$. Arguing as before, by SP, $d_1(v_2, \tilde{v}_{-1}) = 0$ and so, by AN, $u(d_1(v_2, \tilde{v}_{-1}), \tau_1(v_2, \tilde{v}_{-1}); v_2) = u(d_2(v), \tau_2(v); v_2)$. Therefore, $K(v_{-2}) = v_2 + K(v_{-2}) - T(v_{-2}) \implies T(v_{-2}) = v_2 \neq v_1$ and hence, contradiction. Thus, the result follows. \square

Remark 6. From Lemma 2 it follows that at any valuation profile $v \in \mathbb{R}_+^n$, if $d_i(v) = 0$ for some $i \in N$, then $d_j(v) = 0$ for all agents j such that $v_j < v_i$. Similarly, if there exists an $i \in N$ with $d_i(v) = 1$, then $d_j(v) = 1$ for all agents j such that $v_j > v_i$. Athanasiou [2] and Sprumont [17] show the same result in the single object setting without the use of NBD axiom. This is possible because in single object setting, any agent getting an object at any profile implies that no other agent gets an object at that profile. However, in multiple homogeneous objects setting, no such restriction is implicit. The NBD axiom is needed to tackle this complexity.

The following lemma states the restriction imposed by AN, SP and NBD axioms on the decision rule at profiles where all agents have bid the same value. It establishes the existence of a non-negative real number η such that no objects are allocated at any profile

¹²A function of $k \in \mathbb{N}$ variables is said to be *symmetric* if the function value at any k -tuple of arguments is the same as the function value at any permutation of that k -tuple.

where (i) all agents have bid the same value and (ii) this value is less than η . Also, if all agents bid the same value that is greater than η , *at least* one object must be allocated.

Lemma 3. A mechanism (d, τ) satisfies AN, SP and NBD only if $\exists \eta \geq 0$ such that $\forall x \geq 0$,

$$x < \eta \implies \bar{x}^n \in B_0 \text{ and } x > \eta \implies \bar{x}^n \notin B_0$$

Proof: Suppose there exist $0 \leq x < y$ such that $\bar{x}^n \notin B_0$ and $\bar{y}^n \in B_0$. W. l. o. g. suppose that $d_i(\bar{x}^n) = 1$ for all $i = 1, \dots, l$ where $l \in \{1, \dots, m\}$ (that is, l objects are allocated at profile \bar{x}^n). Define the sequence of profiles $(p^k)_{k=1}^l$ where $p^1 = (y, \bar{x}_{-1}^n)$ and for all $2 \leq k \leq l$, $p^k = (y, p_{-k}^{k-1})$. By SP and NBD, for all $1 \leq i \leq l$, $d_i(\bar{x}^n) = 1 \implies d_i(p^1) = 1 \implies d_i(p^2) = 1 \implies \dots \implies d_i(p^l) = 1$ and so, $p^l \notin B_0$. Similarly construct another sequence of profiles $(q^k)_{k=l+1}^n$ such that $q^{l+1} = (x, \bar{y}_{-\{l+1\}}^n)$ and for all $l+2 \leq k \leq n$, $q^k = (x, q_{-k}^{k-1})$. By SP and NBD, $\bar{y}^n \in B_0 \implies q^{l+1} \in B_0 \implies q^{l+2} \in B_0 \implies \dots \implies q^n \in B_0$. By construction, $q^n = p^l$ and hence, contradiction. Therefore, for any $x \geq 0$, if $\bar{x}^n \notin B_0$ and then $\forall y > x$ it must be that $\bar{y}^n \notin B_0$. Thus, if the set $\{x \geq 0 : \bar{x}^n \notin B_0\}$ is non-empty, then the result follows from the choice of $\eta := \inf\{x \geq 0 : \bar{x}^n \notin B_0\}$. If $\{x \geq 0 : \bar{x}^n \notin B_0\} = \emptyset$ then no objects are allocated at any profile where all agents have bid the same value. In this case the result follows by assigning $\eta := \infty$. \square

The following lemma shows that if $\eta > 0$ then no object is allocated at any profile where the highest valuation is strictly less than η .

Lemma 4. A mechanism (d, τ) satisfies AN, SP and NBD only if $\forall v \in [0, \eta]^n$,

$$v \in B_0$$

Proof: Fix any $v \in [0, \eta]^n$. W. l. o. g. assume that $v_1 \geq v_2 \geq \dots \geq v_n$. By definition $M(v) = v_1 < \eta$, and so, by Lemma 3, $\bar{v}_1^n \in B_0$. Construct a sequence of profiles $(p^k)_{k=1}^{n-1}$ such that $p^1 = (v_2, \bar{v}_{1-2}^n)$ and for all $2 \leq k \leq n-1$, $p^k = (v_{k+1}, p_{-\{k+1\}}^{k-1})$. By SP and NBD, $\bar{v}_1^n \in B_0 \implies p^2 \in B_0 \implies \dots \implies p^n \in B_0$. Note that by construction, $p^n = v$ and hence, the result follows. \square

5.2. Proof of Theorem1. Since $m = 1$, by setting $r := \eta$ (as defined in Lemma 3), from Remark 6 and Lemma 4 it follows that for all v and all i , $v_i < \max\{v_{-i}(1), \eta\} \implies d_i(v) = 0$. Therefore, we simply need to show that if $v_i > \max\{v_{-i}(1), \eta\}$, then $d_i(v) = 1$ for all possible choices of v and i .

Fix an arbitrary $i \in N$ and a profile $v \in \mathbb{R}_+^n$ such that $v_i > \max\{v_{-i}(1), \eta\}$. Note that, either $v_i = v(1) > v(2) > \eta$ or $v_i = v(1) > \eta \geq v(2)$. We analyze each of the two cases and show that in each case, $d_i(v) = 1$

Case 1: $v_i = v(1) > v(2) > \eta$

By Lemma 3, $\overline{v(2)}^n \notin B_0$ and so, from Remark 5 and Proposition 3, it follows that $T(\overline{v(2)}^{n-1}) = v(2)$. Construct a sequence of profiles $\{p^k\}_{k=1}^n$ such that $p^1 = \overline{v(2)}^n$, $p^2 = (v_i, p_{-i}^1)$ and $\forall 3 \leq k \leq n$, $p^k = (v_{t_k}, p_{-t_k}^{k-1})$ where $t_k \in \{j \in N | v_j = v(k)\}$ (by the tie-breaking rule, this set is a singleton set). Further, $T(p_{-i}^1) = T(\overline{v(2)}^{n-1}) = v(2)$ and so under the supposition $v_i = v(1) > v(2)$, $d_i(p^2) = 1$. Therefore, from the fact that $m = 1$ it follows that

$d_j(p^2) = 0, \forall j \neq i$. Further, by SP and NBD, for all $j \in N$, $d_j(p^2) = d_j(p^3) = \dots = d_j(p^n)$. Since, by construction, $p^n = v$, we get that $d_i(v) = 1$ and $d_j(v) = 0$ for all $j \neq i$.

Case 2: $v_i = v(1) > \eta \geq v(2)$

Consider the sequence of profiles $\{p^k\}_{k=0}^n$ where $p^0 = \overline{\eta + \epsilon^n}$ where $\epsilon \in (0, v_i - \eta)$ and for all $1 \leq k \leq n$, $p^k = (v_{t_k}, p_{-t_k}^{k-1})$ where $t_k \in \{j \in N | v_j = v(k)\}$ (as mentioned before, this set is a singleton set by the tie-breaking rule). By Remark 5, Lemma 3 and Proposition 3, we get that $p^0 \notin B_0$ which implies that $T(p_{-j}^0) = \eta + \epsilon$ for all $j \in N$. Further, by construction, $p_i^1 = v_i$ and $p_{-i}^0 = p_{-i}^1$. Therefore, $p_i^1 > T(p_{-i}^1) = \eta + \epsilon$ and so, from Result 1 it follows that $d_i(p^1) = 1$. Since $m = 1$, we can then claim that $d_j(p^1) = 0$ for all $j \neq i$. Hence, by SP and NBD, for all $j \in N$, $d_j(p^1) = d_j(p^2) = \dots = d_j(p^n)$. By construction, $p^n = v$ which implies that $d_i(v) = 1$. \square

5.3. Proof of Theorem 3. We set $r = \eta = \inf\{x \geq 0 : \bar{x}^n \notin B_0\}$ and fix any $i \in N$ and any $v \in \mathbb{R}_+^n$. Then we consider the two possible cases $v_{-i}(m) \geq \eta$ and $v_{-i}(m) < \eta$. In the former case we show that $T(v_{-i}) = v_{-i}(m)$ while in the latter case we show that $T(v_{-i}) = \eta$. For economy of notation, henceforth, we denote vector v_{-i} by z in the proof. By Proposition 4, $T(\cdot)$ is symmetric and so, w. l. o. g. assume that $z_1 \geq z_2 \geq \dots \geq z_{n-1}$ and so $z(m) = z_m$. For simplicity of notation, define $\theta := z_m$.

Case 1: $\theta \geq \eta$

By Lemmas 3 and 1, $\bar{\theta}^n \notin B_0$. Therefore, MI implies that $d_j(\bar{\theta}^n) = 1$ for all $j = 1, \dots, m$ and $d_j(\bar{\theta}^n) = 0$ for all $j = m+1, \dots, n$. Consider the sequence of profiles $(q^k)_{k=0}^{n-2}$ such that $q^0 = \bar{\theta}^n$, for all $1 \leq k \leq m-1$, $q^k = (z_k, q_{-\{k\}}^{k-1})$ and for all $m \leq k \leq n-2$, $q^k = (z_{k+1}, q_{-\{k+2\}}^{k-1})$. From SP and NBD it follows that $\forall j \in N$, $d_j(q^0) = d_j(q^1) = \dots = d_j(q^{n-2})$. Therefore, we get that $d_j(q^{n-2}) = 1, \forall j = 1, \dots, m$ and $d_j(q^{n-2}) = 0, \forall j = m+1, \dots, n$. Also, by construction, $q_m^{n-2} = q_{m+1}^{n-2} = \theta$ and $q_{-\{m\}}^{n-2} = q_{-\{m+1\}}^{n-2} = z$. Thus, arguing as in Remark 5, the fact that $d_m(q^{n-2}) = 1, d_{m+1}(q^{n-2}) = 0$ can be used to infer that $\theta \geq T(q_{-\{m\}}^{n-2}) = T(q_{-\{m+1\}}^{n-2}) \geq \theta$ implying that $T(z) = \theta = z_m$.

Case 2: $\theta < \eta$

Fix any $x \geq 0$, define the profile p^x such that $p_1^x = x$ and $p_{-1}^x = z$ and construct the profile p such that $p_k = p^x(k)$ for all $k \in \{1, 2, \dots, n\}$. Therefore, by construction, $p_1 \geq p_2 \geq \dots \geq p_n$. We consider two subcases: $x > \eta$ and $x < \eta$. We shall show that $d_1(p^x) = 1$ in the former case while $d_1(p^x) = 0$ in the latter case. This inference would then be used in conjunction with Result 1 to conclude that $T(p_{-1}^x) = T(z) = \eta$.

Subcase 1. $x > \eta$

Define the agent $g := \{j \in N : p_j \geq \eta \text{ and } p_{j+1} < \eta\}$. Since $z_m < \eta$ and $x > \eta$, agent g is well defined and $g \in \{1, \dots, m\}$. Therefore, p_g is the smallest coordinate of p greater than or equal to η while p_{g+1} is the largest coordinate of p strictly less than η . Consider a sequence of profiles $\{u^k\}_{k=0}^n$ such that $u^0 = \bar{\eta}^n$ and for all $1 \leq k \leq n$, $u^k = (p_k, u_{-k}^{k-1})$. By Lemma 1, $\bar{\eta}^n \notin B_0$ and so, arguing as in Case 1, we can claim from SP and NBD that

for all $j \in N$, $d_j(u^0) = d_j(u^1) = \dots = d_j(u^g)$. Therefore, $d_j(u^g) = 1, \forall j = 1, \dots, m$ and $d_j(u^g) = 0$ and $\forall j = m+1, \dots, n$.

Note that by construction,

$$\begin{aligned} u^g &= (p_1, p_2, \dots, p_g, \eta, \eta, \eta, \eta, \dots, \eta) \\ u^{g+1} &= (p_1, p_2, \dots, p_g, p_{g+1}, \eta, \eta, \eta, \dots, \eta) \end{aligned}$$

Clearly, $u_{-\{g+1\}}^{g+1}(m) \geq \eta$ and so, from Case 1 it follows that $T(u_{-\{g+1\}}^{g+1}) = u_{-\{g+1\}}^{g+1}(m)$. Since, by construction, $p_{g+1} = u_{g+1}^{g+1} < \eta \leq T(u_{-\{g+1\}}^{g+1})$, it must be that $d_{g+1}(u^{g+1}) = 0$. Now, if $d_{g+1}(u^g) = 0$ then by NBD, $d_{g+1}(u^{g+1}) = 0$ implies that $d_j(u^g) = d_j(u^{g+1})$ for all $j \neq g+1$. If $d_{g+1}(u^g) = 1$, then by regularity of the mechanism it follows that $d_j(u^g) = d_j(u^{g+1})$ for all $j \neq g+1$. In either case, we get that

$$d_j(u^{g+1}) = \begin{cases} 1 & \forall j = 1, \dots, g \\ 0 & \forall j = g+1 \end{cases}$$

Consider the next profile in sequence

$$u^{g+2} = (p_1, \dots, p_g, p_{g+1}, p_{g+2}, \eta, \eta, \dots, \eta)$$

Arguing as before, by regularity of (d, τ) and NBD, we get that $p_{g+2} < \eta \implies d_j(u^{g+1}) = d_j(u^{g+2})$ for all $j \neq g+2$. Therefore, $d_{g+1}(u^{g+2}) = 0$. Now, there arise two possibilities: $p_{g+1} > p_{g+2}$ or $p_{g+1} = p_{g+2}$. If $p_{g+1} > p_{g+2}$ then by Lemma 2, $d_{g+1}(u^{g+2}) = 0 \implies d_{g+2}(u^{g+2}) = 0$. If $p_{g+1} = p_{g+2}$ and $d_{g+1}(u^{g+2}) = 0 \neq 1 = d_{g+2}(u^{g+2})$, then AN and SP imply that $T(u_{-\{g+1\}}^{g+2}) = T(u_{-\{g+2\}}^{g+2}) = T(u_{-n}^{g+1}) = p_{g+1}$. Recall that by definition $p_{g+1} < \eta$ and so, Result 1 implies that $d_n(u^{g+1}) = 1$. Since we have already established that $d_j(u^g) = d_j(u^{g+1})$ for all $j \neq g+1$, it must be that $d_n(u^g) = 1$ which leads to a contradiction. Therefore, we get that

$$d_j(u^{g+2}) = \begin{cases} 1 & \forall j = 1, \dots, g \\ 0 & \forall j = g+1, g+2 \end{cases}$$

Arguing similarly for all $\{u^k\}_{k=g+3}^n$, we can show that for $k = g+3, \dots, n$, $d_j(u^k) = 1$ for all $j = 1, \dots, g$ and $d_j(u^k) = 0$ for all $i = g+1, \dots, k$. By construction, $u^n = p$ and so we have established that

$$(5.1) \quad d_j(p) = \begin{cases} 1 & \forall j = 1, \dots, g \\ 0 & \forall j = g+1, \dots, n \end{cases}$$

Since, by supposition $x \geq \eta$, $x \in \{p_1, \dots, p_g\}$ and so the agent bidding x at profile p gets an object. We claim that this implies that $d_1(p^x) = 1$. This is because if $d_1(p^x) = 0$, then by AN, $u(d_1(p^x), \tau_1(p^x); x) = u(d_1(p), \tau_1(p); x)$ where $l \in \{j \in \{1, \dots, g\} : p_j = x\}$. This in turn implies that $K(p_{-1}^x) = x + K(p_{-l}) - T(p_{-l})$. By construction, $p_{-1}^x = p_{-l} = z$ and so, it follows that $x = T(z)$. However, this leads to a contradiction because x was chosen arbitrarily greater than η (with a fixed z vector) and $T(\cdot)$ is function.

Subcase 2. $x < \eta$

If $x < \eta$, there arise two cases: $p_1 < \eta$ and $p_1 \geq \eta$. If $p_1 < \eta$ then, by construction, $p^x \in [0, \eta]^n$ and so, from Lemma 4, $p^x \in B_0$ implying that $d_1(p^x) = 0$. If $\eta \leq p_1$ then agent g is well defined as in possibility 1. Then we can conclude the equation (5.1), by arguing as in case of possibility 1 above. Note that $x < \eta \implies x \in \{p_{g+1}, \dots, p_n\}$ and so any agent bidding x at profile p does not get an object. As before, we can claim from this fact that $d_1(p^x) = 0$. This is because if $d_1(p^x) = 1$ then by AN, $u(d_1(p^x), \tau_1(p^x); x) = u(d_l(p), \tau_l(p); x)$ where $l \in \{j \in \{g+1, \dots, n\} : p_j = x\}$ implying that $x + K(p_{-1}^x) - T(p_{-1}^x) = K(p_{-l})$. Since, by construction, $p_{-1}^x = p_{-l} = z$, we get that $x = T(z)$. Arguing as in possibility 1, we arrive at a contradiction because x is arbitrarily chosen to be less than η and $T(\cdot)$ is a function. \square

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