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Bidding Rings: A Bargaining Approach

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Abstract

We address the issue of bidder ring formation in single and multi-unit Vickrey auctions. We address this issue in a bargaining game set up under the assumption that valuation of bidders is commonly known only amongst themselves. In the single unit case, we show that the equilibrium coalition structure can only be an *order preserving r-ring*, that includes the winner and the top $(r - 1)$ losers. In the multiple units case, we specify sufficient conditions for formation of an interesting class of equilibrium coalition structures, which we call *single winner ring with free riding*, where exactly one winner colludes with all the losers and generates maximum possible bidders' surplus, and, depending on the protocol, the remaining winners free ride either by staying alone or by colluding in pairs.

Keywords: Bidding rings, Bargaining games, Coalition formation, Auctions

JEL Classifications: C71, C72, C78, D44, L41.

1. Introduction

This paper adopts a bargaining approach to analyze the process of bidder ring formation in Vickrey auctions in the absence of resale possibilities. It is an attempt to specify the issues that affect feasibility of bidder collusion, even after elimination of valuation uncertainty within the ring. That is, we consider the grand coalition bidding ring and presume that valuations have become common knowledge as a consequence of some internal mechanism design.¹ We ask whether this ring is immune to further disintegration due to bargaining pressures inherent in the process of coalition formation. Thus we endogenously determine the size and composition of the stable ring.

Avoiding external commitment devices, we presume that any bidding ring formed prior to the auction must settle on a self-enforcing bidding agreement, and then, (i) compute the proceeds that can be

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¹Such mechanisms have been studied in great detail in Graham and Marshall (1987), Marshall and Marx (2007), McAfee and McMillan (1992), Mailath and Zemsky (1991), Hendricks et al. (1999).

generated out of this agreement and (ii) decide how to divide these proceeds. Therefore, we first assume that any ring uses a simple self-enforcing bidding rule that requires the losers participating in the ring to maximize the joint payoff by making zero collusive bids. This ensures that agents' bids constitute a Nash equilibrium, irrespective of the ambient bidding ring structure.² Then, we solve a sequential offer pre-auction bargaining game to predict the *final* equilibrium bidder ring along with the relevant payoffs.³

We consider both the single unit and multiple units cases in which each potential bidder has unit demand. Unlike other papers (see below), we do not assume that the game ends when the first coalition forms. This is especially important in the multiple units setting in which subsequent coalition formation will affect the payoffs to coalitions that have formed earlier. Thus it is important to endogenously determine the bidding ring structure in our setting.

We assume a non-strategic seller who knows neither the true valuations of the bidders nor the distributions of the true valuations of the bidders. The seller simply sets the reservation price. Further, we assume that agents can make offers and counteroffers as fast as possible.⁴ This allows us to visualize an infinite horizon bargaining game with discounting, prior to a stipulated starting date of the Vickrey auction. The discount factor $\delta \in (0, 1)$ simply specifies the probability $(1 - \delta)$ of being caught and the consequent termination of the auction, in which case, all bidders get zero payoffs.⁵ We focus on the asymptotic results as the probability of getting caught goes to 0, and use the notion of sub-game perfect equilibrium in *stationary* strategies for the bargaining game.

In the single unit case, we provide the necessary and sufficient conditions for formation of any bidding ring when the highest valuation bidder proposes first. We also characterize the equilibrium ring structure corresponding to each possible first proposer, in this setting. In the multiple units case, we specify sufficient conditions for the formation of an interesting class of coalition structures where (a) exactly one winner (any one bidder out of those who would win a unit in the non-cooperative play) colludes with all the losers (the bidders who would not win any unit in non-cooperative play) and, (b) depending on the protocol, the remaining winners either stay alone or collude in pairs. Thus the conclusion that exactly one winner will collude with all the losers is independent of the order of bidders in the protocol. We present results for the single unit model and the multiple units model separately since there is a qualitative difference between the two models arising due to the externality in the latter.

²Nash equilibrium bids in the first price auction allows no scope for prior collusion.

³To ensure commitment to the pre-auction cooperative arrangements, we suppose that the payoff payments are made prior to the start of auction.

⁴That is, the time elapsed between an offer, consequent response, and the next offer; can be arbitrarily small. In fact, it may also be endogenous as discussed in Perry and Reny (1993).

⁵Since collusion at auctions is illegal in general, we assume that negotiating agents get (a normalized) zero payoff upon being discovered by regulatory authorities, probably because the auction itself would be scrapped.

1.1. Relevant literature

The sequential bargaining approach, pioneered by Rubinstein (1982) for the two-player bargaining problem was extended to incorporate coalition formation in characteristic function games by Chatterjee et al. (1993). We use their method of analysis in the single unit model. While Chatterjee et al. (1993) provide characterization results for strictly convex games, our game is convex but not necessarily strictly convex. Ray and Vohra (1999) consider similar bargaining games but with externalities, where the entire coalition structure determines a coalition's payoff. However, our analysis differs from Ray and Vohra (1999) in one crucial aspect and that is the absence of binding agreements on actions in the strategic-form game that generates payoffs. They assume that any coalition can write binding agreements that specify (i) the strategies that each member plays at the underlying strategic game and (ii) the contingent payoffs that each member gets. This implies that the resultant strategy profile at the underlying strategic game, after the coalitions have formed, is not necessarily a Nash equilibrium. In our case, no such binding agreements can be enforced.

There is also the very different literature on internal mechanism design of a ring, such as Graham and Marshall (1987), Marshall and Marx (2007), McAfee and McMillan (1992), Mailath and Zemsky (1991), Hendricks et al. (1999). These papers analyze collusion in an ex-ante sense where, at the beginning of the ring formation process, the bidders are yet to know the valuations of their colluding partners. Usually, in these papers, once a coalition forms, the members play a direct mechanism within the coalition; those, if any, outside the given coalition act as singletons. The nature of the coalition formed is therefore not endogenously derived in these papers.

We note also that very few papers discuss collusion in multiple unit auctions, especially in the presence of payoff externalities across coalitions.

As mentioned, coalition formation papers with incomplete information are rare. Okada (2012) does adopt a sequential bargaining approach, but without discounting, so his paper is more akin to the pre-Rubinstein bargaining literature.

The paper that comes closest to ours in approach is Cho et al. (2002). They analyze coalitional bidding in first-price auctions of a single indivisible unit with identical budget constrained players. Cho et al. (2002) too, assume complete information among bidders and pre-auction bargaining. They find that unless the budget constraint is sufficiently acute, in the sense that budgets of all bidders put together is less than the common value for the unit, the grand coalition forms. Note that they assume binding agreements on bidding behavior. In our case, such binding agreements are considered infeasible unless the agreements are self-enforcing.

The paper is organized in the following way. In section 2 we state the general unified structure under which we analyze both the single unit model and the multiple units model. In Section 3 we present our results and, in particular, in subsections 3.1 and 3.2 we state the results for the single unit and the

multiple units case respectively. We conclude in Sections 4. The proofs of all the results are provided in Section 5 that appears after Section 4.

2. The framework

We consider a model of formation of bidding rings in a multi-unit auction with unit demand where $N = \{1, \dots, n\}$ is the set of bidders and $k \in \{1, \dots, n-1\}$ is the number of identical indivisible units. In the auction, as is usual, the seller announces a reserve price (non-strategically, so she announces her true value) and bidders submit their bids simultaneously. The k highest bidders win and pay the $(k+1)^{\text{th}}$ highest bid, provided it is at least as high as the seller's reserve price.

W. l. o. g. suppose that reserve price is 0 and the valuation vector $v = (v_1, v_2, \dots, v_n)$ is such that $v_1 > v_2 > \dots > v_n > v_{n+1} := 0$.⁶ We assume that v is common knowledge among bidders but the seller has no information about it. Let $K = \{1, \dots, k\}$ denote the set of bidders who win a unit at the non-cooperative play of the auction; henceforth, called winners. Similarly, let $L = \{k+1, \dots, n\}$ denote the set of bidders who do not win a unit at the non-cooperative play; henceforth, called losers. The non-cooperative payoffs are $v_i - c$ for each $i \in K$ (where c is the price in excess of the reservation price zero), and 0 for each $i \in L$. Hence, $N = K \cup L$ with $K \cap L = \emptyset$. For any $S \subseteq N$, let $2^S := \{S' : S' \subseteq S\}$ and define $S(L) := \{S_k(m)\}_{m=k+1}^{m=n}$ where, for any given $k = 1, \dots, n-1$, $S_k(k+1) := \{k+1\}$ and $S_k(r) := \{k+1, \dots, r\}$ for all integers $r = k+2, \dots, n$. Also define \bar{m}^S and \underline{m}^S as the agents with the highest and the lowest valuations in any set $S \subseteq N$, respectively.⁷ We now consider the coalition formation phase, prior to the auction.

For each non-empty $S \subseteq N$ (interpreted as a coalition), define the set of all possible partitions on S as $\Pi(S)$. Thus each $\pi_S \in \Pi(S)$ is a collection of mutually exclusive and exhaustive subsets of S . Pick any $\pi_S \in \Pi(S)$ and define $L(\pi_S) := \{T \in \pi_S \mid T \cap K = \emptyset\}$ and $\bar{L}(\pi_S) := \cup_{T \in L(\pi_S)} T$. Therefore, $\bar{L}(\pi_S)$ denotes the union of those members of π_S that do not contain any winner.

The pre-auction ring formation is captured by a bargaining game $G \equiv (N, \bar{w}, p, \delta)$ (see Ray and Vohra (1999)). The function \bar{w} is a partition function that assigns coalitional worths to the members of each possible partition in $\Pi(N)$. The function $p : 2^N \rightarrow N$ is a protocol function which assigns to each set of active bidders (that is, bidders who are yet to form coalitions), a proposer from that set, who carries the game forward. Therefore, $p(T) \in T$ for all $T \subseteq N$. A stage in G is given by a substructure (that is, a partition defined on a strict subset of N) consisting of the coalitions that have formed and left the game. Thus the game begins at the stage \emptyset . The set of all possible stages is $\mathcal{P} := \cup_{S \subset N} \Pi(S)$, which is the set of all possible partitions of all possible strict subsets of N . Therefore, each possible non-terminal history in the bargaining game corresponds to a particular stage in \mathcal{P} . Define $R(\pi) := N \setminus \{\cup_{T \in \pi} T\}$, $\forall \pi \in \mathcal{P}$.

⁶Allowing identical valuations would only lead to multiplicity of equilibria without adding to the qualitative analysis.

⁷Given the non-identical valuations, for any set S , the bidders \bar{m}^S and \underline{m}^S are well defined.

Therefore, $R(\pi)$ is the set of remaining (active) bidders after coalitions in substructure π have formed and left the game.

The pre-auction bargaining game G proceeds as follows. At any stage $\pi \in \mathcal{P}$, the Bidder $p(R(\pi))$ makes a proposal. A proposal is a pair (T, z) with $T \subseteq R(\pi)$, $p(R(\pi)) \in T$ and $z = \{z(\pi')\}_{\pi' \in \Pi(R(\pi) \setminus T)}$ such that: $z(\pi') \in \mathfrak{R}_+^T$ with $\sum_{i \in T} z_i(\pi') = \bar{w}(T; (\pi, T, \pi'))$ for all $\pi' \in \Pi(R(\pi) \setminus T)$. The bidders in $T \setminus \{p(R(\pi))\}$ respond to the proposal sequentially according to the linear order $1 <_0 2 <_0 \dots <_0 n$ (that is, the order of valuations). If all bidders agree to the proposal, the coalition T forms and exits the game (all remaining bidders can observe this and the game proceeds to the stage (π, T)). If some Bidder $j \in T \setminus \{p(R(\pi))\}$ rejects the proposal, then the stage of the game remains unchanged and with probability $\delta \in (0, 1)$ the game moves to the next period when j gets to make a proposal. With probability $1 - \delta$, upon rejection, the game ends with all bidders getting zero payoff (due to termination of the auction upon discovery of the collusive bargaining process). A given coalition structure forms when every individual element of the structure has accepted the proposal made to it. If it forms in period $t + 1$, then all bidders get an expected payoff equal to δ^t times the agreed share of their respective coalitional worth.

In this paper, we focus on pure strategy equilibria. Further, we assume that bidders follow stationary Markovian strategies that depend on a small set of state variables in a way that is insensitive to past history. In particular, they depend on the current set of active bidders, coalition (sub)structure that has already formed and, in case of response, the on-going proposal. Thus effectively, strategy of a Bidder i assigns to each possible stage $\pi \in \mathcal{P}$ with $i \in R(\pi)$: (i) a proposal and (ii) an *acceptance threshold* $a_i(\pi) \in \mathfrak{R}_+$. The acceptance threshold signifies the response decision of i at stage π . That is, at stage π , Bidder i would accept any proposal (T, z) with $i \in T$, only if $z_i(\pi') \geq a_i(\pi)$; where π' is the anticipated (by Bidder i) coalition structure among all the remaining bidders after coalition T forms and exits. At any stage in G , we call a proposal acceptable, if all the responding members of the coalition proposed would accept the proposal given their respective acceptance thresholds for that stage. We solve for the stationary sub-game perfect equilibrium (SSPE) profile of strategies.

We also make the following assumptions.

Assumption 1. *Non-singleton coalitions of worth zero do not form.*

Assumption 2. *Inter-agent payments required to obtain the payoff division agreed upon in the pre-auction bargaining stage, are made prior to the start of the auction.*

Assumption 1 is essentially a selection rule. Relaxing this rule would lead to several uninteresting equilibria where losers form non-singleton coalitions and get the same zero payoff that they would have got by staying alone. Assumption 2 eliminates the possibility of the winners renegeing on their commitments post-auction, after the gains from cooperation have accrued to the winners.⁸ This assumption is essential

⁸As pointed out by the referee, effectively, the game model has three timing points, $t = 1, 2, 3$. At time 1, the sequential

to our paper because we do not assume binding contracts (since such contracts are illegal in nature). It ensures that the losers inside a coalition with winners have no incentive to bid anything other than 0.

Any analysis of pre-auction cooperative agreements involves detailing of bidding rings formed, bidding behavior decided by each ring and division of resultant worth of these rings. As mentioned earlier, any bidding agreement within a ring must be self-enforcing. That is, no agent should have any incentive to deviate from her pre-auction decided bidding behavior at the actual auction. Therefore, a pre-auction ring structure is stable only if the associated bids constitute a Nash equilibrium at the Vickrey auction. We assume that each coalition S specifies an intra-coalition bidding behavior according to the following rule **R**.

R: For any $i \in S$,

$$b_i = \begin{cases} v_i & \text{if } i \in K \text{ or } |S| = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Observe that any coalition structure could, in principle, be associated with several bid profiles. However, a coalition structure would be self-enforcing (in absence of any external device of enforcing illegal contracts) if and only if the associated bid profile constitutes a Nash equilibrium at the consequent Vickrey auction. There can be several such Nash equilibrium bid profiles corresponding to each possible coalition structure. We choose the bid profile implied by rule **R** to be the focal point for three reasons. First, it is simple and intuitive. Second, for any coalition structure, there does not exist any bid profile other than the one specified by **R**, that constitutes a Nash equilibrium at the Vickrey auction and generates greater coalitional worth for at least one member of the coalition. Third, in the single good case, the bidding rule **R** gives the maximum worth to any coalition, irrespective of what other agents outside the coalition are bidding. Finally, we can relax the restriction of bidding rule **R** to consider a pre-auction bargaining game that determines intra-coalition bidding arrangements too. This can be done by modifying the proposal in the bargaining game to not only consist of a coalition and relevant payoff division, but also the bidding arrangement for that coalition (however, these bidding arrangements would not be public information and so, would not enter the state space at any stage). It can be proved that the set of equilibrium coalition structures will remain unchanged.⁹

In Ray and Vohra (1999), the assumption of binding contracts ensures that no member of any coalition plays differently at the underlying game than the play agreed upon by the coalition (that is, cheating is ruled out). In our work, bidding rule **R** takes care of this issue, by ensuring that any deviation from the agreed upon strategy of a coalition, gives the same payoff to an individual as that resulting from any

baragaining game of ring formation is played. At time 2, the agreement of payment, determined in the bargaining game, is implemented. At time 3, Vickrey auction is played under bidding rule **R**. The agreement of payment is binding in the sense that it is perfectly implemented in time 2.

⁹The proof is available from the authors on request.

non-cooperative strategy. That is, bidding rule **R** implies that irrespective of the ring structure formed, the bid profile at the underlying auction is always a Nash equilibrium.

Using **R**, we define the following partition function which assigns a worth to each $S \in \pi_N$, $\forall \pi_N \in \Pi(N)$,

$$\bar{w}(S; \pi_N) = \sum_{j \in S \cap K} \left\{ v_j - \max_{l \in \bar{L}(\pi_N)} v_l \right\}. \quad (2)$$

This function specifies the payoff to any member coalition S of any partition π_N of the set N of bidders.

The particular functional form of the partition function in this setting, follows from the desire of the winners to manipulate the price that they end up paying for the unit. That is, winners want to collude with losers to persuade them to bid lower than their true valuations; and thereby, ensure procurement of the unit at a lower price, in the auction. The extra payoff that accrues to the winner out of this enterprise, is used to compensate the losers suitably. Hence, any worthwhile collusive venture must involve at least one winner, while the losers that are not included in any such venture, cannot benefit by forming coalitions amongst themselves, and so, play non-cooperatively. So, for any $\pi_N \in \Pi(N)$, members of $\bar{L}(\pi_N)$ bid their true valuations. Therefore, the going price at the auction, when a coalition structure π_N has formed, turns out to be $\max_{l \in \bar{L}(\pi_N)} v_l$. The coalitional worth of each coalition in π_N , then, is simply the sum of the payoffs of the winners in that coalition. The following example illuminates on this.

Example 1. Consider $N = \{1, 2, 3\}$, $k = 2$. Given $v_1 > v_2 > v_3$, we have $K = \{1, 2\}$ and $L = \{3\}$. Then, $\bar{w}(\{1\}; \{1\}, \{2, 3\}) = v_1$ and $\bar{w}(\{2, 3\}; \{1\}, \{2, 3\}) = v_2$; $\bar{w}(\{1, 2\}; \{1, 2\}, \{3\}) = v_1 + v_2 - 2v_3$ and $\bar{w}(\{3\}; \{1, 2\}, \{3\}) = 0$; $\bar{w}(\{1, 3\}; \{1, 3\}, \{2\}) = v_1$ and $\bar{w}(\{2\}; \{1, 3\}, \{2\}) = v_2$; $\bar{w}(\{1, 2, 3\}; \{1, 2, 3\}) = v_1 + v_2$ and the non-cooperatively play yields $\bar{w}(\{1\}; \{1\}, \{2\}, \{3\}) = v_1 - v_3$, $\bar{w}(\{2\}; \{1\}, \{2\}, \{3\}) = v_2 - v_3$ and $\bar{w}(\{3\}; \{1\}, \{2\}, \{3\}) = 0$.

For all $i \in K$ and $S \subseteq L \cup \{i\}$, define $w^i(S)$ to be the maximum worth that coalition S can attain in a single unit Vickrey auction with the bidder set $L \cup \{i\}$. It is easy to see that

$$w^i(S) = \begin{cases} v_i - \max_{l \in L \setminus S} v_l & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

3. Results

3.1. The single unit case: $k = 1$

There is a single winner, and so, $K = \{1\}$ and $L = N \setminus \{1\}$. Note that the worth or partition function reduces to a characteristic function $w^1(\cdot)$, that is,

$$\bar{w}(S; S, \pi_{N \setminus S}) = w^1(S) = \begin{cases} v_1 - \max_{l \in N \setminus S} v_l & \text{if } 1 \in S \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

$\forall S \subseteq N, \forall \pi_{N \setminus S} \in \Pi(N \setminus S)$.

Definition 1. A partition $\pi^r \in \Pi(N)$ is an order preserving r -ring if

$$\pi^r = \begin{cases} \{\{1\}, \{2\}, \dots, \{n\}\} & \text{for } r = 1, \\ \{\{1, \dots, r\}, \{r+1\}, \dots, \{n\}\} & \text{for } r = 2, \dots, n-1, \\ \{\{1, \dots, n\}\} & \text{for } r = n. \end{cases}$$

Observe that an order preserving 1-ring is a coalition structure with all singleton coalitions and an order preserving n -ring is the grand coalition. For any other r , an order preserving r -ring includes the top $(r-1)$ losers $\{2, \dots, r\}$ along with Bidder 1 who is the only winner. In particular, if Bidder $l (> 2)$ with valuation v_l is included in the non-singleton member coalition of any order preserving r -ring, then Bidder 1 to Bidder $(l-1)$ with respective valuations v_1 to v_{l-1} (all greater than v_l) are all included in that coalition. It is in this sense that an r -ring is order preserving in Definition 1. The following lemma states that an equilibrium outcome in the pre-auction bargaining game must necessarily induce an order preserving r -ring.

Lemma 1. For any $G = (N, w^1, p, \delta)$, if π^* is an SSPE outcome, then π^* is an order preserving r -ring.

At any stage of the game $G = (N, w^1, p, \delta)$, no active bidder makes an unacceptable proposal and hence, *there can be no delay on the equilibrium path*. The reason is provided in the next two paragraphs.

The $w^1(\cdot)$ function implies that a coalition S generates positive payoff only if S includes Bidder 1. Therefore, if at any stage where Bidder 1 is not active, then, from Assumption 1, all the active bidders stay alone. Consider any stage where Bidder 1 is active and the proposal power is with some $l \in L$. Bidder l will never make an unacceptable proposal because any such proposal, given stationarity, does not change the stage of the game. It simply passes the power of proposal to some other active bidder (because the rejector proposes in our bargaining game). This rejector can either make an acceptable proposal (which must contain Bidder 1 to have a positive worth) and leave the game; or propose unacceptably, in which case, the stage of the game remains unchanged even after two periods of delay. The latter possibility is undesirable to Bidder l as it causes delay without changing the stage of the game. The former possibility gives Bidder l zero payoff if Bidder l is not one of the members to whom an acceptable proposal is made. Even if Bidder l is one of the members of such an acceptable proposal, Bidder l could have always proposed the same thereby saving the cost of delay.

Now, consider the stage where Bidder 1 has the proposal power. Suppose Bidder 1 can get a payoff of x by making an acceptable proposal. As before, Bidder 1 observes that given stationarity, an unacceptable proposal will not change the stage of the game and will only pass the proposal power to some Bidder $l \in L$ in the present stage. By the previous argument Bidder l will never propose unacceptably. Moreover, Bidder l will never leave the game alone (as it will give 0 payoff). Therefore, Bidder l must propose (and have accepted) a coalition containing Bidder 1. This can be done by offering at least δx to Bidder 1. Thus we see that an unacceptable proposal by Bidder 1 gives $\delta^2 x < x$. Hence making an acceptable proposal

strictly dominates any unacceptable proposal. Therefore, at any SSPE outcome, for any $G = (N, w^1, p, \delta)$, there is no delay.

So far we have argued that for any $G = (N, w^1, p, \delta)$, an SSPE outcome π^* is an order preserving r -ring and that at any SSPE outcome π^* there is no delay. Therefore, the next question is what is the order preserving r -ring that will form and what will be the resulting payoffs to the bidders in that ring. We prove this formally in the next theorem for the case where Bidder 1 is the first proposer in the pre-auction bargaining protocol. Here we try and provide an intuitive argument. Assuming $\delta = 1$ it can be shown that any order preserving r -ring that forms in equilibrium must divide its worth equally among all its members. This is because, (i) with $\delta = 1$ agents are no longer averse to delay in resolution of negotiations and (ii) any agent may reject a proposal, obtain the proposal power, and then propose the same proposal. By continuity of the maximization problem faced by each agent at a proposal stage, it follows that the largest average worth maximizing order preserving r -ring continues to be the optimal (acceptable) proposal for δ values sufficiently close to unity. Thus if Bidder 1 is the first proposer, then for sufficiently high δ values the equilibrium ring structure consists of the largest average worth maximizing order preserving r -ring with all other agents playing non-cooperatively. Before stating the theorem we define $AV(i, l) := (v_i - v_l)/(l - k)$ where $i = 1, 2, \dots, k$ and $l = k + 1, k + 2, \dots, n + 1$. Observe that for the single unit game $G = (N, w^1, p, \delta)$, $k = 1$ and hence $w^1(\{1, \dots, t\}) = v_1 - v_{t+1} = tAV(1, t + 1)$ for all $t = 2, \dots, n$. Moreover, $AV(1, t + 1) = \lim_{\delta \rightarrow 1} [w^1(\{1, \dots, t\})/(1 + \delta(t - 1))]$.

Theorem 1. *For any $G = (N, w^1, p, \delta)$ with $p(N) = 1$, there exists $\delta' \in (0, 1)$ such that for all $\delta \in (\delta', 1)$ the SSPE outcome is an order preserving r -ring without any delay, if and only if*

1. $AV(1, r + 1) \geq AV(1, t + 1)$, $\forall t \in \{1, 2, \dots, r - 1\}$ and
2. $AV(1, r + 1) > AV(1, t + 1)$, $\forall t \in \{r + 1, r + 2, \dots, n\}$.

Theorem 1 applies when the highest valuation agents makes the first proposal, that is, $p(N) = 1$. When bidders' valuations are common knowledge, this order of valuations is a natural protocol.¹⁰ Still, the implication of $p(N) \neq 1$ is explained informally using the following example.

Example 2. *Suppose $N = \{1, 2, 3\}$ and $K = \{1\}$ where $v \equiv (v_1 = 70, v_2 = 65, v_3 = 20)$. Note that $AV(1, 2) < AV(1, 3) > AV(1, 4)$. It can be shown that $\forall \delta \in (\frac{2}{3}, 1)$, bidders 1 and 2 propose $\{1, 2\}$, while Bidder 3 proposes $\{1, 2, 3\}$ at stage \emptyset .¹¹ Therefore, if $p(N) \in \{1, 2\}$, then the SSPE outcome is an order preserving 2-ring and, if $p(N) = 3$, then the SSPE outcome is an order preserving 3-ring.*

The intuition in the above example can easily be generalized to obtain Corollary 1 that follows. Corollary 1 states that the largest average worth maximizing order preserving r -ring forms in equilibrium,

¹⁰We are thankful to an anonymous referee for pointing this out.

¹¹In particular, invoking the strategy Σ in the sufficiency proof of the Theorem 1, at the stage \emptyset (that is, at the beginning of the game), we see that $\bar{H}_1(\emptyset) = \bar{H}_2(\emptyset) = \{2\}$ and $\bar{H}_3(\emptyset) = \{2, 3\}$.

if the first proposer $p(N)$ is a member of that ring. If not, the smallest order preserving r -ring containing the first proposer forms in equilibrium.

Corollary 1. *For any $G = (N, w^1, p, \delta)$, there exists $\delta' \in (0, 1)$ such that for all $\delta \in (\delta', 1)$, the SSPE outcome π^* is an order preserving t -ring where $t = \max\{r, p(N)\}$ and $r \in \{2, \dots, n\}$ satisfies the property that $AV(1, 2) \leq \dots \leq AV(1, r) \leq AV(1, r + 1) > AV(1, r + 2) \geq \dots \geq AV(1, n + 1)$.*

It is important to note that Corollary 1 holds *irrespective of the protocol function*. The next corollary specifies a restriction on valuations that leads to formation of the grand coalition N (that is, the order preserving n -ring) without any delay. It shows that if the agents' valuations are sufficiently close to each other, then the grand coalition forms in equilibrium.

Corollary 2. *For any $G = (N, w^1, p, \delta)$, if $n/(n - l) \geq v_1/v_{l+1}$ for all $l = 1, 2, \dots, n - 1$, then there exists $\delta' \in (0, 1)$ such that for all $\delta \in (\delta', 1)$ the SSPE outcome is the order preserving n -ring.*

3.2. The multiple units case: $k \in \{2, \dots, n - 1\}$

Consider the sub-games with the set of active bidders as T such that $L \subseteq T$. For all such sub-games, the substructure formed by the departed bidders (who have formed coalitions and left the game) does not affect the worth of any coalitions that remaining bidders may form in future. That is, at such a stage with active bidder set T with $L \subseteq T$; $\bar{w}(S; \pi_{N \setminus T}, S, \hat{\pi}_{T \setminus S}) = \bar{w}(S; \pi'_{N \setminus T}, S, \hat{\pi}_{T \setminus S})$, $\forall \pi_{N \setminus T}, \pi'_{N \setminus T} \in \Pi(N \setminus T)$, $\forall S \subseteq T$, $\forall \hat{\pi}_{T \setminus S} \in \Pi(T \setminus S)$. At these sub-games, we refer to the stage in the game by the set of active bidders, instead of the substructure consisting of coalitions who have (formed and) left the game.

At any such stage T (with $L \subseteq T$), define $C_i^\delta(T)$ to be the set of coalitions (that is, the non-empty subsets of T containing i) that Bidder i proposes acceptably, in equilibrium, at that stage.¹² Also define $T^i := \{i\} \cup L$, $\forall i \in K$. Further, for all $j \in K$, define $C_i^*(T^j)$ to be the set of coalitions that any Bidder $i \in T^j$ proposes acceptably in equilibrium at stage T^j ; as δ goes to 1 in the limit.¹³

Lemma 2. *For any $i, j \in K$ such that $v_i > v_j$, there exists $\delta' \in (0, 1)$ such that for all $\delta \in (\delta', 1)$; if $[\{k + 1, \dots, m\} \cup \{i\}] \in C_i^\delta(T^i)$, then there exists $m' \geq m$ such that $[\{k + 1, \dots, m'\} \cup \{j\}] \in C_j^\delta(T^j)$.*

Lemma 2 states that when δ is sufficiently high and $v_i > v_j$, if winners i and j separately find themselves at a stage where the remaining set of bidders are T^i and T^j respectively and if i picks a coalition $[\{k + 1, \dots, m\} \cup \{i\}]$ as a best acceptable proposal, then there exists an $m' \in \{m, \dots, n\}$ such that the coalition $[\{k + 1, \dots, m'\} \cup \{j\}]$ is a best acceptable proposal for j .

¹²For economy of notation, whenever the set $C_i^\delta(T)$ set is a singleton, containing only one subset T' (say) of T ; we write $C_i^\delta(T) = T'$ instead of $C_i^\delta(T) = \{T'\}$.

¹³Loosely speaking, for all $j \in K$ and all $i \in T^j$, as δ goes to unity, the set of coalitions $C_i^\delta(T^j)$, that Bidder i proposes acceptably in equilibrium at stage T^j , goes to $C_i^*(T^j)$.

Remark 1. It also follows from Lemma 2 that at the stage T^i (for any $i \in K$); the game $G = (N, \bar{w}, p, \delta)$ reduces to a single unit/single winner bargaining game $G^i = (T^i, w^i, p^i, \delta)$ where $p^i(\cdot)$ is the restriction of the original protocol function $p(\cdot)$ to the set 2^{T^i} . As mentioned earlier, in the bargaining game with single winner, at any stage, no active bidder makes an unacceptable proposal on the equilibrium path. Therefore, $C_j^\delta(T^i)$ is the set of coalitions that Bidder j proposes in equilibrium at stage T^i , in game G ; $\forall j \in T^i, \forall i \in K$.

Therefore, from Theorem 1 it follows that at any such single winner stage T^i , in the limit, winner i chooses an average worth maximizing coalition containing itself. That is, for all $i \in K$,

$$C_i^*(T^i) = \operatorname{argmax}_{S \in \mathcal{S}(L)} [w^i(S \cup \{i\}) / (1 + |S|)]$$

Lemma 3. If $n/(n-l) \geq v_1/v_{l+1}$ for all $l = 1, 2, \dots, n-1$, then $C_i^\delta(T^i) = T^i$ for all $\delta \in (0, 1)$ and all $i \in K \setminus \{1\}$.

Lemma 3 is an obvious consequence of Corollary 2 and Lemma 2. It states that if the winner with highest valuation finds it optimal to collude with all the losers at stage T^1 in the limit (that is, $T^1 \in C_1^*(T^1)$); then, irrespective of the value of δ , all other winners $i \in K \setminus \{i\}$ would find it optimal to collude with all the losers at stage T^i .

For any x , let $[x]$ denote the highest integral value of x not exceeding x . Given $N = K \cup L$ with $k \equiv |K| \geq 1$, we define the *weak winner set* $\bar{K} (\subseteq K)$ in the following way.

$$\bar{K} = \begin{cases} K & \text{if } |K| \in \{1, 2\}, \\ K \setminus \{1, \dots, [(k-1)/2]\} & \text{otherwise.} \end{cases}$$

Therefore, when $k \in \{1, 2\}$ we have $\bar{K} = K$, when $k \in \{3, 4\}$ we have $\bar{K} = K \setminus \{1\}$, when $k \in \{5, 6\}$ we have $\bar{K} = K \setminus \{1, 2\}$ and so on. For any weak winner $j \in \bar{K}$, let $\Pi'(K \setminus \{j\})$ be the set of all possible partitions σ_{-j} of winners in $K \setminus \{j\}$, satisfying the following two properties: (i) if a coalition $S \in \sigma_{-j}$, then $|S| \in \{1, 2\}$ and (ii) $|\{j^* \in K \setminus \{j\} : \{j^*\} \in \sigma_{-j}, v_{j^*} < v_j\}| \in \{0, 1\}$.

Definition 2. A partition $\pi(\sigma_{-j}) \in \Pi(N)$ is a *single winner ring with free riding partition* or simply an *SWRFR partition* if $\pi(\sigma_{-j}) = \{\sigma_{-j}, T^j\}$ where $j \in \bar{K}$ and $\sigma_{-j} \in \Pi'(K \setminus \{j\})$.

Therefore, an SWRFR partition has the property that any one winner j from the weak winner set forms a coalition with all the losers; each remaining winner $j' \in K \setminus \{j\}$ either stays alone or forms a two member coalition by including another member from the remaining winner set $K \setminus \{j, j'\}$; and finally, at most one winner from the set of winners with valuation less than v_j can stay alone. Let $\bar{\Pi} := \left((\pi(\sigma_{-j}))_{\sigma_{-j} \in \Pi'(K \setminus \{j\})} \right)_{j \in \bar{K}}$ denote the set of all possible SWRFR partitions.

Theorem 2. If $n/(n-l) \geq v_1/v_{l+1}$ for all $l = 1, 2, \dots, n-1$, then there exists $\delta' \in (0, 1)$ such that for all $\delta \in (\delta', 1)$:

- (i) For any $G = (N, \bar{w}, p, \delta)$ the SSPE outcome $\pi^* \in \bar{\Pi}$.
- (ii) For every $\pi^* \in \bar{\Pi}$, there exists a protocol p^* such that π^* is the unique SSPE outcome of $G = (N, \bar{w}, p^*, \delta)$.

Theorem 2 (i) specifies the if the valuations are such that $n/(n-l) \geq v_1/v_{l+1}$ for all $l = 1, 2, \dots, n-1$, and if δ is sufficiently high, then, irrespective of the protocol function, the only possible partition that can form in any SSPE is an SWRFR partition. In fact, this result continues to hold for any set of valuations such that $T^1 \in C_1^*(T^1)$. The proof of the theorem uses an inductive argument to solve the game. In particular, the equilibrium SWRFR is a consequence of the optimal proposal strategy of the winners and the losers. The optimal proposal strategy of the winners is the following.

- W(1) If the set of active agents in a sub-game contains one winner and all losers, then the winner acceptably proposes to all active agents.
- W(2) If the set of active agents in a sub-game contains any two winners and all losers, then each winner would choose to stay alone and free ride thus leading to sub-game W(1).
- W(3) If the set of active agents in a sub-game contains more than two winners and all losers, then the highest valuation winner stays alone while the other winners acceptably propose to the highest valuation winner and form a two member coalition. This process continues till we are in a sub-game corresponding to either W(1) or W(2). For example, if the set of active winners is $\{1, 3, 4\}$, then Bidder 1 would choose to stay alone and free ride leading to sub-game W(2). And if either Bidder 3 or Bidder 4 propose, then each would propose a coalition with Bidder 1 which Bidder 1 accepts thus leading to W(1).

The specification of the optimal proposal decisions of the losers in Theorem 2 is a little more subtle.

- L(1) If the set of active agents in a sub-game consists of one winner and all losers, then any loser makes an acceptable proposal. Such an acceptable proposal must include all active agents in that sub-game.
- L(2) At all other sub-games where all losers are active along with at least two winners, losers would make an optimal unacceptable proposal addressed to the winner $k^*(W^0; p) \in W^0$ (given by the recursion (*) in the Appendix) where W^0 is the set of active winners in the sub-game.

The optimal choice of the winner $k^*(W^0; p)$ at whom an unacceptable proposal is addressed; is made in such a way that, given the optimal proposal decisions of the winners (W(1)-W(3)), the game reaches a sub-game with $T^{j^*} = \{j^*\} \cup L$ players where $j^* \in \bar{K}$ is a single winner with highest possible valuation who is active along with all the losers. At this sub-game with T^{j^*} agents, an acceptable proposal will be made (irrespective of the identity of the proposer at this sub-game) that includes all agents in T^{j^*} .

As a result of optimal proposal strategies W(1)-W(3) and L(1)-L(2), the only equilibrium structure

available is an SWRFR partition. This establishes statement (i) of Theorem 2.¹⁴ The statement (ii) of Theorem 2 is proved by suitable design of protocol function p^* for each possible $\pi^* \in \bar{\Pi}$. We provide an example to show that different protocol functions gives rise to different kinds of equilibrium SWRFR partitions.

Example 3. Suppose $N = \{1, 2, 3, 4, 5\}$, and let Bidder 1, Bidder 2 and Bidder 3 be the set of winners so that $K = \{1, 2, 3\}$. Let the protocol function $p(S) = \min_{j \in S} \{<_o\}$, $\forall S \subseteq N$ for some linear order “ $<_o$ ” defined on the bidder set N . For δ sufficiently close to 1, if $\{\frac{v_2}{4}, \frac{v_3}{3}, \frac{v_4}{2}, v_5\} \geq \frac{v_1}{5}$ then $T^1 = \{1, 4, 5\} \in C_1^*(T^1)$. And so, we have the following.

1. If $1 <_o 2 <_o 3 <_o 4 <_o 5$, then the equilibrium SWRFR partition is $\{\{1\}, \{2\}, \{3, 4, 5\}\}$. Since $p(N) = 1$, Bidder 1 wants to stay alone given the optimal strategy $W(1)$ so that the singleton coalition $\{1\}$ forms in the first stage. Given, $p(N \setminus \{1\}) = 2$, Bidder 2 wants to stay alone given the optimal strategy $W(1)$ so that the singleton coalition $\{2\}$ forms in the second stage. Finally, given $T^1 \in C_1^*(T^1)$, from Lemma 3 it follows that in the third stage it is optimal for Bidder 3 to collude with all the losers to form the coalition $\{3, 4, 5\}$.
2. If $3 <_o 2 <_o 5 <_o 4 <_o 1$, then the equilibrium SWRFR partitions $\{\{1, 3\}, \{2, 4, 5\}\}$. Since $p(N) = 3$, Bidder 3 wants to collude with Bidder 1 given the optimal strategy $W(3)$ so that the coalition $\{1, 3\}$ forms in the first stage. Finally, given $T^1 \in C_1^*(T^1)$, from Lemma 3 it follows that in the second stage it is optimal for Bidder 2 to collude with all the losers to form the coalition $\{2, 4, 5\}$.
3. If $4 <_o 1 <_o 5 <_o 2 <_o 3$, then the equilibrium SWRFR partition can either be $\{\{1\}, \{3\}, \{2, 4, 5\}\}$ or it can be $\{\{1, 3\}, \{2, 4, 5\}\}$. Since $p(N) = 4$, the loser (that is, bidder with fourth highest valuation) gets to propose and given the optimal strategy $L(2)$ wants to unacceptably propose to a winner $k^*({1, 2, 3}; p) \in \{1, 2, 3\}$. Can $k^*({1, 2, 3}; p) = 2$? The answer is no. This is because if $k^*({1, 2, 3}; p) = 2$, then, given that the rejector gets to propose in the next stage, by applying the optimal strategies of the winners it follows that the resulting coalition structure will be $\{\{1, 2\}, \{3, 4, 5\}\}$ and the payoff of any loser (in the limit as δ goes to 1) is $v_3/3$. One can easily check that: if $k^*(K \cup L; p) = 1$ then the resulting coalition structure will be $\{\{1\}, \{3\}, \{2, 4, 5\}\}$, and if $k^*(K \cup L; p) = 3$ then the resulting coalition structure will be $\{\{1, 3\}, \{2, 4, 5\}\}$. In both cases, the payoff of any loser (in the limit as δ goes to 1) is $v_2/3 > v_3/3$. Therefore, in the first stage with $p(N) = 4$, the optimal non-unique unacceptable proposal is addressed to $k^*(K; p) \in \{1, 3\}$ so that the resulting equilibrium structure is either $\{\{1, 3\}, \{2, 4, 5\}\}$ or $\{\{1\}, \{3\}, \{2, 4, 5\}\}$.

It is important to note that in any equilibrium SWRFR partition $\{T^j, \sigma_{-j}\}$, a weak winner $j \in \bar{K}$ colludes with all losers L to form T^j and ensures that all the losers bid zero at the auction, thereby

¹⁴Note that Theorem 2 (i) does not claim uniqueness of the SSPE outcome.

reducing the $(k + 1)$ th price to zero. One can show that, in the limit, the other winners $K \setminus \{j\}$ get at least their own valuations by forming a coalition of size two or less. However, Bidder j who colludes with the losers gets $v_j/(n - k + 1)$ (which is strictly less than v_j) in the limit. In other words, *a weak winner j generates the gains from cooperation by colluding with all the losers while the other winners free ride.* One special case of this kind of free ride is the one where the lowest valuation winner k colludes with all the losers in L while all other winners stay alone. The following proposition provides the restriction on the protocol function that characterizes formation of this SWRFR partition in equilibrium.

Proposition 1. *For any $G = (N, p, \bar{w}, \delta)$, if $n/(n - l) \geq v_1/v_{l+1}$ for all $l = 1, 2, \dots, n - 1$, then $\forall \delta \in (\delta', 1)^{15}$; the SSPE outcome is the SWRFR partition $\{\{\{i\}\}_{i \in K \setminus \{k\}}, T^k\}$ if and only if the protocol function $p(\cdot)$ satisfies the property*

$$p(N) = 1, p(N \setminus \{1, \dots, i\}) = i + 1, \forall i \in K \setminus \{k - 1, k\} \quad (5)$$

Finally, note that Theorem 2 and Proposition 5 continue to hold for any valuations such that the highest valuation winner 1 finds it optimal to collude with all the losers at stage T^1 (that is, $T^1 \in C_1^*(T^1)$). The difficulty of relaxing this restriction is addressed in the following example.

Example 4. *Consider the simplest multiple units case where there are two units, that is, $K = \{1, 2\}$ and suppose that $N = \{1, 2, 3, 4, 5, 6\}$. Assume that (a) $\frac{v_i - v_5}{3} > \max\{v_i - v_3, \frac{v_i - v_4}{2}, \frac{v_i - v_6}{4}, \frac{v_i}{5}\}$, for $i = 1, 2$ and (b) $p(N) = 1, p(N \setminus \{1\}) = 2$. Therefore, for sufficiently high δ , by Theorem 1, $C_i^\delta(\{i, 3, 4, 5, 6\}) = \{i\} \cup \{3, 4\}$, for $i = 1, 2$.*

If Bidder 1 stays alone at the stage N , then at the next stage Bidder 2 acceptably proposes $\{2, 3, 4\}$ leading to the coalition structure $\{\{1\}, \{2, 3, 4\}, \{5\}, \{6\}\}$ which gives Bidder 1 a payoff of $v_1 - v_5$. However, if Bidder 1 forms $\{1, 6\}$ at stage N , then Bidder 1's payoff is

$$\begin{cases} \frac{v_1 - v_3}{2} & \text{if Bidder 2 stays alone at the next stage } \{2, 3, 4, 5\}, \\ \frac{v_1 - v_4}{2} & \text{if Bidder 2 forms } \{2, 3\} \text{ at the next stage } \{2, 3, 4, 5\}, \\ \frac{v_1 - v_5}{2} & \text{if Bidder 2 forms } \{2, 3, 4\} \text{ at the next stage } \{2, 3, 4, 5\}, \text{ and} \\ \frac{v_1}{2} & \text{if Bidder 2 forms } \{2, 3, 4, 5\} \text{ at the next stage } \{2, 3, 4, 5\}. \end{cases} \quad (6)$$

Therefore, for Bidder 1 to make the optimal proposal choice at stage N (that is, to evaluate the proposal $\{1, 6\}$ at stage N), Bidder 1 needs to know the proposal choice of Bidder 2 at stage $\{2, 3, 4, 5\}$. Note that our assumption (a) puts no restriction on the ranking of average worths of subsets of $\{2, 3, 4, 5\}$, that Bidder 2 can propose acceptably (keeping in mind that Bidder 6 has already colluded with Bidder 1 and so will bid zero at the auction) at stage $\{2, 3, 4, 5\}$. That is, (a) does not impart any ranking of the numbers $v_2 - v_3, \frac{v_2 - v_4}{2}, \frac{v_2 - v_5}{3}$ (payoffs from forming $\{2\}, \{2, 3\}$ and $\{2, 3, 4\}$ respectively) with respect to

¹⁵The δ' is taken from Theorem 2.

$\frac{v_2}{4}$ (payoff from forming $\{2, 3, 4, 5\}$). Hence the problem becomes fairly intractable, even with two units, once we allow $C_i^\delta(T^i)$ to be strict subset of T^i for all (or some) $i \in K$.

Moreover, in such a case the final coalition structure may or may not have one winner colluding with all the losers, depending upon the protocol function. That is, if we use the protocol function $p(N) = 1, p(N \setminus \{1\}) = 6$; then it is optimal for Bidder 1 to stay alone at stage N since, in the next stage, a loser (Bidder 6) proposes (but has no choice other than to acceptably propose) $\{2, 3, 4, 5, 6\}$, leading to formation of the coalition structure $\{\{1\}, \{2, 3, 4, 5, 6\}\}$ giving Bidder 1 a payoff of v_1 (which is the best that Bidder 1 can get). Therefore, with such a protocol function, irrespective of the valuations, the unique SSPE outcome is $\{\{1\}, \{2, 3, 4, 5, 6\}\}$.

4. Conclusions

In this paper, we study how bargaining pressures inherent in the process of coalition formation impact the structure of bidding rings in a Vickrey auction. For the single unit auction, we completely characterize the bidding ring formation and show that the equilibrium coalition structure can only be an order preserving r -ring, that consists of the winner and the top $(r-1)$ losers in one coalition with all other losers staying alone. Depending on the valuations of all the bidders, the value of r can vary from one to the total number of bidders. In the multiple units case, we specify the sufficient conditions for formation of the class of equilibrium coalition structures, which we call single winner ring with free riding (SWRFR), where exactly one winner colludes with all the losers and generates maximum possible bidders' surplus, and, depending on the protocol, the remaining winners free ride either by staying alone or by colluding in pairs. Our work could also be viewed as a complete information benchmark with regard to collusion at such auctions. Of course, further research needs to be done to extend this line of coalition formation to the incomplete information case.

5. The proofs

Proof of Lemma 1: Let π^* be an SSPE outcome and suppose that it is not an order preserving r -ring. Therefore, π^* must not be an order preserving 1-ring. Hence, there exists $X \in \pi^*$ such that $|X| > 1$. Now, if X does not include Bidder 1, then $w^1(X) = w^1(X \setminus \{i\}) = 0$ for any $i \in X$, that is, the marginal contribution of any Bidder i to the coalition $X \setminus \{i\}$ is zero. But, given Assumption 1, $X \in \pi^*$ implies that in equilibrium Bidder i gets a positive payoff. Therefore, $w^1(X) > 0$, and so, we have a contradiction. Therefore, $1 \in X$ and so, by supposition, there exist $l, l' \in N \setminus \{1\}$ such that $l > l'$, $l \in X$ and $l' \notin X$. As before, $w^1(X) = w^1(X \setminus \{l\})$, and so, the marginal contribution of Bidder l to the coalition $X \setminus \{l\}$ is zero. However, by Assumption 1, equilibrium payoff to l must be positive. This is clearly suboptimal and hence, a contradiction. So for any $l > 2$, if $l \in X$ then $\{1, 2, \dots, l-1\} \subset X$. Also, by definition, $w^1(S) = 0$ for all $S \subseteq N \setminus X$. Hence, from Assumption 1 it follows that, all bidders in $N \setminus X$ must stand alone. Thus $\pi^* = \{X, \{\{i\}\}_{i \in N \setminus X}\}$ is an order preserving r -ring, and so, we have a contradiction. \square

Proof of Theorem 1: We first prove the only if part. Consider a stage π in the game such that Bidder $1 \in R(\pi)$. Since there can be no delay in equilibrium (since no active bidder at any stage proposes unacceptably), the equilibrium acceptance threshold of any $i \in R(\pi)$ must be the one period discounted payoff that i can generate by making the equilibrium proposal, at stage π itself. Therefore, for a given δ ; from Lemma 1 it follows that the equilibrium acceptance thresholds $\{a_i^\delta(\pi)\}_{i \in R(\pi)}$ must be such that

$$\frac{a_i^\delta(\pi)}{\delta} = \max_{T \in [2^{R(\pi)} \cap S(L)], i \in T} \left\{ w(\{1\} \cup T) - \sum_{j \in \{1\} \cup T \setminus \{i\}} a_j^\delta(\pi) \right\}, \forall i \neq 1$$

and,

$$\frac{a_1^\delta(\pi)}{\delta} = \max_{T \in [2^{R(\pi)} \cap S(L)]} \left\{ w(\{1\} \cup T) - \sum_{j \in T} a_j^\delta(\pi) \right\}.^{16}$$

From Chatterjee et al. (1993), it follows that $\forall \pi \in \mathcal{P}$ with $1 \in R(\pi)$; the acceptance thresholds are obtained by the following recursion;

$$(i) \ a_1^\delta(\pi) = \max_{T \in [2^{R(\pi)} \cap S(L)]} \frac{\delta w^1(\{1\} \cup T)}{1 + \delta|T|} \text{ and } a_i^\delta(\pi) = a_1^\delta(\pi) \text{ for all } i \in \bar{H}_1^\delta(\pi) \text{ where } \bar{H}_1^\delta(\pi) := \left[\bigcup_{T \in H_1^\delta(\pi)} T \right]$$

with $H_1^\delta(\pi) := \operatorname{argmax}_{T \in [2^{R(\pi)} \cap S(L)]} \frac{\delta w^1(\{1\} \cup T)}{1 + \delta|T|}$.

(ii) Suppose $(\bar{H}_1^\delta, \bar{H}_2^\delta, \dots, \bar{H}_q^\delta)$ is well defined. If $R(\pi) \setminus [\{1\} \cup \bar{H}_q^\delta(\pi)] \neq \emptyset$, then define

$$H_{q+1}^\delta(\pi) := \operatorname{argmax}_{T \in [2^{R(\pi)} \cap S(L)], \bar{H}_q^\delta(\pi) \subset T} \frac{\delta \left\{ w^1(\{1\} \cup T) - \sum_{j \in \bar{H}_q^\delta(\pi)} a_j^\delta(\pi) - a_1^\delta(\pi) \right\}}{1 + \delta(|T| - |\bar{H}_q^\delta(\pi)| - 1)} \quad (7)$$

As before, $\bar{H}_{q+1}^\delta(\pi) := \left[\bigcup_{T \in H_{q+1}^\delta(\pi)} T \right]$. For all $i \in \bar{H}_{q+1}^\delta(\pi)$, $a_i^\delta(\pi)$ is the maximized value in the definition of $H_{q+1}^\delta(\pi)$.

Note that $\bar{H}_q^\delta(\pi) \subset H_q^\delta(\pi)$, $\forall q$ in the recursion above. This follows from the particular structure of the problem reflected in Lemma 1. The proposal decision at any stage π with $1 \in R(\pi)$ is as follows. Each $i \in R(\pi)$ must belong to some $\bar{H}_q^\delta(\pi)$ and therefore proposes any $M_i \cup \{1\}$ such that $M_i \in H_q^\delta(\pi)$ with $i \in M_i$.

Now, recall that any coalition not containing Bidder 1 has a zero worth. Therefore, at all other stages π' with $1 \notin R(\pi')$, all proposers propose singleton coalitions of themselves and $a_i^\delta(\pi') = 0$, $\forall i \in R(\pi')$.

We, now, make the following claim that for sufficiently high discount factor, $H_1^\delta(\pi)$ consists of a single member set and that set is the largest average worth maximizer among the sets in $\{\{1\}, \{1, 2\}, \{1, 2, 3\}, \dots, N\}$.

Claim (a): $\exists \bar{\delta} \in (0, 1)$ such that $\forall \delta \in (\bar{\delta}, 1)$, $H_1^\delta(\pi)$ contains *only* the largest member of

$$\operatorname{argmax}_{T \in [2^{R(\pi)} \cap S(L)]} AV(1, |T| + 2)$$

¹⁶In case the feasible set of maximizers in the following optimization problem is empty, $a_i^\delta(\pi) := 0$.

for all $\pi \in \mathcal{P}$ with $1 \in R(\pi)$.

Proof of Claim (a): Define $H_1(\pi) := \operatorname{argmax}_{T \in [2^{R(\pi)} \cap S(L)]} AV(1, |T| + 2)$. Note that $\lim_{\delta \rightarrow 1} \frac{\delta w^1(\{1\} \cup T)}{1 + \delta |T|} = AV(1, |T| + 2)$. Therefore, it follows that for values of δ sufficiently close to 1, $H_1^\delta(\pi) \subseteq H_1(\pi)$. Now suppose $\exists T, T' \in H_1(\pi)$. Therefore (A) $\frac{w^1(\{1\} \cup T)}{1 + \delta |T|} - \frac{w^1(\{1\} \cup T')}{1 + \delta |T'|} = 0$ with δ value fixed at 1. Given the structure of the game, it must be that $|T| \neq |T'|$; say $|T| > |T'|$. From (A), $|T| > |T'|$ implies that $w^1(\{1\} \cup T) > w^1(\{1\} \cup T')$ (because we use non-identical valuations). Also, for a ‘‘slight’’ fall in δ value; in the left hand side of (A), the denominator of the first term decreases by more than the second term (since $|T| > |T'|$). Hence, the ‘equals to’ sign in (A), changes to ‘greater than’ for δ values sufficiently close to 1. Therefore, proposal choice of the largest coalition in $H_1(\pi)$ dominates that of the other members of $H_1(\pi)$, for δ sufficiently close to 1. This proves Claim (a).

Since $p(N) = 1$, an order preserving r -ring is formed only if Bidder 1 proposes $\{1, 2, \dots, r\}$ acceptably on the SSPE path. This will happen, for sufficiently high δ , only if $S_1(r) \cup \{1\}$ is the largest coalition in $H_1(\emptyset)$, that is, $S_1(r) \cup \{1\}$ is the largest average worth maximizing coalition. This implies that $AV(1, r + 1) \geq AV(1, t + 1), \forall t < r$ and $AV(1, r + 1) > AV(1, t + 1), \forall t > r$. These two conditions imply results (1) and (2) respectively.

We now prove the if part. Define the following strategy Σ in game G :

- At any stage π with $1 \notin R(\pi)$, all proposers choose to stay alone, and set an acceptance threshold of 0.
- Recall that for any stage π with $1 \in R(\pi)$, $H_1(\pi) := \operatorname{argmax}_{T \in [2^{R(\pi)} \cap S(L)]} AV(1, |T| + 2)$. For all such π , let $\bar{H}_1(\pi)$ be the largest coalition in $H_1(\pi)$. Then, at any stage π with $1 \in R(\pi)$, all $i \in [\bar{H}_1(\pi) \cup \{1\}]$ propose $[\bar{H}_1(\pi) \cup \{1\}]$ and set their acceptance thresholds to be $\frac{\delta w^1(\bar{H}_1(\pi) \cup \{1\})}{1 + \delta |\bar{H}_1(\pi)|}$. If the sequence $(\bar{H}_1, \bar{H}_2, \dots, \bar{H}_q)$ is well defined and $R(\pi) \setminus [\bar{H}_q(\pi) \cup \{1\}] \neq \emptyset$; then

$$H_{q+1}(\pi) := \operatorname{argmax}_{T \in [2^{R(\pi)} \cap S(L)], \bar{H}_q(\pi) \subset T} \frac{w^1(\{1\} \cup T) - w^1(\{1\} \cup \bar{H}_q(\pi))}{|T| - |\bar{H}_q(\pi)|} \quad (8)$$

with $\bar{H}_{q+1}(\pi)$ is defined as before to be the largest coalition in $H_{q+1}(\pi)$. Then all $j \in [\bar{H}_{q+1}(\pi) \cup \{1\}]$ propose $[\bar{H}_{q+1}(\pi) \cup \{1\}]$ and set their acceptance thresholds to be

$$\frac{\delta w^1(\bar{H}_{q+1}(\pi) \cup \{1\}) - \delta w^1(\bar{H}_q(\pi) \cup \{1\})}{1 + \delta (|\bar{H}_{q+1}(\pi)| - |\bar{H}_q(\pi)| - 1)}$$

It can easily be seen that the recursion in strategy Σ is simply the limit version of the recursion given by (i) and (ii) in the proof of necessity. Then, arguing as in Claim (a), for each round q of this recursion; we see that for δ values very close to 1, Σ is SSPE. Therefore, we can find a $\delta' \in (\bar{\delta}, 1)$ such that $\forall \delta \in (\delta', 1)$, Σ is SSPE. Then from conditions (1) and (2) in the statement of the theorem it follows that; when $p(N) = 1$, strategy Σ will lead to formation of an r -ring. Thus the sufficiency is established. \square

Proof of Corollary 2: If $n/(n-k) \geq v_1/v_{k+1}$ for all $k = 1, 2, \dots, n-1$, then $v_1/n \geq (v_1 - v_{k+1})/k$ for all $k = 1, 2, \dots, n-1$. Therefore, $AV(1, n+1) \geq AV(1, k)$ for all $k = 1, 2, \dots, n-1$ implying that the grand coalition $\{N\}$ is an average worth maximizing coalition in the set $\{\{1\}, \{1, 2\}, \dots, \{N\}\}$ and the result follows. \square

Proof of Lemma 2: Observe that at any stage T^i , the sub-game becomes equivalent to a single unit auction where the only winner is Bidder i . This is because the worth of any subset of T^i , irrespective of the substructure formed amongst the bidders who have departed from the game, is given by the $w^i(\cdot)$ function. Hence, using Lemma 1 we infer that for all $i \in K$, if $X \in C_i^\delta(T^i)$ and $X \neq \{i\}$, then $X \setminus \{i\} \in S(L)$, for all $\delta \in (0, 1)$ where the collection $S(L) = \{k+1, k+1, k+2, \dots, k+1, k+2, \dots, n\}$.

Now, from the continuity of the objective functions in the maximization programs of (i) and (ii) in the necessity proof of Theorem 1, it follows that for δ sufficiently close to 1, any Bidder $i \in K$ proposes acceptably the average worth maximizing coalition (containing i) at stage T^i . So, for δ sufficiently close to 1, $\{k+1, \dots, m\} \cup \{i\} \in C_i^\delta(T^i)$ implies that $\{k+1, \dots, m\} \cup \{i\}$ is the average worth maximizing coalition among all subsets of T^i . Therefore, when $v_j < v_i$, it easily follows that: $(v_i - v_{m+1})/(1 + |\{k+1, \dots, m\}|) \geq (v_i - v_{m-l+1})/(1 + |\{k+1, \dots, m-l\}|)$ for all $l = 0, 1, \dots, m-k-1$ implies $(v_j - v_{m+1})/(1 + |\{k+1, \dots, m\}|) \geq (v_j - v_{m-l+1})/(1 + |\{k+1, \dots, m-l\}|)$ for all $l = 0, 1, \dots, m-k-1$. Now, for suitably high δ , any $j \in \{i+1, \dots, k\}$ must also choose the average worth maximizing coalition containing j among the subsets of T^j . Hence, it follows that; for a sufficiently high δ (that is, there exists some $\delta' \in (0, 1)$ such that for all $\delta \in (\delta', 1)$), there exists an $m' \geq m$ with $\{k+1, \dots, m'\} \cup \{j\} \in C_j^\delta(T^j)$. \square

Proof of Lemma 3: Note that if $n/(n-k) \geq v_1/v_{k+1}$ for all $k = 1, 2, \dots, n-1$, then $\{k+1, \dots, n\} \cup \{1\} = T^1 \in C_1^*(T^1)$. Since $v_1 > v_i$, from the arguments of Lemma 2 it follows that, for all $i \in K \setminus \{1\}$, $w^i(\{k+1, \dots, n\} \cup \{i\})/(1 + |\{k+1, \dots, n\}|) \geq w^i(S \cup \{i\})/(1 + |S|)$ for all $S \in S(L)$ and for all $i \in K \setminus \{1\}$. For any $S \in S(L) \setminus \{k+1, \dots, n\}$ and any $i \in K \setminus \{1\}$, define the function $d_i(S, \delta) = [w^i(\{k+1, \dots, n\} \cup \{i\})/(1 + \delta|\{k+1, \dots, n\}|)] - [w^i(S \cup \{i\})/(1 + \delta|S|)]$. By applying proof by contradiction it follows that for any $i \in K \setminus \{1\}$, $d_i(S, \delta) > 0$ for all $\delta \in (0, 1)$ and all $S \in S(L) \setminus \{k+1, \dots, n\}$. Hence $\{k+1, \dots, n\} = \operatorname{argmax}_{S \in S(L)} w^i(S \cup \{i\})/(1 + \delta|S|)$ for any $i \in K \setminus \{1\}$ and hence, by Theorem 1, the result follows. \square

Proof of Theorem 2: *Proof of Theorem 2 (i):* To prove the statement 2(i), we first design the following recursion (*). Then we use this recursion to identify a winner $k^*(T; p)$ for any $T \subseteq K$ with $|T| > 2$ and any protocol function p , and finally establish Theorem 2 (i). Recall that \bar{m}^S (as \underline{m}^S) is defined as the bidder with the highest (the lowest) valuation in any set $S \subseteq N$.

We now define the recursion (*) that is used to optimize the proposal decision of any loser at any stage $T \cup L$ such that $T \subseteq K$. The recursion, essentially, generates the final coalition structure (for a given

protocol function) subject to the choice of a set of winners (one winner from each possible set T' where $T' \subseteq T \subseteq K$). This choice is done under the assumption that at each such stage $T' \cup L$ with $|T'| > 2$; if any winner $j \in T'$ gets to propose, she must propose $\{\bar{m}^{T'}\}$ if $j = \bar{m}^{T'}$ and $\{\bar{m}^{T'}, j\}$ otherwise.

Recursion (*): For any $T \subseteq K$ with $|T| > 2$, define $\mathbf{b}(T; p(\cdot)) \equiv \{b(T'; p(\cdot))\}_{T' \subseteq T}$ to be a sequence of members of T such that (i) $b(T'; p(\cdot)) = \underline{m}^{T'}$ if $|T'| = 2$ and (ii) $b(T'; p(\cdot)) \in T'$ if $|T'| \neq 2$. To simplify the notations, henceforth we ignore the argument for the protocol function when writing the $\mathbf{b}(\cdot)$ expression.

For any such $\mathbf{b}(T; p)$ define the sequence of sets $\{B_t\}_{t=1}^h$ such that

- $\{B_t\}_{t=1}^h$ is a partition of $T \cup L$.
- $B_1 = \begin{cases} \{\bar{m}^T\} & \text{if } b(T; p) = \bar{m}^T \\ \{\bar{m}^T, b(T; p)\} & \text{otherwise} \end{cases}$
- Suppose the sequence of sets $(B_1, B_2, \dots, B_{q-1})$ is well defined. Then, define $D_1 := T \cup L$ and $D_q := [T \cup L] \setminus [\cup_{t=1}^{q-1} B_t]$, $\forall q > 1$.

$$B_q = \begin{cases} D_q & \text{if } |D_q \cap K| = 1 \\ \begin{cases} \{\underline{m}^{D_q \cap K}\} & \text{if } p(D_q) \neq \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}\} & \text{otherwise} \end{cases} & \text{if } |D_q \cap K| = 2 \\ \begin{cases} \{\bar{m}^{D_q \cap K}\} & \text{if } p(D_q) = \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}\} & \text{if } p(D_q) \in L \text{ and } b(D_q \cap K) = \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}, b(D_q \cap K; p)\} & \text{if } p(D_q) \in L \text{ and } b(D_q \cap K; p) \neq \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}, p(D_q)\} & \text{otherwise} \end{cases} & \text{if } |D_q \cap K| > 2 \end{cases} \quad (9)$$

- $B_h = T^{j(\mathbf{b}(T; p))}$ for some $j(\mathbf{b}(T; p)) \in T$.

The last term of the recursion B_h should be a set of all the losers and any one winner j from T . The identity of this winner would depend on the choice of the sequence $\mathbf{b}(T; p)$. That is, for any choice of sequence $\mathbf{b}(T; p)$ we would get a $j(\mathbf{b}(T; p)) \in T$ such that $B_h = T^{j(\mathbf{b}(T; p))}$. Define $\mathbf{b}^*(T; p)$ to be that sequence of winners that maximizes the (valuation of) winner $j(\mathbf{b}(T; p))$ and let $k^*(T; p) := b^*(T; p)$, that is, the winner in sequence $\mathbf{b}^*(T; p)$ corresponding to the set T .¹⁷

Note that if $n/(n-l) \geq v_1/v_{l+1}$ for all $l = 1, 2, \dots, n-1$, then $T^1 \in C_1^*(T^1)$. We now state the following Claim that will help prove the result.

Claim 1: For any $G = (N, \bar{w}, p, \delta)$, if $T^1 \in C_1^*(T^1)$, then there exists $\delta' \in (0, 1)$ such that for all $\delta \in (\delta', 1)$, the SSPE strategies of G are such that $\forall T \subseteq K$ we have the following:

1. if $|T| = 1$, then $C_t^\delta(T \cup L) = T \cup L$ for all $t \in T \cup L$, and all agents in $T \cup L$ propose acceptably.
2. if $|T| = 2$, then $C_t^\delta(T \cup L) = \{t\}$ if $t \in T$, and all winners in T propose acceptably while all losers in L propose unacceptably to \underline{m}^T .

¹⁷It may so happen that we have multiple $k^*(T; \cdot)$ for a given protocol function. In that case, we choose any one.

3. if $|T| > 2$, then

$$C_t^\delta(T \cup L) = \begin{cases} \{\bar{m}^T, t\} & \text{if } t \in T \setminus \{\bar{m}^T\} \\ \{t\} & \text{if } t = \bar{m}^T \end{cases} \quad (10)$$

and all winners in T propose acceptably while all losers in L propose unacceptably to $k^*(T; p)$ where $k^*(T; p)$ follows from Recursion (*).

Proof of Claim 1: Pick any $i \in K$ and consider the stage T^i . At this stage the only winner i and all the losers are active. From Theorem 1, Lemma 2 and Lemma 3 it follows that if $T^1 \in C_1^*(T^1)$ then $\exists \delta(i) \in (0, 1)$ such that $\forall \delta \in (\delta(i), 1)$; $C_j^\delta(T^i) = T^i$, $\forall j \in T^i$, and all agents in T^i propose acceptably. Define $\delta(1) := \max\{\delta(i)\}_{i \in K}$. Therefore, $\forall \delta \in (\delta(1), 1)$, $C_j^\delta(T^i) = T^i$, $\forall j \in T^i$, $\forall i \in K$; and thus result 1 of this claim follows.

Consider any stage $T' = \{i, j\} \cup L$, for any $i, j \in K$. Pick any $\delta \in (\delta(1), 1)$. Then $C_t^\delta(T^j) = T^j$, $\forall t \in T^j$ and $C_t^\delta(T^i) = \{T^i\}$, $\forall t \in T^i$. W.l.o.g. assume $v_i > v_j$. If i has the proposal power then the first possibility is that she chooses to stay alone, so that in the next stage with T^j bidders, the coalition T^j forms (since $C_l^\delta(T^j) = \{T^j\}$ for all $l \in T^j$) and i gets a payoff of v_i . The remaining possibilities do not give Bidder i any more than $\frac{v_i + v_j}{1 + \delta}$.¹⁸ For all $\delta \in \left(\frac{v_j}{v_i}, 1\right)$, $v_i > \frac{v_i + v_j}{1 + \delta}$ and so Bidder i will find it optimal to stay alone. Hence $\forall \delta \in \left(\max\left\{\delta(1), \frac{v_j}{v_i}\right\}, 1\right)$, Bidder i stays alone (that is, $C_i^\delta(T') = \{i\}$). Pick any $\delta \in \left(\max\left\{\delta(1), \frac{v_j}{v_i}\right\}, 1\right)$. As before, if j has the proposal power and she chooses to stay alone then she gets v_j . Otherwise, knowing that winner i can reject any proposal and get a payoff of δv_i , the best Bidder j can achieve, by proposing some coalition that includes i , is no more than $\frac{v_j + (1 - \delta)v_i}{1 + \delta}$. Also any non-singleton coalition excluding i gives j less than $\frac{v_j + (1 - \delta)v_i}{1 + \delta}$. There is also the possibility that Bidder j proposes $\{i, j\}$ acceptably to get $(1 - \delta)v_i + v_j - 2v_{k+1}$. Note that if $v_j \leq 4v_{k+1}$, then $\exists \bar{\delta} \in (0, 1)$ such that $\forall \delta \in (\bar{\delta}, 1)$, $\frac{v_j + (1 - \delta)v_i}{1 + \delta} > (1 - \delta)v_i + v_j - 2v_{k+1}$. If $v_j > 4v_{k+1}$, then $\exists \underline{\delta} \in (0, 1)$ such that $\forall \delta \in (\underline{\delta}, 1)$, $\frac{v_j + (1 - \delta)v_i}{1 + \delta} < (1 - \delta)v_i + v_j - 2v_{k+1}$. Let $\tilde{\delta} := \max\left\{\delta(1), \bar{\delta}, \underline{\delta}, \frac{v_i - 2v_{k+1}}{v_i}, \frac{v_j}{v_i}, \frac{v_i}{v_i + v_j}\right\}$. Therefore $\forall \delta \in (\tilde{\delta}, 1)$, $C_t(T') = \{t\}$, $\forall t = i, j$. Thus none the winners in T' propose unacceptably at the stage $T' \cup L$.

We now consider the possible proposals of any loser for $\delta \in (\tilde{\delta}, 1)$. If any loser $l \in L$ has the proposal power, then she has two choices, (i) to make an acceptable proposal and (ii) to make an unacceptable proposal. If she chooses the former, then $C_l^\delta(T') = \{i \cup j \cup S_k(t)\}$ for some $t = l, \dots, n$. This is because (i) leaving any loser with greater valuation than l out of $C_l^\delta(T')$ makes the marginal contribution of l to coalition $C_l^\delta(T')$ amount to 0; and (ii) for any $t \geq l$, it is better to include both winners along with the set of losers $\{k + 1, \dots, t\}$ in $C_l^\delta(T')$ than including only one winner. Further, note that for each $t \in \{l, l + 1, \dots, n\}$, the loser can attain a payoff of $\frac{(1 - \delta)(v_i + v_j) - 2v_{t+1}}{1 + \delta(t - k - 1)}$. If $\delta \in \left(1 - \frac{2v_n}{v_i + v_j}, 1\right)$, then the

¹⁸Note that this payoff is the outcome of two member bargaining over $v_i + v_j$. Such a payoff will never materialize if both winners form a coalition and exit the game (because the losers have not colluded with any winner and so will bid their true valuations leading to a price equal to the third highest valuation).

maximum attainable payoff (and also the only positive payoff) is $\frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$, resulting from a proposal $T' = \{i, j\} \cup L$. If Bidder l makes an unacceptable proposal, it may either be directed at a winner or a loser. If it is directed at a winner, the winner (say i) would get the proposer power in the next period and given our restriction on δ , would stay alone and exit the game. This would drive the game to the stage T^j where, as mentioned above, the coalition T^j would form giving l a payoff $\frac{\delta v_j}{1+\delta(n-k)}$ in the next period. Observe that given $v_i > v_j$, the loser will never unacceptably propose to i , because she could do better by unacceptably proposing to j and getting a payoff $\frac{\delta v_i}{1+\delta(n-k)}$ in the next period. If the unacceptable proposal is directed at a loser l' , the stage of the game would not change, there would be a period of delay, and in the next period the proposal power would be with loser l' who faces the same options as l with one period delay. Therefore, unacceptable proposal directed to a loser is suboptimal. Thus given $\delta \in \left(1 - \frac{2v_n}{v_i+v_j}, 1\right)$, loser l has two options: either propose T' acceptably and get a payoff $\frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$ or propose unacceptably to j and get one-period discounted payoff $\frac{\delta^2 v_i}{1+\delta(n-k)}$. Define $F(\delta) := \frac{\delta^2 v_i}{1+\delta(n-k)} - \frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$. Note that $F(\delta)$ is strictly increasing and continuous in δ and $\lim_{\delta \rightarrow 1} F(\delta) = \frac{v_i}{n-k+1} > 0$. Therefore, $\exists \bar{\delta} \in \left(1 - \frac{2v_n}{v_i+v_j}, 1\right)$ such that $\forall \delta \in (\bar{\delta}, 1); F(\delta) > 0$, and hence, given the restriction on δ , making unacceptable proposal strictly dominates making acceptable proposal for the loser l at the stage $T' \cup L$. Define $\delta(i, j) := \max\left\{\tilde{\delta}, 1 - \frac{2v_n}{v_i+v_j}, \bar{\delta}\right\}$. Therefore, $\forall \delta \in (\delta(i, j), 1)$, $C_t^\delta(T') = \{t\}$, $\forall t = i, j$ and any loser proposing at stage T' unacceptably proposes to j (the lower valuation winner). Hence, for all $\delta \in (\delta(2), 1)$ condition 2 of this claim follows where $\delta(2) := \max\{\delta(i, j)\}_{i, j \in N, i \neq j}$.

Consider the stage in the game where the set of active agents is $T'' = \{i, j, k\} \cup L$ for any $i, j, k \in K$. W. l. o. g. suppose that $v_i > v_j > v_k$. Fix any $\delta \in (\delta(2), 1)$. It can easily be seen from the discussion of proposal decisions of agents in stages where two or three winners are active along with all the losers that; (i) the minimum payoff that i can get by proposing $\{i\}$ is δv_i and (ii) the maximum payoff that i can get by proposing acceptably to any other agent in T'' is $\frac{v_i+v_j}{1+\delta}$. Clearly, there exists a $\delta_3(i) \in (\delta(2), 1)$ such that for all $\delta \in (\delta_3(i), 1)$, the best acceptable proposal of i at stage T'' , $C_i^\delta(T'') = \{i\}$.

Fix any $\delta \in (\delta_3(i), 1)$ and consider the agent j . Agent j gets the payoff v_j by staying alone and gets the payoff $v_j + (1-\delta)v_i$ by making the acceptable proposal $\{i, j\}$. Any other collusive venture by j will give at most the payoff $\frac{(1-\delta)v_i+v_j+v_k}{1+\delta}$. Note that the expression $\left\{v_j + (1-\delta)v_i - \frac{(1-\delta)v_i+v_j+v_k}{1+\delta}\right\}$ is continuous in δ and has positive the limit as $\delta \rightarrow 1$. Therefore, there exists $\delta_3(j) \in (0, 1)$ such that $\forall \delta \in (\delta_3(j), 1)$, $v_j + (1-\delta)v_i > \frac{(1-\delta)v_i+v_j+v_k}{1+\delta}$ implying that $C_j^\delta(T'') = \{i, j\}$. In the same manner, we can generate a $\delta_3(k) \in (\delta_3(j), 1)$ such that $\forall \delta \in (\delta_3(k), 1)$, $C_k^\delta = \{i, k\}$. Thus for all $\delta \in (\delta_3(k), 1)$, $a_t^\delta(T'') = \delta[v_t + (1-\delta)v_i]$ for both $t = j, k$. Note that $\forall \delta \in \left(\max\{\delta_3(k), \frac{v_j}{v_i}\}, 1\right)$, $a_t^\delta(T'') > v_t$ for both $t = j, k$. This means that for these values of δ , the acceptance threshold of any agent $t \in \{j, k\}$ exceeds the maximum possible marginal contribution that t can make to any coalition containing t (which is t 's own valuation v_t). Therefore, $\forall \delta \in (\delta_3, 1)$ where $\delta_3 = \max\{\delta_3(k), \frac{v_j}{v_i}\}$: (i) $C_i^\delta(T'') = \{i\}$, $C_j^\delta(T'') = \{i, j\}$, $C_k^\delta(T'') = \{i, k\}$, and (ii) *no acceptable proposal can be directed at any $t = j, k$ in equilibrium.*

Fix a $\delta \in (\delta_3, 1)$. We now show that at stage T'' , no winner would make an unacceptable proposal. Recall that, given stationarity, an unacceptable proposal is beneficial only if the new proposer (the rejecting agent) makes an acceptable proposal. Consider the agent i who could make an unacceptable proposal either at another winner $t \in \{j, k\}$ or at any loser $l \in L$. In the former case, any acceptable proposal by $t \in \{j, k\}$ must contain i (since $C_t^\delta(T'') = \{i, t\}$ for both $t = j, k$). Consider the other possibility where i makes an unacceptable proposal directed some $l \in L$. Clearly, any acceptable proposal by l upon getting the proposal power (since the rejector proposes) must include i (since, as shown in the previous paragraph, no acceptable proposal can be directed j and k). Thus in both cases, an unacceptable proposal by i leads to an acceptable proposal being made to i at stage T'' . But then i could have made the same acceptable proposal in the first place at stage T'' ; and thus, could have saved a period of delay. Thus i would never make an unacceptable proposal. Now, consider the possibility that any $t = j, k$ makes an unacceptable proposal. As before, such a proposal is optimal only if it leads to the rejector making an acceptable proposal. If such an unacceptable proposal is directed at i , she would use the proposer power to choose to stay alone. If the unacceptable proposal is directed at a loser, for $\delta > \frac{v_i - v_n}{v_i}$, the only acceptable proposal giving positive payoff to any such loser is $\{i, k+1, k+2, \dots, n\}$ (recall that no acceptable proposal can include either j or k). In the former case, the game moves to a stage where the set active agents consists of two winners and all losers, where the maximum payoff that can be obtained is v_t . In the latter case too, the payoff obtained is v_t . Thus for all $\delta \in \left(\max\left\{\delta_3, \frac{v_i - v_n}{v_i}\right\}, 1\right)$, no winner makes an unacceptable proposal at stage T'' .

Now consider the proposal decision of any loser $l \in L$. Fix a $\delta > \max\left\{\delta_3, \frac{v_i - v_n}{v_i}\right\}$. Then, as argued above, $C_l^\delta(T'') = \{i, k+1, \dots, n\}$ and so l gets maximum payoff $\frac{(1-\delta)v_i}{1+(n-k-1)\delta}$. On the contrary, if l makes an unacceptable proposal to any $t \neq i$, she acceptably propose $\{i, t\}$ and leave the game. As a result, the game would move to a stage with single winner and all losers, where as shown in the first paragraph of the proof, all agents would collude giving l a minimum payoff of $\frac{\delta v_k}{1+(n-k)\delta}$. Since, the difference $\frac{\delta v_k}{1+(n-k)\delta} - \frac{(1-\delta)v_i}{1+(n-k-1)\delta}$ is continuous in δ and positive in the limit as $\delta \rightarrow 1$, there exists a $\delta(3) \in \left(\max\left\{\delta_3, \frac{v_i - v_n}{v_i}\right\}, 1\right)$ such that $\forall \delta \in (\delta(3), 1)$, any loser with proposer power in stage T'' makes an unacceptable proposal directed at some other winner t . The exact identity of t can be obtained from the Recursion (*).

Suppose that at the stage $T''' \cup L$ with $T''' \subset K$ and $3 \leq |T'''| \leq m-1$ result (3) holds $\forall \delta \in (\delta^{m-1}, 1)$. Consider the stage $T \cup L$ where $|T| = m$. Define the winners $\{j_t\}_{t=1}^m$ in T , where $j_1 = \bar{m}^T$ and $j_t = \bar{m}^T \setminus \{j_1, j_2, \dots, j_{t-1}\}$. Fix a $\delta \in (\max\{\delta(3), \delta^{m-1}\}, 1)$. The following STEPS 1 and 2 describe the proposal choice of j_1 and the winners other than j_1 , respectively; when they propose acceptably at stage $T \cup L$. STEP 3 establishes that no winner in T proposes unacceptably at stage $T \cup L$. Finally, STEP 4 describes the proposal choice of the losers at stage $T \cup L$.

STEP 1: Pick the Bidder $j_1 \in T$ ($j_1 = \bar{m}^T$). Now, by staying alone j_1 can get at least $\delta^{m-2}v_{j_1}$. This is because, from our hypothesis (and the specified range of δ) it follows that at all the later stages

(consequent to j_1 staying alone) other than the single winner stage; *only* the winners make acceptable proposals, and all these acceptable proposals are either directed at themselves (that is, they stay alone) or at exactly one active winner (that is, forming a two bidder coalition). This implies that after j_1 has stayed alone, the game must arrive at a single winner stage. From Theorem 1 and Lemma 2 and Lemma 3; it follows that given $T^1 \in C_1^*(T^1)$, all the active bidders in this single winner stage collude amongst themselves (irrespective of the identity of that single winner) and the game ends. Therefore, the final coalition structure yields j_1 a payoff of v_{j_1} .¹⁹ Given our hypothesis, delay can occur along this path if (and only if) at some intermediate stage, an active loser gets to propose. There can be at most $m - 2$ such stages; and so staying alone yields j_1 at least $\delta^{m-2}v_{j_1}$.

The maximum that Bidder j_1 can get by colluding with any other active bidder is given by the number $\max \left\{ \frac{v_{j_1} + v_{j_2}}{1 + \delta}, \dots, \frac{\sum_{t=1}^{m-1} v_{j_t}}{1 + (m-2)\delta} \right\}$.²⁰ For any $t' = 2, \dots, m-1$, the difference $\left[\delta^{m-2}v_{j_1} - \frac{\sum_{t=1}^{t'} v_{j_t}}{1 + (t'-1)\delta} \right]$ is continuous and strictly increasing in δ with the $\delta \rightarrow 1$ limit being positive. Therefore, for δ sufficiently close to 1, this difference is positive. Thus $\exists \delta_1 \in (\max\{\delta(3), \delta^{m-1}\}, 1)$ such that $\delta^{m-2}v_{j_1} > \max \left\{ \frac{\sum_{t=1}^{t'} v_{j_t}}{1 + (t'-1)\delta} \right\}_{t'=2}^{m-1} \forall \delta \in (\delta_1, 1)$. Hence, $C_{j_1}^\delta(T \cup L) = \{j_1\}$, $\forall \delta \in (\delta_1, 1)$, if agent j_1 proposes acceptably at the stage $T \cup L$.

STEP 2: Fix a $\delta \in (\delta_1, 1)$ and consider the bidder j_2 . For such a δ , our hypothesis implies that if j_2 stays alone, then the maximum payoff (attained if no delay occurs in the intermediate stages) she can get is v_{j_2} and the minimum payoff (attained if there is delay at each of the intermediate stages) that she can get is $\delta^{m-3}v_{j_2}$. If j_2 acceptably proposes $\{j_1, j_2\}$, she gets at least, $\delta^{m-3}[v_{j_2} + (1 - \delta)v_{j_1}]$. Any other collusive venture gives j_2 a maximum possible payoff of $\max \left\{ \frac{(1-\delta)v_{j_1} + \sum_{t=2}^{t'} v_{j_t}}{1 + (t'-2)\delta} \right\}_{t'=3}^{m-1}$. Like in the previous case, there exists a $\bar{\delta}_2 \in (\delta_1, 1)$ such that $\forall \delta \in (\bar{\delta}_2, 1)$, $\delta^{m-3}[v_{j_2} + (1 - \delta)v_{j_1}] > \max \left\{ v_{j_2}, \max \left\{ \frac{(1-\delta)v_{j_1} + \sum_{t=2}^{t'} v_{j_t}}{1 + (t'-2)\delta} \right\}_{t'=3}^{m-1} \right\}$. The reason is the following. For any $t' = 3, \dots, m-1$, the difference $\delta^{m-3}v_{j_2} - \frac{(1-\delta)v_{j_1} + \sum_{t=2}^{t'} v_{j_t}}{1 + (t'-2)\delta}$ is continuous and strictly increasing in δ with a positive value in the limit (tends to 1). Therefore for δ sufficiently close to 1, the difference is always positive. Hence for δ sufficiently close to 1, the difference $\left[\delta^{m-3}v_{j_2} - \max \left\{ \frac{(1-\delta)v_{j_1} + \sum_{t=2}^{t'} v_{j_t}}{1 + (t'-2)\delta} \right\}_{t'=3}^{m-1} \right]$ is positive and so staying alone strictly dominates formation of any coalition other than $\{j_1, j_2\}$. However, the difference $\delta^{m-3}[v_{j_2} + (1 - \delta)v_{j_1}] - v_{j_2} = (1 - \delta) [\delta^{m-3}v_{j_1} - (1 + \delta + \delta^2 + \dots + \delta^{m-4})v_{j_2}]$ is positive if and only if $H(\delta) > \frac{v_{j_2}}{v_{j_1}}$ where $H(\delta) := \frac{\delta^{m-3}}{1 + \delta + \delta^2 + \dots + \delta^{m-4}}$. Since $\frac{v_{j_2}}{v_{j_1}} \in (0, 1)$ and $H(\delta)$ is a strictly increasing function of δ , once again the for sufficiently high δ , the difference $\delta^{m-3}[v_{j_2} + (1 - \delta)v_{j_1}] - v_{j_2}$ is positive. Thus a $\bar{\delta}_2$ can indeed be found. That is, $C_{j_2}^\delta(T \cup L) = \{j_1, j_2\}$ if j_2 proposes acceptably at the stage $T \cup L$, and $a_{j_2}^\delta(T \cup L) \geq \delta^{m-2}[v_{j_2} + (1 - \delta)v_{j_1}]$.²¹

¹⁹This follows from our worth of partition function; where any singleton (winner) member of a partition gets her valuation as payoff, if that partition contains another member set where all the losers collude with one or more winners.

²⁰Bidder j_1 attains the payoff of $\frac{v_{j_1} + \dots + v_{j_t}}{1 + (t-1)\delta}$, for any $t < m$; when j_1 acceptably proposes $\{j_1, j_2, \dots, j_t\}$ at this stage and the remaining winners (or winner) colludes with all the losers in the next stage.

²¹Recall that j_2 can always reject a proposal, incur a period of delay, and then acceptably propose $\{j_1, j_2\}$.

Note that we can also find a $\delta_2 \in (\bar{\delta}_2, 1)$ such that $\forall \delta \in (\delta_2, 1)$, $\delta^{m-2}[v_{j_2} + (1 - \delta)v_{j_1}] > v_{j_2}$. This means that for this range of discount factor; any acceptable proposal directed at j_2 must give him at least a payoff greater than v_{j_2} (which is the maximum possible marginal contribution that j_2 can make to any coalition containing it). This is always suboptimal and therefore, for this range of δ , no acceptable proposal directed at j_2 is ever made in equilibrium. In this manner we can generate a sequence $\{\delta_t\}_{t=3}^m$ such that $\delta_3 \in (\delta_2, 1)$ and $\delta_{t+1} \in (\delta_t, 1)$, $\forall t$ with the property that if all active winners in stage $T \cup L$ propose acceptably, then (i) $C_t^\delta(T \cup L) = \{j_1, j_t\}$, $\forall \delta \in (\delta_t, 1)$, $\forall t \geq 3$; and (ii) no acceptable proposal will be directed to any member in $\{j_2, \dots, j_t\}$ in equilibrium (since their acceptance thresholds exceed their valuation which is the maximum marginal contribution that they can make to any coalition containing them), if the discount factor exceeds $\bar{\delta}_2$.²²

STEP 3: Fix a $\delta \in (\delta_m, 1)$. In this step, we show that every winner in T proposes acceptably at the stage $T \cup L$. Consider any $j \in T \setminus \{j_1\}$ and suppose that j makes an unacceptable proposal. This is optimal only if, this leads to transfer of proposal power to some other active Bidder $j' \in T \cup L$ who makes an acceptable proposal $S^{j'}$ excluding j .

If $j' \in L$, then from (ii) in STEP 2, $S^{j'} \in \{\{j_1\} \cup S_k(t)\}_{t=j'}^n$.²³ For any $t \in \{j', \dots, n\}$, the acceptable proposal $[\{j_1\} \cup S_k(t)]$ gives j' a maximum possible payoff $\frac{(1-\delta)v_{j_1} - v_{t+1}}{1+(t-k-1)\delta}$. Observe that if $\delta > \frac{v_{j_1} - v_n}{v_{j_1}}$, then only acceptable proposal giving j' a positive (and hence, maximum possible) payoff $\frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is $[\{j_1\} \cup S_k(n)]$. Then $\forall \delta \in \left(\max\left\{\delta_m, \frac{v_{j_1} - v_n}{v_{j_1}}\right\}, 1\right)$, $C_{j'}^\delta(T \cup L) = [\{j_1\} \cup S_k(n)]$ for all $j' \in L$. But then, if loser j' does acceptably propose $[\{j_1\} \cup S_k(n)]$, then the game goes to stage $T \setminus \{j_1\}$, where the maximum possible payoff that j (the original winner in $T \setminus \{j_1\}$, who made the unacceptable proposal directed at the loser j') can get is $\delta[v_j + (1 - \delta)v_{\bar{m}^{T \setminus \{j_1\}}}]$.²⁴ It can be easily seen that $\exists \delta'_m \in \left(\max\left\{\delta_m, \frac{v_{j_1} - v_n}{v_{j_1}}\right\}, 1\right)$ such that $\forall \delta \in (\delta'_m, 1)$, $\delta^{m-2}[v_{j_2} + (1 - \delta)v_{j_1}] > \delta[v_j + (1 - \delta)v_{\bar{m}^{T \setminus \{j_1\}}}]$ (since $v_{j_1} > \bar{m}^{T \setminus \{j_1\}}$). Therefore, making an acceptable proposal dominates making an unacceptable proposal directed at a loser in L ; for any winner $j \in T \setminus \{j_1\}$ at the stage $T \cup L$.

Fix any $\delta \in (\delta'_m, 1)$ and consider the other possibility that agent $j \neq j_1$ makes an unacceptable proposal to a fellow active winner $j' \in T \setminus \{j\}$ (who in turn makes an acceptable proposal $S^{j'}$). If $j' \in T \setminus \{j\}$, then either $j' = j_1$ or $j' \neq j_1$. If $j' = j_1$, then from STEP 1, $S^{j'} = C_{j_1}^\delta(T \cup L) = \{j_1\}$, and so the game goes to the stage $[T \setminus \{j_1\}] \cup L$ with $m-1$ winners. Then from the induction hypothesis, j must propose acceptably in the stage $[T \setminus \{j_1\}] \cup L$, and this acceptable proposal is given by $C_j^\delta([T \setminus \{j_1\}] \cup L) = \{\bar{m}^{T \setminus \{j_1\}}, j\}$. Recall that for the given the range of δ , from STEP 2, the best proposal for agent j (in case she chooses to propose acceptably in stage $T \cup L$) is given by $C_j^\delta(T \cup L) = \{j_1, j\}$. Therefore, the payoff to j from proposing $\{j_1, j\}$ acceptably at stage $T \cup L$ exceeds that from proposing $\{\bar{m}^{T \setminus \{j_1\}}, j\}$ acceptably at stage

²²This is because the expression $\delta^{m-2}[x + (1 - \delta)v_{j_1}] - x$ is decreasing in x .

²³Recall that $S_k(t) := \{k+1, k+2, \dots, t\}$.

²⁴As in STEP 2, we can show that it is suboptimal for j to acceptably propose to any other winner in $T \setminus \{j_1, \bar{m}^{T \setminus \{j_1\}}\}$, at stage $T \setminus \{j_1\}$.

$T \cup L$ (which, in turn, is weakly greater than doing the same at stage $[T \setminus \{j_1\}] \cup L$). Therefore, making an acceptable proposal dominates making an unacceptable proposal directed at the highest valuation winner j_1 ; for any agent $j \in T \setminus \{j_1\}$ in stage $T \cup L$. Consider the other possibility where an agent $j \in T \setminus \{j_1\}$ makes an unacceptable proposal directed at some fellow winner $j' \neq j_1$. From STEP 2, $S^{j'} = C_{j'}^\delta(T \cup L) = \{j_1, j'\}$ and so, the game proceeds to the next stage $[T \setminus \{j_1, j'\}] \cup L$ with $m-2$ winners. Then, from induction hypothesis, $C_j^\delta([T \setminus \{j_1, j'\}] \cup L) = \{\bar{m}^{T \setminus \{j_1, j'\} \cup L}, j\}$. Given the range of δ , from STEP 2, $C_j^\delta(T \cup L) = \{j_1, j\}$; and so, arguing as before, for any winner $j \in T \setminus \{j_1\}$, making an acceptable proposal dominates making an unacceptable proposal directed to any other winner in $T \setminus \{j_1, j\}$ at stage $T \cup L$.

Finally, consider the possibility that $j = j_1$. Then, $j' \neq j_1$. Therefore, $\forall \delta \in (\delta'_m, 1)$, as mentioned before, $j' \in L \Rightarrow C_l^\delta(T \cup L) = [\{j_1\} \cup S_k(n)]$ and $j' \in T \setminus \{j_1\} \Rightarrow C_{j'}^\delta(T \cup L) = \{j_1, j'\}$. But for both these cases, j_1 could have proposed the same coalition acceptably, in the first place; thereby saving a period of delay (and getting the (higher) proposer's share out of the worth of $[\{j_1\} \cup S_k(n)]$, in case of $j' \in L$). Hence, proposing unacceptably turns out to be sub-optimal for j_1 at stage $T \cup L$. Therefore, $\forall \delta \in (\delta'_m, 1)$, no winner in T makes an unacceptable proposal at stage $T \cup L$.

STEP 4: If any loser l proposes acceptably at stage $T \cup L$, then, from STEP 3, $\forall \delta \in (\delta'_m, 1)$, $C_l^\delta(T \cup L) = [\{j_1\} \cup S_k(n)]$ and l gets a maximum possible payoff of $\frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is $[\{j_1\} \cup S_k(n)]$. On the other hand, like in the two winner stage $\{i, j\} \cup L$, given the specified range of δ and our induction hypothesis, an unacceptable proposal by l to some winner, yields at least $\frac{\delta v_{jm}}{1+(n-k)\delta}$ in the final single winner stage, at most $m-1$ periods later. That is, the least l gets by making an unacceptable proposal when $\delta \in (\delta'_m, 1)$ is $\frac{\delta^m v_{jm}}{1+(n-k)\delta}$. The difference $\frac{\delta^m v_{jm}}{1+(n-k)\delta} - \frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is continuous and strictly increasing in δ and this difference is positive in the limit. Therefore, $\exists \underline{\delta} \in (\delta'_m, 1)$ such that for all $\delta \in (\underline{\delta}, 1)$ the difference is positive, that is, making unacceptable proposal is the optimal action. The particular identity of the winner in T to whom any l must unacceptably propose is given Recursion (*).

Define $\delta^m := \max \left\{ \delta'_m, \frac{v_{j_1} - v_n}{v_{j_1}}, \underline{\delta} \right\}$. Then, $\forall \delta \in (\delta^m, 1)$; at the stage $T \cup L$ such that $|T| = m$, all losers make an unacceptable proposal at some active winner (given by Recursion (*)), all winners make acceptable proposals and $C_t^\delta(T \cup L) = \{j_1, t\} = \{\bar{m}^T, t\}$, $\forall t \neq j_1 = \bar{m}^T$ with $C_{\bar{m}^T}^\delta(T \cup L) = \{\bar{m}^T\}$. We can continue such a recursion to get a sequence of $\{\delta^m\}_{m=4}^n$ such that result (3) follows by simply choosing $\delta^m\}_{m=4}^n$. This completes the proof of Claim 1.

Claim 1 implies that for sufficiently high values of δ , if $n/(n-k) \geq v_1/v_{k+1}$ for all $k = 1, 2, \dots, n-1$, then the SSPE outcome ring structure π^* , irrespective of the protocol function, must be such that (a) all losers collude with any one *weak winner*, and (b) all other winners either stay alone or form pairs. That is, $\pi^* = \{T^j, \sigma_{-j}\}$ for some $j \in \bar{K}$ and some $\sigma_{-j} \in \Pi'(K \setminus \{j\})$. To complete the proof of Theorem 2 (i) we have to show that $\pi^* = \{T^j, \sigma_{-j}\}$ must also satisfy the property that $|\{t \in K \setminus \{j\} : \{t\} \in \sigma_{-j}, v_t < v_j\}| \in \{0, 1\}$. Suppose not. That is, there exists a set of weak winners $\tilde{K} \subset \bar{K}$ such that $|\tilde{K}| \geq 2$ and for

all $j' \in \tilde{K}$, $v_{j'} < v_j$ and $\{j'^*\}$.²⁵ From Claim 1 it follows that coalition T^j forms at stage T^j , that is, the single winner stage (after which the game ends). This means that there exists $j', j'' \in \tilde{K}$ such that either $\{j'\}$ or $\{j''\}$ must have formed at some stage T' such that $\{\{j\} \cup \{j'\} \cup \{j''\}\} \subseteq T'$. In either case, this is in contradiction to the equilibrium proposal decisions in Claim 1 for the specified range of δ . Hence if π^* is an equilibrium coalition structure, then (c) $|\{t \in K \setminus \{j\} \mid \{t\} \in \sigma_{-j}, v_t < v_j\}| \in \{0, 1\}$. From (a), (b) and (c) it follows that the equilibrium coalition structure π^* must be an SWFR partition, that is, $\pi^* \in \bar{\Pi}$.

Proof of Theorem 2 (ii): Fix an arbitrary $\pi^* \in \bar{\Pi}$. W. l. o. g. suppose that $\pi^* = \{T_1, T_2, \dots, T_r\}$ where $T_r = T^{j^{\pi^*}} = [\{j^{\pi^*}\} \cup L]$ for some $j^{\pi^*} \in \tilde{K}$. Therefore, for all $h = 1, \dots, r-1$, $|T_h| \in \{1, 2\}$ and $\cup_{h=1}^{r-1} T_h = K \setminus \{j^{\pi^*}\}$. Further define the ordered set $A(\pi^*) := (a_1, a_2, \dots, a_r)$ such that $a_r := j^{\pi^*}$ and for all $1 \leq h \leq r-1$, $a_h := \underline{m}^{T_h}$. Finally, define the protocol function p^* such that $p(N) = a_1$ and for all $1 < h \leq r-1$, $p(N \setminus \{a_1, \dots, a_h\}) = a_{h+1}$. From Claim 1 it follows that for all $\delta \in (\delta', 1)$, π^* is the unique SSPE outcome ring structure in $G = (N, \bar{w}, p^*, \delta)$. \square

Proof of Proposition 1: The sufficiency of condition 5 follows quite easily. To establish the necessity, consider the member $T^k = \{k, k+1, \dots, n\}$. For T^k to have formed; on the equilibrium path, at some stage \hat{T} (such that $T^k \subseteq \hat{T}$), some member $i \in T^k$ must have acceptably proposed T^k . Now if $|\hat{T} \cap K| \geq 2$ then, given the specified range of δ , irrespective of whether $i = k$ or $i \in L$, we get a contradiction to the equilibrium strategies defined in Theorem 2. Hence $\hat{T} = T^k$.

Now consider the singleton coalition $\{k-1\}$. Since T^k must have formed at the stage T^k itself, $\{k-1\}$ must have formed at a stage \bar{T} such that $\{\{k-1\} \cup T^k\} \subseteq \bar{T}$. Given the range of δ , the only possibility where Bidder $k-1$ would choose to stay alone without contradicting our findings in Claim 1 in Appendix; is when $\bar{T} = \{\{k-1\} \cup T^k\}$. Now if $p(\bar{T}) \in L$, then it must unacceptably propose to the lower value winner k , who would then stays alone. If $p(\bar{T}) = k$, then it is optimal for k to stay alone so that T^{k-1} forms in the next stage. Therefore in either case we have a contradiction. Therefore, $p(\{k-1\} \cup T^k) = k-1 \Rightarrow p(N \setminus \{1, \dots, k-2\}) = k-2+1 = k-1$. Continuing in this manner, for the rest of the singleton coalitions, $\{k-2\}, \{k-3\}, \dots, \{1\}$; the result follows.²⁶ \square

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²⁵Note that if $j \in \tilde{K}$, then $\forall j' \neq j$ with $v_{j'} < v_j$, $j' \in \tilde{K}$.

²⁶Note that $p(T^k) = p(N \setminus \{1, \dots, k-1\})$ is free from any restriction because any bidder in T^k proposes T^k optimally.

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