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Assignment Games with Externalities

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August 27, 2013

Abstract

We introduce externalities into a two-sided, one-to-one assignment game by letting the values generated by pairs depend on the behavior of the other agents. Extending the notion of blocking to this setup is not straightforward; a pair has to take into account the possible reaction of the residual agents to be able to assess the value it could achieve. We define blocking in a rather general way that allows for many behavioral considerations or beliefs. The main result of the paper is that a stable outcome in an assignment game with externalities always exists if and only if all pairs are pessimistic regarding the others' reaction following a deviation. The relationship of stability and optimality is also discussed, as is the structure of the set of stable outcomes.

Keywords: two-sided matching, assignment game, externalities, stability

JEL Classification: C71, C78, D62

1 Introduction

Matching markets have been extensively analyzed ever since the publication of the pioneering paper of Gale and Shapley (1962). Our focus here is on the two-sided matching

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problems, and we refer the reader to Roth and Sotomayor (1992) for a thorough survey. More precisely, we concentrate on assignment games, introduced by Koopmans and Beckmann (1957) and Shapley and Shubik (1971), where matched pairs create some value to distribute among themselves and the agents aim to maximize their payoff.

It is implicitly assumed in the vast majority of the literature that agents' preferences - or values in case of assignment games - are independent of how the other agents are matched. However, in many applications, for instance in case of labor markets, this assumption does not seem to be realistic. On the contrary, it is natural to assume that firms competing in the same market care about which workers are hired by their rival firms. Thus, the preferences and respective profits of firms should depend on the matchings of the rest of the agents.

Analyzing externalities in a matching framework is also important from a theoretical point of view, since it is well known that the core of a cooperative game might be empty in the presence of externalities (Funaki and Yamato, 1999; Kóczy, 2007). In addition, Mumcu and Saglam (2007) show that the core of the housing market with externalities also may be empty. Thus, it is interesting to examine whether one can find a nonempty set of stable outcomes in this environment.

Externalities in matching problems began to receive attention following the wave of papers on cooperative games with externalities. The key aspect of the presence of externalities in a matching problem is that when a pair is considering whether to block a certain matching or not, it has to take into account how the rest of the agents are going to react. These considerations are typically referred to as *residual behavior* in the cooperative game theory literature. Whereas these reactions do not play a role in problems without externalities, different assumptions on residual behavior lead to different solution concepts and different sets of stable outcomes when externalities are present.

In the literature, one can detect a number of ways to model externalities in matching environments. Agents may experience externalities from the way that other agents are matched or from the payoffs of the others. Moreover, externalities may be incorporated by extending the preference profile, by imposing behavioral assumptions, or by enabling agents to have beliefs or expectations about the occurrence of outcomes. We summarize these solutions and their implications below.

Li (1993) was the first to introduce externalities into the one-to-one, two-sided matching market by assuming that each agent has strict preferences over the set of all possible

matchings. He finds that equilibrium may not exist in general, but does if externalities are small enough: more specifically, if an agent's preferences over matchings is lexicographically determined, first and foremost by his partner and then by how the other agents are matched. Similarly, Sasaki and Toda (1996) also find non-existence when expectation about residual behavior is determined endogenously. They show that there always is a stable matching if estimation functions on the set of possible outcomes are exogenously given. They are the first to examine assignment games with externalities and find that a stable matching exists if agents find all matchings to be possible. Taking this approach one step further, Hafalir (2008) introduces endogenous beliefs depending on the preferences. He confirms the anticipation of Sasaki and Toda (1996) that rational expectations do not guarantee existence. He introduces the notion of sophisticated expectations, determined via an algorithm, inducing a game without externalities at the end, and shows that the resulting set of stable matchings is nonempty. To achieve nonemptiness, he assumes that there is no commitment; that is, a blocking pair can split up if they can get better off by blocking again through a different pair. Eriksson, Jansson, and Vetander (2011) consider assignment games where agents experience negative externalities from the payoffs of the agents on the same side of the market in form of ill will. They define a new, stronger notion of stability assuming bounded rationality and show that such stable outcomes always exist.

It is clear from the results summarized above that the introduction of externalities causes many issues which need to be resolved. For instance, it is not unambiguous how to generalize the notion of blocking and how to define stability. Moreover, the set of stable outcomes may generally be empty. In the existing literature there are two main ways to solve this problem: *(i)* put restrictions on agents' preferences, or *(ii)* use a stronger notion of stability. All papers cited above produce only sufficient conditions for existence.

In this paper we introduce externalities into assignment games by allowing the values of matched pairs to depend on how the rest of the agents are matched. We look for stable outcomes in the standard sense: an outcome is *stable* if it is individually rational and has no blocking pairs. However, it is not straightforward though how one should define the notion of blocking in this environment. When a pair is considering blocking a given outcome, in this setup, their future value will depend on how the remaining agents react. The cooperative game theory literature discusses many assumptions that can be made on residual behavior, ranging from the pessimistic approach of Aumann and Peleg (1960) to the optimistic one of Shenoy (1980). Here, we aim to cover as many behavioral assumptions

or beliefs as possible by applying a very *general definition of blocking*. When agents are deciding whether to form a blocking pair, they take the values for all contingencies into account. According to their attitude towards risk or beliefs about the other agents, they calculate a threshold based on the possible outcomes and form a blocking pair whenever this threshold exceeds the sum of their current payoffs. By using this general definition we manage to avoid imposing any initial assumption on beliefs or residual behavior. In turn, we can distinguish different types of agents based on how they determine their threshold.

To facilitate proving the nonemptiness of the set of stable outcomes we introduce an artificial assignment game based on our definition of blocking; it is a transformation of a game with externalities to one without. We then show a relation between the set of stable outcomes in the two games. This finding is still independent of behavioral assumptions; the transformation can be made for any types of agents.

The main result of the paper shows that a stable outcome in an assignment game with externalities *always* exists if and only if all agents are pessimistic. Remember that previous results in the literature only provide sufficient conditions for nonemptiness, while ours is a necessary and sufficient one. It also follows from our proof that the slightest optimism, meaning that any pair puts a positive probability on an optimistic outcome, could lead to nonexistence.

We will repeatedly relate the properties of the assignment game with externalities to the known properties of assignment games without externalities. For instance, it is well known in the case of assignment games without externalities that stability and efficiency go hand in hand, in the sense that the total value generated by the agents is maximized at stable matchings. It is not difficult to see that this relation does not extend to our setup. For instance, if all matched pairs create roughly the same value at the efficient matching, then it may be unstable if some other pair creates very large values at other, less balanced matchings. We also show that stable outcomes may be inefficient. However, if all agents are pessimistic, then we can show that there always exist a *Pareto optimal* stable outcome.

Shapley and Shubik (1971) show that the core of the assignment game, and hence the set of stable outcomes, forms a complete lattice. Examining its generalization, we find that the result does not carry over completely to games with externalities. However, for *a given matching*, we show that the set of corresponding stable payoffs still does form a complete lattice. In addition, we show that if there are multiple stable matchings, then their respective sets of stable payoffs may be disjoint.

The outline of the paper is as follows. We introduce the notation and define stable outcomes in assignment games with externalities in Section 2. We characterize our notion of stability in Section 3; we show existence of stable outcomes in Section 3.1, discuss its relation to optimality in Section 3.2 and the lattice property in Section 3.3. Finally, Section 4 concludes.

2 Preliminaries

2.1 The Assignment Game

We consider a finite set of agents N , consisting of m firms F and $n = m$ workers W .¹ We reserve i to denote a typical firm, j a typical worker. A firm can employ at most one worker and no two can employ the same. A *matching* is a bijection $\mu : N \rightarrow N$ such that for all $(i, j) \in F \times W$, $\mu(i) \neq i \Leftrightarrow \mu(i) \in W$, $\mu(j) \neq j \Leftrightarrow \mu(j) \in F$, $\mu(\mu(i)) = i$ and $\mu(\mu(j)) = j$. Additionally, if $\mu(i) = j$, we say that $(i, j) \in \mu$. The set of matchings for F and W is $M(F, W)$. If i employs j , they generate a value or monetary benefit $\alpha_{ij} \in \mathbb{R}_+$. If i hires no worker or j is unemployed, their values are zero. Let $\alpha = (\alpha_{ij})_{i,j \in N}$ be the collection of values for all pairs of agents.

Definition 2.1. An *assignment game* Γ (without externalities) is completely described by a triplet $\Gamma = (F, W, \alpha)$.

An *outcome* of Γ is a matching and a pair of payoff vectors, $(\mu, u, v) \in M(F, W) \times \mathbb{R}^m \times \mathbb{R}^n$, such that

$$\sum_{i \in F} u_i + \sum_{j \in W} v_j = \sum_{(i,j) \in \mu} \alpha_{ij}.$$

An outcome is *stable* if it is *individually rational*, $u_i \geq 0$ and $v_j \geq 0$, and has no *blocking pairs*, $u_i + v_j \geq \alpha_{ij}$. It can be shown that $u_i + v_j = \alpha_{ij}$ for all $(i, j) \in \mu$ if (μ, u, v) is stable (Shapley and Shubik, 1971).

A matching $\mu \in M(F, W)$ is *efficient* in Γ if, for all matchings $\mu' \in M(F, W)$,

$$\sum_{(i,j) \in \mu} \alpha_{ij} \geq \sum_{(i,j) \in \mu'} \alpha_{ij}.$$

¹That $n = m$ is without loss of generality as we can create "null-agents" to balance the count, i.e. agents k such that $\alpha_{ik} = 0$ and $\alpha_{kj} = 0$ for all i and j .

Shapley and Shubik (1971) show that every efficient matching is compatible with a stable payoff and that every stable outcome is efficient.

Assignment games are examples of cooperative games with transferable utility. Stability in this setup can be captured by the notion of the core, which is the set of those allocations that cannot be blocked by any coalition. It is well known that the set of stable outcomes coincide with the set of core allocations (Shapley and Shubik, 1971).

2.2 Introducing Externalities

We would like to capture the phenomenon that agents' values in the assignment game may well depend on the behavior of other agents. To be able to model this interdependence we need to introduce externalities. We do this by allowing the value of a pair to depend on how the other agents are matched.

For each matching $\mu \in M(F, W)$ and each pair $(i, j) \in \mu$, let $\alpha_{ij}^\mu \in \mathbb{R}_+$ denote the value or monetary benefit i and j jointly can achieve given the matching μ . Let $A = (\alpha^\mu)_{\mu \in M(F, W)}$ be the collection of values for all the possible matchings.

Definition 2.2. An *assignment game with externalities*, Γ_e , is completely described by a triplet $\Gamma_e = (F, W, A)$.

An *outcome* of Γ_e is a matching and a pair of payoff vectors, $(\mu, u, v) \in M(F, W) \times \mathbb{R}^m \times \mathbb{R}^n$, such that

$$\sum_{i \in F} u_i + \sum_{j \in W} v_j = \sum_{(i, j) \in \mu} \alpha_{ij}^\mu.$$

An outcome is stable if, again, it is individually rational and there are no blocking pairs. However, it is not immediately clear *when* a pair actually should engage in blocking. In the absence of externalities, this is natural. Here, note that the value of the blocking pair may well depend on the behavior of the rest of the agents. This is illustrated in the following example.

Example 1. Consider an assignment game with externalities Γ_e with $m = n = 3$ firms and workers. Suppose the current matching is $\mu_1 = \{(1, 1), (2, 2), (3, 3)\}$. Table 1 displays the values created by the different pairs; all others are assumed to be zero.

Hence, at μ_2 the pair $(2, 3)$, firm 2 and worker 3, generate a value of $\alpha_{23}^{\mu_2} = 0$. Say the current outcome is (μ_1, u, v) , with $u = (1, 1, 1)$, $v = (1, 1, 0)$. Then $u_2 + v_3 = 1$, and we

Matching	Pair 1	Pair 2	Pair 3
$\mu_1 = \{(1, 1), (2, 2), (3, 3)\}$	2	2	1
$\mu_2 = \{(1, 2), (2, 3), (3, 1)\}$	2	0	2
$\mu_3 = \{(1, 1), (2, 3), (3, 2)\}$	2	2	1

Table 1: Values for Example 1. The second line shows that the pair (1, 2) is matched at μ_2 and generates a value of 2. At the same matching, the second pair, (2, 3), get 0. Values for all pairs at all matchings not mentioned in the table are zero.

have $\alpha_{23}^{\mu_2} < u_2 + v_3 < \alpha_{23}^{\mu_3}$. In other words, it is sensible for agents (2, 3) to form a blocking pair if the matching formed thereupon is μ_3 , but not if it is μ_2 .

Importantly, blocking a matching may set in motion a chain of events leading up to a completely different matching with completely different values associated to it, which should be taken into account by the agents. We construct a general notion of blocking without imposing any assumptions on residual behavior in the very beginning. When agents are deciding whether to form blocking pairs, they take the values for all contingencies into account. According to their attitude towards risk or beliefs about the other agents, they calculate a threshold $d_{ij} \in \mathbb{R}$ based on the possible outcomes, and block whenever d_{ij} exceeds the sum of their current payoffs.

We can distinguish different types of agents based on how they determine d_{ij} :

Definition 2.3. The pair (i, j) is *optimistic* if $d_{ij} = o_{ij} = \max_{\mu' \ni (i,j)} \alpha_{ij}^{\mu'}$.

Hence, the threshold is set to the highest value that the pair can hope to generate. This pair is very opportunistic and difficult to satisfy; they block as soon as they see a chance of benefiting from it.

In contrast, careful or pessimistic agents form blocking pairs only if it guarantees them a preferable outcome:

Definition 2.4. The pair (i, j) is *pessimistic* if $d_{ij} = p_{ij} = \min_{\mu' \ni (i,j)} \alpha_{ij}^{\mu'}$.

The values p_{ij} and o_{ij} are natural bounds on the threshold d_{ij} . If we would have $d_{ij} < p_{ij}$, then there are cases ($d_{ij} < u_i + v_j < p_{ij}$) when the pair does not form a blocking pair even though it surely would benefit from doing so. On the other hand, if $d_{ij} > o_{ij}$

then the pair may form a blocking pair even though it surely cannot benefit. Thus, it is arguably reasonable to assume that $p_{ij} \leq d_{ij} \leq o_{ij}$ for all pairs. We maintain this assumption throughout the paper.

Assumption 2.5. For all pairs $(i, j) \in F \times W$, $p_{ij} \leq d_{ij} \leq o_{ij}$.

Note that this covers all types of beliefs on residual behavior and attitude towards risk one might think of. In particular, expected utility maximization, rational expectations (for instance in the spirit of Li, 1993), or any kind of recursive reasoning (Kóczy, 2007) are all contained in this notion. None of these behavioral concepts result in thresholds outside the bounds of p_{ij} and o_{ij} .²

We are now ready to define stable outcomes in assignment games with externalities.

Definition 2.6 (Stability). An outcome is *stable* if it is *individually rational*, $u_i \geq 0$ and $v_j \geq 0$, and has no *blocking pairs*, $u_i + v_j \geq \alpha_{ij}^\mu$ if $(i, j) \in \mu$ and $u_i + v_j \geq d_{ij}$ if $(i, j) \notin \mu$.

Observe that for this concept to be applicable, the informational requirement is very limited. Agents need to know a very small part of the whole collection of values A . Firm i needs to be aware only of the values α_{ij}^μ for $j \in W$ and $\mu \in M(F, W)$. Similarly, worker j only needs to know α_{ij}^μ for $i \in F$ and $\mu \in M(F, W)$.

We illustrate the thresholds and the difference between stable outcomes with optimistic and pessimistic agents in the following example.

Example 2. Consider an assignment game with externalities Γ_e with $m = n = 3$ firms and workers. Among the workers, 1 and 2 are considered competent whereas 3 is less so. Likewise, the firms have different characteristics. The first, a , is innovative: it takes larger risks, but if it employs a competent worker it will be very successful. This is of particular use for the second firm b : this firm can adopt the leading technology quickly. Hence, it benefits from firm a hiring a competent worker. Finally, the third firm c knows its way around the laws of corporate finance, generating a negative externality through patents and legal disputes. Values are displayed in Table 2.

²These bounds are for instance used by Sasaki and Toda (1996) as well. Translating their setup into ours, the thresholds fall within p_{ij} and o_{ij} , though in a more restricted way. In particular, suppose the pair (i, j) creates a value of 0 or 1 depending on the matching. Then Sasaki and Toda's thresholds are either 0 or 1, whereas ours can fall anywhere inbetween.

Matching	Pair 1	Pair 2	Pair 3
$\mu_1 = \{(a, 1), (b, 2), (c, 3)\}$	8	6	1
$\mu_2 = \{(a, 1), (b, 3), (c, 2)\}$	7	4	3
$\mu_3 = \{(a, 2), (b, 1), (c, 3)\}$	6	7	2
$\mu_4 = \{(a, 2), (b, 3), (c, 1)\}$	4	3	5
$\mu_5 = \{(a, 3), (b, 1), (c, 2)\}$	2	5	4
$\mu_6 = \{(a, 3), (b, 2), (c, 1)\}$	1	3	6

Table 2: Values for Example 2.

We will focus on the individually rational outcome (μ_1, u, v) with $u = (u_a, u_b, u_c) = (3, 3, 1)$, $v = (v_1, v_2, v_3) = (5, 3, 0)$. To see if it is stable, we construct the matrices for pessimistic and optimistic pairs and compare it to the sum of the agents' respective payoffs $u + v$. Row i , column j relates to firm i ($= a, b, c$) and worker j ($= 1, 2, 3$). In the case of the rightmost matrix, it shows the sum $u_i + v_j$.

$$p = \begin{pmatrix} 7 & 4 & 1 \\ 5 & 3 & 3 \\ 5 & 3 & 1 \end{pmatrix} \quad o = \begin{pmatrix} 8 & 6 & 2 \\ 7 & 6 & 4 \\ 6 & 4 & 2 \end{pmatrix} \quad u + v = \begin{pmatrix} 8 & 6 & 3 \\ 8 & 6 & 3 \\ 6 & 4 & 1 \end{pmatrix}$$

Let us examine the top row, second column of p and o . These values relate to the pair $(a, 2)$. If they match, we reach either μ_3 , where the pair gets a value of 6, or μ_4 where they earn 4. Hence, the pair is sure to get $p_{a2} = 4$ when blocking. Similarly, the second value shows what the pair can get at best, $o_{a2} = 6$. When we compare these values to the pair's current total payoffs, top row, second column of the matrix $u + v$, we see that o_{a2} does not exceed $u_a + v_2 = 3 + 3 = 6$. Hence, $(a, 2)$ cannot benefit compared to the current outcome by matching.

Moving a step down and to the right, we can see that the pair $(b, 3)$ is assured 3 (in case of μ_4), at best 4 (if μ_2), and currently gets $u_b + v_3 = 3 + 0 = 3$. Hence, unless the pair is pessimistic, it has incentives to form a blocking pair.

Finally, note that the rightmost matrix $u + v$ has no entries smaller than the corresponding ones in p ; the interpretation of this is that if all pairs are pessimistic, then the outcome (μ_1, u, v) is stable. It turns out that the matchings μ_2 and μ_3 also can be made

stable with such pairs,³ whereas no matching is stable if all pairs are optimistic.

3 Characterization

Next, we show that we can find stable outcomes for the assignment game with externalities by inspecting an artificial assignment game without externalities. This finding will be essential later in showing the existence of a stable outcome.

We start by defining the values of the artificial assignment game. Consider an assignment game with externalities $\Gamma_e = (F, W, A)$. For each $\mu \in M(F, W)$, let

$$d_{ij}^\mu = \begin{cases} \alpha_{ij}^\mu & \text{if } (i, j) \in \mu \\ d_{ij} & \text{otherwise.} \end{cases}$$

Hence, d_{ij}^μ is either the value created by a matched pair at μ or the blocking threshold for an unmatched pair. Using these values, we can generate an assignment without externalities, $\Gamma = (F, W, d^\mu)$. There is an important relation between the stable outcomes of these two games.

Proposition 3.1. *An outcome $(\mu, u, v) \in M(F, W) \times \mathbb{R}^m \times \mathbb{R}^n$ is stable in $\Gamma_e = (F, W, A)$ if and only if it is stable in $\Gamma = (F, W, d^\mu)$.*

Proof. As (μ, u, v) is an outcome of Γ_e ,

$$\sum_{i \in F} u_i + \sum_{j \in W} v_j = \sum_{(i,j) \in \mu} \alpha_{ij}^\mu.$$

By construction, $d_{ij}^\mu = \alpha_{ij}^\mu$ for $(i, j) \in \mu$. Hence,

$$\sum_{(i,j) \in \mu} \alpha_{ij}^\mu = \sum_{(i,j) \in \mu} d_{ij}^\mu.$$

Taken together, we find that (μ, u, v) indeed is an outcome of Γ :

$$\sum_{i \in F} u_i + \sum_{j \in W} v_j = \sum_{(i,j) \in \mu} d_{ij}^\mu.$$

Next, (i, j) forms a blocking pair in Γ_e whenever they do so in Γ :

$$\begin{aligned} u_i + v_j &< \alpha_{ij}^\mu = d_{ij}^\mu & \text{if } (i, j) \in \mu \\ u_i + v_j &< d_{ij} & \text{if } (i, j) \notin \mu. \end{aligned}$$

³For instance, $u = (3, 3, 1)$ is stable with $v = (v_1, v_2, v_3) = (4, 1, 2)$ for μ_2 and with $v = (4, 3, 1)$ for μ_3 .

It follows that the outcome is stable in Γ_e whenever it is stable in Γ . \square

Be mindful not to draw the conclusion that this result implies that there exists stable outcomes in the game with externalities. Surely, as noted before, $\Gamma = (F, W, d^\mu)$ has stable outcomes. However, if all of these use matchings other than μ , then none of them is stable in the game with externalities. Observe also that the relationship between the stable outcomes is independent of behavioral assumptions: it holds no matter how d_{ij} is calculated.

3.1 Existence

The second and main result of the paper gives a necessary and sufficient condition for the existence of a stable outcome in assignment games with externalities. This result is particularly important given that the existing literature has only provided sufficient conditions so far (Sasaki and Toda, 1996). We find that stability can be guaranteed if and only if agents are pessimistic regarding residual behavior and hence careful in forming blocking pairs.

Theorem 3.2. *There exists a stable outcome in all assignment games with externalities if and only if all pairs are pessimistic.*

Proof. Assume first that all pairs are pessimistic. Note that $\Gamma = (F, W, d)$, where $d = (d_{ij})_{i,j \in N}$ is the blocking thresholds, is an assignment game without externalities. Hence, there exists a stable outcome (μ, u, w) in Γ (Shapley and Shubik, 1971). Define v such that for all $(i, j) \in \mu$,

$$v_j = w_j + \alpha_{ij}^\mu - d_{ij}.$$

Note that $\alpha_{ij}^\mu \geq d_{ij}$ as (i, j) is pessimistic. We claim that (μ, u, v) is stable in $\Gamma_e = (F, W, A)$. First, it is feasible:

$$\sum_{i \in F} u_i + \sum_{j \in F} v_j = \sum_{i \in F} u_i + \sum_{j \in F} w_j + \sum_{(i,j) \in \mu} \alpha_{ij}^\mu - \sum_{(i,j) \in \mu} d_{ij} = \sum_{(i,j) \in \mu} \alpha_{ij}^\mu.$$

The last equality follows as (μ, u, w) is an outcome in Γ ,

$$\sum_{i \in F} u_i + \sum_{j \in W} w_j = \sum_{(i,j) \in \mu} d_{ij}.$$

Moreover, as (μ, u, w) is stable, it is individually rational. As $v_j \geq w_j$, so is (μ, u, v) . Finally, we show that there are no blocking pairs. Consider first $(i, j) \in \mu$. As (μ, u, w) is stable, we have $u_i + w_j = d_{ij}$ if $(i, j) \in \mu$. Then

$$u_i + v_j = u_i + w_j + \alpha_{ij}^\mu - d_{ij} = \alpha_{ij}^\mu.$$

Thus, the agents do not form a blocking pair. For $(i, j) \notin \mu$, we reach the same conclusion:

$$u_i + v_j \geq u_i + w_j \geq d_{ij}.$$

Hence, the outcome is stable in Γ_e .

Now we prove the other direction. Assume there exists a pair which is not pessimistic. Without loss of generality, suppose the pair $(1, 1)$ puts positive weight $\lambda \in (0, 1]$ on the optimistic outcome, $d_{11} = \lambda o_{11} + (1 - \lambda)p_{11}$. Consider (F, W, A) such that $\alpha_{ij}^\mu = 0$ in all cases except for the following:

$$\begin{aligned} o_{11} = \alpha_{11}^\mu &= 1/\lambda + 1 \text{ for } \mu = \{(1, 1), (2, 2), \dots, (n, n)\} \\ \alpha_{12}^\mu &= 1 \text{ for } \mu \ni (1, 2) \\ \alpha_{23}^\mu &= 1 \text{ for } \mu \ni (2, 3). \end{aligned}$$

With $p_{11} = 0$, we get $d_{11} = 1 + \lambda$. Then, no matter if the other pairs are pessimistic or not,

$$d_{ij} = \begin{cases} 1 + \lambda & \text{if } (i, j) = (1, 1) \\ 1 & \text{if } (i, j) = (1, 2), (2, 3) \\ 0 & \text{otherwise.} \end{cases}$$

Then the matching $\mu = \{(1, 1), (2, 2), \dots, (m, m)\}$ can be blocked by $(2, 3)$. For all matchings $\mu \not\ni (1, 1)$, $(1, 1)$ can block as $v_1 = 0$ and u_1 cannot exceed $1 < 1 + \lambda$. For all remaining matchings, $(1, 2)$ can block. Hence, there exists no stable outcome. \square

This result shows that even the slightest optimism (in the example, λ close to zero) can lead to unstable instances.

3.2 Efficiency

In contrast to the case without externalities, efficiency and stability no longer go hand in hand when there are externalities. In the following example an inefficient matching is stable whereas the efficient matching is not.

Example 3. Consider (F, W, A) such that $\alpha_{ij}^\mu = 0$ in all cases except for the following:

$$\alpha_{11}^{\mu_1} = 2 \text{ for } \mu_1 = \{(1, 1), (2, 2), (3, 3)\}$$

$$\alpha_{23}^{\mu_2} = 1 \text{ for } \mu_2 = \{(1, 1), (2, 3), (3, 2)\}$$

$$\alpha_{23}^{\mu_3} = 1 \text{ for } \mu_3 = \{(1, 2), (2, 3), (3, 1)\}.$$

Then the matching μ_1 is efficient, though $(2, 3)$ forms a blocking pair. Instead, the matching μ_2 is stable.⁴

However, if all pairs are pessimistic we can still attain a form of optimality.

Definition 3.3. An outcome (μ', u', v') of Γ_e is a *Pareto improvement* to an outcome (μ, u, v) if $u'_i \geq u_i$ for all $i \in F$ and $v'_j \geq v_j$ for all $j \in W$ with at least one strict inequality. An outcome (μ, u, v) is *Pareto optimal* if there is no Pareto improvement to it.

Proposition 3.4. *Assume that all pairs are pessimistic. Let (μ', u', v') be stable in Γ_e , but not Pareto optimal. Then there exists a stable Pareto improvement (μ, u, v) to (μ', u', v') .*

Proof. By contradiction, suppose there exists a blocking pair (i, j) for (μ, u, v) . Assume first $(i, j) \in \mu$. We show that there exists a different outcome (μ, u'', v'') that still Pareto dominates (μ', u', v') but for which (i, j) is not a blocking pair. As $(i, j) \in \mu$, $u_i + v_j < \alpha_{ij}^\mu$. Define u'' and v'' equal to u and v with the exception that $u''_i + v''_j = \alpha_{ij}^\mu$. Then (μ, u'', v'') is a Pareto improvement with one less blocking pair. By repetition, we find that it is without loss of generality to assume no $(i, j) \in \mu$ can block the outcome.

Now assume $(i, j) \notin \mu$ is a blocking pair; we show that this contradicts (μ', u', v') being stable. If $(i, j) \in \mu'$, $d_{ij}^{\mu'} = \alpha_{ij}^{\mu'} \geq d_{ij}^\mu$ as pairs are pessimistic. If $(i, j) \notin \mu'$, then $d_{ij}^{\mu'} = d_{ij}^\mu$. As by assumption the pair (i, j) forms a blocking pair for μ , $d_{ij}^{\mu'} > u_i + v_j$. As (μ, u, v) is a Pareto improvement to (μ', u', v') , $u_i + v_j \geq u'_i + v'_j$. In conclusion, $d_{ij}^{\mu'} > u'_i + v'_j$. Hence, the pair (i, j) forms a blocking pair for (μ', u', v') , which is a contradiction. \square

Hence, if all pairs are pessimistic, there always exists a Pareto optimal stable outcome. The following example shows the necessity of pairs being pessimistic.

Example 4. Assume that there is exactly one pair which is not pessimistic. Suppose the pair $(1, 1)$ puts positive weight $\lambda = 1/2$ on the optimistic outcome, $d_{11} = (o_{11} +$

⁴Stable payoffs include $u = (0, 1, 0)$ with $v = (0, 0, 0)$. If a matched pair is allowed to break up and rematch, then the outcome is stable only if the pair $(1, 1)$ is pessimistic. In that case, so is (μ_3, u, v) .

Matching	Pair 1	Pair 2	Pair 3
$\mu_1 = \{(1, 1), (2, 2), (3, 3)\}$	1	1	1
$\mu_2 = \{(1, 1), (2, 3), (3, 2)\}$	9	0	0
$\mu_3 = \{(1, 2), (2, 1), (3, 3)\}$	2	2	2

Table 3: Values for Example 4.

$p_{11})/2$. Consider (F, W, A) such that $\alpha_{ij}^\mu = 0$ in all cases except for the ones displayed in Table 3. Then the matching μ_1 is Pareto dominated by μ_3 . However, μ_1 is the unique stable matching (compatible with any individually rational payoffs), whereas μ_3 is blocked by $(1, 1)$.

3.3 The Structure of the Set of Stable Outcomes

For the assignment model without externalities, Shapley and Shubik (1971) show that the core of the game has a special structure. In particular, the set of stable outcomes is a complete lattice with two extreme points: one where all firms achieve their highest possible stable payoffs and a corresponding worker-optimal outcome. The following proposition shows that this property does not generalize to our setup.

Proposition 3.5. *The set of stable payoff vectors for an assignment game with externalities may not form a lattice.*

Proof. The proof is by counter example. Assume all pairs are pessimistic and values are as in Table 4. There are two stable matchings, μ_1 and μ_4 . For μ_1 , the set of stable outcomes forms a lattice with a (firm-) minimal element of $u = (1, 0, 0)$ together with $v = (7, 6, 4)$. For μ_4 , the stable outcomes form a lattice disjoint from the former set. The minimal element is $u' = (0, 0, 1)$ with $v' = (5, 4, 8)$. For neither matching $u'' = (0, 0, 0)$ is stable. Hence, the entire set of stable outcomes does not form a lattice. See Figure 1 for a graphical illustration. \square

Without externalities, all stable matchings are compatible with all stable payoff vectors (Shapley and Shubik, 1971). That is, if (μ, u, v) and (μ', u', v') are stable, then we can interchange the matchings: (μ', u, v) is also stable. As is immediate from the example in

Matching	Pair 1	Pair 2	Pair 3
$\mu_1 = \{(1, 1), (2, 2), (3, 3)\}$	8	6	4
$\mu_2 = \{(1, 1), (2, 3), (3, 2)\}$	3	4	5
$\mu_3 = \{(1, 2), (2, 1), (3, 3)\}$	6	4	4
$\mu_4 = \{(1, 2), (2, 3), (3, 1)\}$	4	8	6
$\mu_5 = \{(1, 3), (2, 1), (3, 2)\}$	5	2	5
$\mu_6 = \{(1, 3), (2, 2), (3, 1)\}$	5	2	5

Table 4: Values for the example in the proof of Proposition 3.5.

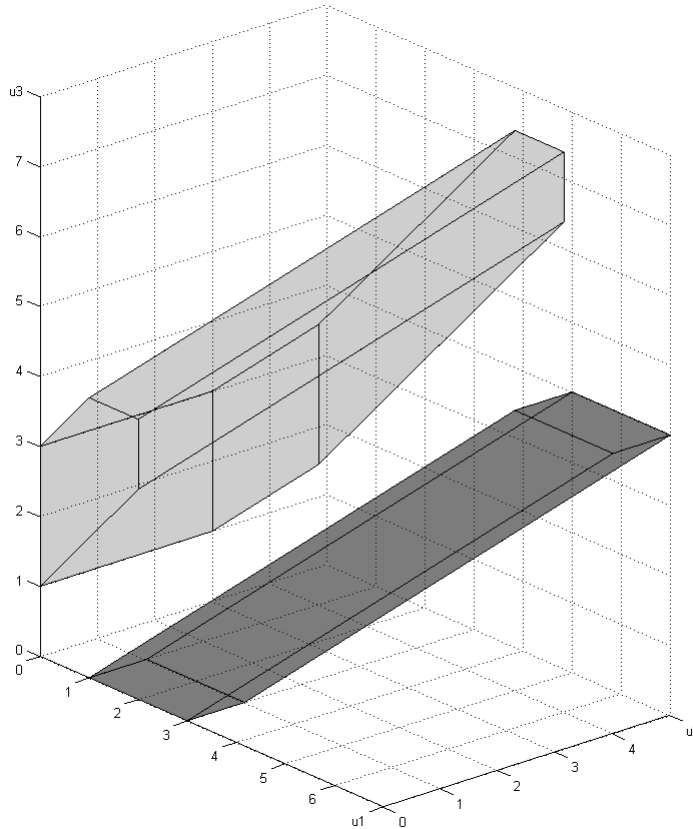


Figure 1: The dark grey area shows the lattice structure of the stable payoffs compatible with the matching μ_1 in the example of Proposition 3.5. The light grey area consists of stable payoffs compatible with μ_4 . Importantly, the two areas are disjoint.

the proof of Proposition 3.5, this property no longer holds in the generalized model with externalities.

Corollary 3.6. *If (μ, u, v) and (μ', u', v') are stable outcomes in an assignment game with externalities Γ_e , then (μ', u, v) need not be a stable outcome in Γ_e .*

Finally, let us return to the lattice property and instead focus on a specific matching μ . Then the set of payoffs that are stable in combination with μ , if any, actually do form a lattice. Let Ω denote the set of stable outcomes in Γ_e and $\Omega^\mu \subseteq \Omega$ the set of stable payoffs compatible with μ , that is, all u such that (μ, u, v) is stable.

Proposition 3.7. *For any matching μ , the set of stable payoffs compatible with μ , Ω^μ , forms a complete lattice.*

The result follows from the characterization in Proposition 3.1. There we found that the set of stable outcomes for a given matching for the problem with externalities was identical to the set of stable outcomes for an artificial problem without externalities. As already noted, the latter set has the lattice property (Shapley and Shubik, 1971).

4 Discussion

There are numerous ways to trivially strengthen and generalize the results. First and foremost, we can allow matched agents to form blocking pairs; that is, to break up and rematch. This does not affect the results regarding pessimistic agents as a pessimistic pair would never exercise this option. Secondly, we can allow the blocking thresholds to be matching dependent. That is, a pair (i, j) can have a different threshold d_{ij} depending on which matching currently is in place. Again, this has no effect on the pessimistic pairs as their thresholds would still be the same at all matchings (by the definition of pessimism). Thirdly, we need not consider the full set of matchings $M(F, W)$. Our model is equipped to handle that some matchings are not allowed, say for legal reasons.

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