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Department of Economics
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Cycles and Third-Party Payments in the Partnership Formation Problem

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Cycles and third-party payments in the partnership formation problem[☆]

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Abstract

We characterize the core of the partnership formation problem (Talman and Yang, *Journal of Mathematical Economics* 47, 2011) using cycles in the solution of a linear programming problem. The cycles also lead us to a new and intuitive sufficient condition for the existence of equilibrium, generalizing known results. We then introduce the partnership-stable equilibrium, a weakened solution concept in the sense that each equilibrium is also a partnership-stable equilibrium, and find sufficient and necessary conditions for its existence. Finally, we conduct a simulation study to see how often existence is an issue and which factors affect the rate at which problems have solutions.

Keywords: Partnership, equilibrium, linear programming, cycles, third-party payments

JEL: C62, D02, D60

1. Introduction

In a world of social interaction and competitive business, the formation of partnerships among individuals and likewise among firms plays a significant role. A cornerstone, frequent in practice and attractive in its simplicity, is partnerships in pairs. In this paper we examine the following situation. There is a group of economic agents, each and every one having the option to enter a joint venture with another agent. All the individual agents as well as all the potential joint ventures generate profits, and every agent wants to maximize his share of these. Whom, if anyone, an agent chooses to cooperate with comes down to how the profits are split in the various joint ventures – even if you and I make a million dollar profit, I do not want to work with you unless I get a sufficiently large share of it! For every agent there is a trade-off: the larger the share of the profit he demands, the less likely he is to attract a partner.

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An intuitive way of modelling this is as an *assignment game* (Koopmans and Beckmann, 1957; Shapley and Shubik, 1971). Originally and most frequently in the literature, the model has been used with two-sided markets in mind: the agents are explicitly divided into, say, one set of firms and another set of workers. We will consider the more general one-sided version of the model. Hence, all agents are “the same” – say business agents who want to form joint ventures. Central in the model is the *value function*, which associates a number to every agent and every pair of agents. For intuition, think of these values as the profits of the joint ventures. Larger values are then likely associated to bigger companies, and larger differences between what agents get on their own compared to what they get jointly reflect that agents complement each other well. We can think of the agents as going through a process of bargaining on how to split the joint values succeeded by the actual formation of a number of partnerships. The market is said to be in equilibrium when no agent gets a smaller profit than he is able to generate on his own, and no pair of agents can make themselves both better off (compared to their payoffs) by sharing their joint profit among themselves. The set of equilibrium outcomes coincides with the *core* (see also Sotomayor, 2005); we use the terms interchangeably. With the above interpretation in mind, we denote the problem the *partnership formation problem* (as do Talman and Yang, 2011). Note however that it has been given other labels in the literature: the “one-sided assignment game” (Klaus and Nichifor, 2010; Sotomayor, 2005) and the “roommate problem with transferable utility” (Chiappori et al., 2012; Eriksson and Karlander, 2001).

In this paper we aim to make several contributions to the existing literature – all related to and based on the fact that not all partnership formation problems possess equilibrium outcomes. First, by investigating the problem as a linear program, we find results that complement those of Talman and Yang (2011) and Chiappori et al. (2012). Both results rely on certain cycles obtained when solving the LP problems. We show (Proposition 3) how the existence of equilibrium can be expressed in terms of cycles. Based on this result, we then (Theorem 1) both generalize and provide new intuition to a sufficient condition for the existence of equilibrium by Talman and Yang (2011, Theorem 1). Furthermore, by using the cycles we can bound the solution of one LP problem in terms of the solution of another (Proposition 4), a result complementing Chiappori et al. (2012, Theorem 1). It can be noted that cycles play a similarly important role in the *roommate problem* (Gale and Shapley, 1962) for the existence of stable roommate matchings (Chung, 2000; Gudmundsson, 2013).

Another contribution is to introduce and analyse a weakening of the core that may say something about the market in instances without equilibrium. Intuitively, if the stability concept deems an instance unstable when there is a plausibly stable outcome, then a weaker condition that incorporates this may be more appropriate. As a motivational example, think of the following three-agent problem. Suppose agents 1 and 2 are matched, though their partnership is fragile: there is another agent 3 who makes tempting offers to both agents 1 and 2, thereby jeopardizing their current partnership. It may well be in their (1 and 2’s) interest to compensate agent 3 simply to reduce the risk of their partnership falling apart. However, that in itself would be an objection to the core: if agents 1 and 2 use even the slightest fraction of their joint profit to compensate agent 3, the outcome is deemed unstable. The same idea can be extended to the viewpoint of agents 1 and 3: is it intuitively unstable if they (1 and 3) would spend a fraction of their joint profit to compensate and persuade agent 2 into being single?

It is not new to the literature that the core has these properties. It is even explicitly pointed out by Shapley and Shubik (1971, p. 112): “The core solution, as it turns out, excludes third-party payments”. In the partnership formation problem, the core requires the profit that is created to be divided *in its entirety* among its creators – there is no value left to compensate other agents. As stated by Roth and Sotomayor (1990, p. 206) on the two-sided model, “... this is an implication of stability, not an assumption of the model”. Hence, *if* we want to be able to solve problems by third-party payments where matched agents compensate other agents, it is *unavoidable* that we turn away from the core. In particular, some groups of agents simply have to be content with getting less than they could produce on their own – and in this paper, those groups are chosen to be the matched agents. If agents are risk averse and want to minimize the risk of their partnership collapsing, these agents are likely the ones most willing to part with some of their profits.

We propose a new solution concept: a *partnership-stable equilibrium*. This is similar to an equilibrium as discussed thus far in all aspects but one: it is not immediately deemed unstable if two *matched* agents in total get less than their joint profit, opening up for problems to be solved using third-party payments. It portrays a different kind of stability than that of the core: in a partnership-stable equilibrium, no agents have incentives to break up or form new partnerships. If we are looking for *stability in the market as such* – in the sense that no one wants to change the structure of the partnerships – then this may be better captured by the partnership-stable equilibrium than by the core. First we argue by example why this may indeed be an attractive concept, and then we use our LP setup to derive sufficient and necessary conditions for its existence in Theorem 2. Finally, in Proposition 5 we find an upper bound on the maximum amount of money that needs to be added to the problem (say, as a subsidy) for there to exist a partnership-stable equilibrium.

Every step away from the standard solution concept should be taken with care, and so it is in this case: every core outcome is a partnership-stable equilibrium. Hence, as long as there exists a core outcome, third-party payments are not needed; what we add to the existing literature is a suggestion of a different and weaker solution concept when the core turns out to be too restrictive.

The core has been central in a number of related papers. Klaus and Nichifor (2010) give a characterization of the core in terms of a consistency axiom. It is known that the core is non-empty if we replicate the economy such that there is an even number of agents of the same type (Chiappori et al., 2012), where two agents are of the same type if they create identical values in the different partnerships. There also exists an equilibrium if we avoid value functions that satisfy a graph theoretical forbidden minors condition (Eriksson and Karlander, 2001). Finally, Andersson et al. (2013) provide an algorithm that identifies an equilibrium whenever one exists, analysing the problem as an auction.

The paper is outlined as follows. In the next section, we introduce the model. In Section 3, we formulate and investigate the LP problems. In Section 4, we define cycles and show how they relate to the existence of equilibrium. We introduce the new solution concept in Section 5. Finally, to get a better idea of the extent to which the non-existence problem actually is a problem, we conduct a simulation study in Section 6.

2. Model and definitions

There is a finite set of agents $N = \{1, 2, \dots, n\}$. A matching is a one-to-one mapping on N such that $\mu(i) = j \Leftrightarrow \mu(j) = i$ for all $i, j \in N$. We denote the set of all matchings \mathcal{M} . The set of all singles and pairs is $\mathcal{N} = \{S \subseteq N : \#S \leq 2\}$.¹ To each element of \mathcal{N} we associate a real value reflecting the profit created by that specific agent or joint venture. This is the (characteristic) value function $v : \mathcal{N} \rightarrow \mathbb{R}$. For our purposes it is without loss of generality to assume $v(\{i\}) = 0$ for all $i \in N$ and $v(S) \geq 0$ for all $S \in \mathcal{N}$. A partnership formation problem is defined as a pair (N, v) . We denote the set of all partnership formation problems \mathcal{P} .

With some abuse of notation, the total value created by the agents N at a matching μ in a problem (N, v) is denoted $v(\mu)$,²

$$v(\mu) = \frac{1}{2} \sum_{i \in N} v(\{i, \mu(i)\}).$$

A payoff vector is $p \in \mathbb{R}^n$ such that $i \in N$ receives p_i out of the total value. In equilibrium, no value is wasted, no agent prefers being single to his match, and no pair of agents can share their joint value in a way that is beneficial for both of them compared to their current payoffs.

Definition 1. An equilibrium in $(N, v) \in \mathcal{P}$ is a pair $(\mu, p) \in \mathcal{M} \times \mathbb{R}^n$ such that

1. $p_1 + p_2 + \dots + p_n = v(\mu)$
2. $p_i \geq 0$ for all $i \in N$
3. $p_i + p_j \geq v(\{i, j\})$ for all $i, j \in N, j \neq i$.

We refer to the second condition as *individual rationality* and the third as *pairwise stability*.³ The following example presented by Talman and Yang (2011, p. 208) shows that there need not exist an equilibrium in the partnership formation problem:

Example 1 (No equilibrium). Let $N = \{1, 2, 3\}$, $v(\{1, 2\}) = v(\{1, 3\}) = 3$, and $v(\{2, 3\}) = 5$. For each pairwise stable payoff vector p we have $p_1 + p_2 \geq 3$, $p_1 + p_3 \geq 5$, and $p_2 + p_3 \geq 3$. If all inequalities are satisfied, then $2(p_1 + p_2 + p_3) \geq 11$ and $p_1 + p_2 + p_3 \geq 5.5$. However, no matching μ generates a total value of 5.5: we have $v(\mu) \leq 5$ for all $\mu \in \mathcal{M}$. Hence, there exists no equilibrium.

If we instead focus on instances that have an equilibrium, it is known (Sotomayor, 2005, Proposition 3) that they consist of so called optimal matchings:

¹By $\#S$ we denote the cardinality of the set S .

²Dividing by two corrects the double counting of $v(\{i, j\})$ and $v(\{j, i\})$. If $\mu(i) = i$, then i contributes nothing to the total value as $v(\{i, \mu(i)\}) = v(\{i\}) = 0$.

³This is the reason for normalizing v . By individual rationality, if $i, j \in N$ bargain over splitting $v(\{i, j\})$, they get at least $v(\{i\})$ and $v(\{j\})$ in equilibrium. Hence, the bargain really concerns the surplus, $e(\{i, j\}) = v(\{i, j\}) - v(\{i\}) - v(\{j\})$, and is irrelevant if said surplus is negative. Naturally, $e(\{i\}) = 0$ for all $i \in N$ as there is no surplus from cooperating with yourself compared to being single. Hence, for every value function v there exists a corresponding (in terms of existence of equilibrium) normalized value function e .

Definition 2. $\mu \in \mathcal{M}$ is optimal in $(N, v) \in \mathcal{P}$ if, for all $\mu' \in \mathcal{M}$,

$$v(\mu) \geq v(\mu').$$

The optimal value is denoted $v(N) = \max_{\mu \in \mathcal{M}} v(\mu)$.

Proposition 1 (Sotomayor, 2005). *If $(\mu, p) \in \mathcal{M} \times \mathbb{R}^n$ is an equilibrium in $(N, v) \in \mathcal{P}$, then μ is optimal in (N, v) .*

For the two-sided model, Shapley and Shubik (1971) showed that optimality was crucial to stability – a matching can be made stable if and only if it is optimal. In the partnership formation problem optimality is necessary, but not sufficient.

3. The duality of the partnership formation problem

We now move on to introduce the LP problems that will be analysed in the remainder of this paper. In addition, we briefly discuss the intuition behind a result of Talman and Yang (2011, Lemma 3). More to the point, they provide an exact characterization of when there exists an equilibrium in terms of the solution to the following LP problem:

$$\begin{aligned} P(v) = \text{minimize} \quad & \sum_{i \in N} p_i \\ \text{subject to} \quad & p_i \geq 0 \quad \forall i \in N; \quad p_i + p_j \geq v(\{i, j\}) \quad \forall i, j \in N: j \neq i \end{aligned} \tag{Primal}$$

If $v(N) < P(v)$, no matching μ produces a sufficient total value $v(\mu) \leq v(N)$ to cover the expenses $P(v)$ needed to make the payoff vector individually rational and pairwise stable. Hence, there cannot exist an equilibrium. By similar but opposing arguments, there exists an equilibrium whenever $v(N) \geq P(v)$. This can in turn be restated using the solution to the corresponding dual problem:

$$\begin{aligned} D(v) = \text{maximize} \quad & \frac{1}{2} \sum_{i \in N} \sum_{j \in N} \lambda_{ij} v(\{i, j\}) \\ \text{subject to} \quad & \sum_{i \in N} \lambda_{ij} = 1 \quad \forall j \in N; \quad \sum_{j \in N} \lambda_{ij} = 1 \quad \forall i \in N; \quad \lambda_{ij} \geq 0 \quad \forall i, j \in N \end{aligned} \tag{Dual}$$

In particular, we have $P(v) = D(v)$. The choice variable λ is an $n \times n$ doubly stochastic matrix with multiple interpretations of a typical element λ_{ij} : one can for instance think of it as the fraction of agent j assigned to agent i or as a probability.

Before continuing, we can gain some insight by returning to Example 1. The primal problem has a unique solution $p = (5/2, 1/2, 5/2)$ with $P(v) = 11/2$. The absence of equilibrium follows from $v(N) = 5 < 11/2 = P(v)$. On the other hand, there are many solutions to the dual problem. Two of them are permutation matrices, λ and λ^T , and the remaining ones are all convex combinations of λ and λ^T , including the symmetric matrix λ' defined below.⁴

$$\lambda = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda' = \frac{\lambda + \lambda^T}{2} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

⁴A permutation matrix has exactly one 1 in each row and each column and 0's otherwise. A matrix is symmetric if it equals its transpose.

These properties are no peculiarities of this specific problem: there always, for instance, exists a permutation matrix that solves the dual problem (Dantzig, 1963; Shapley and Shubik, 1971). As the value function is symmetric, $v(\{i, j\}) = v(\{j, i\})$, if λ solves the dual problem then so does its transpose λ^T . By the linearity of the problem, every convex combination of λ and λ^T , including the symmetric $\lambda' = (\lambda + \lambda^T)/2$, is a solution. Note however that λ' need not be a permutation matrix – more precisely, it is so if and only if λ is a symmetric permutation matrix. When we examine $v(N)$ it becomes clear that whether λ is such a matrix is essential also to whether there exists an equilibrium or not:

$$\begin{aligned} v(N) = \underset{\lambda}{\text{maximize}} \quad & \frac{1}{2} \sum_{i \in N} \sum_{j \in N} \lambda_{ij} v(\{i, j\}) \\ \text{subject to} \quad & \sum_{i \in N} \lambda_{ij} = 1 \quad \forall j \in N; \quad \lambda_{ij} \in \{0, 1\} \quad \forall i, j \in N; \quad \lambda_{ij} = \lambda_{ji} \quad \forall i, j \in N \end{aligned}$$

Importantly, note that the feasible set contains exactly the symmetric permutation matrices. As the constraints are more restrictive than for the dual problem,⁵ we have $D(v) \geq v(N)$, with equality exactly when a symmetric permutation matrix solves the dual problem. By combining the inequalities, there exists an equilibrium whenever $v(N) \geq P(v) = D(v) \geq v(N)$. These findings can be summarized as follows:

Proposition 2 (Talman and Yang, 2011). *Let $(N, v) \in \mathcal{P}$. The following statements are equivalent.*

1. *There exists an equilibrium in (N, v) .*
2. *A symmetric permutation matrix solves the dual problem.*
3. *$D(v) = v(N) = P(v)$.*

Notice how Proposition 1 follows: if (μ, p) is an equilibrium, then $v(N) \geq v(\mu) \geq P(v) = v(N)$, and hence μ is optimal.

Finally, to round of this section we prove a linearity lemma related to the dual problem. It is used in numerous proofs in Section 4.

Lemma 1. *Assume λ is a solution to the dual problem for $(N, v) \in \mathcal{P}$. Let $L = (1, 2, \dots, k)$ for $k \leq n$ and let λ' and λ'' be symmetric $k \times k$ permutation matrices. If, for all $i, j \in L$, we have*

$$\lambda_{ij} = \frac{\lambda'_{ij} + \lambda''_{ij}}{2},$$

then $\tilde{\lambda}$ is a solution to the dual problem, defined as

$$\tilde{\lambda}_{ij} = \begin{cases} \lambda'_{ij} & \text{if } i, j \in L \\ \lambda_{ij} & \text{otherwise.} \end{cases}$$

⁵This relates to the work of Birkhoff (1946) and von Neumann (1953), who show that the doubly stochastic matrices is the convex hull of the set of permutation matrices.

Proof. Note that $\tilde{\lambda}$ is feasible in the dual problem⁶ and generates the same value as λ for agents in $N - L$. The contribution to the total value by the agents in L is

$$\frac{1}{2} \sum_{i \in L} \sum_{j \in L} \lambda_{ij} v(\{i, j\}) = \frac{1}{2} \left(\frac{1}{2} \sum_{i \in L} \sum_{j \in L} \lambda'_{ij} v(\{i, j\}) + \frac{1}{2} \sum_{i \in L} \sum_{j \in L} \lambda''_{ij} v(\{i, j\}) \right).$$

As the average is no larger than the largest of its components, the fact that λ solves the dual problem implies that all three sums are equal,

$$\sum_{i \in L} \sum_{j \in L} \lambda_{ij} v(\{i, j\}) = \sum_{i \in L} \sum_{j \in L} \lambda'_{ij} v(\{i, j\}) = \sum_{i \in L} \sum_{j \in L} \lambda''_{ij} v(\{i, j\}).$$

Hence, $\tilde{\lambda}$ generates the same value as λ for all agents and therefore solves the dual problem. \square

4. Cycles and subproblems

With the basics in place we are now ready to analyse the dual problem in detail. This will lead us to an alternative characterization of Proposition 2, followed by a new upper bound of $P(v)$ in relation to $v(N)$ to complement a similar result by Chiappori et al. (2012). Moreover, we generalize a sufficient condition for the existence of equilibrium by Talman and Yang (2011). All of these results rely on “cycles” in the matrices:

Definition 3. Let λ be an $n \times n$ matrix and $L = (1, 2, \dots, k)$ be an ordered list of $k \geq 3$ agents. L is a cycle in λ if, for all $i \in L$ taken modulo k , we have

$$\lambda_{i, i+1} = 1/2.$$

Some cycles are more problematic than others; to distinguish between these cycles, we use the following “dominance” concept:

Definition 4. Let λ and λ' be solutions to the dual problem. If each cycle in λ' is a cycle in λ and L is a cycle in λ but not in λ' , then L is an avoidable cycle.

Example 2. Let $N = \{1, 2, 3\}$ and construct the value function v such that $v(\{i, j\})$ is the (i, j) th entry of the leftmost matrix v below. Then λ and λ' are solutions to the dual problem. As λ has no cycles, the cycle $(1, 2, 3)$ in λ' is avoidable, as is its reverse $(3, 2, 1)$.

$$v = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \quad \lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda' = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

There always exists a solution without avoidable cycles,⁷ and we show next that it never has a cycle of even length. The idea is similar to that of even cycles not being problematic for the existence of stable roommate matchings (see e.g. Chung, 2000; Gudmundsson, 2013).

⁶We use “dual problem” throughout rather than “dual problem for (N, v) ” when there is no risk of confusion.

⁷As there is a finite number of agents, each solution has a finite number of cycles. If, by contradiction, all of them have avoidable cycles, then for any solution there is another with one less cycle. Then there must exist a solution entirely without (avoidable) cycles, contradicting the absence of such a solution.

Lemma 2. *If λ is a solution to the dual problem without avoidable cycles, then λ has no even cycles.*

Proof. By contradiction, suppose $L = (1, 2, \dots, k)$ is an even cycle in λ . Hence, for all $i \in L$ taken modulo k , $\lambda_{i, i+1} = 1/2$. Define the $k \times k$ symmetric permutation matrices λ' and λ'' such that $\lambda'_{i, i+1} = 1$ for all odd $i \in L$ and $\lambda''_{i, i+1} = 1$ for all even $i \in L$. Then $\lambda_{ij} = (\lambda'_{ij} + \lambda''_{ij})/2$ for all $i, j \in L$. By Lemma 1, $\tilde{\lambda}$ is a solution to the dual problem, with

$$\tilde{\lambda}_{ij} = \begin{cases} \lambda'_{ij} & \text{if } i, j \in L \\ \lambda_{ij} & \text{otherwise.} \end{cases}$$

L is not a cycle in $\tilde{\lambda}$, and $\tilde{\lambda}$ has no more cycles than λ . Hence, L is avoidable, a contradiction. \square

Recall that there always exists a permutation matrix λ that solves the dual problem as well as a symmetric solution $\lambda' = (\lambda + \lambda^T)/2$, whereas the absence of equilibrium is equivalent to no solution being both. Note also that when λ is not symmetric, there necessarily exists $i, j \in N$ such that $\lambda'_{ij} = 1/2$, the primitives of a cycle. These are used in the upcoming result to characterize the existence of equilibrium. We do it in terms of symmetric half-integer solutions λ , that is, $\lambda_{ij} = \lambda_{ji} \in \{0, 1/2, 1\}$ for all $i, j \in N$.

Proposition 3. *Let $(N, v) \in \mathcal{P}$. The following statements are equivalent.*

1. *There exists an equilibrium in (N, v) .*
2. *A symmetric half-integer matrix without odd cycles solves the dual problem.*

Proof. $1 \Rightarrow 2$. If there exists an equilibrium, then, by Proposition 2, a symmetric permutation matrix λ solves the dual problem. As λ is integral, it is half-integral and has no cycles.

$1 \Leftarrow 2$. Assume there exists no equilibrium. Let λ be a symmetric half-integral solution to the dual problem. As there is no equilibrium, λ is not integral by Proposition 2, and hence $\lambda_{ij} = 1/2$ for some $i, j \in N$. By contradiction, assume λ has no cycle.

Assume first $\lambda_{ii} \in \{0, 1\}$ for all $i \in N$. Without loss of generality, assume $\lambda_{12} = 1/2$. By symmetry, $\lambda_{21} = 1/2$. As λ is doubly stochastic, $\sum_{j \in N} \lambda_{2j} = 1$, there exists $i \neq 1$ such that $\lambda_{2i} = 1/2$. As $\lambda_{22} \in \{0, 1\}$, $i \neq 2$. Without loss of generality, assume $i = 3$. As $\lambda_{21} + \lambda_{23} = 1$ and λ is doubly stochastic, $\lambda_{2i} = \lambda_{i2} = 0$ for all $i \neq 1, 3$. Again by symmetry, $\lambda_{32} = 1/2$. There must now exist $i \neq 2, 3$ such that $\lambda_{3i} = 1/2$. Suppose $i = 1$. Then $(1, 2, 3)$ is a cycle, which completes the proof. Suppose instead $i = 4$. By symmetry, $\lambda_{43} = 1/2$, and there must exist $i \neq 2, 3, 4$ such that $\lambda_{4i} = 1/2$. Repeat the same arguments. As the set of agents is finite, eventually we reach an agent k such that $\lambda_{k1} = 1/2$, which completes the cycle $(1, 2, \dots, k)$, a contradiction.

Assume instead $\lambda_{ii} = 1/2$ for some $i \in N$. Without loss of generality, assume $\lambda_{11} = \lambda_{12} = \lambda_{21} = 1/2$. If $\lambda_{22} = 1/2$, then λ for agents $\{1, 2\}$ is the average of the symmetric permutation matrices λ' and λ'' such that

$$\lambda' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \lambda'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If instead $\lambda_{22} = 0$, we have, say, $\lambda_{23} = \lambda_{32} = 1/2$. If $\lambda_{33} = 1/2$, then λ for agents $\{1, 2, 3\}$ is the average of the symmetric permutation matrices λ' and λ'' such that

$$\lambda' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \lambda'' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As there is a finite set of agents, we eventually will reach k such that $\lambda_{kk} = 1/2$. At that point, λ for agents $\{1, 2, \dots, k\}$ will be the average of two symmetric permutation matrices. By Lemma 1 there exists a solution $\tilde{\lambda}$ that is integral for agents $\{1, 2, \dots, k\}$. As long as $\tilde{\lambda}_{ii} = 1/2$ for some $i \in N$, repeat the above. Eventually we reach a symmetric permutation matrix that solves the dual problem, a contradiction as there then exists an equilibrium according to Proposition 2.

Therefore, every symmetric half-integer solution has a cycle. Supposing one of them only has even cycles, Lemma 2 assures that all those cycles are avoidable. Consequently, there would exist a solution without cycles, a contradiction. Hence, every such solution has odd cycles. \square

As noted in Proposition 2, the existence of equilibrium can be explained in terms of $v(N)$ and $P(v)$. Chiappori et al. (2012) examine the difference between these and find that

$$P(v) - v(N) \leq n^2 \max_{i,j \in N} v(\{i, j\}).$$

It should be noted that they primarily have larger problems in mind where multiple agents are of the same ‘‘type’’; n really corresponds to the number of types, not the number of agents. Our bound is hence tighter when agents differ (even just the slightest), whereas theirs is tighter for certain problems with many agents of the same type and few types.⁸

Proposition 4. *Let $(N, v) \in \mathcal{P}$. Then⁹*

$$v(N) \leq P(v) \leq v(N) + \frac{1}{2} \left\lceil \frac{n}{3} \right\rceil \max_{i,j \in N} v(\{i, j\}).$$

Proof. For the first inequality, see the discussion leading up to Proposition 2.

For the second, assume $v(N) < P(v)$. By Proposition 2, there exists no equilibrium. By Proposition 3, every symmetric half-integral solution to the dual problem has an odd cycle. Let λ be such a solution, and let $L = (1, 2, \dots, k)$ be an odd cycle in λ . Let λ' be symmetric and equal to λ except for $\lambda'_{1k} = 0$ and $\lambda'_{11} = \lambda'_{kk} = 1/2$. Then

$$\frac{1}{2} \sum_{i \in N} \sum_{j \in N} \lambda_{ij} v(\{i, j\}) - \frac{1}{2} \sum_{i \in N} \sum_{j \in N} \lambda'_{ij} = \frac{v(\{1, k\}) + v(\{k, 1\}) - v(\{1\}) - v(\{k\})}{4} = \frac{v(\{1, k\})}{2}.$$

Moreover, λ' can be written as the average of two permutation matrices for agents in L as in the second part of the proof of Proposition 3. Hence, ‘‘removing’’ the cycle L can be done by

⁸Say there are t types with k agents of each type, either t or k divisible by 3. Our bound is tighter whenever $k < 6t$, that is, if there are fewer agents per type than six times the number of types.

⁹By $\lceil x \rceil$ we denote the largest integer no larger than x .

decreasing the total value created by $v(\{1, k\})/2 \leq \max_{i, j \in N} v(\{i, j\})/2$. As cycles are of at least length 3, we need to “remove” at most $\lfloor n/3 \rfloor$ cycles to find a matrix without cycles.¹⁰ Hence, $v(N)$, which relates to a matrix λ free cycles, is at most

$$\frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \max_{i, j \in N} v(\{i, j\})$$

smaller than $P(v)$. □

Finally, we will reinterpret and generalize a result by Talman and Yang (2011, Theorem 1). To do so, we introduce the concept of a “subproblem”:

Definition 5. A subproblem of $(N, v) \in \mathcal{P}$ is $(M, w) \in \mathcal{P}$ such that $M \subseteq N$ and $w(\{i, j\}) = v(\{i, j\})$ for all $i, j \in M$.

Hence, a subproblem is a different partnership formation problem where some agents have been removed, though no changes have been made to the values created by the agents that remain.

Lemma 3. Let $(N, v) \in \mathcal{P}$ and suppose λ is a solution to the dual problem without avoidable cycles. Assume $L = (1, 2, \dots, k)$ is an odd cycle in λ . Then the following holds.

1. The subproblem $(L, w) \in \mathcal{P}$ has no equilibrium.
2. $v(\{i-1, i\}) + v(\{i, i+1\}) \geq v(\{i-1, i+1\})$ for all $i \in L$ taken modulo k .
3. $v(\{i, i+1\}) + v(\{j, j+1\}) \geq v(\{i, j+1\}) + v(\{j, i+1\})$ for all $i, j \in L$ taken modulo k .

Proof. Part 1. By contradiction, suppose (L, w) has an equilibrium. By Proposition 2, a symmetric permutation matrix λ' solves the dual problem for (L, w) . Then $\tilde{\lambda}$ such that

$$\tilde{\lambda}_{ij} = \begin{cases} \lambda'_{ij} & \text{if } i, j \in L \\ \lambda_{ij} & \text{otherwise} \end{cases}$$

is also a solution to the dual problem for (N, v) , with no more cycles than λ . Moreover, as L is not a cycle in $\tilde{\lambda}$, it must be avoidable, contradicting λ having no avoidable cycles.

Part 2. By contradiction, suppose there exists $i \in L$ such that

$$v(\{i-1, i\}) + v(\{i, i+1\}) < v(\{i-1, i+1\}).$$

Without loss of generality, suppose $i = 2$. Then $v(\{1, 2\}) + v(\{2, 3\}) < v(\{1, 3\})$. Construct the symmetric matrix $\tilde{\lambda}$ equal to λ with the exception of agents 1, 2, and 3, where $\tilde{\lambda}_{13} = 1/2$, $\tilde{\lambda}_{22} = 1$ and $\tilde{\lambda}_{12} = \tilde{\lambda}_{23} = 0$. Then $\tilde{\lambda}$ is feasible and produces a strictly greater value than λ for the dual problem in (N, v) . This contradicts λ being a solution to the dual problem.

¹⁰If L is a cycle, then the reverse of L is a cycle. Hence, an agent is part of exactly 0 or 2 cycles. By “removing” one cycle, we remove the reverse as well.

Part 3. By contradiction, suppose there exists a pair $i, j \in L$ such that

$$v(\{i, i+1\}) + v(\{j, j+1\}) < v(\{i, j+1\}) + v(\{j, i+1\}).$$

Similar to Part 2, construct the symmetric $\tilde{\lambda}$ equal to λ except for agents $i, i+1, j$, and $j+1$, where $\tilde{\lambda}_{i,j+1} = \tilde{\lambda}_{j,i+1} = 1/2$ and $\tilde{\lambda}_{i,i+1} = \tilde{\lambda}_{j,j+1} = 0$. Again, $\tilde{\lambda}$ is feasible and produces a strictly greater value for the dual problem. This contradicts λ being a solution to the dual problem. \square

We want to relate the above results to the following assumption presented by Talman and Yang (2011, Assumption 1). Note that they present it slightly differently due to not making the normalizing assumption $v(\{i\}) = 0$ for all $i \in N$ (see footnote 3).

Assumption 1 (Talman and Yang, 2011). *For any ordered list $L = (1, 2, \dots, k)$ of $k \geq 3$ agents and k odd, either*

1. $v(\{i-1, i\}) + v(\{i, i+1\}) < v(\{i-1, i+1\})$ for some $i \in L$ taken modulo k or
2. $v(\{i, i+1\}) + v(\{j, j+1\}) < v(\{i, j+1\}) + v(\{j, i+1\})$ for some $i, j \in L$ taken modulo k .

Our contribution is a different condition on (N, v) , which instead relates to the various subproblems of it.

Assumption 2. *Each subproblem (M, w) of (N, v) has an equilibrium.*

Surely, as Assumption 2 holds for $M = N$, it is sufficient for the existence of an equilibrium, though neither of the assumptions is necessary.¹¹ Moreover, the following relation holds:

Theorem 1. *Assumption 2 is a strictly less restrictive sufficient condition for the existence of an equilibrium in $(N, v) \in \mathcal{P}$ than Assumption 1.*

Proof. That Assumption 2 is sufficient holds by construction.

Suppose (N, v) satisfies Assumption 1. According to Theorem 1 of Talman and Yang (2011), there exists an equilibrium in (N, v) . By Proposition 2, there exists a symmetric permutation matrix that solves the dual problem for (N, v) . By contradiction, suppose a subproblem (M, w) of (N, v) does not have an equilibrium, say for $M = \{1, 2, \dots, m\}$. Let λ be a solution without avoidable cycles to the dual problem for (M, w) . By Proposition 3, there exists a cycle in λ , say $L = (1, 2, \dots, k)$. By Lemma 2, k is odd. But then the second and third part of Lemma 3 contradict Assumption 1 for L . Hence, λ has no cycle, in turn contradicting that (M, w) has no equilibrium. Hence, if Assumption 1 holds, then Assumption 2 holds as well. The relation is strict: in Example 2 all subproblems have an equilibrium (hence, Assumption 2 holds), but Assumption 1 is not satisfied for $(1, 2, 3)$. \square

¹¹Let $N = \{1, 2, 3, 4\}$ and v be as in Example 1 with the additions $v(\{2, 4\}) = 1$ and $v(\{i, 4\}) = 0$ for all $i \neq 2$. Then (μ, p) such that $\mu(1) = 3, \mu(2) = 4, p = (3, 1, 2, 0)$ is an equilibrium. However, the subproblem for $M = \{1, 2, 3\}$ is exactly Example 1, lacking an equilibrium. Neither assumption is satisfied even though there exists an equilibrium.

A final remark: Talman and Yang point out that two-sided assignment games are special cases of the one-sided model. They provide a way of incorporating a two-sided problem into the one-sided framework by carefully constructing certain parts of the value function. Just as they note that every two-sided problem satisfies their Assumption 1, it is not difficult to see that all such problems satisfy our more general Assumption 2 as well. All two-sided assignment games have an equilibrium (Shapley and Shubik, 1971), and hence all corresponding subproblems of a given two-sided problem must have an equilibrium.

5. A new solution concept: partnership-stable equilibrium

In this section, we introduce a new solution concept for the partnership formation problem, a partnership-stable equilibrium. This is a weakening of the equilibrium concept defined in Definition 1 in the sense that each equilibrium is also a partnership-stable equilibrium. Moreover, for instances that lack equilibria it more often than not suggests a solution. We first explore the concept through examples, and then analyse it using the LP framework developed in previous sections.

The core, through pairwise stability, enforces a specific way of dividing the values among the agents. Suppose there are four agents who at an equilibrium are divided into two pairs; the total value created is then the sum of the values for the two pairs. Assume now the first pair in total gets more than it produces. Then the other pair gets the rest – which is less than it produces. This is not pairwise stable and hence contradicts it being an equilibrium. Therefore, we have that the value created is split entirely among the agents creating it. Formally, if (μ, p) is an equilibrium in (N, v) , then, for all $i \in N$, we have $p_i = 0$ if $\mu(i) = i$ and $p_i + p_j = v(\{i, j\})$ if $\mu(i) = j \neq i$. Hence, the only transfers that are made are between agents that are matched. Here we want to use third-party payments as a tool to reach stable outcomes. To achieve this, we must let matched agents compensate other agents; hence, we can no longer require that they get at least what they produce. Therefore we propose a solution concept much in line with that of an equilibrium, but where the third requirement, pairwise stability, is weakened:¹²

Definition 6. A partnership-stable equilibrium in $(N, v) \in \mathcal{P}$ is a pair $(\mu, p) \in \mathcal{M} \times \mathbb{R}^n$ such that

1. $p_1 + p_2 + \dots + p_n = v(\mu)$
2. $p_i \geq 0$ for all $i \in N$
3. $p_i + p_j \geq v(\{i, j\})$ for all $i, j \in N$ such that $j \neq i$ and $j \neq \mu(i)$

A partnership-stable equilibrium should be interpreted as a state where no agents have incentives to break up or form new partnerships. It is straightforward that if there exists an equilibrium – hence *all* pairwise stability inequalities are satisfied – then it is also a partnership-stable equilibrium. The reverse is not true. A repetition of Example 1 will serve as an illustration of this:

¹²It can be argued that it, however, is not as much in line with the fundamental idea of the core. The core requires that every coalition – including those with more than two agents – should get at least what it produces, $\sum_{i \in S} p_i \geq v(S)$. By constructing the values $v(S)$ for these larger coalitions as in the paper by Shapley and Shubik (1971), we need only consider singles and pairs (as in the definition of equilibrium). A partnership-stable equilibrium then requires the same but only for coalitions $S \subseteq N$ such that $\{i, \mu(i)\} \not\subseteq S$ for all $i \in N$. Again, this is necessary if we want agents i and $\mu(i)$ to be able to compensate agents outside of S .

Example 3. Recall from Example 1 that $N = \{1, 2, 3\}$, $v(\{1, 2\}) = v(\{2, 3\}) = 3$, and $v(\{1, 3\}) = 5$. Consider the matching μ of agents 1 and 3 with agent 2 remaining single. This is not stable according to Definition 1, as agent 2 always can lure at least one of agents 1 and 3 away from the other. However, when constructing a partnership-stable equilibrium (μ, p) we need not satisfy $p_1 + p_3 \geq 5$: there is room for agents 1 and 3 to compensate agent 2. Therefore, suppose agent 2 receives one unit out of $v(\{1, 3\})$. He can now no longer make an offer to another agent that is beneficial for both of them. Hence, (μ, p) such that $p = (2, 1, 2)$ is a partnership-stable equilibrium.

Unfortunately there are also instances that lack partnership-stable equilibria (see Example 4). An interesting aspect of the partnership-stable equilibrium is that we in these cases can get a measure of “the cost of stability” for an instance by adding an artificial pair of agents. This cost could be interpreted as, for instance, how much value needs to be added to the market (say as a subsidy) for it to be stable.

Definition 7. Let $(N, v) \in \mathcal{P}$. Define the extended problem $(M, w) \in \mathcal{P}$ for $M = N \cup \{A, B\}$ and

1. $w(\{i, j\}) = v(\{i, j\})$ for all $i, j \in N$,
2. $w(\{A\}) = w(\{B\}) = w(\{A, i\}) = w(\{B, i\}) = 0$ for all $i \in N$, and
3. $w(\{A, B\}) = C$.

The cost of partnership-stability in (N, v) is the smallest $C \geq 0$ such that there exists a partnership-stable equilibrium (μ, p) with $\mu(A) = B$ in (M, w) .

Example 4 (No partnership-stable equilibrium and cost of stability). Let $N = \{1, 2, \dots, 5\}$ and suppose $v(\{i, j\}) = 2$ whenever $i \neq j$. By contradiction, suppose there exist a partnership-stable equilibrium (μ, p) in this problem. By symmetry, it is without much loss of generality to assume $\mu(1) = 2$, $\mu(3) = 4$ and $\mu(5) = 5$. The total value generated is $v(\mu) = 4$. With the weakened pairwise stability condition we need p to satisfy the following inequalities:

$$\begin{array}{cccc} p_1 + p_3 \geq 2 & p_1 + p_4 \geq 2 & p_1 + p_5 \geq 2 & p_2 + p_3 \geq 2 \\ p_2 + p_4 \geq 2 & p_2 + p_5 \geq 2 & p_3 + p_5 \geq 2 & p_4 + p_5 \geq 2. \end{array}$$

Adding all inequalities, we get $3(p_1 + p_2 + \dots + p_5) + p_5 \geq 16$. As the agents share exactly $v(\mu) = 4$ among themselves, this becomes $12 + p_5 \geq 16$, and $p_5 \geq 4$. Hence, agent 4 must get *all* profits that are created, whereas every other agent gets zero. This is not stable, as agents 1 and 3 can make themselves better off by splitting their joint profit of 2 among themselves. Hence, (μ, p) cannot be a partnership-stable equilibrium. This is a contradiction, and hence there cannot exist a partnership-stable equilibrium.

Suppose next a “dummy couple” is added, agents A and B , such that $v(\{A, i\}) = v(\{B, i\}) = 0$ for all agents i except for $v(\{A, B\}) = 1$. If we now extend μ above with $\mu(A) = B$, we find that (μ, p) such that $p = (1, 1, 1, 1, 0, 0)$ is a partnership-stable equilibrium. By the symmetric nature of the problem, it should be intuitively clear that no $v(\{A, B\}) < 1$ can achieve partnership-stability. Hence, the market can be made stable through a subsidy of no less than 1 unit.

The next example should be compared to Proposition 1, where it is stated that any equilibrium is optimal in the partnership formation problem. It no longer holds true for partnership-stable equilibria. However, this is not entirely unexpected – moving away from the core induces a trade-off between stability and optimality also in the paper by Farrell and Scotchmer (1988). In their paper, agents commit to sharing joint profits equally in partnerships, and inefficiency arises as more able agents are not willing to include less able agents in their partnership, as the more able would have to compensate the less.

Example 5 (Trade-off between stability and optimality). Let $N = \{1, 2, \dots, 5\}$ and suppose v is such that $v(\{i, j\})$ is the $(i, j)^{\text{th}}$ entry of the matrix below.

$$v = \begin{pmatrix} 0 & 9 & 2 & 8 & 8 \\ 9 & 0 & 9 & 6 & 8 \\ 2 & 9 & 0 & 9 & 0 \\ 8 & 6 & 9 & 0 & 6 \\ 8 & 8 & 0 & 6 & 0 \end{pmatrix}$$

The optimal matching is μ such that $\mu(1) = 2$, $\mu(3) = 4$, and $\mu(5) = 5$, with $v(N) = 18$. However, there exists no payoff vector p such that (μ, p) is an partnership-stable equilibrium. The “smallest” individually rational payoff vector satisfying the pairwise stability inequalities is $p = (5, 5, 4, 3, 3)$, adding up to 20.

On the other hand, (μ', q) such that $\mu'(1) = 5$, $\mu'(2) = 2$, and $\mu'(3) = 4$ together with $q = (2, 8, 1, 6, 0)$ is partnership-stable with $v(\mu') = 17$. Hence, μ' is not optimal, but nevertheless part of a partnership-stable equilibrium. Further illustrating the trade-off between stability and optimality, the gain in total value $18 - 17 = 1$ is less than the cost of making the optimal matching stable, $20 - 18 = 2$.

Let us now return to the LP framework. We can use the same method to find a partnership-stable equilibrium. It should be intuitively clear that if we fix a matching μ , then we can find the smallest amount needed to satisfy individual rationality and pairwise stability in a similar way to the first minimization problem (the primal), though without the constraints $p_i + p_j \geq v(\{i, j\})$ whenever $j = \mu(i)$. What may not be as clear is that this amount actually can be found by solving the primal for a different artificial value function. Specifically, the value function $v^\mu : \mathcal{N} \rightarrow \mathbb{R}$ such that, for all $i, j \in N$, we have

$$v^\mu(\{i, j\}) = \begin{cases} v(\{i, j\}) & \text{if } j \neq \mu(i) \\ 0 & \text{otherwise.} \end{cases}$$

When solving the primal problem for (N, v^μ) ,

$$\begin{aligned} P(v^\mu) = \underset{p \in \mathbb{R}^n}{\text{minimize}} \quad & \sum_{i \in N} p_i \\ \text{subject to} \quad & p_i \geq 0 \quad \forall i \in N; \quad p_i + p_j \geq v^\mu(\{i, j\}) \quad \forall i, j \in N: j \neq i \end{aligned}$$

the pairwise stability constraint for two matched agents i and $\mu(i)$ will never affect the solution as this constraint is already satisfied by the agents' respective individual rationality constraints.

Hence, the solution to the primal problem will turn out to be exactly the solution to the problem without this pairwise stability constraint. It then is just a matter of comparing this to the total value produced (using the actual value function v).

Theorem 2. *Let $(N, v) \in \mathcal{P}$. The following statements are equivalent.*

1. *There exists a partnership-stable equilibrium in (N, v)*
2. *$v(\mu) \geq P(v^\mu)$ for some $\mu \in \mathcal{M}$.*

Proof. $1 \Rightarrow 2$. Assume (μ, p) is a partnership-stable equilibrium in (N, v) . Then

1. $p_1 + p_2 + \dots + p_n = v(\mu)$,
2. $p_i \geq 0$ for all $i \in N$, and
3. $p_i + p_j \geq v(\{i, j\})$ for all $i, j \in N$ such that $j \neq i$ and $j \neq \mu(i)$.

Define $v^\mu : \mathcal{N} \rightarrow \mathbb{R}$ such that

$$v^\mu(\{i, j\}) = \begin{cases} 0 & \text{if } j = \mu(i) \\ v(\{i, j\}) & \text{otherwise.} \end{cases}$$

As $p_i + p_j \geq v(\{i, j\}) \geq v^\mu(\{i, j\})$ for all $i, j \in N$, p is feasible in the following LP problem:

$$\begin{aligned} P(v^\mu) = \text{minimize} \quad & \sum_{i \in N} p_i \\ & \text{subject to } p_i \geq 0 \forall i \in N; \quad p_i + p_j \geq v^\mu(\{i, j\}) \forall i, j \in N: j \neq i. \end{aligned}$$

Consequently, $v(\mu) = \sum_{i \in N} p_i \geq P(v^\mu)$.

$2 \Leftarrow 1$. Suppose there exists $\mu \in \mathcal{M}$ such that $v(\mu) \geq P(v^\mu)$. Assume p' solves the primal problem, $P(v^\mu) = \sum_{i \in N} p'_i$. Then $p'_i \geq 0$ for all $i \in N$ and $p'_i + p'_j \geq v^\mu(\{i, j\})$ for all $i, j \in N$ such that $j \neq i$. Define $p = p'$ with the exception that $p_1 = p'_1 + v(\mu) - P(v^\mu)$. Then $p_i \geq p'_i \geq 0$ for all $i \in N$ and $p_i + p_j \geq p'_i + p'_j \geq v^\mu(\{i, j\}) = v(\{i, j\})$ for all $i, j \in N$ such that $j \neq i$ and $j \neq \mu(i)$. Moreover,

$$\sum_{i \in N} p_i = \sum_{i \in N} p'_i + v(\mu) - P(v^\mu) = P(v^\mu) + v(\mu) - P(v^\mu) = v(\mu).$$

Hence, (μ, p) is a partnership-stable equilibrium in (N, v) . □

When applied to Example 1 with the matching μ such that $\mu(1) = 3$ and $\mu(2) = 2$, we have

$$v = \begin{pmatrix} 0 & 3 & 5 \\ 3 & 0 & 3 \\ 5 & 3 & 0 \end{pmatrix} \quad \text{and} \quad v^\mu = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}.$$

The unique solution to the primal problem is $P(v^\mu) = 3$ with $p = (0, 3, 0)$. As $P(v^\mu) = 3 < 5 = v(\mu)$, there exists a partnership-stable equilibrium.

Finally, we can calculate the cost of partnership-stability for any problem (recall, this cost is zero whenever a partnership-stable equilibrium exists):

Proposition 5. *If there does not exist a partnership-stable equilibrium in $(N, v) \in \mathcal{P}$, then the cost of partnership-stability C is*

$$C = \min_{\mu \in \mathcal{M}} \{P(v^\mu) - v(\mu)\} \leq \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \max_{i, j \in N} v(\{i, j\}).$$

Proof. Define the extended problem $(M, w) \in \mathcal{P}$ for $M = N \cup \{A, B\}$ such that

1. $w(\{i, j\}) = v(\{i, j\})$ for all $i, j \in N$,
2. $w(\{A\}) = w(\{B\}) = w(\{A, i\}) = w(\{B, i\}) = 0$ for all $i \in N$, and
3. $w(\{A, B\}) = C \geq 0$.

For each matching μ on N , let μ_e denote the extension to M that includes $\mu_e(A) = B$. Define also w^{μ_e} such that

$$w^{\mu_e}(\{i, j\}) = \begin{cases} 0 & \text{if } j = \mu_e(i) \\ w(\{i, j\}) & \text{otherwise.} \end{cases}$$

The following primal problem adds only trivial restriction to the corresponding primal problem for v^μ .

$$\begin{aligned} P(v^{\mu_e}) = \text{minimize} & \sum_{i \in M} p_i \\ & \text{subject to } p_i \geq 0 \forall i \in M; \quad p_i + p_j \geq w^{\mu_e}(\{i, j\}) \forall i, j \in M: j \neq i \end{aligned}$$

Hence, we have $P(v^{\mu_e}) = P(v^\mu)$. On the other hand, the total value has increased by C :

$$w(\mu_e) = \frac{1}{2} \sum_{i \in M} w(\{i, \mu_e(i)\}) = \frac{1}{2} \sum_{i \in N} v(\{i, \mu(i)\}) + C = v(\mu) + C.$$

By Theorem 2, (M, w) has a partnership-stable equilibrium (μ_e, p) if and only if

$$P(v^{\mu_e}) \leq w(\mu_e) \Leftrightarrow P(v^\mu) \leq v(\mu) + C \Leftrightarrow C \geq P(v^\mu) - v(\mu).$$

The smallest such C is

$$C = \min_{\mu \in \mathcal{M}} \{P(v^\mu) - v(\mu)\}.$$

Part 2. Let μ denote the optimal matching. Then, by Proposition 4, we have

$$C = \min_{\mu \in \mathcal{M}} \{P(v^\mu) - v(\mu)\} \leq P(v^\mu) - v(\mu) \leq P(v) - v(\mu) = P(v) - v(N) \leq \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \max_{i, j \in N} v(\{i, j\}). \quad \square$$

6. Simulations

In this section, we approximate by computer simulations how often the partnership formation problem is solvable. We run 1000 simulations for each set of parameters. Table 1 displays the number of solvable instances. The number of agents varies from 4 to 12. The joint values $v(\{i, j\})$ are drawn from the discrete uniform distribution with support on either $\{-20, \dots, 10\}$, $\{-10, \dots, 20\}$, or $\{0, \dots, 30\}$.¹³ All negative $v(\{i, j\})$'s are set to zero. Hence, for the first distribution

Number of agents	Solution concept									
	Equilibrium			Optimal, partnership-stable			Partnership-stable			
4	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
5	940	653	211	1000	984	879	1000	989	895	
6	983	976	975	1000	1000	1000	1000	1000	1000	1000
7	890	464	81	1000	977	846	1000	990	890	
8	957	946	957	1000	1000	1000	1000	1000	1000	1000
9	843	300	37	1000	978	810	1000	989	852	
10	949	924	928	1000	1000	1000	1000	1000	1000	1000
11	784	191	12	1000	966	814	1000	986	852	
12	922	906	897	1000	1000	1000	1000	1000	1000	1000
Lower bound	-20	-10	0	-20	-10	0	-20	-10	0	
Upper bound	10	20	30	10	20	30	10	20	30	

Table 1: For each set of parameters, 1000 instances of the partnership formation problem were simulated. The number of agents varies from 4 to 12, as specified by the leftmost column. The joint values are drawn from the discrete uniform distribution with support on $\{-20, \dots, 10\}$, $\{-10, \dots, 20\}$, and $\{0, \dots, 30\}$, as indicated on the final two lines. For instance, “653” in the second row, second column, means that 653 out of 1000 instances had an equilibrium. Moreover, these instances consisted of five agents, and the joint values were drawn from the discrete uniform distribution with support on $\{-10, \dots, 20\}$. Three columns to the right, 984 of the instances had an optimal partnership-stable equilibrium, and, another three columns to the right, five more had a partnership-stable equilibrium, though none optimal.

roughly one third of the partnerships will be fruitful (positive joint values). The corresponding fraction for the second and third distribution is two thirds and one.

Based on Table 1, we can draw a number of conclusions. First, whether an instance has a solution or not depends clearly on whether there is an even or odd number of agents. With an odd number of agents, someone has to be single, while everyone can find a match if there is an even number of agents. The results for the partnership-stable equilibrium in the case of even number of agents may be somewhat misleading. Surely, it is *unlikely* that such an instance lacks a partnership-stable equilibrium, but it nevertheless is possible. For instance, if we add an agent who creates a value of zero in all partnerships to an unsolvable instance with an odd number of agents, we get an unsolvable instance with an even number of agents.

Second, with some minor exceptions, considering problem only with odd (or only even) number of agents, the more agents there are, the fewer problems are solvable. Intuitively, a larger market is more difficult to coordinate than a smaller.

Third, the more “competitive” the market is, in the sense that a larger fraction of the partnerships are meaningful, the fewer instances are solvable. The case of low competition, where

¹³The simulations were repeated with these distributions expanded by a factor of 10, 100, and 1000. The differences were too small to be safely deemed anything but random.

in expectation roughly one third of the joint values are positive, a lot of the pairwise stability constraints are trivially satisfied. Again, the results for the partnership-stable equilibrium may be deceptive: it is not probable, but still possible for an instance with low competition to lack a partnership-stable equilibrium.

Fourth, if there exists a partnership-stable equilibrium, then there exists an optimal one roughly 98.8 % of the times. This tells us that the trade-off between efficiency and stability exists but is far from severe; see the discussion leading up to Example 5.

Fifth, for odd (even) number of agents, there existed an equilibrium in 45.1 % (95.5 %) of the instances. For the partnership-stable equilibrium, the corresponding percentages are 95.4 % and 100 %. Hence, lack of equilibrium is indeed a significant issue, and is resolved close to always by third-party payments.

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