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A Review of the Existence of Stable Roommate Matchings

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A Review of the Existence of Stable Roommate Matchings

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A review of the existence of stable roommate matchings

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Abstract

We compare different preference restrictions that ensure the existence of a stable roommate matching. Some of these restrictions are generalized to allow for indifferences as well as incomplete preference lists, in the sense that an agent may prefer remaining single to matching with some agents. We also introduce a new type of cycles and in greater detail investigate the domain of preferences that have no such cycles. In particular, we show how the absence of these cycles relates to the “symmetric utilities hypothesis” by Rodrigues-Neto (Journal of Economic Theory 135, 2007) when applied to roommate problems with weak preferences.

Keywords: Roommate problem, existence, stable matching, cycles

JEL: C62, C78

1. Introduction

When Gale and Shapley (1962) introduced the roommate problem, they did so merely to highlight an important difference to the two-sided marriage problem: some instances do not have stable outcomes. The essentials of the roommate problem is a set of agents, allowed to pair up in couples or stay single, together with individual-specific preferences over partners. A stable matching is a subdivision of the agents into pairs and singles such that no agent prefers being single to being matched with his partner, and no two agents prefer one another over their respective partners. The following example not only illustrates the absence of a stable outcome, but also sheds light on the cyclical preference structure that is central throughout the paper.

Suppose there are three agents, 1, 2, and 3, with preferences as follows. Agent 1 prefers 2 to 3; agent 2 prefers 3 to 1; and agent 3 prefers 1 to 2. Each of them rather finds a partner than remains single. Suppose now agents 1 and 2 match while agent 3 stays single. Then agent 2 prefers 3 to 1, and agent 3 prefers 2 to being single. Hence, the matching is not stable: agents 2 and 3 can mutually benefit by deviating from it. By symmetry, the argument extends to the other matchings. Hence, there exists no stable matching.

We will consider a more general model than the one originally put forth by Gale and Shapley (1962). First, agents may remain single. We can therefore analyse models with an odd number

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of agents. Second, agents need not prefer every potential partner to being single.¹ Third, an agent may be indifferent between two different agents, in contrast to the case where each agent strictly orders any two different partners. These changes capture, for example, that people tend to have limited networks. If an agent only is interested in collaborating with his friends, he can group all the others at the bottom of his preference list, below remaining single.

Our first contribution is to compare some preferences restrictions that grant the existence of stable matchings.² These include “no odd rings” by Chung (2000), “stable partitions” by Tan (1991), and a generalized form of “ α -reducible” preferences introduced by Alcalde (1995).³ Our second contribution is to examine preferences that have no so called “weak cycles”. The main result shows that such preferences are equivalent to those satisfying a generalization of the “symmetric utilities hypothesis” (Rodrigues-Neto, 2007) and to those of roommate problems with “globally ranked pairs” (Abraham et al., 2008).

The remainder of this paper is organized as follows. We introduce the model in the next section. In Section 3, we examine different restrictions on preferences that admit stable matchings. Preferences without weak cycles are investigated in Section 4.

2. Model

There is a finite set of agents $N = \{1, 2, \dots, n\}$. We denote arbitrary agents i , j , and k . A matching $\mu : N \rightarrow N$ is a one-to-one mapping on N such that $\mu(i) = j \Leftrightarrow \mu(j) = i$ for all $i, j \in N$. Agents may be single (i.e., matched to themselves). \mathcal{M} denotes the set of matchings. Each i has preferences \succsim_i over N . Each agent may be indifferent between different agents and he may prefer being single to matching with some agents. We use the following notation: i finds j as least as good as k if $j \succsim_i k$. If in addition $k \succsim_i j$, then $j \sim_i k$. Otherwise, if $j \succsim_i k$ and $k \not\succeq_i j$, then $j >_i k$. We collect the preferences in $\succsim = (\succsim_i)_{i \in N}$. A roommate problem is defined by a pair (N, \succsim) . \mathcal{R} denotes the set of roommate problems. Moreover, $\mathcal{S} \subset \mathcal{R}$ denotes the set of roommate problems with strict preferences, that is, $(N, \succsim) \in \mathcal{S}$ whenever $(N, \succsim) \in \mathcal{R}$ and $j \sim_i k \Leftrightarrow j = k$ for all $i, j, k \in N$.

Definition 1. $\mu \in \mathcal{M}$ is stable in $(N, \succsim) \in \mathcal{R}$ if, for all $i, j \in N$,

$$j >_i \mu(i) \Rightarrow \mu(j) \succsim_j i.$$

Plainly, a matching is stable if there is no agent who strictly prefers being single to being matched with his current partner, nor any pair of agents who strictly prefer one another to their respective partners. Note *strictly*: one agent being better off while the other is indifferent is not an objection to stability. This can for instance be interpreted as there being a small cost to breaking up or entering a partnership.⁴

¹This has been denoted “incomplete preference lists” (Irving and Manlove, 2002).

²We merely mention the concepts here; they are defined and discussed in detail in the upcoming sections.

³This is a selection, not an exhaustive list: stable matchings have for instance also been characterized in terms of “simple matchings” (Sotomayor, 2005).

⁴The notion of “weak stability” is common for weak preferences. See Chung (2000), Demange and Gale (1985), Gale and Sotomayor (1985), and Pycia (2012) among others. See Irving and Manlove (2002) for alternative stability concepts.

Agent	Preferences \succsim	Preferences \succsim'
1	2 3 1	2 3 1
2	3 2 1	<u>1</u> 3 2
3	1 2 3	1 2 3

Table 1: Preferences ranging from most to least preferred for Example 1. Underline indicates indifference. Bold font marks a stable matching.

3. Restrictions on preferences that admit stable matchings

In their paper, Gale and Shapley (1962) specifically point out that it is not the case that every roommate problem has a stable matching. This has spurred a significant amount of research aimed at identifying preference restrictions that grant the existence of stable roommate matchings. We will compare and in some cases generalize various such restrictions, commonly revolving around limiting cycles in the preferences in one way or another.

Definition 2. Let $(N, \succsim) \in \mathcal{R}$ and $L = (1, 2, \dots, k)$ be an ordered list of $k \geq 3$ agents.⁵ Below each i is taken modulo k .

1. L is a weak cycle in (N, \succsim) if
 - (a) $\forall i \in L, i+1 \succsim_i i-1$
 - (b) $\exists i \in L: i+1 \succ_i i-1$.
2. L is a strict cycle in (N, \succsim) if
 - (a) $\forall i \in L, i+1 \succ_i i-1$.
3. L is a ring in (N, \succsim) if
 - (a) $\forall i \in L, i+1 \succsim_i i-1 \succsim_i i$
 - (b) $\forall i \in L, i+1 \succ_i i-1$ if i is odd.
4. L is a strict ring in (N, \succsim) if
 - (a) $\forall i \in L, i+1 \succ_i i-1 \succ_i i$.

Note that every strict cycle and every ring is a weak cycle – or, reversing the argument – if there are no weak cycles, there can be neither strict cycles nor rings. Strict cycles and rings are less related: as can be seen in the following example, there are strict cycles that are not rings, and rings that are not strict cycles.

Example 1 (Cycles and rings). Consider agents $N = \{1, 2, 3\}$ and preferences \succsim, \succsim' as in Table 1. Here, $(1, 2, 3)$ is a weak cycle in both (N, \succsim) and (N, \succsim') , a strict cycle only in (N, \succsim) , and a ring only in (N, \succsim') . Both problems have stable matchings; in (N, \succsim) agent 2 is single, in (N, \succsim') agent 3.

When looking for preference restrictions that grant the existence of a stable matching, it is natural to (to some extent) eliminate cycles. This is informally explained and exemplified in the

⁵We avoid naming the agents x_1 through x_k simply to avoid the aesthetically displeasing double subscripts. We let also L denote the set $\{1, 2, \dots, k\}$. This should cause no confusion.

leftmost graph of Figure 1. The five vertices represent five agents, and the directed arcs represent a cycle (1, 2, 3, 4, 5). In other words, every agent (weakly) prefers the agent he is pointing at to the one pointing at him. The twisted lines represent a matching of agents 1 and 2 and agents 3 and 4. If we divide an odd number of agents into pairs, of course someone – here agent 5 – has to be single. But then the agent before him in the cycle (agent 4) prefers him (agent 5) to his match, agent 3. Hence, if agent 5 prefers the agent before him (agent 4) ahead of staying single, the matching is unstable.

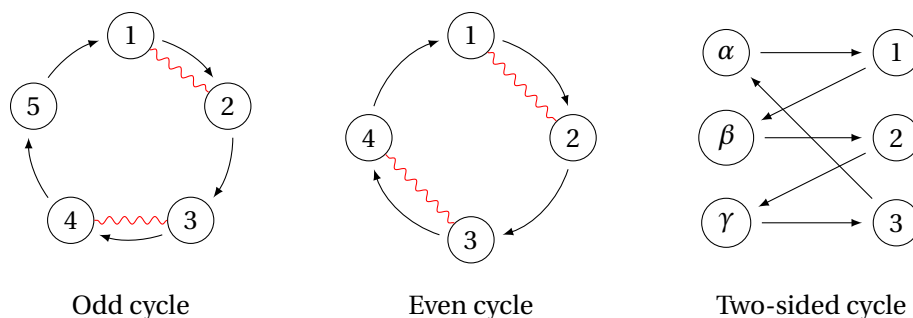


Figure 1: The leftmost graph illustrates the odd cycle (1, 2, 3, 4, 5). Every vertex is an agent pointing at the next agent in the cycle. The center graph illustrates the even cycle (1, 2, 3, 4). The rightmost graph illustrates that cycles in the two-sided version of the problem have to be even.

A first elementary result, that we examine further in Section 4, is the following:

Claim 1. *If there exists no weak cycle in $(N, \succ) \in \mathcal{R}$, then there exists a stable matching in (N, \succ) .*

This result is not new to the literature; we will see that it follows as a corollary to some previously known results (e.g. Proposition 1).

It is somewhat unsatisfactory to rule out cycles altogether as not all cycles are problematic. This is illustrated in the center graph of Figure 1, where the cycle (1, 2, 3, 4) instead consists of an even number of agents.⁶ In this case, no one is “forced” to be single when we divide the agents into pairs. It turns out that these even cycles always can be resolved:

Claim 2. *If there exists no odd weak cycle in $(N, \succ) \in \mathcal{R}$, then there exists a stable matching in (N, \succ) .*

Again, this claim is stated without a proof as it follows by Proposition 1. It also serves as an explanation to why stability is not a problem in two-sided markets. In the rightmost graph of Figure 1, the workers 1, 2, and 3 are on one side of the market, whereas the firms α , β , and γ are looking to hire them. Any cycle on such a market follows the pattern “firm, worker, firm, worker, ..., firm, worker”. Hence, it is even – and consequently poses no issues for stability (see also Chung, 2000, p. 120).

⁶Note that the stable matching does not have to be as depicted; we may have to resolve the cycle (1, 2, 3, 4) by matching agents 1 and 3, 2 and 4 instead for instance.

Recall, in order for it to be an objection to stability, agent 5 must prefer agent 4 to being single in the leftmost cycle. This is usually referred to as agent 4 being *acceptable* to agent 5 (Chung, 2000; Klaus and Klijn, 2010). If this is not the case, then the odd weak cycle is not problematic. This is captured by rings:

Proposition 1 (Chung, 2000). *If there exists no odd ring in $(N, \succ) \in \mathcal{R}$, then there exists a stable matching in (N, \succ) .*

Note that the two previously stated claims follow immediately: if there are no weak cycles – and hence no odd weak cycle, in particular no odd ring – then there exists a stable matching. Note also that the restriction is not necessary: in Example 1, (N, \succ') has a stable matching even though there exists an odd ring. To explain the existence of a stable matching in this particular problem, we can first observe that it is of the following type:

Definition 3. $(N, \succ) \in \mathcal{R}$ is weakly α -reducible if, for all $M \subseteq N$, $M \neq \emptyset$, there exists $i, j \in M$ such that

$$j \succ_i k \text{ and } i \succ_j k \text{ for all } k \in M.$$

In other words, a roommate problem is weakly α -reducible if in every subset of agents there are two who prefer one another ahead of the others (or one who prefers being single). The concept originated in a paper by Alcalde (1995), but then in a more restrictive form.⁷ He shows that if a roommate problem is “ α -reducible”, then there exists a stable matching. We consider a larger domain of preferences, and hence this proposition extends the result of Alcalde (1995):⁸

Proposition 2. *If $(N, \succ) \in \mathcal{R}$ is weakly α -reducible, then there exists a stable matching in (N, \succ) .*

Proof. Initialize $t = 0$ and $M^0 = N$. As (N, \succ) is weakly α -reducible, there exists $i, j \in M^t$ such that $j \succ_i k$ and $i \succ_j k$ for all $k \in M^t$. Set $\mu(i) = j$, $\mu(j) = i$, $M^{t+1} = M^t - \{i, j\}$, $t = t + 1$ and repeat until $M^t = \emptyset$.

For every $i, j \in N$, there exists t such that $i, j \in M^t$ and either $i \notin M^{t+1}$ or $j \notin M^{t+1}$. Assume, without loss of generality, $i \notin M^{t+1}$. Then $\mu(i) \succ_i k$ for all $k \in M^t \ni j$, implying $\mu(i) \succ_i j$. Hence, for no $i, j \in N$, we have both $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$. Consequently, μ is stable in (N, \succ) . \square

It should be clear from the proof that we need not impose the constraint on all subsets $M \subseteq N$. It is enough if we can construct, as in the proof, a finite sequence of sets M^0, M^1, \dots that have the property that two agents have one another as favourites.

Example 1 illustrates that weak α -reducibility is not a special case of Chung’s (2000) “no odd rings” condition:⁹ (N, \succ') is weakly α -reducible and $(1, 2, 3)$ is an odd ring. It also shows that the reverse is not true: (N, \succ) is not weakly α -reducible even though there is no odd ring.

⁷Alcalde (1995) only considers roommate problems $(N, \succ) \in \mathcal{S}$ such that every agent finds all agents acceptable. Under these assumptions, (N, \succ) is α -reducible if, for each subset of at least two agents $M \subseteq N$, there exist agents $i, j \in M$, $i \neq j$, such that $j \succ_i k$ for all agents $k \neq j$ in M and $i \succ_j k$ for all agents $k \neq i$ in M .

⁸Alcalde’s (1995) result is also generalized by Banerjee et al. (2001) to a model where agents can form larger coalitions consisting of more than two agents.

⁹Neither is Alcalde’s (1995) original α -reducibility. Consider $N = \{1, 2, 3, 4, 5\}$ and \succ such that

$$3 \succ_1 2 \succ_1 5 \succ_1 4 \succ_1 1 \quad 4 \succ_2 3 \succ_2 1 \succ_2 5 \succ_2 2 \quad 1 \succ_3 4 \succ_3 2 \succ_3 5 \succ_3 3 \quad 2 \succ_4 5 \succ_4 3 \succ_4 1 \succ_4 4 \quad 2 \succ_5 1 \succ_5 4 \succ_5 3 \succ_5 5.$$

Then $(1, 2, 3, 4, 5)$ is a strict cycle and an odd ring even though (N, \succ) is α -reducible.

An instance of the roommate problem in $\mathcal{R} - \mathcal{S}$ must have an agent who is indifferent between two different agents. These ties in the preference list can be broken in different ways: if i is indifferent between j and k , then the tie can be broken by putting j in front of k or k in front of j . By repeatedly breaking ties, we ultimately obtain a roommate problem in \mathcal{S} :

Definition 4. $(N, \succ'_i) \in \mathcal{S}$ is obtained from $(N, \succ) \in \mathcal{R}$ by breaking ties if, for all $i, j, k \in N$,

$$j \succ'_i k \Rightarrow j \succ_i k.$$

The following proposition is based on the fact that the stability concept requires both agents to strictly improve in order to deviate from a matching.

Proposition 3 (Irving and Manlove, 2002). *There exists a stable matching in $(N, \succ) \in \mathcal{R}$ if and only if there exists a stable matching in some $(N, \succ'_i) \in \mathcal{S}$ obtained from (N, \succ) by breaking ties.*

Taken together with Proposition 1, as long as there exists one way of breaking the ties that removes all odd rings, we are sure that there exists a stable matching. This is straightforward for (N, \succ'_i) in Example 1: break the tie $1 \sim'_2 3$ in favour of 1 ahead of 3.

By combining the above result by Irving and Manlove (2002) with the paper by Tan (1991) on strict preferences, we can find a sufficient and necessary condition for existence of stable roommate matchings under weak preferences. First however, we introduce the main idea presented by Tan (1991) for problems in \mathcal{S} .

Intuitively, one can think of a matching as a partitioning of the agents into sets of singles and pairs. Suppose we extend this idea to take strict rings into account. Let A_1, A_2, \dots, A_k be such that each $i \in N$ is in exactly one A_j , and each A_j is either a single agent, a pair of agents, or a ring. It will be convenient to represent this using a function $\phi : N \rightarrow N$ defined as follows. Take an arbitrary $i \in N$ and suppose $i \in A_j$. If A_j consists only of a single agent (i.e., i), let $\phi(i) = i$. If A_j is a pair, let $\phi(i)$ be the other agent in the pair. Finally, if A_j is a ring, let $\phi(i)$ be the agent immediately before agent i . In case i is the first agent of the ring, $\phi(i)$ is the last. Hence, if $A_3 = (2, 4, 5)$, then $\phi(4) = 2$ and $\phi(2) = 5$. Note that if there are no rings, then ϕ is a matching in \mathcal{M} . Denote the set of such functions \mathcal{H} . Introducing the terminology of Tan (1991):

Definition 5. Consider $\phi \in \mathcal{H}$ and $i, j \in N$.

1. j is superior for i at ϕ if $j \succ_i \phi(i)$
2. j is inferior for i at ϕ if
 - (a) $\phi(i) \succ_i j$ if i is single or in a pair, $\phi(\phi(i)) = i$
 - (b) $\phi(i) \succ_i j$ if i is in a ring, $\phi(\phi(i)) \neq i$.

In the words of Tan (1991), the following “stable partition” generalizes the notion of stable matchings:

Definition 6. $\phi \in \mathcal{H}$ is a stable partition in $(N, \succ) \in \mathcal{S}$ if, for all $i, j \in N$,

$$i \text{ is superior for } j \text{ at } \phi \Rightarrow j \text{ is inferior for } i \text{ at } \phi.$$

An immediate but nevertheless worthwhile observation is that a stable partition without rings is exactly a stable matching. It turns out that rings of even length are of no concern as they can be decomposed into pairs, and hence we have the following:

Proposition 4 (Tan, 1991). *There exists a stable matching in $(N, \succ) \in \mathcal{S}$ if and only if there exists a stable partition without odd rings in (N, \succ) .*

Tan (1991) shows that there always exists a stable partition, and that each stable partition contains all odd rings, in case there are any. Hence, if one stable partition contains an odd ring, then all of them do, and there cannot exist a stable matching. Taken together with Proposition 3, we immediately get the following:

Theorem 1. *There exists a stable matching in $(N, \succ) \in \mathcal{R}$ if and only if there exists a stable partition without odd rings in $(N, \succ') \in \mathcal{S}$ obtained from (N, \succ) by breaking ties.*

4. Preferences without weak cycles

We will devote the remainder of this paper to the preference restriction mentioned in the very first claim – the case of preferences entirely without weak cycles. This is a strictly smaller domain of preferences than those that we have considered so far (i.e., compared to weakly α -reducible problems and preferences without odd rings), but it nevertheless contains some interesting results. We will show that it can be characterized in two different ways and that preferences without weak cycles induce roommate problems that are weakly α -reducible.

The first way of characterizing the preference domain requires some additional definitions and terminology:

Definition 7. *Let $(N, \succ) \in \mathcal{R}$. $u : N \times N \rightarrow \mathbb{R}$ represents \succ if, for all $i, j, k \in N$,*

$$u(i, j) \geq u(i, k) \Leftrightarrow j \succ_i k.$$

This is a purely ordinal utility function: a higher utility is equivalent to a more preferred partner. The symmetric utilities hypothesis is introduced by Rodrigues-Neto (2007):

Definition 8. *$(N, \succ) \in \mathcal{R}$ satisfies the symmetric utilities hypothesis if there exists $u : N \times N \rightarrow \mathbb{R}$ that represents \succ such that, for all $i, j \in N$,*

$$u(i, j) = u(j, i).$$

It turns out that the symmetric utilities hypothesis is a special case of weak α -reducibility:

Proposition 5. *If $(N, \succ) \in \mathcal{R}$ satisfies the symmetric utilities hypothesis, then (N, \succ) is weakly α -reducible. The contrary is not true.*

Proof. Let $u : N \times N \rightarrow \mathbb{R}$ represent \succ such that $u(i, j) = u(j, i)$ for all $i, j \in N$. For all $M \subseteq N$, $M \neq \emptyset$, there exists $i, j \in M$ such that $u(i, j) \geq u(k, l)$ for all $k, l \in M$ as N and hence M is finite. In particular, $u(i, j) \geq u(i, k)$ and $u(j, i) \geq u(j, k)$ for all $k \in M$. Then $j \succ_i k$ and $i \succ_j k$ for all $k \in M$. Hence, (N, \succ) is weakly α -reducible.

Example 1 provides counter-examples for the opposite direction: both instances are weakly α -reducible, but none satisfies the symmetric utilities hypothesis. \square

Rodrigues-Neto (2007) shows that a roommate problem in \mathcal{S} satisfies the symmetric utilities hypothesis if and only if there are no strict cycles in the preferences. As can be seen in Example 1 this does not readily generalize to \mathcal{R} (there is no symmetric representation of \succsim' even though it contains no strict cycles). Instead, the following proposition extends the result of Rodrigues-Neto (2007) to the larger class of problems \mathcal{R} :

Proposition 6. $(N, \succsim) \in \mathcal{R}$ satisfies the symmetric utilities hypothesis if and only if there are no weak cycles in (N, \succsim) .

Proof. (i) Suppose that there exists a weak cycle in (N, \succsim) . Without loss of generality, assume it is $L = (1, 2, \dots, k)$ and $2 \succ_1 k$. By contradiction, suppose (N, \succsim) satisfies the symmetric utilities hypothesis and $u: N \times N \rightarrow \mathbb{R}$ represents \succsim with $u(i, j) = u(j, i)$. Then, by alternating between $u(i, i+1) \geq u(i, i-1)$, as L is a weak cycle, and $u(i+1, i) = u(i, i+1)$, as (N, \succsim) satisfies the symmetric utilities hypothesis, we have

$$u(k, 1) \geq u(k, k-1) = u(k-1, k) \geq u(k-1, k-2) \geq \dots u(2, 3) \geq u(2, 1) = u(1, 2).$$

Finally, as $2 \succ_1 k$, we get $u(k, 1) \geq \dots \geq u(1, 2) > u(1, k)$, contradicting that the symmetric utilities hypothesis is satisfied. Hence, it is satisfied only if there are no weak cycles in the preferences.

(ii) Suppose that (N, \succsim) has no weak cycles. An *augmented* weak cycle is an ordered list of $k+2$ agents $L = (0, 1, \dots, k, k+1)$, such that

1. $i+1 \succsim_i i-1$ for all $i \in \{1, 2, \dots, k\}$ and
2. $i+1 \succ_i i-1$ for some $i \in \{1, 2, \dots, k\}$.

Note that if agents $0 = k$ and $1 = k+1$, then $(1, 2, \dots, k)$ forms a weak cycle. Define the binary relation $<$ on $N \times N$ such that

$$(i, j) < (k, l) \Leftrightarrow \text{there exists an augmented weak cycle } L = (i, j, \dots, k, l).$$

We say that $(i, j) < (k, l)$ “by” L . We use “+” to denote joint augmented weak cycles:

$$(x_0, x_1, \dots, x_p) + (y_0, y_1, \dots, y_q) = (x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q).$$

First, note that $<$ is a partial strict order (transitive and irreflexive). For transitivity, suppose $(i, j) < (k, l)$ by L and $(k, l) < (m, n)$ by L' . Then $(i, j) < (m, n)$ by $L + L'$, as $k \sim_l k$ and $l \sim_k l$. To show that $<$ is irreflexive, suppose the contrary, that $(i, j) < (i, j)$ by L . As we noted immediately after the definition of an augmented weak cycle, removing the first (i) and last (j) elements of L leaves a weak cycle (j, \dots, i) – which we have assumed do not exist. Hence, the assumption of $<$ being reflexive is incorrect.

The relation $<$ exhibits other useful properties. If $(i, j) < (m, n)$ by L and $j \sim_i k$, then $(i, k) < (m, n)$ by $(i, k) + L$. Also, $j \succ_i k$ if and only if $(i, k) < (i, j)$: first, if $j \succ_i k$ then $(i, k) < (i, j)$ by (i, k, i, j) . Second, if $(i, k) < (i, j)$ by L and by contradiction $k \not\succsim_i j$, then removing the first (i) and last (j) elements of L leaves a weak cycle. Next, if $(i, j) < (k, l)$ by L , then $(j, i) < (k, l)$ by $(j) + L + (j)$. Hence, $<$ is “symmetric” in the sense that $(i, j) < (k, l) \Leftrightarrow (j, i) < (k, l)$.

We can iteratively create a utility function u from $<$. Initialize $t = 0$ and $M^0 = N \times N$. Let

$$T = \{(i, j) \in M^t : (i, j) \not\prec (k, l) \text{ for all } (k, l) \in M^t\}.$$

As there is a finite number of pairs in M^t and $<$ is a partial strict order, $T \neq \emptyset$.¹⁰ Set $u(i, j)$ equal to the number of elements of M^t for all $(i, j) \in T$. Note that $(i, j) \in T \Leftrightarrow (j, i) \in T$, and hence $u(i, j) = u(j, i)$: u is symmetric. Moreover, if $j \sim_i k$, then $(i, j) \in T \Leftrightarrow (i, k) \in T \Leftrightarrow u(i, j) = u(i, k)$. Also, if $j \succ_i k$, then $(i, k) < (i, j)$ and $u(i, k) < u(i, j)$. On the other hand, if $u(i, k) < u(i, j)$, then there exists (m, n) such that $(i, k) < (m, n)$ by $L = (i, k, \dots, m, n)$, but $(i, j) \not\prec (m, n)$. In particular, $(i, j) + L$ is not an augmented weak cycle. This can only be the case if $k \not\prec_i j$, that is, if $j \succ_i k$.

Therefore, we have $u(i, j) > u(i, k) \Leftrightarrow j \succ_i k$ and $u(i, j) = u(i, k) \Leftrightarrow j \sim_i k$. Hence, u represents \succsim . Finally, as u is symmetric, (N, \succsim) satisfies the symmetric utilities hypothesis. \square

Essentially, the second half of the proof shows that pairs of agents can be arranged into a partial strict order. This is related to a characteristic introduced by Abraham et al. (2008). They use a so called ranking function – a function that to every agent and to every pair of agents assigns a number corresponding to the particular “ranking” of that agent or couple. An instance of the roommate problem is said to have “globally ranked pairs” if the preferences of the agents are based on this ranking function, in the sense that agent i prefers agent j to agent k whenever the couple $\{i, j\}$ is ranked higher than the couple $\{i, k\}$.

Definition 9. $(N, \succsim) \in \mathcal{R}$ has globally ranked pairs if there exists $r : \{S \subseteq N : \#S \leq 2\} \rightarrow \mathbb{R}$ such that, for all $i, j, k \in N$,¹¹

$$j \succsim_i k \Leftrightarrow r(\{i, j\}) \geq r(\{i, k\}).$$

The relation to the symmetric utilities hypothesis is straightforward: if we interpret the ranking function as a utility function (i.e., $u(i, j) = r(\{i, j\})$ for all $i, j \in N$), then it follows immediately that u is a symmetric utility representation of the preferences. To summarize, we have the following.

Theorem 2. Let $(N, \succsim) \in \mathcal{R}$. The following statements are equivalent.

1. (N, \succsim) has no weak cycles.
2. (N, \succsim) satisfies the symmetric utilities hypothesis.
3. (N, \succsim) has globally ranked pairs.

Figure 2 provides a visual summary, illustrating how the different preference domains are interrelated. Recall that \mathcal{R} is the set of all roommate problems with weak preferences, whereas \mathcal{S} is the subset of \mathcal{R} only considering strict preferences.

Examined in Proposition 1, we can think of Chung’s (2000) preference restriction as applying to instances in the area “No odd rings”. Alcalde’s (1995) “ α -reducible” problems are then in $B \cup D \cup E$, whereas our generalization of this, investigated in Proposition 2, applies to instances in the area “Weakly α -reducible”. The “symmetric utilities hypothesis”, as considered by Rodrigues-Neto (2007), then holds for problems in $A \cup B$. Finally, our generalization examined in Theorem 2, equivalent to preferences without weak cycles and roommate problems with “globally ranked pairs” (Abraham et al., 2008), applies to the area $A \cup B \cup C$.

¹⁰Otherwise, through transitivity we can generate a cycle of related pairs contradicting that $<$ is irreflexive.

¹¹By $\#S$ we denote the cardinality of the set S .

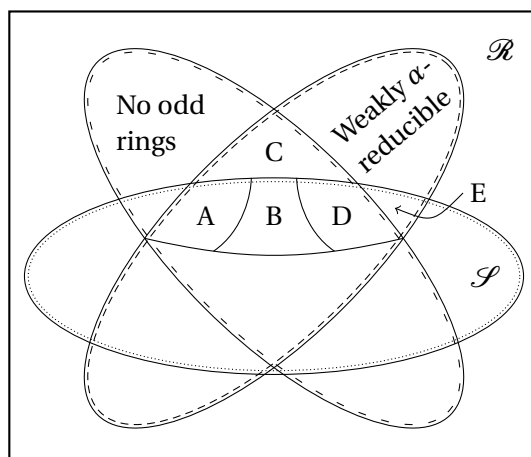


Figure 2: An illustration of the preference domains for various sets of problems and restrictions.

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