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NON-MANIPULABLE HOUSE ALLOCATION WITH RENT CONTROL

TOMMY ANDERSSON AND LARS-GUNNAR SVENSSON*

In many real-life house allocation problems, rents are bounded from above by price ceilings imposed by a government or a local administration. This is known as rent control. Because some price equilibria may be disqualified given such restrictions, this paper proposes an alternative equilibrium concept, called rationing price equilibrium, tailored to capture the specific features of housing markets with rent control. An allocation rule that always selects a rationing price equilibrium is defined, and it is demonstrated to be constrained efficient and (group) non-manipulable for “almost all” preference profiles. In its bounding cases, the rule reduces to a number of well-known mechanisms from the matching literature. In this sense, the housing market with rent control investigated in this paper integrates several of the predominant matching models into a more general framework.

KEYWORDS: House allocation, rent control, rationing, rationing price equilibrium, constrained efficiency, non-manipulability.

1. INTRODUCTION

HOUSE ALLOCATION is a classical problem in the mechanism design literature. The aim is to allocate a number of houses (or some other indivisible items such as jobs or tasks) among a group of agents given that each agent is interested in renting or buying at most one house. Several suggestions have been proposed for this problem. For example, when rents are exogenously given, and in the absence of existing tenants,¹ [Hylland and Zeckhauser \(1979\)](#) proposed the Serial Dictatorship Mechanism as a solution to this problem. This

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¹The problem with existing tenants is not considered in this paper. It was first investigated by [Shapley and Scarf \(1974\)](#). See also [Abdulkadiroğlu and Sönmez \(1998\)](#), [Ma \(1994\)](#), [Pápai \(2000\)](#), [Roth \(1982a\)](#), [Roth and Postlewaite \(1977\)](#), and [Svensson \(1999\)](#) among others. Also, the “kidney exchange problem” resembles a housing market with existing tenants, see [Roth et al. \(2004\)](#) for a detailed discussion. The house allocation model with both existing tenants and new applicants was first considered by [Abdulkadiroğlu and Sönmez \(1999\)](#).

deterministic rule assigns each agent a priority according to some criteria (random, queue, etc.) and the agents have to choose a house from the set of “remaining houses” when the agents with higher priorities have made their choices. [Svensson \(1994\)](#) demonstrated that a weaker version of this mechanism (where agents are allowed to have weak preferences) is Pareto efficient and non-manipulable. [Ergin \(2002\)](#) considered a more general priority structure, where agents are allowed to have diverse priorities for different houses, and provided necessary and sufficient conditions for an allocation rule to be Pareto efficient and non-manipulable.²

Even if the assumption of exogenously given and fixed rents is restrictive, it is satisfied in many real-life applications. Examples include on-campus housing and public housing. However, a priority-based fixed-price mechanism cannot take into consideration the existence of agents who are willing to accept higher rents to receive some other house in the allocation process. In this sense, there are potential welfare gains by allowing for more flexible rents. The obvious solution to this problem is to adopt a competitive price mechanism for the allocation procedure. This idea has been advocated by [Crawford and Knoer \(1981\)](#), [Demange and Gale \(1985\)](#), [Demange et al. \(1986\)](#), [Leonard \(1983\)](#), and [Shapley and Shubik \(1972\)](#) among others. The type of housing markets where such mechanisms are applicable are common in metropolitan areas. Also here, Pareto efficient and non-manipulable allocation rules exist. More precisely, because the set of equilibrium prices forms a complete lattice ([Demange and Gale, 1985](#); [Shapley and Shubik, 1972](#)), the existence of a unique minimal equilibrium price vector is guaranteed. Then by using this price vector as a direct mechanism for allocating the houses, a Pareto efficient outcome where no agent has any incentive to misrepresent his preferences is assured as demonstrated by [Demange and Gale \(1985\)](#).³

Fixed-price priority-based mechanisms and the competitive price mechanism can be regarded as two polar cases for allocating houses when there are no existing tenants as rents are totally non-flexible in the former and fully flexible in the latter. A third and intermediate practice for allocating houses is when a government or a local administration, by laws or ordinances, imposes a price ceiling and thereby allows for a limited flexibility in the rents. Even if there has been a widespread agreement among economists that this type of rent control generates a mismatch between houses and tenants, discourages new construction, retards maintenance, etc. (see, e.g., [Arnott, 1995](#); [Turner and Malpezzi, 2003](#), for a detailed discussion), this practice is widely used. As of 2011, legislated rent control existed in approximately 40 countries around the world.

A general problem in housing markets with rent controls is that there will typically be an unbalanced relationship between supply and demand (this is obviously also a problem for housing markets where fixed-price mechanisms are used). Thus, the phenomenon of a price rigidity arises and a rationing mechanism is normally needed to facilitate the distribution of houses among agents in addition to the rent leverage. This situation is studied in the

²For additional results and characterizations, see e.g., [Balinski and Sönmez \(1999\)](#), [Ehlers \(2002\)](#), [Erdil and Ergin \(2008\)](#), [Larsson and Svensson \(2005\)](#), [Pápai \(2000\)](#), and [Zhou \(1990\)](#).

³See also [Andersson and Svensson \(2008\)](#), [Leonard \(1983\)](#), [Sun and Yang \(2003\)](#), and [Svensson \(2009\)](#).

classical paper by Drèze (1975) where a variant of a competitive equilibrium based on rationing was introduced.⁴ To model such a housing market, the rents (or prices) must be bounded to belong to an exogenously given set that specifies the price ceilings as well as the minimal acceptable rent (price) for the landlord (the “reservation rent”). This is also the type of housing market considered in this paper. However, the model is not restricted to this type of interpretation as explained later in this section.

Because the set of price equilibria that respects the price ceilings may be empty for some preference profiles, a different equilibrium notion than the concept of a price equilibrium is needed to analyze the housing market with rent control. In this notion, the rationing mechanism will play a role as explained in the above. This paper assumes that the rationing mechanism is given by a priority structure, i.e., for each house, there is a strict priority-order that ranks the agents in terms of priority. This may, for example, reflect a situation where families are given higher priorities to three-bedroom apartments, whereas singles are given higher priorities to studio apartments. In the special case when the priority-order is identical for all houses, the priority structure collapses to the one adopted by, e.g., Hylland and Zeckhauser (1979) and Svensson (1994) for the Serial Dictatorship Mechanism.

As is well-known, the above type of priority structure have implications on which notion of efficiency that can be incorporated in the equilibrium concept. For example, (unconstrained) Pareto efficiency is in general incompatible with a priority respecting allocation of the houses (see, e.g., Balinski and Sönmez, 1999; Ergin, 2002, or the more detailed discussion in Section 4). However, this does not mean that it is impossible to base the notion of efficiency on Pareto improvements. In fact, the efficiency notion adopted in this paper essentially states that the assignment of the houses to the agents is “efficient” (or “constrained efficient” to use the language of this paper), at a given price vector, if the priority structure is respected, and if it is not possible to make a Pareto improvement by reallocating the houses among the agents, and at the same time, respect the priority structure. This efficiency notion is the key component in the definition of a rationing price equilibrium. The other components are the standard requirements (see, e.g., Talman and Yang, 2008; Zhu and Zhang, 2011) that the prices must belong to the price space, the price of any unassigned house must equal its lower price bound and the rationing mechanism should only be effective in the special case when the rent that is attached to a house equals its exogenously given ceiling.

Given a domain restriction where “almost all” preference profiles are considered, it is demonstrated that the set of rationing price equilibrium price vectors always contains a unique minimal vector. Using this insight, a rule where the minimal rationing price equilibrium vector is used as a direct mechanism for allocating the houses among the agents is defined. This rule is called the minimal RPE mechanism, and it is demonstrated to be constrained efficient and (group) non-manipulable for “almost all” preference profiles.⁵ In the limiting cases where rents are fully flexible or where rents are fixed, the outcome of the

⁴See also Cox (1980), Dehez and Drèze (1984), Kurz (1982), and van der Laan (1980), among others.

⁵For the remaining profiles, it is an open question how to define the allocation rule as the set of rationing price equilibrium vectors need not have a unique minimal element. See Example 3 and the discussion

mechanism is the same as for the Competitive Price Mechanism (Demange and Gale, 1985) and the (agent proposing) Deferred Acceptance Algorithm⁶ (Gale and Shapley, 1962), respectively.

As stated in the above, the framework introduced in this paper can be applied to other types of problems than house allocation with rent control. This is next illustrated with the aid of two real-life examples. The first is based on the skyscraper *HSB Turning Torso* located in Malmö, Sweden. This unique skyscraper has won numerous awards for its ground-breaking architecture.⁷ The owners of the building (HSB) decided to allocate the 147 apartments in the building using a fixed-price mechanism and a well-defined priority-order based on, e.g., membership and savings in the HSB fund. However, a combination of two circumstances made it very difficult for the owners to determine the fixed rents. More explicitly, the cost to build the house was more than twice as high as expected (1.6 billion SEK instead of 700 million SEK), and the uniqueness of the building made it very difficult to predict willingness-to-pay for potential tenants. The first of these facts suggests “high” rents whereas the latter is a warning that if the rents are “too high,” it will be difficult to find tenants. In the end, it was decided to set the rents below the break-even level. However, at the inauguration on August 27, 2005, all 147 apartments had a tenant, and there were around 350 persons in the queue willing to lease an apartment. This raises the following question: were the rents too low considering that many people obviously were interested in leasing an apartment and given the fact that costs not were covered? The answer to this question is probably yes. An alternative way of assigning the apartments to the potential tenants is to use the framework suggested in this paper, namely, to impose upper and lower bounds on the rents and then let the reported preferences determine the rents and the assignment of the apartments for the given priority structure.

A second example, and an alternative interpretation of the model, is the case when the indivisible items are jobs or positions. In many such markets, wages are bounded from below by legislated minimum wages (corresponding to price ceilings in the rent control model) and from above by the employers’ maximum willingness to pay for a worker (corresponding to the reservation rents of the landlord in the rent control model). As of 2011, this type of legislation was present in around 200 countries. On these markets, it is well established that workers often are paid above the minimum wage, e.g., members of the union (Shaviro, 1997) or workers employed by the local government (Wilkinson, 2004), which makes it relevant to study labor markets with minimum wages but where wages may exceed the legislated lower bound. The model introduced in this paper provides a framework for analyzing this type of labor market.

The paper is outlined as follows. Section 2 relates the concept of a rationing price equilibrium and the minimal RPE mechanism to various solution concepts and mecha-

following Definition 5.

⁶Note that houses are merely objects to be consumed in our framework exactly as colleges are merely objects to be consumed in the “student placement problem” (see, e.g., Balinski and Sönmez, 1999, for a detailed discussion).

⁷The process of erecting the building was covered in the documentary *Extreme Engineering* broadcast for the first time on *Discovery Channel* in July 2004.

nisms proposed in the literature. Section 3 introduces the house allocation model with rent control. Section 4 defines the priority structure and a notion of efficiency. The concept of a rationing price equilibrium is formally defined and analyzed in Section 5. A domain restriction is defined and thoroughly discussed in Section 6 where, in particular, the set of rationing price equilibrium vectors is demonstrated to contain a unique minimal price vector given the domain restriction. Section 7 defines the minimal RPE mechanism and demonstrates that it is (group) non-manipulable for “almost all” preference profiles. This mechanism is also related to several well-known mechanisms from the literature. Some concluding remarks are gathered in Section 8. All proofs, technical definitions, and technical results are relegated to the Appendix.

2. RELATED LITERATURE

To the best of our knowledge, this paper is the first to define an constrained efficient and (group) non-manipulable allocation rule for a housing market with rent control and with an exogenously given priority structure. This does, of course, not mean that this type of market has not been considered earlier. As already explained in Section 1, many classical papers consider such a market (see, e.g., [Drèze, 1975](#), or footnote 4), but the main focus in those papers is the weakening of the price equilibrium concept in exchange economies to handle the case with price rigidities when items are perfectly divisible. The case with indivisibilities on housing markets with price rigidities has recently been considered by [Talman and Yang \(2008\)](#) where the concept of a “constrained Walrasian equilibrium” (CWE, henceforth) is introduced. It is straightforward to demonstrate that any rationing price equilibrium (RPE, henceforth) can be transformed into an CWE by basing the “rationing scheme” (as defined in [Talman and Yang, 2008](#)) on the assignment of the houses. However, it is in general impossible to transform all CWE states into RPE states as the RPE states have to respect an exogenously given priority structure while no such restriction prevails for the CWE states. Hence, the set of RPE states is a proper subset of the set of CWE states. Also, as the CWE states have no built-in efficiency requirement, there are CWE states where it is possible to make a Pareto improving reallocation of the houses among the agents, and at the same time, respect all requirements of the concept for *any* given priority structure. This is not possible at any RPE state as an RPE state must be constrained efficient.

The efficiency concern is also the main motivation behind the study by [Zhu and Zhang \(2011\)](#) where a weakening of the CWE concept is introduced to include so-called “market efficient assignments.” This notion of efficiency is based on maximizing a specific sum of individual house valuations and upper price ceilings subject to that preferences are represented by quasi-linear utility functions. However, even if we restrict the preferences to be quasi-linear, the solution concept suggested by [Zhu and Zhang \(2011\)](#) is fundamentally different from an RPE as, again, a “market efficient” assignment need not respect a given priority structure and Pareto improvements that respect all requirements of their concept may be possible for *any* given priority structure.

Both [Talman and Yang \(2008\)](#) and [Zhu and Zhang \(2011\)](#) provide algorithms that identify their respective solution concepts. In these dynamic processes, the rationing scheme

is endogenously determined by the process. On the other hand, neither [Talman and Yang \(2008\)](#) nor [Zhu and Zhang \(2011\)](#) investigates if their respective mechanisms reach a non-manipulable end state or not. In addition, their sets of “equilibrium price vectors” will, in general, not contain a unique minimal element. Hence, the core problems of the present paper are not considered in [Talman and Yang \(2008\)](#) and [Zhu and Zhang \(2011\)](#).

The allocation rule considered in this paper is also one of few in the literature that integrates existing “efficient” and non-manipulable rules into a more general framework. A well-known example is the You-Request-My-House-I-Get-Your-Turn mechanism, introduced by [Abdulkadiroğlu and Sönmez \(1999\)](#), which integrates the Serial Dictatorship Mechanism with strict preferences ([Hylland and Zeckhauser, 1979](#)) and the Top-Trading Cycles Mechanism ([Shapley and Scarf, 1974](#)). Another example is the Kidney Exchange Mechanism, first investigated by [Roth et al. \(2004\)](#), where a generalized version of the Top-Trading Cycles Mechanism ([Shapley and Scarf, 1974](#)) that also takes trading chains into account is considered. A third example is [Pápai \(2000\)](#) where a large class of mechanisms that solve the house allocation problem is proposed. These rules are called Hierarchical Exchange Rules, and they can be regarded as a generalization of the Top-Trading Cycles Mechanism ([Shapley and Scarf, 1974](#)), even if no initial ownership is assumed in the model. An important difference between those mechanisms and the one considered in this paper is that the former mechanisms only work in the absence of monetary transfers whereas the rule considered here can be applied to housing markets independently of if monetary transfers can be carried out or not.

3. THE HOUSING MARKET WITH RENT CONTROL

There is a finite set of *houses* and a finite set of *agents* denoted by $H = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$, respectively. In some examples, the letters a, b, c, d , and e will be used to denote elements in H . The houses in H do not have any existing tenants by assumption. In some housing markets, it is natural to assume that there are several owners of the houses, but since our results do not require this, it is, for notational simplicity, assumed that there is a single owner of all houses (our arguments extend with only a few modifications to the case with multiple owners). Agents wish to buy, or rent, at most one house and have an option not to buy, or rent, a house at all. This outside option is formally represented by a *null house*, denoted by 0. These houses are available in an unlimited number of copies.

An *assignment* is a mapping $\mu : N \rightarrow H \cup \{0\}$ such that $\mu_i = \mu_{i'}$ for $i \neq i'$ only if $\mu_i = 0$. Hence, two distinct agents can not be assigned the same house in H . Denote by μ_0 the set of houses that is not assigned to any agent at assignment μ . The null house is always included in this set as its supply is unlimited. Hence:

$$\mu_0 = \{h \in H : \mu_i \neq h \text{ for all } i \in N\} \cup \{0\}.$$

Let $p \in \mathbb{R}^{m+1}$ be a *price vector*. A coordinate in p is denoted by p_h and it represents the price, or rent, of house $h \in H \cup \{0\}$. The price of the null house is, without loss of generality, always assumed to equal zero, i.e., $p_0 = 0$. A *state* is a pair (p, μ) consisting of a price vector and an assignment.

Price vectors are assumed to be restricted by exogenously given lower and upper bounds denoted by $\underline{p} \in \mathbb{R}^{m+1}$ and $\bar{p} \in \overline{\mathbb{R}}^{m+1}$, respectively. Here, $\overline{\mathbb{R}} = [-\infty, \infty]$ represents the extended real line, and it is assumed that $\underline{p} \leq \bar{p}$. Note that $\bar{p} \in \overline{\mathbb{R}}^{m+1}$ since we do not exclude the case when there is no upper price limit for some houses. The lower bounds can be thought of as the owner's reservation prices or rents, and the upper bounds as a legislated rent control. Also note that as p_0 is always zero, it is clear that $\underline{p}_0 = \bar{p}_0 = 0$. The *price space* is given by:

$$\Omega = \{p \in \mathbb{R}^{m+1} : \underline{p}_h \leq p_h \leq \bar{p}_h \text{ for } h \in H \cup \{0\}\}.$$

Each agent $i \in N$ has (indirect) *preferences* on pairs of houses and prices. Denote by R_i agent i 's preference relation on the set of houses and prices $(H \cup \{0\}) \times \mathbb{R}$. The corresponding strict preference and indifference relations are denoted by P_i and I_i , respectively. The following notation will be used: if $h, h' \in H \cup \{0\}$ and $p_h R_i p_{h'}$, then agent $i \in N$ weakly prefers house h at price p_h to house h' at price $p_{h'}$.⁸ When the price vector is fixed during the analysis (i.e., when $p = p'$), the simplified notation $h R_i h'$ is employed.

Preferences are assumed to be *rational* and *monotonic* for all agents $i \in N$, i.e., R_i is a complete and transitive binary relation on $(H \cup \{0\}) \times \mathbb{R}$ and $p_h P_i p_{h'}$ if $p_h < p_{h'}$. It is also, for all agents $i \in N$, assumed that $p_h P_i p_{h'}$ if $h = 0$ and if $p_{h'}$ is "sufficiently large." Further, preferences are assumed to be *continuous*, i.e., the sets $\{p_h \in \mathbb{R} : p_h R_i p_{h'}\}$ and $\{p_h \in \mathbb{R} : p_{h'} R_i p_h\}$ are closed for each $i \in N$ and all $h, h' \in H \cup \{0\}$ and all $p_{h'} \in \mathbb{R}$. All preference relations R_i satisfying the above properties for agent $i \in N$ are gathered in the set \mathcal{R}_i . A preference profile, or for short a *profile*, is a list $R = (R_1, \dots, R_n)$ of the agents' preferences. This list belongs to the set $\mathcal{R} = \mathcal{R}_1 \times \dots \times \mathcal{R}_n$. Finally, we adopt the notational convention of writing a profile $R \in \mathcal{R}$ as $R = (R_C, R_{-C})$ for $C \subset N$.

4. RATIONING AND CONSTRAINED EFFICIENCY

It is well-known that a price equilibrium⁹ exists under very general conditions (see, e.g., Demange and Gale, 1985; Shapley and Shubik, 1972). However, if there are finite price ceilings, the intersection between the set of equilibrium price vectors and the price space Ω is empty for some profiles $R \in \mathcal{R}$. Hence, a different notion than the concept of price equilibrium is needed to analyze housing markets with rent control, and this weakening must contain some kind of rationing mechanism as prices alone cannot solve the allocation problem. Here, it is assumed that the rationing mechanism is based on a *priority structure* π , i.e., for each house $h \in H$, there is an exogenously given strict *priority-order* π_h . Formally, $\pi_h : N \rightarrow N$ is a bijection where the highest-ranked agent $i \in N$ is the agent with $\pi_{ih} = 1$, the second highest ranked agent i' has $\pi_{i'h} = 2$, and so on. A special type

⁸Note that p_h has a dual meaning. It represents the real number p_h as well as the pair (h, p_h) . This will not cause any confusion as p_h is equivalent to the pair (h, p_h) if and only if it is written in connection to a preference relation.

⁹A price equilibrium will be formally introduced and defined in Section 5. See also Crawford and Knoer (1981), Demange and Gale (1985), Demange et al. (1986), Leonard (1983), Mishra and Talman (2010), Shapley and Shubik (1972), or Svensson (1983).

of priority structure, which is a prerequisite for analyzing the Serial Dictatorship Mechanism (Hylland and Zeckhauser, 1979; Svensson, 1994), is when $\pi_h = \pi_{h'}$ for all $h, h' \in H$. Throughout this section, prices are assumed to be fixed. In this case, we recall, from Section 3, that the simplified notation $hR_i h'$ will be employed instead of $p_h R_i p_{h'}$.

An assignment μ is priority respecting if there is no agent $i \in N$ who strictly prefers some house h to μ_i and house h is assigned to some other agent $j \in N$ who has lower priority for house h than agent i , and, furthermore, all agents weakly prefer their assigned house to any unassigned house.¹⁰

DEFINITION 1 For a given profile $R \in \mathcal{R}$ and a given priority structure π , assignment μ is *priority respecting* if for all $i, i' \in N$: (i) $\mu_{i'} P_i \mu_i$ only if $\pi_{i' \mu_{i'}} < \pi_{i \mu_i}$, and (ii) $\mu_i R_i h$ if $h \in \mu_0$.

A priority respecting assignment μ need not be (unconstrained) Pareto efficient as there may exist some other assignment that Pareto dominates μ but where the priority structure is not respected. In fact, Pareto efficiency is only guaranteed for specific priority structures. Ergin (2002) demonstrated that *acyclicity* of the priority structure is sufficient for Pareto efficiency (given fixed prices). This condition roughly means that no agent can “block” a potential settlement between two other agents without affecting his own assignment.

EXAMPLE 1 Let $N = \{1, 2, 3\}$, $H = \{a, b\}$, $\pi_{1a} < \pi_{2a} < \pi_{3a}$ and $\pi_{2b} < \pi_{3b} < \pi_{1b}$. Let also $bP_1 aP_1 0$, $aP_2 0P_2 b$, and $aP_3 bP_3 0$. In this case, assignment $\mu = (a, 0, b)$ is priority respecting and assignment $\mu' = (b, 0, a)$ Pareto dominates μ . However, assignment μ' does not respect the priority structure π as $\mu'_3 P_2 \mu'_2$ and $\pi_{2a} < \pi_{3a}$, i.e., agent 2 can “block” the settlement between agents 1 and 3. \square

The above insight leads to a “second best” notion of efficiency where only priority respecting assignments are compared.¹¹

DEFINITION 2 For a given profile $R \in \mathcal{R}$ and a given priority structure π , a priority respecting assignment μ is *constrained efficient* at profile R if there is no priority respecting assignment μ' that Pareto dominates μ at profile R , i.e., there is no priority respecting assignment μ' such that $\mu'_i R_i \mu_i$ for all $i \in N$ and $\mu'_i P_i \mu_i$ for some $i \in N$.

¹⁰In, e.g., Balinski and Sönmez (1999), condition (i) of Definition 1 is called *fairness* and condition (ii) is the combination of *individual rationality* and *non-wastefulness*. Note also that an assignment is *stable* (Balinski and Sönmez, 1999; Gale and Shapley, 1962) if and only if it is fair, individually rational, and non-wasteful (Balinski and Sönmez, 1999, Lemma 2). Hence, any stable assignment is priority respecting and vice versa. Finally, condition (i) of Definition 1 is also known as *adaptability*.

¹¹In some special cases, constrained efficiency (as defined in Definition 2) coincides with (unconstrained) Pareto efficiency. This is, e.g., the case when agents have strict preferences and the priority structure of Hylland and Zeckhauser (1979) is adopted, i.e., when $\pi_h = \pi_{h'}$ for all $h, h' \in H$. Note, however, that if agents are allowed to have weak preferences (as is our framework), it is not always possible to attain (unconstrained) Pareto efficiency, even for this simple priority structure, since a potential settlement between two agents may be “blocked” by a third agent (i.e., the priority structure must to be respected).

We end this section by stating three important remarks. First, the set of constrained efficient assignments is non-empty for any $R \in \mathcal{R}$ because there is a finite number of assignments and since there always exists a priority respecting assignment.¹²

Second, the notion of efficiency in Definition 2 is rather weak in the sense that there may be several constrained efficient assignments at a specific profile. This feature will be important when prices are introduced into the model. In fact, if there are several constrained efficient assignments at a given profile R (and a given price vector), not all of them may be compatible with our notion of equilibrium (Definition 3). This point is later illustrated in Example 2 in Section 5.

Finally, note that we adopt a “straightforward” interpretation of the priority structure, π , in the sense that agents are not allowed to exchange priorities. This interpretation is different from the one considered in, e.g., Abdulkadiroğlu and Sönmez (1998, 2003) and Pápai (2000). There, Pareto efficient outcomes are guaranteed for any priority structure if a generalization of the Top Trading Cycles Mechanism (Shapley and Scarf, 1974) is adopted. In our framework, where agents are not allowed to exchange priorities, (unconstrained) Pareto efficient outcomes can generally not be obtained as assignments have to be priority respecting. This can be seen in Example 1. If exchange would be allowed, agents 1 and 2 (i.e., the agents with highest priorities for houses a and b , respectively) would trade priorities as they both are made strictly better off by this trade, and the resulting assignment would be $\mu'' = (b, a, 0)$. This is a Pareto improvement compared to $\mu = (a, 0, b)$ from Example 1. Assignment μ'' is, however, not allowed in our framework since π must be respected, $\mu''_1 P_3 \mu''_3$, and $\pi_{3b} < \pi_{1b}$.

5. RATIONING PRICE EQUILIBRIA

As explained in the previous section, the standard notion of price equilibrium cannot be used to analyze the housing model with rent control as there may be an unbalanced relationship between supply and demand when prices are restricted to belonging to Ω . This imbalance also suggests that any equilibrium notion used to analyze this type of market must have some incorporated rationing mechanism. This section proposes an equilibrium notion called rationing price equilibrium where the built-in rationing mechanism is given by the priority structure π from Section 4.

DEFINITION 3 For a given profile $R \in \mathcal{R}$ and a given priority structure π , a state (p, μ) is a *rationing price equilibrium* (RPE) if the following conditions hold:

¹²A priority respecting assignment can be obtained by arbitrary breaking ties in R (and thereby obtaining the strict profile \hat{P}) and by applying the (agent proposing) Deferred Acceptance Algorithm (Gale and Shapley, 1962) given \hat{P} and π (exactly as in the “student placement problem,” see Balinski and Sönmez, 1999). This algorithm generates an assignment μ where (a) there is no pair $(i, h) \in N \times H$ where $h \hat{P}_i \mu_i$ and $\pi_{ih} < \pi_{i'h}$ if $\mu_{i'} = h$, and (b) no agent $i \in N$ with $h \hat{P}_i \mu_i$ if $\mu_i \in \mu_0$, see, e.g., Roth (2007, pp. 3–4). Since the breaking of indifferences does not switch the positions of any two houses in the preference relation R_i , conditions (a) and (b) are satisfied also for profile R . But if conditions (a) and (b) are satisfied for R , then conditions (i) and (ii) of Definition 1 are satisfied for R , respectively. Hence, μ is a priority respecting assignment.

- (i) for all $h \in \mu_0$, $p_h = \underline{p}_h$,
- (ii) $\underline{p}_h \leq p_h \leq \bar{p}_h$ for all $h \in H \cup \{0\}$,
- (iii) for all $i, i' \in N$, $p_{\mu_{i'}} = \bar{p}_{\mu_{i'}}$ if $p_{\mu_{i'}} P_i p_{\mu_i}$, and
- (iv) μ is a constrained efficient assignment.

Before relating this concept to the notion of a price equilibrium and related suggestions in the literature, the intuitive ideas behind the various conditions of Definition 3 are explained. Conditions (i) and (ii) are the standard requirements that the price of any unassigned house must equal its lower price bound and that the price vector must belong to the price space, respectively.

Requirement (iii) is based on Drèze (1975) and a series of subsequent papers (where Talman and Yang, 2008; Zhu and Zhang, 2011, are the most closely related to this study) where it is argued that the rationing mechanism should only be put to use when a specific set of houses, with prices equal to the upper price bound, is overdemanded.¹³ The underlying motivation is the standard textbook argument that states that prices on houses that are overdemanded should increase to approach a price equilibrium (of course, given that reaching a price equilibrium is the ultimate goal). However, when prices are bounded from above, it may not be possible to increase the prices in such fashion that all overdemanded sets of houses are eliminated, which is necessary for obtaining a price equilibrium (see, e.g., Demange et al., 1986; Hall, 1935; Mishra and Talman, 2010).

Requirement (iv) guarantees that the assignment at any RPE state is “efficient” and that the priority structure is respected. As already explained in Section 4, the “cost” of this guarantee is that the state need not be (unconstrained) Pareto efficient. Condition (iv) is also dependent on condition (iii), and this dependence will give further insights to why a stronger efficiency notion cannot be adopted.¹⁴ More specifically, the efficiency notion must be weak enough to guarantee that condition (iii) of Definition 3 is always satisfied for *some* “efficient” assignment. This need not be the case as there may be several constrained efficient assignments at a given price vector and not all of them are necessarily compatible with condition (iii). This conflict is illustrated in the following example.

EXAMPLE 2 Let $N = \{1, 2, 3\}$, $H = \{a, b, c\}$, and $\pi_{ih} = i$ for all $i \in N$ and $h \in H$. For each agent $i \in N$, preferences are represented by a quasi-linear utility function $u_{ih}(p) = v_{ih} - p_h$ where the values v_{ih} are represented by real numbers. Let $v_{i0} = 0$ for all $i \in N$ and:

$$\begin{pmatrix} v_{1a} & v_{1b} & v_{1c} \\ v_{2a} & v_{2b} & v_{2c} \\ v_{3a} & v_{3b} & v_{3c} \end{pmatrix} = \begin{pmatrix} 11 & 10 & 0 \\ 10 & 0 & 5 \\ 0 & 10 & 5 \end{pmatrix}.$$

¹³That is when the number of agents that only demand houses from some set $H' \subset H$ is strictly larger than the number of elements in the set H' , see, e.g., Demange et al. (1986) or Mishra and Talman (2010).

¹⁴From a technical point of view, the set of “quasi rationing price equilibria” (see Appendix B) need not be closed for all profiles $R \in \mathcal{R}$ if stronger efficiency notions are applied (See Lemma 2 in Appendix B). Without a closed set there may not exist a “minimal” RPE vector, the existence of which is necessary to obtain a well-defined allocation rule (see the discussion in Sections 6 and 7).

Let the price space Ω be given by the condition $0 \leq p_h \leq 1$ for all $h \in H$. Consider the price vector p , where $p_a = 1$ and $p_b = p_c = 0$, and note that there are two constrained efficient assignments at this price vector: $\mu = (a, c, b)$ and $\mu' = (b, a, c)$. However, only state (p, μ) is an RPE state as Definition 3(iii) is violated for state (p, μ') since $0 = p_b < \bar{p}_b = 1$ and $p_{\mu'_1} P_3 p_{\mu'_3}$. \square

We next remark on the relation between an RPE and a *price equilibrium*. The latter is a state (p, μ) where each agent has been assigned his most preferred house among all houses at the prevailing prices p , i.e., $p_{\mu_i} R_i p_h$ for all $i \in N$ and all $h \in H \cup \{0\}$, and where the price of any unassigned house equals its lower price bound. The former of these conditions guarantees that the state also is *envy-free* (Foley, 1967). However, if the state (p, μ) is envy-free, then μ must be constrained efficient as μ can be selected for any priority structure. Consequently, an envy-free RPE is also a price equilibrium. Conversely, a price equilibrium (p, μ) is an RPE if $p \in \Omega$, and, therefore, a price equilibrium is always an RPE if $\bar{p}_h = \infty$ for all $h \in H$.

The final remark concerning Definition 3 is its relation to alternative equilibrium concepts. Talman and Yang (2008) proposed a notion of equilibrium for housing markets with rent control called *Constrained Walrasian Equilibrium* (CWE). Their concept is not based on an exogenously given priority structure but rather on a rationing scheme Q that specifies which houses in H the agents in N can and cannot be assigned. For a given rationing scheme Q , a CWE satisfies conditions identical to conditions (i)–(iii) of Definition 3. Because Q can be created arbitrarily as long as these requirements are satisfied, it is easy to demonstrate that for any RPE state (p, μ) there exists a rationing scheme Q such that (p, μ) is a CWE state. However, if (p, μ) is a CWE state, there need not exist a priority structure such that (p, μ) constitutes an RPE state. The underlying reason for this is that a CWE need not be constrained efficient, i.e., at a given CWE, it may be possible to make a Pareto improving reallocation of the houses among the agents and at the same time respect all requirements of the concept for *any* given priority structure. In a related paper, Zhu and Zhang (2011) provided a weaker equilibrium concept than CWE and introduced a notion of efficiency that can be applied whenever preferences are represented by quasilinear utility functions. However, their notion of efficiency is based on maximizing a sum of valuations and upper price limits so it is not based on Pareto efficiency or on Pareto improvements. In fact, if state (p, μ) satisfies their notion of equilibrium, it may be Pareto improved for *any* priority structure.

We end this section by introducing some notation. A price vector p is an RPE vector if there is an assignment μ such that the state (p, μ) is an RPE. For a given profile $R \in \mathcal{R}$, the set of RPE states is denoted by Σ_R and the corresponding set of price vectors is denoted by Π_R , i.e.: $\Pi_R = \{p \in \Omega : (p, \mu) \in \Sigma_R \text{ for some assignment } \mu\}$.

6. MINIMAL RPE VECTORS

The basic idea for constructing a (group) non-manipulable allocation rule for housing markets with rent control is the same as in the standard competitive model where a Pareto efficient and (group) non-manipulable allocation rule can be obtained if the unique minimal

equilibrium price vector is used as a (direct) mechanism for allocating the houses among the agents (Andersson and Svensson, 2008; Demange and Gale, 1985). Therefore we are, in the rent control framework, searching for minimal price vectors in the set Π_R .

DEFINITION 4 For a given profile $R \in \mathcal{R}$ and a given priority structure π , the price vector $p^* \in \Pi_R$ is a *minimal RPE vector* if for $p \in \Pi_R$, $p \leq p^*$ only if $p = p^*$.

THEOREM 1 For any profile $R \in \mathcal{R}$ and any given priority structure π , there is a minimal RPE vector $p^* \in \Pi_R$.

A minimal RPE vector need not be unique as illustrated in the following example.

EXAMPLE 3 Let $N = \{1, 2, 3, 4, 5\}$, $H = \{a, b, c, d, e\}$, $\pi_{1h} = 1$ for all $h \in H$, $\pi_{2a} = 2$, and $\pi_{3b} = 2$ (the remaining priorities may be selected arbitrarily). For each agent $i \in N$, preferences are represented by a quasi-linear utility function $u_{ih}(p) = v_{ih} - p_h$ where the values v_{ih} are represented by real numbers. Let $v_{i0} = 0$ for all $i \in N$, and:

$$\begin{pmatrix} v_{1a} & v_{1b} & v_{1c} & v_{1d} & v_{1e} \\ v_{2a} & v_{2b} & v_{2c} & v_{2d} & v_{2e} \\ v_{3a} & v_{3b} & v_{3c} & v_{3d} & v_{3e} \\ v_{4a} & v_{4b} & v_{4c} & v_{4d} & v_{4e} \\ v_{5a} & v_{5b} & v_{5c} & v_{5d} & v_{5e} \end{pmatrix} = \begin{pmatrix} 10 & 10 & 0 & 0 & 0 \\ 10 & 0 & 0 & 5 & 0 \\ 0 & 10 & 0 & 0 & 5 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{pmatrix}.$$

Let also $0 \leq p_h \leq 1$ for all $h \in H$. In this example, there are two minimal price vectors p and p' where $p_a = p_b = p_e = 1$, $p_c = p_d = 0$, $p'_a = p'_b = p'_d = 1$, and $p'_c = p'_e = 0$. The vector p is obtained at the assignment $\mu = (b, a, e, d, c)$, and the vector p' is obtained at the assignment $\mu' = (a, d, b, c, e)$. \square

To obtain uniqueness of a minimal RPE vector, it is necessary to exclude some profiles from the domain \mathcal{R} . The domain restriction that will be employed when proving the uniqueness of a minimal RPE vector, as well as the (group) non-manipulability result in the next section, is a subset $\tilde{\mathcal{R}} \subset \mathcal{R}$ where $\tilde{\mathcal{R}}$ is the set of profiles such that *no two houses are connected by indifference* (NCBI) at any price vector $p \in \Omega$.

DEFINITION 5 For a given profile $R \in \mathcal{R}$, two houses, h_1 and h_t , in $H \cup \{0\}$ are *connected by indifference* (CBI) if there is a price vector $p \in \Omega$, a sequence of agents (i_1, \dots, i_{t-1}) , and a sequence of houses (h_1, \dots, h_t) where $t \geq 2$, $h_j \neq h_k$ for some houses in the sequence, and:

- (i) $p_{h_1} = \bar{p}_{h_1}$ and $p_{h_t} = \bar{p}_{h_t}$,
- (ii) $p_{h_j} I_{i_j} p_{h_{j+1}}$ for $1 \leq j \leq t-1$, and
- (iii) if $h_1 = h_t$, then $t \geq 3$ and $i_j \neq i_k$ for some agents in the sequence.

In Example 3, houses a and b are CBI as long as $v_{1a} = 10$. However, if $v_{1a} = 10 - \varepsilon$ for some “small” $\varepsilon \neq 0$, the profile belongs to $\tilde{\mathcal{R}}$. Because $\tilde{\mathcal{R}}$ only contains profiles such that no two houses are CBI at any price vector, it is relevant to have a discussion about whether

this is a severe domain restriction or not, and what implications the NCBI assumption have on the individual preference domains $\tilde{\mathcal{R}}_i$, i.e., when $\tilde{\mathcal{R}}$ is a product space of individual domains. Three examples are used to shed some light on this. The first example is based on a fixed-price mechanism, and it demonstrates that the assumption of NCBI is equivalent to assuming strict preferences. Because it is well-known that one cannot go much beyond strict preferences if one insists on “efficiency” and non-manipulability (Ehlers, 2002), the NCBI assumption must, in this case, be regarded as a mild domain restriction. The second example shows that if there is no upper price bound, then the assumption of NCBI does not reduce the preference domain at all, i.e., $\tilde{\mathcal{R}} = \mathcal{R}$.

EXAMPLE 4 Suppose that $\underline{p}_h = \bar{p}_h$ for all $h \in H$ and let $R \in \tilde{\mathcal{R}}$. If $\bar{p}_h I_i \bar{p}_{h'}$ for some $i \in N$ and some $h, h' \in H$ ($h \neq h'$), then houses h and h' are CBI, which contradicts the assumption that $R \in \tilde{\mathcal{R}}$. Hence, preferences must be strict. \square

EXAMPLE 5 Suppose that $\bar{p}_h = \infty$ for all $h \in H$ and let $R \in \tilde{\mathcal{R}}$. Then it will never be the case that $p_h I_i \bar{p}_{h'}$ for some agent $i \in N$ and some houses $h, h' \in H$ as preferences are monotonic, $p_h \in \mathbb{R}$, and $\bar{p}_h \notin \mathbb{R}$. Hence, $\tilde{\mathcal{R}} = \mathcal{R}$.

The next example investigates the domain containing only quasi-linear profiles $\mathcal{R}^q \subset \mathcal{R}$. This commonly adopted domain¹⁵ has previously been described in Examples 2 and 3. From that description, it is apparent that for each $R \in \mathcal{R}^q$, it holds that $v_{ih} - v_{ih'} \in \mathbb{R}$ for any agent $i \in N$ and any two distinct houses $h, h' \in H$, and the main insight is that two houses can be CBI. Example 6, below, demonstrates that it is possible to define a restricted domain $\tilde{\mathcal{R}}_i^q$ for $i \in N$ of quasi-linear preferences such that $\tilde{\mathcal{R}}^q = \tilde{\mathcal{R}}_1^q \times \dots \times \tilde{\mathcal{R}}_n^q \subset \tilde{\mathcal{R}}$, where $v_{ih} - v_{ih'} \in \Delta_i$ for any agent $i \in N$ and Δ_i is a *tight* subset of \mathbb{R} . Consequently, the NCBI assumption must be regarded as a mild assumption when only considering quasi-linear profiles.

EXAMPLE 6 Suppose that $\bar{p}_h \in \mathbb{Q}$ for all $h \in H$ where \mathbb{Q} is the set of rational numbers. Let also \mathbb{I} denote the set of irrational numbers. Assume next that $\alpha_1, \dots, \alpha_n$ are n “independent” irrational numbers in the sense that $\alpha_i \in \mathbb{I}$ for all $i \in N$ and $\alpha_i \neq \sum_{j \neq i} r_j \alpha_j$ if $r_j \in \mathbb{Q}$. Define:

$$\Delta_i = \{x \in \mathbb{R} : x = y\alpha_i \text{ where } y \in \mathbb{Q} \text{ and } y \neq 0\} \text{ for each } i \in N.$$

Clearly, Δ_i is a *tight* subset of \mathbb{R} and $\Delta_i \cap \mathbb{Q} = \emptyset$. Suppose next that the preference domain for agent $i \in N$ is given by:

$$\tilde{\mathcal{R}}_i^q = \{\text{quasi-linear utility functions} : v_{ih} - v_{ih'} \in \Delta_i \text{ for all } h, h' \in H \cup \{0\}, h \neq h'\}.$$

Consider now a sequence of agents (i_1, \dots, i_{t-1}) and a sequence of houses (h_1, \dots, h_t) where $t \geq 2$, $p_{h_1} = \bar{p}_{h_1}$, and $p_{h_t} = \bar{p}_{h_t}$. To see that houses h_1 and h_t cannot be connected by

¹⁵See, e.g., Crawford and Knoer (1981), Demange et al. (1986), Leonard (1983), Mishra and Talman (2010), Talman and Yang (2008), and Zhu and Zhang (2011) among others.

indifference, suppose that $p_{h_j} I_{i_j} p_{h_{j+1}}$ for all $1 \leq j \leq t-1$, i.e., $v_{i_j h_j} - p_{h_j} = v_{i_j h_{j+1}} - p_{h_{j+1}}$ for all $1 \leq j \leq t-1$. By adding these equalities and by simplifying, we obtain:

$$(1) \quad \sum_j (v_{i_j h_j} - v_{i_j h_{j+1}}) = p_{h_1} - p_{h_t}.$$

But $p_{h_1} - p_{h_t} \in \mathbb{Q}$ while $\sum_j (v_{i_j h_j} - v_{i_j h_{j+1}}) \notin \mathbb{Q}$. Hence, equation (1) cannot hold, and therefore houses h_1 and h_t cannot be connected by indifference. \square

A final observation is that for any profile $R \in \tilde{\mathcal{R}}$, any sequence of agents (i_1, \dots, i_{t-1}) , and any sequence of houses (h_1, \dots, h_t) where $t \geq 2$, the prices p_{h_j} may be chosen so that $p_{h_1} = \bar{p}_{h_1}$ and $p_{h_j} I_{i_j} p_{h_{j+1}}$ for $1 \leq j \leq t-1$. In that case, p_{h_t} will be uniquely determined by continuity and monotonicity of the preferences. However, if preferences are chosen randomly, the probability is zero that $p_{h_t} = \bar{p}_{h_t}$ since \bar{p} and \underline{p} are given exogenously (similarly, if $h_1 = h_t$ and $i_j \neq i_k$ for some agents in the sequence, then the probability that $p_{h_t} = \bar{p}_{h_t}$ is zero). Consequently, one may argue that very few profiles are excluded in $\tilde{\mathcal{R}}$ compared to \mathcal{R} , and, therefore, that “almost all” profiles are included in $\tilde{\mathcal{R}}$.

THEOREM 2 For any profile $R \in \tilde{\mathcal{R}}$ and any given priority structure π , a minimal price vector $p^* \in \Pi_R$ is unique.

7. NON-MANIPULABILITY

This section defines a (group) non-manipulable allocation rule that implements an RPE state. Let $\mathcal{R}^* \subset \mathcal{R}$. An allocation mechanism is a function f with domain \mathcal{R}^* that selects an RPE state as an outcome, i.e., $f(R) \in \Sigma_R$ for $R \in \mathcal{R}^*$. This paper employs the following definition of (group) non-manipulability.

DEFINITION 6 Let $\mathcal{R}^* \subset \mathcal{R}$. An allocation mechanism f with domain \mathcal{R}^* is *group manipulable* at a profile $R \in \mathcal{R}^*$ by a (nonempty) group of agents $C \subset N$ if there are preferences R'_i for agents $i \in C$ such that $R' = (R'_C, R_{-C}) \in \mathcal{R}^*$, and there are two states $f(R) = (p, \mu)$ and $f(R') = (p', \mu')$ such that $p'_{\mu'_i} P_i p_{\mu_i}$ for all $i \in C$. If the mechanism f is not group manipulable by any group $C \subset N$, at any profile $R \in \mathcal{R}^*$, it is (weakly) *group non-manipulable*.

Note that a group of agents can manipulate an allocation mechanism only if all members of the group are made *strictly* better off by misrepresenting their preferences.

For the remaining part of this paper, let $\mathcal{R}^* = \tilde{\mathcal{R}}$. Note then that the above definition of (non-)manipulability means that an agent’s choice of preferences to manipulate an allocation mechanism is restricted by the choices of the other agents. However, Examples 4–6 illustrate cases where $\tilde{\mathcal{R}}$ is a product space of agents’ preference domains and, consequently, where an agent makes his choice of preferences to manipulate independently of the choices of the other agents. Also, as argued in Section 6, “almost all” preference profiles are included in $\tilde{\mathcal{R}}$.

From Theorem 2, it is clear that the following mechanism is well-defined for any $R \in \tilde{\mathcal{R}}$.

DEFINITION 7 The *minimal RPE mechanism* f is defined to be a mapping of profiles $R \in \tilde{\mathcal{R}}$ to a rationing price equilibrium (p^*, μ) , where p^* is the unique minimal price vector in Π_R and μ is any selection of assignment such that $(p^*, \mu) \in \Sigma_R$.

THEOREM 3 Let f be a minimal RPE mechanism with domain $\tilde{\mathcal{R}}$. Then f is group non-manipulable.

We next remark on two polar cases of the minimal RPE mechanism, and consequently also two polar cases of Theorem 3. The first is when there are no upper price bounds (i.e., when $\bar{p}_h = \infty$ for all $h \in H$). In this special case, no two houses are connected by indifference (see Example 5), and $p_h < \bar{p}_h$ for all $h \in H$ at any RPE state (p, μ) . Consequently, $p_{\mu_i} R_i p_h$ for all $i \in N$ and all $h \in H \cup \{0\}$ by Definition 3(iii). Hence, the minimal RPE mechanism always selects a price equilibrium containing the unique minimal equilibrium price vector (Shapley and Shubik, 1972; Demange and Gale, 1985). As this price vector must also be the minimal vector in Π_R , it follows that the Competitive Price Mechanism (Demange and Gale, 1985) reduces to a special case of the minimal RPE mechanism. Consequently, the main (group) non-manipulability result in Demange and Gale (1985, Theorem 2), as well as its generalization in Andersson and Svensson (2008, Theorem 1), is covered by Theorem 3.

In the other extreme case when the upper and lower price bounds coincide (i.e., when $\underline{p}_h = \bar{p}_h$ for all $h \in H$), the minimal RPE mechanism reduces to a fixed-price mechanism. As explained in Example 4, the NCBI assumption implies that preferences are strict. Because the (agent proposing) Deferred Acceptance Algorithm (Gale and Shapley, 1962) is the only non-manipulable and constrained efficient (or stable, see footnote 10) mechanism in this case (see, e.g., Alcade and Barbera, 1994; Balinski and Sönmez, 1999), it follows, by Theorem 3, that the minimal RPE mechanism reduces to a (direct mechanism) version of the (agent proposing) Deferred Acceptance Algorithm. Therefore, the theorem also reproduces some well-known (group) non-manipulability results in the matching literature where monetary transfers are not allowed, including, e.g., Dubins and Freedman (1981, Section 3 “Coalitions”) and Roth (1982b, Theorem 5). Also, in the case when the priority-order is the same for all houses, the outcome of the minimal RPE mechanism is the same as the one of a Serial Dictatorship Mechanism (Hylland and Zeckhauser, 1979), and therefore non-manipulability results related to this mechanism are also covered by Theorem 3, e.g., the result in Svensson (1994, Theorem 1) given strict preferences.

8. CONCLUDING REMARKS

This paper has introduced the concept of a rationing price equilibrium (RPE), and it has been demonstrated that the set of RPE states is nonempty for any given profile in \mathcal{R} , and, moreover, that the set of RPE price vectors may contain several minimal price vectors. However, if the domain $\tilde{\mathcal{R}}$, where no two houses are connected by indifference at any given price vector, is considered, then the minimal RPE price vector is unique. This result implies that the minimal RPE mechanism is well-defined for all profiles in $\tilde{\mathcal{R}}$, as its definition requires the existence of a unique minimal RPE vector (recall from Example 3 that the

minimal RPE vector need not be unique for all profiles in \mathcal{R}). It remains an open question how to define the minimal RPE mechanism when considering the full preference domain. Consequently, it is also an open question how to define a non-manipulable allocation rule for the full preference domain. However, the non-manipulability result presented in this paper is valid for “almost all” preference profiles.

Also, the minimal RPE mechanism is defined as a direct mechanism. Another interesting extension of this paper is to define a dynamic process that converges to a state that is selected by the minimal RPE mechanism in a finite number of iterations.¹⁶ One can imagine that such a process starts at the lower price bounds and that each agent reports his most preferred houses at the given prices. Then the prices of all houses in some cleverly selected set of overdemanded houses S (if such a set exists) are increased until one of the agents that demand only houses from the set S obtains an indifference to some house outside of the set S , or the price of some house in S equals its upper limit. Then, by Example 2, each constrained efficient assignment needs to be identified at the given prices to ensure that an RPE can be identified if there is one at the given prices. If, at the given prices p , there is an assignment μ such that the state (p, μ) is an RPE, the process terminates. Otherwise a new overdemanded set is cleverly selected, and the process is repeated. The upper bounds guarantee that the process eventually terminates. However, even if such a dynamic process appears to be a promising candidate, several obstacles must be overcome. For example, it is not clear how to identify all constrained efficient assignments at any given price vector when indifferences are allowed as an exhaustive search is computationally infeasible (Manlove et al., 2002). Here, some recent papers (including, e.g., Erdil and Ergin, 2006, 2008) may be helpful. Second, it is not clear how to select the set S as it cannot include any house with a price equal to the upper price bound, and as it is also not clear if and how the preferences should be reported whenever the price of a house equals its upper price bound.¹⁷ Even if the problem of finding an “auction type” dynamic process that generates an outcome identical to the minimal RPE mechanism is important, it is left for future research.

Other possible extensions of the model includes the possibility for initial ownership of the houses as in, e.g., Shapley and Scarf (1974), Abdulkadiroğlu and Sönmez (1998) and Pápai (2000), and priority structures that allow for weak priorities as in, e.g., Abdulkadiroğlu et al. (2009), Ehlers and Erdil (2010), and Erdil and Ergin (2006, 2008).

¹⁶In, for example, the auction literature, the dynamic processes are central. See, e.g., Ausubel (2004, 2006), Ausubel and Milgrom (2002), Bikhchandani and Ostroy (2002), Bikhchandani et al. (2011), Gul and Stacchetti (2000), Mishra and Parkes (2007, 2009), Perry and Reny (2005), or Sankaran (1994).

¹⁷In Talman and Yang (2008), this problem is solved by matching overdemanded houses (with prices equal to the upper price bound) to agents that demand these houses based on a lottery mechanism, and then by removing all matched houses and agents from the problem. This trick does not work for the type of housing markets investigated in this paper (even if the lottery mechanism is replaced by the priority structure) as it need not result in an RPE because a constrained efficient assignment must be selected when all prices are known to guarantee that the minimal RPE price vector is identified (a similar point has previously been illustrated in Example 2).

APPENDIX A: PRELIMINARY RESULTS AND DEFINITIONS

In Appendix A, the concept of an isolated set (Definition 13) is introduced, and a sufficient condition for its nonemptiness is provided (Proposition 1). The latter result is a key ingredient in the proof of Theorem 3. However, from a technical viewpoint, it will sometimes be convenient to work with a weaker notion of efficiency than constrained efficiency. Also this notion is introduced in Appendix A (Definition 11) together with a number of related concepts and results that will be helpful in the subsequent appendices.

The entire net trade between any two states can be decomposed into a number of unique trading cycles as explained in the following definition.¹⁸

DEFINITION 8 Let (p, μ) and (p', μ') be two states. A sequence $G = (i_1, \dots, i_t)$ of distinct agents constitutes a *trading cycle* from μ to μ' if $\mu_{i_j} \in H$ for $2 \leq j \leq t$, $\mu'_{i_j} = \mu_{i_{j+1}}$ for $1 \leq j < t$, and either:

- (i) $\mu_{i_1} \in H$ and $\mu'_{i_t} = \mu_{i_1}$ (*closed trading cycle*), or
- (ii) $\mu_{i_1} \in \mu'_0$ and $\mu'_{i_t} \in \mu_0$ (*open trading cycle*).

The above definition is stated somewhat more generally than necessary in this appendix. In fact, in the remaining part of this appendix, prices are assumed to be fixed. In this case, we recall from Section 3, that the simplified notation $hR_i h'$ will be employed instead of $p_h R_i p_{h'}$.

The set of *unenvied agents* and the set of *envied agents* at profile $R \in \mathcal{R}$ are defined as $F = \{i \in N : \mu_{i'} R_i \mu_i \text{ for all } i' \in N\}$ and $E = N \setminus F$, respectively. An assignment μ is *envy-free* at profile $R \in \mathcal{R}$ if E is empty and all agents in N prefer their assigned house to any unassigned house (Foley, 1967).

DEFINITION 9 For a given profile $R \in \mathcal{R}$ and a given priority structure π , agent $i' \in N$ has *priority to envy* agent $i \in E$ at assignment μ if:

- (i) $\mu_i P_{i'} \mu_{i'}$, and
- (ii) for all $i'' \in N$ and $i'' \neq i'$, $\mu_i P_{i''} \mu_{i''}$ only if $\pi_{i'} \mu_i < \pi_{i''} \mu_{i''}$.

DEFINITION 10 For a given profile $R \in \mathcal{R}$ and a given priority structure π , let μ and μ' be two priority respecting assignments. Then μ' is a *priority respecting improvement* of μ if there is a trading cycle $G = (i_1, \dots, i_t)$ from μ to μ' such that:

- (i) $\mu'_i R_i \mu_i$ for all $i \in N$ and $\mu'_i P_i \mu_i$ for some $i \in N$,
- (ii) if $\mu'_i = \mu_{i'}$ and $\mu_{i'} P_i \mu_i$ for some $i, i' \in N$, then agent i has priority to envy the agent i' , and
- (iii) $\mu'_i = \mu_i$ for all $i \in N \setminus \{i_1, \dots, i_t\}$.

DEFINITION 11 For a given profile $R \in \mathcal{R}$ and a given priority structure π , let μ be a priority respecting assignment. Then μ is *weakly constrained efficient* at profile R if there is no priority respecting improvement of μ .

Obviously, a constrained efficient assignment μ is also weakly constrained efficient, while the converse is not necessarily true.

¹⁸We remark that an alternative presentation of Definition 8 is to describe the net trade by a number of directed arcs from agents to houses, and from houses to agents. In this case, the description of closed and open trading cycles would be more closely related to the description of the Top-Trading Cycles Mechanism (Shapley and Scarf, 1974) and *w-chains* (Roth et al., 2004), respectively. Note, however, that top-trading cycles as well as *w-chains* are mechanisms used to reallocate items among the agents in the case of initial ownership whereas trading cycles, as defined in this paper, only are used as tools to describe allocative distinctions between any two given states. Also, initial ownership is not assumed in this paper.

LEMMA 1 For a given profile $R \in \mathcal{R}$ and a given priority structure π , let μ be a weakly constrained efficient assignment. Then the set of unenvied agents F is nonempty.

PROOF: Suppose that μ is weakly constrained efficient. To obtain a contradiction, assume that $F = \emptyset$. But then, $E = N$ by construction. First note that if $\mu_i = 0$ for some $i \in N$, then $i \in F$. Hence, $\mu_i \neq 0$ for all $i \in N$. Because $E = N$, let $i_1 \in E$ and define recursively a sequence of agents (i_1, \dots, i_t) such that i_{j+1} has priority to envy i_j for $1 \leq j < t$. Since $F = \emptyset$, there are indices t and k such that i_k has priority to envy i_t and $k < t$. But then there is a priority respecting improvement of μ , denoted by μ' , defined by a closed trading cycle where: $\mu'_{i_j} = \mu_{i_{j-1}}$ for $k < j \leq t$, $\mu'_{i_k} = \mu_{i_t}$, and $\mu'_{i_j} = \mu_{i_j}$ for the remaining agents. But this contradicts the assumption that μ is weakly constrained efficient. Hence, $F \neq \emptyset$. *Q.E.D.*

DEFINITION 12 For a given profile $R \in \mathcal{R}$ and a given priority structure π , suppose that assignment μ is weakly constrained efficient and that the corresponding set E is nonempty. The correspondence $\varphi : F \rightarrow 2^F$ is then defined as follows. For $i \in F$, $i' \in \varphi(i)$ if there is a sequence (i_1, \dots, i_t) of agents and an index k , $2 \leq k \leq t-1$, such that:

- (i) $i_1 = i$ and $i_t = i'$,
- (ii) $i_j \in E$ if and only if $2 \leq j \leq k$,
- (iii) i_j has priority to envy i_{j+1} for $1 \leq j < k$, and
- (iv) $\mu_{i_j} I_{i_j} \mu_{i_{j+1}}$ for $k \leq j < t$.

The purpose of the correspondence φ is to get means to make priority respecting improvements. The construction of φ starts with an agent $i_k \in E$ such that $\mu_{i_k} I_{i_k} \mu_{i_{k+1}}$ and $i_{k+1} \in F$. Then agent i_{k-1} is chosen to be the agent that has priority to envy agent i_k , agent i_{k-2} is chosen to be the agent that has priority to envy agent i_{k-1} , and so on until the beginning of the sequence is reached where $i_2 \in E$ and $i_1 \in F$. The choice of i_j for $k < j \leq t$ is such that $i_j \in F$ and $\mu_{i_j} I_{i_j} \mu_{i_{j+1}}$ for $k \leq j < t$. Note that the definition of φ presupposes that F is nonempty. But this follows from Lemma 1 as μ is constrained efficient by assumption. Also, $\varphi(i)$ may be empty for some $i \in F$.

A set of houses S is isolated, at a given assignment, if the assignment is envy-free among the agents who are assigned a house in S , and all agents who are not assigned a house in S strictly prefer their assigned house to any house in the set S .

DEFINITION 13 For a given profile $R \in \mathcal{R}$, a set of houses $S \subset H$ ($S \neq H$) is *isolated* at assignment μ if for all $h \in S$: (i) $\mu_i R_i h$ for all $i \in N$, and (ii) $\mu_i P_i h$ for all $i \in N$ with $\mu_i \notin S$.

PROPOSITION 1 Let $R \in \mathcal{R}$ be a given profile and let π be a given priority structure. If μ is a weakly constrained efficient assignment and there is an agent $i \in N$ such that $\mu_{i'} P_i \mu_i$ for some $i' \in N$, then there is an isolated set $S \subset H$ of houses. In addition, $S \setminus \mu_0 \neq \emptyset$.

PROOF: Suppose, as in the proposition, that $E \neq \emptyset$ and that μ is weakly constrained efficient. Let, in addition, the correspondence $\varphi : F \rightarrow 2^F$ be defined as in Definition 12. To obtain a contradiction, suppose that there is no isolated set.

First, it is demonstrated that $\cup_{i \in F} \varphi(i) = F$. By contradiction, suppose that $i' \in F$ but $i' \notin \cup_{i \in F} \varphi(i)$, and let $F' \subset F$ be defined as: $i \in F'$ if and only if there is a sequence (i_1, \dots, i_t) of agents $i_j \in F$ such that $i_1 = i$, $i_t = i'$ and $\mu_{i_j} I_{i_j} \mu_{i_{j+1}}$ for $1 \leq j < t$. Since there is no isolated set, by assumption, there is an agent $i'' \in E$ such that $\mu_{i''} I_{i''} \mu_i$ for some $i \in F'$. But then there is also a sequence (i_q, \dots, i_0) of agents, where $q < 0$, such that:

- $i_q \in F$ and $i_j \in E$ if $j > q$,
- i_j has priority to envy i_{j+1} for $q \leq j \leq -1$, and

- $i_0 = i''$.

Note that the existence of $i_q \in F$ follows as μ is weakly constrained efficient, i.e., if always $i_j \in E$, then there would be a priority respecting improvement assignment where only agents in E are trading, contradicting that μ is weakly constrained efficient. But we now have a sequence $(i_q, \dots, i_0, i_1, \dots, i_t)$ that satisfies properties (i)–(ii) in Definition 12. Hence, $i' = i_t \in \varphi(i_q)$. This shows that $\cup_{i \in F} \varphi(i) = F$ must be the case.

Let now $F^* = \{i \in F : \varphi(i) \neq \emptyset\}$. Then, $F = \cup_{i \in F} \varphi(i) = \cup_{i \in F^*} \varphi(i)$. Hence, for each $i \in F^*$ there is an $i' \in F^*$ such that $i \in \varphi(i')$. Then there are sequences (i_1, \dots, i_t) , $i_j \in F^*$, of agents such that $i_j \in \varphi(i_{j+1})$ for $1 \leq j \leq t-1$. Consider some $i_1 \in F^*$ and choose t as large as possible. This means that $i_t \in \varphi(i_l)$ for some $l < t$. According to the definition of φ , for each $j \leq t$, there are sequences of agents satisfying properties (i)–(iv) of Definition 12 in the following way:

$$(i_{1j}, i_{2j}, \dots, i_{q_j j}) \text{ with } i_{1j} = i_{j+1} \text{ and } i_{q_j j} = i_j.$$

Thus for $j \geq l$:

$$\begin{aligned} & (i_{1t}, i_{2t}, \dots, i_{q_t t}) \text{ with } i_{1t} = i_l \text{ and } i_t = i_{q_t t} \in \varphi(i_{1t}), \\ & (i_{1t-1}, i_{2t-1}, \dots, i_{q_{t-1} t-1}) \text{ with } i_{1t-1} = i_t \text{ and } i_{t-1} = i_{q_{t-1} t-1} \in \varphi(i_{1t-1}), \\ & \vdots \qquad \qquad \qquad \vdots, \\ & (i_{1l+1}, i_{2l+1}, \dots, i_{q_{l+1} l+1}) \text{ with } i_{1l+1} = i_{l+2} \text{ and } i_{l+1} = i_{q_{l+1} l+1} \in \varphi(i_{1l+1}), \\ & (i_{1l}, i_{2l}, \dots, i_{q_l l}) \text{ with } i_{1l} = i_{l+1} \text{ and } i_l = i_{q_l l} \in \varphi(i_{1l}). \end{aligned}$$

Now construct one sequence and rename the agents according to:

$$\begin{aligned} (i'_1, i'_2, \dots, i'_{t'}) = \\ (i_{1t}, i_{2t}, \dots, i_{q_t t}, i_{2t-1}, i_{3t-1}, \dots, i_{q_{t-1} t-1}, \dots, i_{2l+1}, i_{3l+1}, \dots, i_{q_{l+1} l+1}, i_{2l}, i_{3l}, \dots, i_{q_l l}). \end{aligned}$$

Here, $i'_1 = i_{1t} = i_l$ and $i'_{t'} = i_{q_l l} = i_l$. Then, since $i'_1 = i'_{t'}$, there must be indices k and p such that the subsequence $(i'_k, i'_{k+1}, \dots, i'_{p-1})$ contains only distinct agents and $i'_k = i'_p$. From the construction of the sequence, it also follows that some agent in the subsequence belongs to E . But then we can define a priority respecting improvement μ' according to $\mu'_{i'_{j+1}} = \mu'_{i'_j}$ for $j < p$, $\mu'_{i'_1} = \mu'_{i'_p}$ and $\mu'_i = \mu_i$ for the remaining agents. This is a contradiction to μ being weakly constrained efficient. Hence, there must be an isolated set.

Finally, if the assignment $\mu : N \rightarrow H$ is weakly constrained efficient and the set of envied agents E is nonempty, the same presumptions are valid for the assignment $\tilde{\mu} : N \rightarrow H \setminus \mu_0$ where $\tilde{\mu}_i = \mu_i$ for all $i \in H$. Hence, the assignment $\tilde{\mu}$ has an isolated set \tilde{S} where $\tilde{S} \cap \mu_0 = \emptyset$. The set \tilde{S} is obviously also an isolated set at assignment μ . Q.E.D.

APPENDIX B: THE PROOF OF THEOREM 1

The first step in the proof of Theorem 1 is to consider a superset of Π_R , denoted by $\bar{\Pi}_R$, and show the existence of a minimal price vector p^* in $\bar{\Pi}_R$. In the second step, it is demonstrated that p^* belongs to Π_R , and, consequently, that p^* is a minimal price vector in Π_R . The proof of Theorem 1 is proceeded by some notation and two lemmas.

A state (p, μ) that satisfies conditions (ii)–(iv) of Definition 3, at a given profile $R \in \mathcal{R}$, is called a *quasi rationing price equilibrium* (quasi RPE). For a given profile $R \in \mathcal{R}$, denote by $\bar{\Sigma}_R$ the set of quasi RPE and by $\bar{\Pi}_R$ the corresponding set of price vectors. Clearly, $\Sigma_R \subset \bar{\Sigma}_R$ and $\Pi_R \subset \bar{\Pi}_R$.

LEMMA 2 For any given profile $R \in \mathcal{R}$ and any given priority structure π , the set $\bar{\Pi}_R$ is nonempty, bounded from below, and closed.

PROOF: From the assumptions on R_i (see Section 3), it follows that $p_h P_i p_{h'}$ for all $i \in N$ and for all $h' \in H$ if $h = 0$ and if $p_{h'}$ is “sufficiently large.” Hence, $p \in \bar{\Pi}_R$ if $p_{h'}$ is “sufficiently large” for all $h' \in H$ as then $(p, \mu) \in \bar{\Sigma}_R$ if $\mu_i = 0$ for all $i \in N$. Consequently, $\bar{\Pi}_R \neq \emptyset$. The set $\bar{\Pi}_R$ is bounded from below as $p \in \Omega$ and as Ω is bounded from below by \underline{p} . To prove that $\bar{\Pi}_R$ is closed, suppose that $R \in \mathcal{R}$ and let $(p^j)_{j=1}^\infty$ be a convergent sequence of price vectors such that $p^j \in \bar{\Pi}_R$ and $p^j \rightarrow p$ as $j \rightarrow \infty$. Consider next the state $(p^j, \mu^j) \in \bar{\Sigma}_R$. Since there is only a finite number of distinct assignments, it is, without loss of generality, assumed that $\mu^j = \mu$ for all j . The following three observations complete the proof:

- $p \in \Omega$. This follows since Ω is closed.
- There is a constrained efficient assignment μ^* . This assignment can be identified using the following procedure. If μ is envy-free, then μ is constrained efficient by definition. Hence, $\mu^* = \mu$. Suppose instead that μ is not envy-free, i.e., that $p_{\mu_{i'}} P_i p_{\mu_i}$ for some distinct agents i and i' in N . Then $p_{\mu_{i'}}^j P_i p_{\mu_i}^j$ for j “sufficiently large” by continuity of the preferences. Hence, $\pi_{i'} \mu_{i'} < \pi_i \mu_i$ as $(p^j, \mu) \in \bar{\Sigma}_R$. Consequently, μ is priority respecting. Now, since there is only a finite number of assignments, there is a constrained efficient assignment μ^* that dominates μ in case μ is not constrained efficient.
- $p_h = \bar{p}_h$ if $p_h P_i p_{\mu_i^*}$ for some $i \in N$ and some $h \in H$. This follows because if $p_h P_i p_{\mu_i^*}$, then $p_h P_i p_{\mu_i}$ since μ^* dominates μ . But then $p_h^j P_i p_{\mu_i}^j$ for j “sufficiently large” and, hence, $p_h^j = \bar{p}_h$ for j “sufficiently large.” Thus, $p_h = \bar{p}_h$.

Q.E.D.

LEMMA 3 (Alkan et al., 1991, Perturbation Lemma) Let (p, μ) be a state and $R \in \mathcal{R}$ a given profile. If $p_{\mu_i} R_i p_h$ for all $i \in N$ and for all $h \in H \cup \{0\}$, and $p_h > \underline{p}_h$ for all $h \in H$, then for each “sufficiently small” $\varepsilon > 0$ there exists another state (p', μ') , where $p'_{\mu'_i} R_i p'_h$ for all $i \in N$ and for all $h \in H \cup \{0\}$, and where $\underline{p}_h \leq p_h - \varepsilon < p'_h < p_h$ for all $h \in H$.

THEOREM 1 For any profile $R \in \mathcal{R}$ and any given priority structure π , there is a minimal RPE vector $p^* \in \bar{\Pi}_R$.

PROOF: Let $(p^*, \mu) \in \bar{\Sigma}_R$, and suppose that p^* is a minimal price vector in $\bar{\Pi}_R$ and, moreover, that the assignment μ is chosen so that the number $\#H' = \#\{h \in \mu_0 : p_h^* > \underline{p}_h\}$ is minimal. The existence of a minimal price vector $p^* \in \bar{\Pi}_R$ follows since $\bar{\Pi}_R$ is nonempty, bounded from below, and closed by Lemma 2. To prove the statement, we show that $\#H' = 0$ because then p^* is also minimal in Π_R . This conclusion follows directly from Definition 3(i), the assumption that p^* is minimal in $\bar{\Pi}_R$, and the fact that $\Pi_R \subset \bar{\Pi}_R$. To obtain a contradiction, suppose that $H' \neq \emptyset$, i.e., that $\#H' > 0$. Let now:

$$H'' = \left\{ \begin{array}{l} h \in H : \text{there is a sequence } (i_1, \dots, i_t) \text{ of agents such that } \mu_{i_t} = h \\ \text{and } p_{h'}^* R_{i_1} p_{\mu_{i_1}}^* \text{ for some } h' \in H' \text{ and } p_{\mu_{i_{j-1}}}^* R_{i_j} p_{\mu_{i_j}}^* \text{ for } 2 \leq j \leq t. \end{array} \right\}$$

It is first demonstrated that $p_{\mu_i}^* R_i p_h^*$ for all $i \in N$ and for all $h \in H' \cup H''$. Note that if $h \in H'$, the conclusion follows directly by the definition of a quasi RPE. To obtain a contradiction, suppose instead that there is an agent $i \in N$ and a house $h \in H''$ such that $p_h^* P_i p_{\mu_i}^*$. But then there is also an agent $l \in N$, houses $h' \in H'$, $h'' \in H''$ where $p_{h''}^* P_l p_{\mu_l}^*$, and a sequence (i_1, \dots, i_t) of agents such that:

- $p_{h'}^* I_{i_1} p_{\mu_{i_1}}^*, p_{\mu_{i_{j-1}}}^* I_{i_j} p_{\mu_{i_j}}^*$ for $2 \leq j \leq t$, and $h'' = \mu_{i_t}$,
- $p_{\mu_i}^* R_i p_{i_j}^*$ for all $i \in N$, and all j where $1 \leq j \leq t$, and
- agent l is the agent with priority to envy agent i_t at the assignment μ .

But given such a sequence, the assignment μ' is a priority respecting improvement of μ if $\mu'_{i_j} = \mu_{i_{j-1}}$ for $2 \leq j \leq t$, $\mu'_{i_1} = h'$, $\mu'_l = \mu_{i_t}$, and $\mu'_i = \mu_i$ for the remaining agents i . This is a contradiction to the assumption that μ is constrained efficient. Hence, $p_{\mu_i}^* R_i p_h^*$ for all $i \in N$ and for all $h \in H' \cup H''$.

Now, if $p_h^* = \underline{p}_h$ for some $h \in H''$, then it is possible to make a priority respecting reassignment of the houses such that the number $\#H'$ decreases, contradicting the assumption that the number $\#H'$ is minimal. Hence, $p_h^* > \underline{p}_h$ for all $h \in H' \cup H''$. But then $H' \cup H''$ is an isolated set with $p_h^* > \underline{p}_h$ for all $h \in H' \cup H''$ by the above conclusions. By Lemma 3, it is then possible to decrease the price p_h^* for all $h \in H' \cup H''$ and obtain a new quasi RPE. But this contradicts that p^* is a minimal price vector in $\bar{\Pi}_R$. Hence, $H' = \emptyset$, i.e., $\#H' = 0$. Q.E.D.

COROLLARY 1 Let $R \in \mathcal{R}$ be a given profile and π a given priority structure. A price vector p^* is a minimal price vector in $\bar{\Pi}_R$ if and only if p^* is a minimal price vector in Π_R .

PROOF: The proof of Theorem 1 shows that if p^* is minimal in $\bar{\Pi}_R$, then p^* is minimal in Π_R .

Let now $\tilde{p} \in \Pi_R$ be minimal in Π_R . Then $\tilde{p} \in \bar{\Pi}_R$ since $\Pi_R \subset \bar{\Pi}_R$. Suppose that \tilde{p} is not minimal in $\bar{\Pi}_R$. Then there is a minimal price vector p^* in $\bar{\Pi}_R$ where $p^* \leq \tilde{p}$ and $p^* \neq \tilde{p}$. But then p^* is minimal in Π_R by Theorem 1. Consequently, \tilde{p} is not minimal in Π_R , which is a contradiction. Q.E.D.

APPENDIX C: THE PROOF OF THEOREM 2

The proof of Theorem 2 is preceded by some notation and two lemmas. Because the outcome of these lemmas also will be useful in the proof of Theorem 3, they are formulated somewhat more general than necessary at this point. More precisely, the lemmas are based on two given profiles $R \in \tilde{\mathcal{R}}$ and $R' \in \tilde{\mathcal{R}}$ where $R' = (R'_C, R'_{-C})$ for some (possibly empty) set $C \subset N$, and two states $(p, \mu) \in \Sigma_R$ and $(p', \mu') \in \Sigma_{R'}$ where $p'_{\mu'_q} P_q p_{\mu_q}$ for all $q \in C$. All results presented in this appendix hold for $C = \emptyset$ (i.e., for $R = R'$) and this is exactly what is needed in the proof of Theorem 2. Later, in the proof of Theorem 3, it will be assumed that C is nonempty.

NOTATION 1 Let $R \in \tilde{\mathcal{R}}$ and $R' \in \tilde{\mathcal{R}}$ be two given profiles where $R' = (R'_C, R'_{-C})$ for some $C \subset N$, and consider the two states $(p, \mu) \in \Sigma_R$ and $(p', \mu') \in \Sigma_{R'}$ where $p'_{\mu'_q} P_q p_{\mu_q}$ for all $q \in C$. Define $S_1 = \{h \in H : p'_h < p_h\}$, $S_2 = \{h \in H : p'_h = p_h\} \cup \{0\}$, and $S_3 = \{h \in H : p'_h > p_h\}$.¹⁹

LEMMA 4 Let $R \in \tilde{\mathcal{R}}$ and $R' \in \tilde{\mathcal{R}}$ be two given profiles where $R' = (R'_C, R'_{-C})$ for some $C \subset N$, and consider the two states $(p, \mu) \in \Sigma_R$ and $(p', \mu') \in \Sigma_{R'}$ where $p'_{\mu'_q} P_q p_{\mu_q}$ for all $q \in C$. Let also S_1 and S_3 be defined as in Notation 1. Then (i) $p'_{\mu'_i} P_i p_{\mu_i}$ if $\mu_i \in S_1$, and (ii) $p_{\mu_i} P_i p'_{\mu'_i}$ if $\mu'_i \in S_3$.

PROOF: Note that part (i) holds by definition if $i \in C$. Suppose therefore that $i \notin C$, i.e., $R_i = R'_i$. In this case, $p'_{\mu'_i} R_i p_{\mu_i}$ by Definition 3(iii) since $(p', \mu') \in \Sigma_{R'}$ and $\mu_i \in S_1$ (i.e., $p'_{\mu'_i} < p_{\mu_i}$). But then $p'_{\mu'_i} P_i p_{\mu_i}$, by monotonicity, as $p'_{\mu'_i} < p_{\mu_i}$. Part (ii) follows by symmetric arguments. Q.E.D.

LEMMA 5 Let $R \in \tilde{\mathcal{R}}$ and $R' \in \tilde{\mathcal{R}}$ be two given profiles where $R' = (R'_C, R'_{-C})$ for some $C \subset N$, and consider the two states $(p, \mu) \in \Sigma_R$ and $(p', \mu') \in \Sigma_{R'}$ where $p'_{\mu'_q} P_q p_{\mu_q}$ for all $q \in C$. Let also (i_1, \dots, i_t) be a trading cycle where $p'_{\mu'_{i_l}} P_{i_l} p_{\mu_{i_l}}$ for some i_l in the trading cycle, and let S_1 and S_2 be defined as in Notation 1. Then $p'_{\mu'_{i_j}} R_{i_j} p_{\mu_{i_j}}$ and $\mu'_{i_j} \in S_1 \cup S_2$ for $1 \leq j \leq t$.

PROOF: To obtain a contradiction, suppose that there is a trading cycle $G = (i_1, \dots, i_t)$ from μ to μ' with $p_{\mu_{i_k}} P_{i_k} p'_{\mu'_{i_k}}$ for some agent $i_k \neq i_l$ in the trading cycle. Without loss of generality, it is assumed that $l < k$. In case $l < k$ is not possible for the cycle G , we can instead consider some other trading cycle with trade from μ' to μ . Now it is also possible, without loss of generality, to choose $l = 1$ and $k > 1$ so that:

$$p'_{\mu'_{i_1}} P_{i_1} p_{\mu_{i_1}}, p'_{\mu'_{i_j}} I_{i_j} p_{\mu_{i_j}} \text{ for } 1 < j < k, \text{ and } p_{\mu_{i_k}} P_{i_k} p'_{\mu'_{i_k}}.$$

¹⁹Note that the null house cannot belong to S_1 or S_3 as $p'_0 = p_0 = \underline{p}_0 = \bar{p}_0 = 0$ by assumption.

Suppose now that $k > 2$. From Lemma 4(i) and the above assumptions, it follows that $\mu'_{i_1} = \mu_{i_2} \notin S_1$ and $\mu'_{k-1} = \mu_k \notin S_1$. Moreover, from Lemma 4(ii) and the above assumptions, it follows that $\mu'_{i_1} = \mu_{i_2} \notin S_3$ and $\mu'_{k-1} = \mu_k \notin S_3$. Hence, $\mu'_{i_1} = \mu_{i_2} \in S_2$ and $\mu'_{k-1} = \mu_k \in S_2$. But then $p_{\mu_{i_2}} = \bar{p}_{\mu_{i_2}}$ and $p_{\mu_{i_k}} = \bar{p}_{\mu_{i_k}}$, by Definition 3(iii), as $p_{\mu_{i_2}} P_{i_1} p_{\mu_{i_1}}$ and $p_{\mu_{i_k}} P_{i_k} p'_{\mu'_{i_k}}$, respectively. Consequently, houses μ_{i_2} and μ_{i_k} are CBI at profile R which contradicts that $R \in \tilde{\mathcal{R}}$. Hence, $k = 2$.

Since μ and μ' are constrained efficient at states (p, μ) and (p', μ') , respectively, it is clear that $\pi_{i_2 \mu_{i_2}} < \pi_{i_1 \mu_{i_2}}$ and $\pi_{i_1 \mu'_{i_1}} < \pi_{i_2 \mu'_{i_1}}$. But this is a contradiction since $k = 2$, i.e., $\mu'_{i_1} = \mu_{i_2}$. Hence, there is no agent $i_k \neq i_l$ with $p_{\mu_{i_k}} P_{i_k} p'_{\mu'_{i_k}}$.

Finally, the statement $\mu'_{i_j} \in S_1 \cup S_2$ for $1 \leq j \leq t$ follows directly from Lemma 4.

Q.E.D.

Given the result in Lemma 5, each agent in N can be assigned to one of two disjoint sets, defined below in Notation 2, whenever two RPE states are compared.

NOTATION 2 Let $R \in \tilde{\mathcal{R}}$ and $R' \in \tilde{\mathcal{R}}$ be two given profiles where $R' = (R'_C, R_{-C})$ for some $C \subset N$, and consider the two states $(p, \mu) \in \Sigma_R$ and $(p', \mu') \in \Sigma_{R'}$ where $p'_{\mu'_q} P_q p_{\mu_q}$ for all $q \in C$. Denote by $N^+ \subset N$ the set of agents belonging to a trading cycle where at least one agent receives a strictly higher utility in the change from (p, μ) to (p', μ') . Let N^- represent the remaining agents, i.e., $N^- = N \setminus N^+$.

THEOREM 2 For any profile $R \in \tilde{\mathcal{R}}$ and any given priority structure π , a minimal price vector $p^* \in \Pi_R$ is unique.

PROOF: This proof uses Notation 2 for the case when $C = \emptyset$, i.e., for the case when $R = R'$. Let now $R \in \tilde{\mathcal{R}}$, and let $p', p'' \in \Pi_R$ be two minimal price vectors. The two corresponding RPE states are denoted by $(p', \mu') \in \Sigma_R$ and $(p'', \mu'') \in \Sigma_R$. Let also p be a price vector defined by $p_h = \min\{p'_h, p''_h\}$ for all $h \in H$. The proof demonstrates that there is an assignment μ such that the state (p, μ) is a quasi RPE (defined earlier in Appendix B). But then a minimal price vector in Π_R is unique. Consequently, a minimal price vector in Π_R is unique by Corollary 1. As p obviously belongs to Ω , we need only to demonstrate that μ is well-defined and constrained efficient, and that Definition 3(iii) is satisfied for (p, μ) .

Consider the trading cycles from (p', μ') to (p'', μ'') , and define N^+ and N^- as in Notation 2. Let now $\hat{\mu}$ be defined by:

$$\hat{\mu}_i = \mu''_i \text{ if } i \in N^+, \text{ and } \hat{\mu}_i = \mu'_i \text{ if } i \in N^-.$$

To show that $\hat{\mu}$ is a well-defined assignment, let $i, l \in N$ ($i \neq l$) be two agents such that $\hat{\mu}_i = \hat{\mu}_l$. If $i, l \in N^+$, then $\mu''_i = \mu''_l$. Hence, $\mu''_i = \mu''_l = 0$. This cannot be the case when $i, l \in N^+$. By the same arguments, if $i, l \in N^-$ then $\mu'_i = \mu'_l = 0$, and, consequently, $\hat{\mu}_i = \hat{\mu}_l = 0$. Finally, if $i \in N^+$ and $l \in N^-$, then $\mu''_i = \mu'_l$. But this means that i and l belong to the same trading cycle, and hence both belong to N^+ or both belong to N^- , which is a contradiction. Hence, $\hat{\mu}$ is a well-defined assignment.

We next show that $\hat{\mu}$ is priority respecting (Definition 1), i.e., that for all $i, l \in N$: (i) $p_{\hat{\mu}_i} P_l p_{\hat{\mu}_l}$ only if $\pi_{i \hat{\mu}_i} < \pi_{l \hat{\mu}_l}$, and (ii) $\mu_i R_i h$ if $h \in \mu_0$. Note first that if $i, l \in N^+$ or if $i, l \in N^-$, then condition (i) is satisfied since μ' and μ'' are priority respecting. If, on the other hand, $i \in N^+$ and $l \in N^-$, then $p_{\hat{\mu}_i} P_l p_{\hat{\mu}_l}$ means that $p''_{\mu'_i} P_l p'_{\mu'_l}$. Hence, $p''_{\mu'_i} P_l p'_{\mu'_l}$ since $l \in N^-$. But then $\pi_{i \mu'_i} < \pi_{l \mu'_l}$ and $\mu''_i = \hat{\mu}_i$. Consequently, condition (i) is satisfied (the remaining case, $p_{\mu_i} P_l p_{\mu_l}$, follows by symmetrical arguments). That condition (ii) is satisfied follows directly since μ' and μ'' are priority respecting. Hence, $\hat{\mu}$ is priority respecting.

Note now that if $\hat{\mu}$ is not constrained efficient, then there is another priority respecting assignment that Pareto dominates $\hat{\mu}$. Hence, after a finite number of improvements, a constrained efficient assignment μ is obtained.

Finally, to demonstrate that Definition 3(iii) holds for (p, μ) , assume that $p_{\mu_i} < \bar{p}_{\mu_i}$ for some $i \in N$. Suppose also that $p_{\mu_i} = p'_{\mu_i}$, and, hence, that $p'_{\mu_i} < \bar{p}_{\mu_i}$. Consider first an agent $l \in N^+$. Then:

$$p_{\mu_i} R_l p_{\hat{\mu}_l} I_l p_{\mu'_l} I_l p''_{\mu'_l} R_l p'_{\mu'_l} R_l p'_{\mu'_l} I_l p_{\mu_i}.$$

This sequence follows from the definitions of p , μ , $\hat{\mu}$, N^+ , and Definition 3(iii) since $p'_{\mu_i} < \bar{p}_{\mu_i}$. If $l \in N^-$, the corresponding sequence becomes:

$$p_{\mu_l} R_l p_{\hat{\mu}_l} I_l p_{\mu'_l} I_l p'_{\mu'_l} R_l p'_{\mu_i} I_l p_{\mu_i}.$$

In both cases, we obtain $p_{\mu_l} R_l p_{\mu_i}$ if $p'_{\mu_i} < \bar{p}_{\mu_i}$. Hence, $p_{\mu_i} = \bar{p}_{\mu_i}$ if $p_{\mu_i} P_l p_{\mu_l}$ must be the case. The case $p_{\mu_i} = p''_{\mu_i}$ now follows by symmetrical arguments. Q.E.D.

APPENDIX D: THE PROOF OF THEOREM 3

THEOREM 3 Let f be a minimal RPE mechanism with domain $\tilde{\mathcal{R}}$. Then f is group non-manipulable.

PROOF: Suppose that the agents in $C \subset N$ can manipulate the minimal RPE mechanism at profile $R \in \tilde{\mathcal{R}}$. Then there is a profile $R' = (R'_C, R_{-C}) \in \tilde{\mathcal{R}}$ and two states $(p, \mu) = f(R)$ and $(p', \mu') = f(R')$ such that $p'_{\mu'_q} P_q p_{\mu_q}$ for all $q \in C$ by Definition 6. Let S_1, S_2 , and S_3 be defined as in Notation 1, and let N^+ and N^- be defined as in Notation 2. Now $N^+ \neq \emptyset$ as $q \in N^+$ for all $q \in C$ by assumption (i.e., $N^- \cap C = \emptyset$). To prove the theorem, it is sufficient to show that there is an isolated set $S \subset S_1$. Then it is possible to decrease p_h for all $h \in S$ by a “small” $\varepsilon > 0$ and obtain a new RPE according to Lemma 3, which contradicts that (p, μ) is selected by the minimal RPE mechanism. Note also that there may be cases where $S \cap S_1 = \emptyset$ (e.g., when $\underline{p} = \bar{p}$), but then a contradiction is obtained before Lemma 3 is being used.

Consider now the restriction μ^+ of μ to N^+ . That is, $\mu^+ : N^+ \rightarrow H^+$ and $\mu_i^+ = \mu_i$, where H^+ is the range of μ^+ , i.e., $H^+ = \{h \in H \cup \{0\} : h = \mu_i \text{ for some } i \in N^+\}$. By Lemma 5, $H^+ \subset S_1 \cup S_2$.

We will, in a series of steps, prove that the restriction μ^+ of μ is weakly constrained efficient. Suppose that the restriction μ^+ is not weakly constrained efficient. Then there is a sequence of agents $G = (i_1, \dots, i_t)$ where $i_j \in N^+$ for $1 \leq j \leq t$ that defines a trading cycle which is a priority respecting improvement of μ^+ but can be “blocked” by an agent $k \in N^-$. This means that there is an agent i_l in the sequence G such that $p_{\mu_{i_l}} P_k p_{\mu_k}$ and $\pi_{k\mu_{i_l}} < \pi_{i_{l-1}\mu_{i_l}}$. But then $p_{\mu_{i_l}} = \bar{p}_{\mu_{i_l}}$ since (p, μ) is an RPE. Moreover, since $k \in N^-$ and $\mu_{i_l} \in S_1 \cup S_2$ it follows by monotonicity that $p'_{u_{i_l}} P_k p'_{u_k}$. Consequently, $p'_{\mu_{i_l}} = \bar{p}_{\mu_{i_l}}$ as (p', μ') is an RPE. Hence, $p_{\mu_{i_l}} = p'_{\mu_{i_l}} = \bar{p}_{\mu_{i_l}}$ and $\mu_{i_l} \in S_2$.

We first prove that the agents i_l and k can be chosen so that i_{l-1} is the agent in N^+ with priority to envy agent i_l . To prove this, note that if $p_{\mu_{i_l}} P_{i_{l-1}} p_{\mu_{i_{l-1}}}$, then agent i_{l-1} is the agent in N^+ with priority to envy agent i_l according to Definition 9. Hence, to obtain a contradiction, it is assumed that $p_{\mu_{i_l}} I_{i_{l-1}} p_{\mu_{i_{l-1}}}$ (note that it cannot be the case that $p_{\mu_{i_{l-1}}} P_{i_{l-1}} p_{\mu_{i_l}}$ as $i_{l-1} \in N^+$).

Since the sequence G defines a priority respecting improvement, there is an agent $i_{l'}$ in the sequence such that $p_{\mu_{i_{l'+1}}} P_{i_{l'}} p_{\mu_{i_{l'}}$. Now, if $l' < l - 1$ or if G is a closed trading cycle, l' can be chosen so that $p_{\mu_{i_j}} I_{i_j} p_{\mu_{i_{j+1}}}$ for all j , $l' < j \leq l - 1$, after renumbering the agents and the houses. But then houses $\mu_{i_{l'+1}}$ and μ_{i_l} are connected by indifference, which is a contradiction. Assume instead that $l' \geq l$ and that G is an open trading cycle. Then $p_{\mu_{i_j}} I_{i_j} p_{\mu_{i_{j+1}}}$ for all $j < l$. In this case, choose l as large as possible such that $p_{\mu_{i_l}} P_k p_{\mu_k}$ for some $k \in N^-$ and let k be the highest ranked agent with this property. Then an assignment μ'' is a priority respecting improvement for the entire set N of agents, if $\mu''_k = \mu_k$, $\mu''_{i_j} = \mu_{i_{j+1}}$, for $j \geq l$, and $\mu''_i = \mu_i$ for the remaining agents. To show this, we need only to check that the priorities are respected. But this follows since $\pi_{k\mu_{i_l}} < \pi_{i_{l-1}\mu_{i_l}}$ and $\pi_{i_{l-1}\mu_{i_l}} < \pi_{i_j\mu_{i_l}}$ if $p_{\mu_{i_l}} P_{i_j} p_{\mu_{i_j}}$ and $j < l - 1$. Hence, a contradiction to the assumption that the assignment μ is constrained efficient. We, therefore, conclude that $p_{\mu_{i_l}} P_{i_{l-1}} p_{\mu_{i_{l-1}}}$ must be the case, and, consequently, that agent i_{l-1} is the agent in N^+ with priority to envy agent i_l .

Consider now the state (p', μ') and recall from the above that $p_{\mu_{i_l}} = p'_{\mu_{i_l}} = \bar{p}_{\mu_{i_l}}$. But then $\mu_{i_l} \notin \mu'_0$ as (p', μ') is an RPE. Consequently, there is an agent $r \in N^+$ such that $\mu'_r = \mu_{i_l}$ since $\mu_{i_l} \in H^+$. Note also that $r \neq i_l$ because if $r = i_l$, then agent i_l belongs to a trading cycle containing only himself which contradicts that $i_l \in N^+$ as $\mu_{i_l} \in S_2$ by the above conclusions. Now, either $p_{\mu_{i_l}} P_r p_{\mu_r}$ or $p_{\mu_r} R_r p_{\mu_{i_l}}$ must

prevail. We will demonstrate that neither of these cases hold which establishes a contradiction to the assumption that the restriction μ^+ of μ is not weakly constrained efficient.

- Suppose that $p_{\mu_{i_l}} P_r p_{\mu_r}$. Because $p_{\mu_{i_l}} P_k p_{\mu_k}$ and $\mu'_r = \mu_{i_l}$, by assumption, it is clear that $p_{\mu'_r} P_k p_{\mu_k}$. Further, because $\mu'_r = \mu_{i_l} \in S_1 \cup S_2$, by Lemma 5, monotonicity yields $p'_{\mu'_r} P_k p_{\mu_k}$. Moreover, $p_{\mu_k} R_k p'_{\mu'_k}$ as $k \in N^-$. Consequently, $p'_{\mu'_r} P_k p'_{\mu'_k}$. Hence, $\pi_{r\mu_{i_l}} < \pi_{k\mu_{i_l}}$ since μ is priority respecting. Because agent i_{l-1} is the agent in N^+ with priority to envy i_l and $p_{\mu_{i_l}} P_r p_{\mu_r}$, it must be the case that $\pi_{l-1\mu_{i_l}} < \pi_{r\mu_{i_l}}$. But this, together with $\pi_{r\mu_{i_l}} < \pi_{k\mu_{i_l}}$, is a contradiction to $\pi_{k\mu_{i_l}} < \pi_{i_{l-1}\mu_{i_l}}$.
- Suppose that $p_{\mu_r} R_r p_{\mu_{i_l}}$. Note first that $p_{\mu_r} P_r p_{\mu_{i_l}}$ cannot be the case since $r \in N^+$, $\mu'_r = \mu_{i_l}$, and $\mu_{i_l} \in S_2$. Hence, $p_{\mu_r} I_r p_{\mu_{i_l}}$. Let now $H' \subset H$ be a set of houses that can be linked to μ_{i_l} by a sequence of indifferences, and let $h' = \mu_{i_l}$ and $i' = i_l$ to avoid the same index for two different agents. More precisely, $h \in H'$ if there is a sequence of distinct houses (h_1, \dots, h_t) and a sequence of distinct agents (i_1, \dots, i_{t-1}) such that:

$$p_{h_{j+1}} I_{i_j} p_{h_j} \text{ for } 1 \leq j < t, h = h_1, \text{ and } h_t = \mu_{i_l}.$$

By definition $\mu_{i_l} \notin H'$, but $H' \neq \emptyset$ as $\mu_r \in H'$. Since $h_t = \mu_{i_l}$, $p_{i_l} = \bar{p}_{\mu_{i_l}}$, and no two houses are CBI, it must be the case that $\underline{p}_{h_j} < p_{h_j} < \bar{p}_{h_j}$ for all $1 \leq j < t$. Moreover, if $\mu_i \notin H'$ and $\mu_i \neq h'$ then $p_{\mu_i} P_i p_h$ for all $h \in H'$ by the construction of H' . Note first that if also $p_{\mu_{i'}} P_{i'} p_h$ for all $h \in H'$, then it is possible to decrease p_h for all $h \in H'$ by a “small” $\varepsilon > 0$ and obtain a new RPE according to Lemma 3, which contradicts that (p, μ) is selected by the minimal RPE mechanism. Note next that if $p_{\mu_{i'}} I_{i'} p_h$ for some $h \in H'$, we have a contradiction to the NCBI assumption as $p_{i'} = p_{i_l} = \bar{p}_{\mu_{i_l}}$.

We conclude that the restriction μ^+ of the assignment μ to the group N^+ is weakly constrained efficient. But then by Proposition 1, there is an isolated set $S \subset S_1 \cup S_2$ such that:

- (1) $S \neq \emptyset$ and $S \cap \mu_0 = \emptyset$,
- (2) if $N^{++} = \{i \in N^+ : \mu_i \in S\}$ then $p_{\mu_i} R_i p_{\mu_l}$ for all $i, l \in N^{++}$, and
- (3) if $i \in N^+ \setminus N^{++}$ then $p_{\mu_i} P_i p_h$ for all $h \in S$.

This follows because if some agent in N^+ is envied by some agent in N^+ , then there is an isolated set $S \subset (S_1 \cup S_2) \cap \mu_0$ by Proposition 1, and if no agent in N^+ is envied by an agent in N^+ , the set $S = \{h \in S_1 \cup S_2 : h = \mu_i \text{ for some } i \in N^+\}$ satisfies points (1)–(3) from the above. Note also that if S is isolated, then the set $S \cap \mu_0$ is isolated.

It is next demonstrated that $S \cap S_2 = \emptyset$. Suppose that $S \cap S_2 \neq \emptyset$. Then, there is an agent i' such that $\mu_{i'} \in S \cap S_2$ where the set S satisfies points (1)–(3) from the above. Let (i_1, \dots, i_t) be a trading cycle from μ to μ' such that $i_j = i'$ for some $1 \leq j \leq t$. Then $i_j \in N^+$ for all $1 \leq j \leq t$, by Lemma 5 and Notation 2, as $i' \in N^+$ is implied by the above construction of S . We next consider two cases and prove that in each case a contradiction is obtained, implying that $S \cap S_2 = \emptyset$.

- (i) If $\mu_{i_j} \in S \cap S_2$ for all $1 \leq j \leq t$, then $p_{\mu_{i_j}} I_{i_j} p_{\mu_{i_{j+1}}}$ for all $1 \leq j \leq t$ as S is envy-free. This contradicts that $i_j \in N^+$ because at least one strict preference is required by Notation 2.
- (ii) If $\mu_{i_j} \notin S \cap S_2$ for some $1 \leq j \leq t$, then there is an agent i_l such that $\mu_{i_l} \notin S \cap S_2$ and $\mu'_{i_l} = \mu_{i_{l+1}} \in S \cap S_2$ (or $\mu'_{i_l} \in \mu_0$). Then $\mu_{i_l} \in S_1$. Moreover, $p_{\mu_{i_{l+1}}} P_{i_l} p_{\mu_{i_l}}$ by Lemma 4. This is a contradiction to (2) above.

We conclude that $S \cap S_2 = \emptyset$ and $S \subset S_1$ must be the case. Now if $S_1 = \emptyset$ (e.g., if $p = \bar{p}$), then also $S = \emptyset$, which is a contradiction to (1) above. On the other hand, if $S \neq \emptyset$, then for all $h \in S$:

- $p_{\mu_i} R_i p_h$ for all $i \in N^+$ with $\mu_i \in S$ since S is isolated,
- $p_{\mu_i} P_i p_h$ for all $i \in N^+$ with $\mu_i \notin S$ since S is isolated,

- $p_{\mu_i} P_i p_h$ for all $i \in N^-$. This follows since $p'_{\mu'_i} R_i p'_h$ by the definition of RPE as $p'_h < p_h \leq \bar{p}_h$. Hence, $p'_{\mu'_i} P_i p_h$ by monotonicity. Then once more by monotonicity, $p_{\mu_i} P_i p_h$ since $i \in N^-$.

The above three bullet points demonstrate that $S \subset S_1$ is an isolated set for the entire set H of houses. But then the proof is complete by the arguments stated in the beginning of the proof. Q.E.D.

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