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# Idiosyncratic Risk and Higher-Order Cumulants\*

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## Abstract

We show that, when allowing for general distributions of dividend growth in a Lucas economy with multiple “trees,” idiosyncratic volatility will affect expected returns in ways that are not captured by the log linear approximation. We derive an exact expression for the risk premia for general distributions. Assuming growth rates are Normal Inverse Gaussian (NIG) and fitting the distribution to the data used in Mehra and Prescott (1985), the coefficient of relative risk aversion required to match the equity premium is more than halved compared to the finding in their article.

*Keywords:* idiosyncratic risk, idiosyncratic volatility, risk premia,  
cumulants, NIG distribution

*JEL Codes:* C13, G12

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# 1 Introduction

According to textbook financial theory, exposure to idiosyncratic (asset-specific, unique) risk should not be rewarded on the market. This is because rational investors can eliminate idiosyncratic risk from their portfolios through diversification. The risk that cannot be diversified away is termed *systematic risk*. In order for risk-averse investors to hold a positive supply of stocks, exposure to systematic risk would have to be rewarded.

Although this story seems convincing, recent empirical research has presented evidence that idiosyncratic risk does affect risk premia, but it might not be in the direction that one would first expect. Counter intuitively, Ang, Hodrick, Xing, and Zhang (2006) find a negative relation between lagged idiosyncratic volatility and returns. They write that their results represent “a substantive puzzle” (Ang, Hodrick, Xing, and Zhang, 2006, p. 262).<sup>1</sup> As a theoretical motivation for their empirical setup, they refer to works that use log linear approximations or assume a log normal structure (Campbell, 1993, 1996; Chen, 2002).

This paper derives an exact relation between volatilities and expected returns. Employing an exchange-only Lucas (1978) economy in which we allow for general distributions of dividend growth rates, we find that, because idiosyncratic volatility will affect expected returns in ways that are not captured by the log linear approximation, it will automati-

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<sup>1</sup>However, Fu (2009) argues that “The lagged idiosyncratic volatility might not be a good measure of expected idiosyncratic volatility” (Fu, 2009, p. 25). Instead using EGARCH to capture the time-varying features of idiosyncratic risk, he finds a *positive* relation between conditional idiosyncratic volatilities and returns.

cally appear as though idiosyncratic risk is priced. Moreover, we show that assuming a log Normal Inverse Gaussian (NIG) structure may help mitigate the equity premium puzzle of Mehra and Prescott (1985): using their data set, the coefficient of relative risk aversion required to match the equity premium is more than halved compared to the finding in their article. For high levels of risk aversion, the difference in terms of generated equity premia compared with the log linear approximation (log Normal case) is striking.

From a technical point of view, our paper is related to Martin (2009, 2010), who also expresses equilibrium quantities in terms of cumulant generating functions. Martin (2010) considers a Lucas economy with a single risky asset (*tree*), whereas Martin (2009) considers the case of multiple risky assets. Lillestøl (1998) explores the possibility of using the multivariate NIG distribution within the areas of portfolio choice and risk analysis, and he also briefly considers equilibrium conditions assuming constant absolute risk aversion (CARA) utility. However, none of these works specifically addresses the relation between risk premia and idiosyncratic volatilities.

The remainder of the paper is organized as follows. Section 2 presents our model. In Section 3, we present our theoretical results and, finally, Section 4 concludes the paper.

## 2 Model

To prove our main point, we consider a simple exchange-only Lucas (1978) economy in which there are several risky assets and one risk-free asset. The risk-free asset is in zero net supply. We explicitly model the dividends from  $n$  of the risky assets and the aggregate

dividends. The future dividends from assets 1 through  $n$  are given by

$$D_i = D_{0i}e^{g_i}, i = 1, 2, \dots, n \quad (1)$$

and the future aggregate dividend is  $D_A = D_{0A}e^{g_A}$ . Here,  $g_1, g_2, \dots, g_n$  and  $g_A$  are dividend growth rates, and the vector of growth rates,  $(g_1, g_2, \dots, g_n, g_A)$ , follows a joint probability distribution. The difference  $D_A - \sum_{i=1}^n D_i$  represents the cash flows from the other assets ("the rest of the economy").<sup>2</sup>

There are  $N$  agents, having constant relative risk aversion: Their utility of consumption is given by

$$u(C) = \frac{C^{1-\gamma} - 1}{1 - \gamma}, \quad (2)$$

where  $\gamma > 1$ .<sup>3</sup> They maximize their expected utility of current and future consumption.

That is, they seek to maximize

$$u(C_0^j) + \beta E \left[ u \left( \tilde{C}^j \right) \right], \quad j = 1, 2, \dots, N, \quad (3)$$

where  $C_0^j$  denotes agent  $j$ 's current consumption,  $\beta$  is a time-preference parameter and  $\tilde{C}^j$  denotes his final consumption. All agents share the same beliefs and have access to the same information. Each of the  $N$  agents is endowed with  $1/N$  shares of each risky asset, and this constitutes each agent's sole endowment.

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<sup>2</sup>Similar modeling approaches can be found in Bakshi and Chen (1997), Bansal and Yaron (2004), Yan (2007) and Zhang (2008), among others.

<sup>3</sup>The restriction  $\gamma > 1$  is imposed in order to avoid having solutions to the portfolio choice problem that yield an infinite expected utility.

### 3 Results

The agents are identical, so we can solve for equilibrium prices by considering a representative consumer with an endowment equal to the average aggregate endowment,<sup>4</sup>

$$S_i = E \left[ \frac{\beta u' (D_A/N)}{u' (D_{0A}/N)} D_i \right], \quad i = 1, 2, \dots, n \quad (4)$$

and

$$B = E \left[ \frac{\beta u' (D_A/N)}{u' (D_{0A}/N)} \right], \quad (5)$$

where  $S_i$  is the price of stock  $i$  and  $B$  is the price of the bond.

In particular, under the assumed CRRA preferences, we can express the asset prices as

$$S_i = \beta D_{0i} E[e^{g_i - \gamma g_A}] = \beta D_{0i} M_{[g_i - \gamma g_A]}(1), \quad i = 1, 2, \dots, n \quad (6)$$

and

$$B = \beta E[e^{-\gamma g_A}] = \beta M_{-\gamma g_A}(1), \quad (7)$$

where  $M_X$  denotes the moment-generating function for the random variable  $X$ .

Thus, the expected gross return on stock  $i$  is given by

$$E[(1 + R_i)] = E \left[ \frac{D_i}{S_i} \right] = \frac{D_{0i} E[e^{g_i}]}{\beta D_{0i} M_{[g_i - \gamma g_A]}(1)} = \frac{M_{g_i}(1)}{\beta M_{[g_i - \gamma g_A]}(1)}, \quad (8)$$

and the risk-free rate is

$$1 + R_f = \frac{1}{B} = \frac{1}{\beta M_{-\gamma g_A}(1)}. \quad (9)$$

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<sup>4</sup>For a general aggregation theorem, see Rubinstein (1974).

The difference (in logs) between the two is given by

$$rp_i \equiv \ln(E[(1 + R_i)]) - \ln(1 + R_f) = k_{g_i}(1) + k_{-\gamma g_A}(1) - k_{[g_i - \gamma g_A]}(1), \quad (10)$$

where  $k$  is the *cumulant-generating function*, defined by  $k_X(t) \equiv \ln(M_X(t))$ . We call  $rp_i$  the *risk premium of asset  $i$* .

If the cumulant-generating function exists in an open interval containing 0, then it is infinitely differentiable in this interval and thus, making a Taylor expansion around 0, we can write the cumulant-generating function of the random variable  $X$  as

$$k_X(t) = \sum_{m=1}^{\infty} \kappa_{m,X} \frac{t^m}{m!}, \quad (11)$$

where  $\kappa_{m,X} \equiv k_X^{(m)}(0)$  is referred to as the *cumulant*.

Since  $k_X(t) \equiv \ln M_X(t)$ , there is an obvious relation between cumulants and moments.

For example, the first four cumulants are

$$\kappa_{1,X} \equiv k'_X(0) = E[X], \quad (12)$$

$$\kappa_{2,X} \equiv k''_X(0) = \text{Var}[X], \quad (13)$$

$$\kappa_{3,X} \equiv k_X^{(3)}(0) = \text{Skew}[X], \quad (14)$$

$$\kappa_{4,X} \equiv k_X^{(4)}(0) = \text{Kurt}[X] - 3 \text{Var}[X]^2, \quad (15)$$

where  $\text{Skew}[X] \equiv E[(X - E[X])^3]$  is the third central moment (which we call *skewness*) and  $\text{Kurt}[X] \equiv E[(X - E[X])^4]$  is the fourth central moment (which we call *kurtosis*).

It is also possible to define cumulant-generating functions and cumulants for multivariate random variables. In the bivariate case, one can define a cumulant-generating

function  $k_{(X,Y)}(t_1, t_2) \equiv \ln M_{(X,Y)}(t_1, t_2)$  with joint cumulants

$$\kappa_{(m,n),(X,Y)} \equiv \frac{\partial^m \partial^n k_{(X,Y)}}{\partial t_1^m \partial t_2^n}(0, 0). \quad (16)$$

Further, it can be shown that, if  $Z \equiv a_1 X + a_2 Y$ , where  $a_1$  and  $a_2$  are constants, then

$$\kappa_{m,Z} = \sum_{j=0}^m \binom{m}{j} a_1^{m-j} a_2^j \kappa_{(m-j,j),(X,Y)} \quad (17)$$

(McCullagh, 1987).

Hence, the risk premium of asset  $i$  can be written as

$$\begin{aligned} rp_i &= \gamma \text{Cov}(g_i, g_A) + \sum_{m=3}^{\infty} \frac{1}{m!} (\kappa_{m,g_i} + \gamma^m \kappa_{m,g_A} - \kappa_{m,(g_i - \gamma g_A)}) \\ &= \gamma \text{Cov}(g_i, g_A) \\ &\quad + \sum_{m=3}^{\infty} \frac{1}{m!} \left( (\gamma^m + (-\gamma)^m) \kappa_{m,g_A} + \sum_{j=1}^{m-1} \binom{m}{j} (-\gamma)^j \kappa_{((m-j),j),(g_A,g_i)} \right). \end{aligned} \quad (18)$$

If the vector  $(g_1, g_2, \dots, g_n, g_A)$  follows a joint normal distribution, then  $g_i$ ,  $g_A$ , and  $(g_i - \gamma g_A)$  are normally distributed. It is well known that, for a normally distributed random variable, the cumulants of order three and higher are zero. Thus, in the case when  $(g_1, g_2, \dots, g_n, g_A)$  follows a joint normal distribution, we obtain the familiar expression

$$rp_i = \gamma \text{Cov}(g_i, g_A), \quad (19)$$

where  $\text{Cov}(g_i, g_A)$  is said to capture systematic risk (cf. Mehra and Prescott, 2003). A log linear approximation also suggests that the relation in (19) holds approximately (see the Appendix). However, the normal distribution is the only distribution with a finite number of nonzero cumulants (Marcinkiewicz, 1938). Thus, unless higher-order terms cancel out



in (18), the result in (19) does not hold in general.<sup>5</sup> Below, we discuss the case in which  $(g_1, g_2, \dots, g_n, g_A)$  follows a multivariate Normal Inverse Gaussian (NIG) distribution and we find that the relation in (19) does not hold in this case.

For a more general heuristic discussion, we can focus on the first two additional terms in the infinite series in (18):

$$\begin{aligned}
rp_i &= \gamma \text{Cov}(g_i, g_A) + \frac{1}{2} \left( \gamma^2 \kappa_{(1,2),(g_A, g_i)} - \gamma \kappa_{(2,1),(g_A, g_i)} \right) \\
&\quad + \frac{1}{12} \left( \gamma^4 \kappa_{4, g_A} - 2\gamma \kappa_{(3,1),(g_A, g_i)} + 3\gamma^2 \kappa_{(2,2),(g_A, g_i)} - 2\gamma^3 \kappa_{(1,3),(g_A, g_i)} \right) \\
&\quad + \text{higher order terms}
\end{aligned} \tag{20}$$

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<sup>5</sup>A well-known result is that CAPM holds when returns follow an elliptical distribution (Owen and Rabinovitch, 1983; Ingersoll, 1987). Indeed, it follows from the analysis in Hamada and Valdez (2008) that, if we let  $D_i = D_{0i}(1 + \bar{g}_i)$  for all risky assets in the economy, where the growth rates  $(\bar{g}_i:s)$  follow a joint elliptical distribution, then CAPM would hold. However, in order to avoid negative consumption, we model the log of dividend growth. Of course, the circumstance that a random variable is log-elliptically distributed does not imply that it is elliptically distributed (e.g., the log-normal distribution does not belong to the elliptic class). In particular, assuming that  $(g_1, g_2, \dots, g_n, g_A)$  follows a Laplace distribution (which is elliptical) and using the results in (8) and (9), we get some additional terms compared to (19). Interestingly, we obtain a consumption CAPM result for continuously compounded returns assuming log-normal growth rates (19) even though the log-normal distribution does not belong to the elliptic class.

In the above expression, the second term can be written as<sup>6</sup>

$$\begin{aligned} \text{second term} &= \frac{1}{2} (\gamma^2 (\text{Cov}(g_i^2, g_A) - 2\mu_{g_i} \text{Cov}(g_i, g_A)) \\ &\quad - \gamma (\text{Cov}(g_i, g_A^2) - 2\mu_{g_A} \text{Cov}(g_i, g_A))) \end{aligned} \quad (21)$$

while the third term can be written as

$$\begin{aligned} \text{third term} &= \frac{1}{12} (\gamma^4 (\text{Kurt}[g_A] - 3 \text{Var}[g_A]^2) \\ &\quad - 2\gamma (3 (\mu_{g_A}^2 - \text{Var}[g_A]) \text{Cov}(g_i, g_A) + \text{Cov}(g_i, g_A^3) - 3\mu_{g_A} \text{Cov}(g_i, g_A^2)) \\ &\quad + 3\gamma^2 (4\mu_{g_i} \mu_{g_A} \text{Cov}(g_i, g_A) + \text{Cov}(g_i^2, g_A^2) - 2\text{Cov}(g_i, g_A)^2 - 2\mu_{g_A} \text{Cov}(g_i^2, g_A) - 2\mu_{g_i} \text{Cov}(g_i, g_A^2)) \\ &\quad - 2\gamma^3 (3 (\mu_{g_i}^2 - \text{Var}[g_i]) \text{Cov}(g_i, g_A) + \text{Cov}(g_i^3, g_A) - 3\mu_{g_i} \text{Cov}(g_i^2, g_A))) \end{aligned} \quad (22)$$

The second term depends on the expected (log) dividend growth rate of the individual stock and the expected (log) growth rate of the aggregate dividend, while the third term depends on the variances of the (log) aggregate and individual dividend growth rates. Thus, looking at the third term and using the common interpretations, it appears as though idiosyncratic risk is priced. In addition, Equation (22) tells us that the direction of the effect of idiosyncratic volatility on the third term in the expression for the risk premium of stock  $i$  depends on the covariance between its dividend growth rate and the growth rate of the aggregate endowment.

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<sup>6</sup>Here, and also later, when we reformulate the third term, we use the CumulantToCentral and CentralToRaw functions in the Mathematica add-on mathStatica.

### Example: Multivariate NIG distribution

Now, let  $(g_1, g_2, \dots, g_n, g_A)$  be distributed according to a multivariate Normal Inverse Gaussian (NIG) distribution.<sup>7</sup> The moment generating function of a NIG distributed  $d$ -dimensional random variable  $\mathbf{X}$  is given by

$$M_{\mathbf{X}}(\mathbf{t}) = \exp \left\{ \mathbf{t}'\boldsymbol{\mu} + \delta \left( \sqrt{\alpha^2 - \mathbf{h}'\Phi\mathbf{h}} - \sqrt{\alpha^2 - (\mathbf{h} + \mathbf{t})'\Phi(\mathbf{h} + \mathbf{t})} \right) \right\}, \quad (23)$$

where  $\delta$  is a scale parameter ( $\delta > 0$ ),  $\alpha$  is a parameter controlling tail thickness ( $\alpha > 0$ ),  $\boldsymbol{\mu}$  is a  $d \times 1$  vector controlling location,  $\mathbf{h}$  is a  $d \times 1$  vector controlling the asymmetry of the distribution, and  $\Phi$  is a  $d \times d$  positive definite symmetric matrix with determinant 1, which is related to covariance. With the help of the moment generating function in (23), we can find the exact relation between  $\Phi$  and the variance-covariance matrix ( $\boldsymbol{\Sigma}$ ):

$$\boldsymbol{\Sigma} = \delta (\alpha^2 - \mathbf{h}'\Phi\mathbf{h})^{-1/2} \left[ \Phi + (\alpha^2 - \mathbf{h}'\Phi\mathbf{h})^{-1} \Phi\mathbf{h}\mathbf{h}'\Phi \right] \quad (24)$$

(Lillestøl, 1998; Øigård, Hanssen, Hansen, and Godtliebsen, 2005).

Further, the moment generating function of the random variable  $Y = \mathbf{w}'\mathbf{X}$ , where  $\mathbf{X}$  is a

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<sup>7</sup>Not all choices of preference and distribution parameters will give rise to well-defined equilibria. In order to ensure that the equilibria are well-defined, we may impose the following restrictions:  $\gamma > 1$  (as discussed earlier) and  $\max\{\mathbf{h}'\Phi\mathbf{h}, ((\mathbf{h} + \mathbf{o}_{iA})'\Phi(\mathbf{h} + \mathbf{o}_{iA}))_{i=1}^n, (\mathbf{h} + \mathbf{o}_A)'\Phi(\mathbf{h} + \mathbf{o}_A), ((\mathbf{h} + \mathbf{o}_i)'\Phi(\mathbf{h} + \mathbf{o}_i))_{i=1}^n, (\mathbf{h} + \mathbf{v}_A)'\Phi(\mathbf{h} + \mathbf{v}_A), (\mathbf{h} + (1 - \gamma)\mathbf{v}_A)'\Phi(\mathbf{h} + (1 - \gamma)\mathbf{v}_A)\} \leq \alpha^2$ , where  $\mathbf{v}_A$  is an  $(n + 1) \times 1$  vector with 1 as its last entry and zeros in all other entries, and other scalars and vectors are as defined below.

NIG distributed  $d$ -dimensional random variable and  $\mathbf{w}$  is a  $d \times 1$  vector, is given by

$$M_Y(t) = M_{\mathbf{X}}(t\mathbf{w}) = \exp \left\{ t\mathbf{w}'\boldsymbol{\mu} + \delta \left( \sqrt{\alpha^2 - \mathbf{h}'\Phi\mathbf{h}} - \sqrt{\alpha^2 - (\mathbf{h} + t\mathbf{w})'\Phi(\mathbf{h} + t\mathbf{w})} \right) \right\}. \quad (25)$$

Thus, in the case when  $(g_1, g_2, \dots, g_n, g_A)$  is distributed according to a multivariate NIG distribution, it follows that the risk premium in (10) is exactly equal to

$$rp_i = \delta \left( \sqrt{\alpha^2 - \mathbf{h}'\Phi\mathbf{h}} + \sqrt{\alpha^2 - (\mathbf{h} + \mathbf{o}_{iA})'\Phi(\mathbf{h} + \mathbf{o}_{iA})} \right. \\ \left. - \sqrt{\alpha^2 - (\mathbf{h} + \mathbf{o}_A)'\Phi(\mathbf{h} + \mathbf{o}_A)} - \sqrt{\alpha^2 - (\mathbf{h} + \mathbf{o}_i)'\Phi(\mathbf{h} + \mathbf{o}_i)} \right), \quad (26)$$

where  $\mathbf{o}_{iA}$  is an  $(n+1) \times 1$  vector with 1 in its  $i$ th entry and  $-\gamma$  in its last entry and zeros in all other entries,  $\mathbf{o}_A$  is an  $(n+1) \times 1$  vector with  $-\gamma$  as its last entry and zeros in all other entries, and  $\mathbf{o}_i$  is an  $(n+1) \times 1$  vector with 1 as its  $i$ th entry and zeros in all other entries.

Now, in order to gain some intuition, consider the *semi-symmetric* case in which  $\mathbf{h} = \mathbf{0}$ .

In this case,

$$rp_i = \delta \left( \alpha + \sqrt{\alpha^2 - \phi_{ii} + \gamma(\phi_{iA} + \phi_{Ai}) - \gamma^2\phi_{AA}} - \sqrt{\alpha^2 - \gamma^2\phi_{AA}} - \sqrt{\alpha^2 - \phi_{ii}} \right). \quad (27)$$

Given that  $\mathbf{h} = \mathbf{0}$ , the variance-covariance matrix simplifies to  $\Sigma = \frac{\delta}{\alpha}\Phi$  (see Eq. (24)), so

we can rewrite the above expression as

$$rp_i = \delta \left( \alpha + \sqrt{\alpha^2 - \frac{\alpha}{\delta}(\sigma_i^2 + \gamma^2\sigma_A^2 - 2\gamma\sigma_{iA})} - \sqrt{\alpha^2 - \frac{\alpha}{\delta}\gamma^2\sigma_A^2} - \sqrt{\alpha^2 - \frac{\alpha}{\delta}\sigma_i^2} \right), \quad (28)$$

where  $\sigma_i^2 = \text{Var}[g_i]$ ,  $\sigma_A^2 = \text{Var}[g_A]$  and  $\sigma_{iA} = \text{Cov}(g_i, g_A)$ . That is, the risk premium of an arbitrary asset is affected by its idiosyncratic volatility ( $\sigma_i$ ) and the volatility of aggregate consumption ( $\sigma_A$ ).

By studying the risk premium of a claim to the aggregate dividends/aggregate consumption, we get to the equity premium. Comparing the exact expression for the equity premium to its log linear approximation (the log normal case) both exposes the weakness of the log linear approximation and shows that assuming a log NIG structure may help mitigate the equity premium puzzle.

From (25), we know that the marginal distributions are also NIG: the moment generating function for the growth rate of aggregate dividends/aggregate consumption can be written as

$$M_{g_A}(t) = \exp \left\{ t\mu_A + \delta_A \left( \sqrt{\alpha_A^2 - h_A^2} - \sqrt{\alpha_A^2 - (h_A + t)^2} \right) \right\}, \quad (29)$$

where  $\delta_A = \sqrt{\phi_{AA}}\delta$ ,  $h_A = \sum_j \phi_{Aj}h_j/\phi_{AA}$ ,  $\alpha_A = \sqrt{\eta_A^2 + h_A^2}$  where  $\eta_A = \sqrt{(\alpha^2 - \mathbf{h}'\Phi\mathbf{h})/\phi_{AA}}$  and  $\mu_A = \mathbf{v}'_A\mu$  where  $\mathbf{v}_A$  is an  $(n+1) \times 1$  vector with 1 as its last entry and zeros in all other entries.

Thus, by (8) and (9), the equity premium is given by

$$ep \equiv \ln E[(1 + R_A)] - \ln(1 + R_f) = \delta_A \left( \sqrt{\alpha_A^2 - h_A^2} + \sqrt{\alpha_A^2 - (h_A + 1 - \gamma)^2} - \sqrt{\alpha^2 - (h_A + 1)^2} - \sqrt{\alpha_A^2 - (h_A - \gamma)^2} \right). \quad (30)$$

Instead using a log linear approximation, the equity premium is given by  $ep \approx \gamma\sigma_A^2$  (see the Appendix).<sup>8</sup>

We estimate the NIG parameters by maximum likelihood using two data sets: the first is the one used in the original article by Mehra and Prescott (1985), and it is obtained from Prof. Rajnish Mehra's homepage ([www.academicwebpages.com/preview/mehra](http://www.academicwebpages.com/preview/mehra)); the

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<sup>8</sup>Under log normality, the approximation becomes exact.

second is a revised and updated version of the one used in Chapter 26 in Shiller (1989), and it is obtained from Prof. Robert Shiller's homepage ([www.econ.yale.edu/~shiller](http://www.econ.yale.edu/~shiller)).<sup>9</sup> The estimates of the NIG parameters are quite similar for the two data sets (see Table 1). A Jarque-Bera test rejects normally distributed growth rates at the 5% level for both data sets.

Having estimated the parameters of the NIG distribution, the equity premium and the variance of consumption growth, we compare the relative risk aversion required to match the equity premium, under a log normal and a log NIG structure, respectively. As seen in Table 2, the differences are striking: assuming a log NIG structure more than halves the required coefficients of relative risk aversion. Thus, we have demonstrated that assuming a log NIG structure helps mitigate the equity premium puzzle.

Further, as seen in Figure 1, the log normal approximation seems to work well for small values on the coefficient of relative risk aversion whereas, for larger values on this coefficient, there is a large discrepancy between the exact equity premium in (30) and its log linear approximation. In fact, for a relative risk aversion of 19, the error in the log linear approximation is a whole 576 basis points a year.

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<sup>9</sup>We thank Prof. Rajnish Mehra and Prof. Robert Shiller for making their data publicly available.

Table 1. Maximum likelihood estimates of the parameters of the Normal Inverse Gaussian (NIG) distribution when fitted to the continuously compounded growth in aggregate consumption. The "Mehra-Prescott" data is the same as the one used in the original article on the equity premium puzzle by Mehra and Prescott (1985), and it has been downloaded from Prof. Rajnish Mehra's homepage ([www.academicwebpages.com/preview/mehra](http://www.academicwebpages.com/preview/mehra)). The "Shiller" data is a revised and updated version of the one used in Chapter 26 in Shiller (1989), which we have obtained from Prof. Robert Shiller's homepage ([www.econ.yale.edu/~shiller](http://www.econ.yale.edu/~shiller)). The numbers within parentheses are the standard errors.

Data	Period	NIG parameters	$\mu_A$	$\alpha_A$	$\delta_A$	$h_A$
Mehra-Prescott	1889 – 1978	0.0255 (0.00627)	26.4 (13.2)	0.0325 (0.0115)	–6.31 (6.37)	
Shiller	1889 – 2009	0.0261 (0.00696)	33.6 (14.2)	0.0406 (0.0135)	–4.98 (6.51)	

Table 2. Relative risk aversion required to match the equity premium when growth rates are assumed to follow a Normal (N) and a Normal Inverse Gaussian (NIG) distribution, respectively. The numbers reported in the "Equity Premium" column are sample analogs of the definition in (30):  $ep \equiv \ln E[(1 + R_A)] - \ln(1 + R_f)$ . The "Variance of cons. growth" column reports the sample variance of continuously compounded growth rates in aggregate consumption. As for the "Mehra-Prescott" data, the numbers in the "Equity Premium" and "Variance of cons. growth" columns are taken from Mehra and Prescott (2003). In the case of the "Shiller" data, we have computed the corresponding quantities using a revised and extended version of the data used in Shiller (1989), which we have obtained from Prof. Robert Shiller's homepage ([www.econ.yale.edu/~shiller](http://www.econ.yale.edu/~shiller)).

Data	Period	Equity Premium	Variance of		Required $\gamma$	
			cons. growth	NIG	N	NIG
Mehra-Prescott	1889 – 1978	5.95%	0.00125		47.6	17.9
Shiller	1889 – 2009	6.39%	0.00124		51.5	24.5



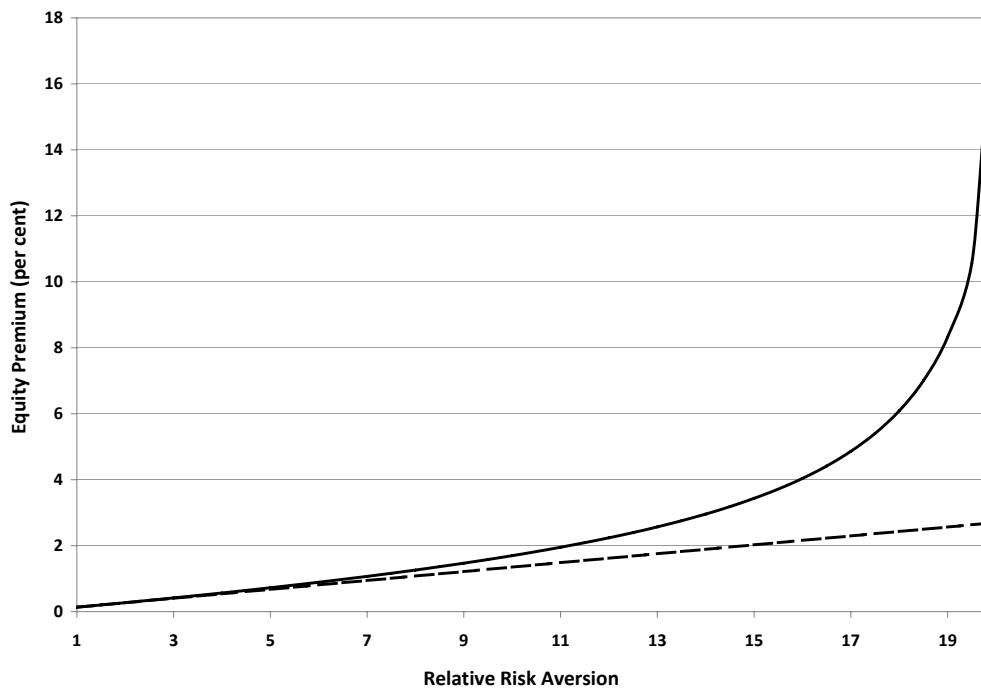


Figure 1: The figure shows the equity premium as a function of the coefficient of relative risk aversion under the NIG distribution (solid line). The dashed line is the log linear approximation. The parameter values are based on our NIG estimation using the same data as in Mehra and Prescott (1985):  $\alpha_A = 26.4$ ,  $\delta_A = 0.0325$  and  $h_A = -6.31$  (see Table 1). Since we assume that the data is distributed according to a NIG distribution, we let  $\sigma_A^2$  equal the model-implied variance ( $\sigma_A^2 = \delta_A \alpha_A^2 / (\alpha_A^2 - h_A^2)^{3/2} = 0.00135$ ).

## 4 Conclusions

In this paper, we demonstrate that, allowing for general distributions of dividend growth rates in a Lucas economy with multiple trees, idiosyncratic volatility will generically appear to be priced, because it affects risk premia in ways that are not captured by a log linear approximation. Our findings suggest that one needs to be careful in interpreting empirical results that build on log linear approximations or assume a log normal structure.

## Appendix: A Log Linear Approximation

Here, we derive a log linear approximation of the risk premium in (10).

The risk premium in (10) can be expressed as

$$rp_i = \ln E[e^{g_i}] + \ln E[e^{-\gamma g_A}] - \ln E[e^{g_i - \gamma g_A}]. \quad (31)$$

Now, making a second-order Taylor expansion of  $e^x$  around the expected value of the random variable  $X$ ,  $E[X]$ , we obtain

$$\begin{aligned} e^x &\approx e^{E[X]} + e^{E[X]}(x - E[X]) + \frac{1}{2}e^{E[X]}(x - E[X])^2 \\ &= e^{E[X]} \left( 1 + (x - E[X]) + \frac{1}{2}(x - E[X])^2 \right). \end{aligned} \quad (32)$$

For the function  $e^X$  of the random variable  $X$ , we thus have that

$$e^X \approx e^{E[X]} \left( 1 + (X - E[X]) + \frac{1}{2}(X - E[X])^2 \right). \quad (33)$$

Formally taking expectations on both sides, we get

$$E[e^X] \approx e^{E[X]} \left( 1 + \frac{1}{2} \text{Var}[X] \right). \quad (34)$$

Applying the above approximation to all the terms in (31), we have that

$$rp_i \approx \ln \left( 1 + \frac{1}{2} \text{Var}[g_i] \right) + \ln \left( 1 + \frac{1}{2} \gamma^2 \text{Var}[g_A] \right) - \ln \left( 1 + \frac{1}{2} \text{Var}[g_i - \gamma g_A] \right). \quad (35)$$

A first-order Taylor expansion of  $\ln(1+x)$  around  $x=0$  gives that  $\ln(1+x) \approx x$ . Hence, applying this approximation to all the terms in (35), we find that

$$rp_i \approx \gamma \text{Cov}(g_i, g_A). \quad (36)$$

If the vector of growth rates follows a normal distribution, this relation will be exact.

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