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The multi-item bisection auction*

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Abstract

This paper proposes an iterative sealed-bid auction for selling multiple heterogeneous items with unit-demand agents. It generalizes the single item *bisection auction* (Grigorieva et al., 2007) to the environment with multiple heterogeneous items. We show that it elicits a minimal amount of information on preferences required to find the Vickrey-Clark-Groves outcome (Clarke, 1971; Groves, 1973; Vickrey, 1961), when there are two items for sale and an arbitrary number of agents.

JEL classification: D44, C72

Keywords: Bisection Auction, Multi-item, Unit-demand, Sealed-bid.

1 Introduction

Recent research in auction theory has produced a number of papers on iterative auctions (Mishra and Parkes, 2009; Perry and Reny, 2005; Ausubel, 2004). In an iterative auction the auctioneer announces a price and bidders submit their bids. Then the price is updated based upon the submitted bids. The process is repeated until an allocation is determined. This is in contrast to the approach with direct mechanisms, where agents submit their

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preferences and then an allocation is determined. There are several reasons for focusing on iterative auctions. In iterative auctions bidders may not reveal all information regarding their private valuations. This is beneficial, since it has been shown that full revelation of preferences can be problematic (Rothkopf et al., 1990; Engelberecht-Wiggans and Kahn, 1991). Partial revelation of preferences can lead to less communication and thereby decrease the amount of data required for the computation of an allocation. Another argument for looking closer at iterative auctions is the prevalence of them in real world auctions (e.g. English auction, Dutch auction, etc.).

One property of importance when analyzing an auction is efficiency. The objective is to find an auction with an efficient assignment in the equilibrium of the game induced by the auction. An efficient assignment maximizes the sum of valuations. The benchmark for the environment with private valuations is the Vickrey-Clarke-Groves mechanism (Clarke, 1971; Groves, 1973; Vickrey, 1961), henceforth VCG. The VCG mechanism is a direct mechanism with truth-telling as a weakly dominant strategy and the equilibrium outcome is efficient. Another classical mechanism is the English auction. It is an iterative open bid ascending auction for selling one item, and its ascending counterpart is the Dutch auction. The English auction is strategically equivalent to the Vickrey auction. Hence it is a weakly dominant strategy to bid truthfully and the resulting equilibrium is efficient.

The single item *bisection auction* presented and analyzed by Grigorieva et al. (2007) is another example of an iterative auction. In contrast to the English auction it is a sealed-bid auction. It elicits a limited amount of information on preferences but still reaches the VCG outcome. Furthermore it has fewer rounds than the English auction. In other words, both from a strategic- and from a privacy preserving perspective it is satisfactory. For multiple heterogeneous items things get more complicated. It is a complex problem to solve in the most general setting where agents are allowed to bid on any packages of items. In a seminal paper by Demange et al. (1986) an iterative auction for multiple items with unit-demand agents was presented. The auction results in the VCG outcome.

In this paper we propose a multi-item bisection auction. It generalizes the single item bisection auction (Grigorieva et al., 2007). We keep the assumption from the single item bisection auction of unit-demand agents and we do not consider the case for bidding on packages of items. It is a multi-item sealed-bid auction for the environment with unit-demand agents and private valuations. In other words it is a standard assignment problem. A description of the single item bisection auction and the properties of it is

found in the Section 3. To illustrate the idea behind the multi-item bisection auction it is enough to consider the case with two items for sale. The auction is then called the duo-item bisection auction. The duo-item bisection auction is later modified to allow for a more selective elicitation of information on preferences.

All our results are given for the environment with two items for sale. However, many of them are straightforward to generalize to a setting with more than two items for sale. The first result establish an upper limit on the number of iterations for the duo-item bisection auction. Then we proceed by showing that the duo-item bisection auction reaches the VCG outcome under the assumption of truthful bidding. This is not as restrictive as it first may look. There are general results on incentives for dynamic auction mechanisms implementing the VCG outcome. Loosely speaking the results establish truthful bidding as a weakly dominant strategy when bidding strategies are constrained to maximize utility in each step by taking prices as given (Gul and Staccetti, 2000; Parkes, 2001). Our last result concerns the modification of the duo-item bisection auction. We prove that the modification attains the VCG outcome, while eliciting the minimal amount of information on preferences.

With one item to sell it is fairly easy to describe the rules of the single item bisection auction. However, already with two items the generalized bisection auction gets involved and more effort is required to describe it. This is not specific for this auction mechanism. Using auctions in solving an assignment problem with an arbitrary number of items and agents is a complex problem. There are both computational and theoretical obstacles to overcome. The multi-item bisection auction can also be seen as a computational alternative for solving an assignment problem with known valuations.

The organization in the rest of the paper is as follows. In section 2 the model is presented together with some preliminaries. In the subsequent third section the bisection auction mechanisms' are described. They are presented as algorithms and we begin by describing the single item bisection auction and then move on to the duo-item bisection auction and the modification of it. Section 3 ends with a short discussion of the multi-item bisection auction. Section 4 contains the main results and section 5 concludes the paper.

2 Model and preliminaries

There is a set $M = \{1, 2, \dots, m\}$ with indivisible items and a set $N = \{1, 2, \dots, n\}$ with agents. It is an unit-demand environment, each agent is interested in buying at most one item. Every agent $i \in N$ has private valuations over each item $j \in M$, let v_{ij} be agent i 's valuation of item $j \in M$. Assume that agents' valuations v_{ij} are integer valued, drawn randomly from the interval $[0, 2^R)$ for some positive integer R . Assuming integer valuations is reasonable since there is always a smallest quantity in which the value needs to be specified in. Construct the matrix V of all valuations v_{ij} with n rows and m columns. Row i contains agent i 's valuations and column j contains all valuations for the j -th item. The matrix V represents the preferences' of the agents. The utility of any agent i for obtaining item $j \in M$ and paying price p_j is given by $v_{ij} - p_j$. There is a null-item denoted by 0. The value and the price of it is zero and it can be assigned to any number of agents. For notational convenience let $\tilde{M} = M \cup \{0\}$. The standard notation of N_{-i} is employed, it stands for the set N excluding agent i . The demand correspondence for agent i at price vector p is defined as,

$$D_i(p) = \{j \in \tilde{M} : v_{ij} - p_j \geq v_{ij'} - p_{j'} \text{ for all } j' \in \tilde{M}\}.$$

This unit-demand environment with indivisible items is a standard *assignment* problem. Using auctions is one way to solve the assignment problem and obtain an allocation. Let the pair $(x, p) \in \tilde{M}^n \times \mathbb{N}_+^{m+1}$ denote an allocation. The first component x is an assignment and p is a price vector, where x_i indicates which object individual i is assigned and p_j is the price of object j . An allocation is *feasible* if:

- (i) $x_i \neq x_{i'}$ for all $i \neq i'$ and $x_i, x_{i'} \in M$
- (ii) if $p_j > 0$ then $x_i = j$ for some $i \in N$.

An allocation (x, p) is *efficient* if:

$$\sum_{i \in N} v_{ix_i} \geq \sum_{i \in N} v_{iy_i} \text{ for all feasible allocations } (y, p).$$

The agents' in the set N can be partitioned into m partitions, one partition T_j for every item $j \in M$. Each partition T_j is based on valuations among the agents for item j . It is constructed in the following manner. Define for every item $j \in M$ a set $T_{j,1}$, consisting of all agents (could be one or more) with the highest valuation of item j , i.e. $T_{j,1} = \arg \max_{i \in N} v_{ij}$. This gives the first subset $T_{j,1}$ in the partition T_j . Consecutively define the subset $T_{j,2}$ in the partition T_j consisting of agents (could be one or more) with the highest valuation for item j amongst remaining agents in $N \setminus T_{j,1}$, formally

$T_{j,2} = \arg \max_{i \in N \setminus T_{j,1}} v_{ij}$. Repeating this procedure until all agents are assigned into a subset $T_{j,k}$ creates the partition T_j . For each step in the process the group of agents not yet assigned into a subset is decreasing with at least one. Subset $T_{j,k}$ in the partition T_j consists of agents with the highest valuation for item j among the remaining agents not yet assigned into a subset at step $1, 2, \dots, k-1$. To express formally the subsets in partition T_j let $T_j^{k-1} = \cup_{m=1}^{k-1} T_{j,m}$. Now, the subset $T_{j,k}$ can be expressed as $T_{j,k} = \arg \max_{i \in N \setminus T_j^{k-1}} v_{ij}$. Denote by i_{jk} an agent belonging to $T_{j,k}$. Agent i_{jk} has the $|T_j^{k-1}| + 1$ highest valuation for item j .

There are several possible ways to measure the degree of preference elicitation. In this paper a sort of binary measure is employed. This approach resembles the one taken in Andersson and Andersson (2010); Hudson and Sandholm (2004). They also measure the degree of elicitation of preferences in relation to full revelation. An agent's valuation for an item is considered elicited if the exact value is known to the auctioneer. The total number of valuations are nm , full revelation of preferences means that all nm valuations are elicited, and the measure equals one. At the other extreme with no information at all about preferences the value of the measure is zero. In all other cases the measure lies between zero and one.

Definition 1. The measure on preference elicitation is defined as the ratio between the number of elicited valuations and the total number of valuations nm .

The multi-item bisection auction to be proposed elicits valuations from one item at a time, and then an allocation is computed after the last round. The auction mechanism can be viewed as method of iteratively eliciting information on preferences to compute the VCG outcome. Let us formulate what we mean by iteratively eliciting information on preferences.

Definition 2. A sequential elicitation method elicits valuations from one item at a time, and once it continues to the next item it cannot elicit any more valuations from previous items.

3 A generalized bisection auction

Before looking at the duo-item bisection auction, a description and discussion of the single item bisection auction (Grigorieva et al., 2007) is given. Each auction is described as an

algorithm. Within the algorithms agents submit answers at each iteration and the price is updated based on their answers. A *yes* answer for item j stands for being willing to buy at price p_j^t and *no* the contrary.

3.1 Single-item bisection auction

The following description of the single item bisection auction differs from the presentation in the paper by Grigorieva et al. (2007). The reason for this different approach is that the generalization to the multi-item environment becomes less technical involved and the idea behind stands out clearer. Before describing the auction some preliminaries are required. Let p_L^t and p_H^t denote the lower bound respectively the upper bound in the price interval at iteration t . Below is the algorithm describing the single item bisection auction. From Algorithm 1 a price is generated and the assignment is based upon the agents' answers (yes or no) in the process of the algorithm.

Algorithm 1. *The single item bisection auction.*

Start the process with $p_L^0 = 0$ and $p_H^0 = 2^R$ and $A^0 = N$. For each iteration $t = 1, 2, \dots, R$:

1. Update $p^t = (p_L^{t-1} + p_H^{t-1})/2$, if $t < R$ collect answers from the set of active agents in A^{t-1} , else if $t = R$ the auction terminates.
 - (i) If two or more agents report yes, set $p_L^t = p^t$, $p_H^t = p_H^{t-1}$,
 $A^t = \{i \in A^{t-1} : i \text{ reported yes}\}$, $t = t + 1$ and repeat from step 1.
 - (ii) If every agent report no, set $p_L^t = p_L^{t-1}$, $p_H^t = p^t$, $A^t = A^{t-1}$, $t = t + 1$ and repeat from step 1.
 - (iii) If only agent i report yes, set $p_L^t = p_L^{t-1}$, $p_H^t = p^t$, $A^t = A^{t-1} \setminus \{i\}$, $t = t + 1$ and move on to step 2.
2. Update $p^t = (p_L^{t-1} + p_H^{t-1})/2$, if $t < R$ collect answers from the set of active agents in A^{t-1} , else if $t = R$ the auction terminates.
 - (i) If one or more agents report yes, set $p_L^t = p^t$, $p_H^t = p_H^{t-1}$,
 $A^t = \{i \in A^{t-1} : i \text{ reported yes}\}$, $t = t + 1$ and repeat from step 2.

- (ii) If every agent report no, set $p_L^t = p_L^{t-1}$, $p_H^t = p^t$, $A^t = A^{t-1}$, $t = t + 1$ and repeat from step 2.

Convergence in Algorithm 1 is no issue since the process always has R number of iterations. If the auction terminates at step 2 the winner of the auction is the agent with a yes answer at step 1.(iii), i.e. the agent willing to buy for the highest associated price in the auction. Otherwise the winner of the auction is the agent with the lowest index among the agents with a yes answer at price p^R .¹ The item is given to the winner of the auction and she pays the price p equal to the lower bound p_L^R of the final interval at step R . At this price there are at least two agents who are willing to buy the item. The price p is uniquely determined in the process of the auction. Because for each iteration the length of the interval is decreased by a factor of one half. After R numbers of iterations and cuttings of an interval with an initial length of 2^R the length of the interval is 1.

Example 1. Suppose there are four agents α , β , γ , and δ participating in the auction. They have the following valuations for the item: $v_\alpha = 13$, $v_\beta = 9$, $v_\gamma = 11$ and $v_\delta = 6$, drawn from the interval $[0, 16)$. There will be 4 number of rounds in the auction before the winner and the price can be determined. In all of our examples agents bid truthfully. With the following strategy. Reporting *yes* for item j if it belongs to the demand correspondence at the current stated price vector and *no* otherwise. In the single item case the report from an agent is yes if her valuation is above or equal to the current stated price. If the price is above her valuation she reports no.

Table 1: The single item bisection auction

Round	Price	Agent α	Agent β	Agent γ	Agent δ
1	8	yes	yes	yes	no
2	12	yes	no	no	-
3	10	-	no	yes	-
4	11	-	-	yes	-

The process starts with the lower bound of $p_L^0 = 0$ and the upper bound of $p_H^0 = 16$. Every agent belong to the set of active agents in the first round, i.e. $A^0 = \{\alpha, \beta, \gamma, \delta\}$. In the

¹Using the lowest index is one example of a tie-break rule. As a matter of fact any tie-break rule could have been used, the results later in the paper are still valid.

first round the price is set to $p^1 = (p_L^0 + p_H^0)/2 = 8$ and answers are collected from every agent. There are three yes answers and one no answer in the first round. With more than two yes answers we end up at step 1.(i). First the price bounds are updated, $p_L^1 = p^0 = 8$ and $p_H^1 = p_H^0 = 16$, then the set of active agents for the next round is formed. It consists of those with a yes answer, using the notation from Algorithm 1, $A^1 = \{\alpha, \beta, \gamma\}$. The second round starts at step 1 with updating the price $p^2 = (p_L^1 + p_H^1)/2 = 12$ and then collecting answers from agents in A^1 . At the price of 12, α is the sole agent with a yes answer, and α is the winner of the auction. We are now at step 1.(iii), the price bounds are updated $p_L^2 = p_L^1 = 8$ and $p_H^2 = p^1 = 12$. Round number 3 begins at step 2 in Algorithm 1, the price is updated to $p^3 = (p_L^2 + p_H^2)/2 = 10$ and answers are collected from β and γ . With one yes answer from γ and one no answer from agent β , we are in 2.(i) and the price bounds are updated, $p_L^3 = p^3 = 10$, $p_H^3 = p_H^2 = 12$. In the fourth and last step the price is updated to $p^4 = (p_L^3 + p_H^3)/2 = 11$. A yes answer is collected from γ , the only agent still active, and the auction terminates. Agent α is the winner of the auction. She gets the item and pays the price of 11. This is the VCG outcome.

3.2 Duo-item bisection auction

The duo-item bisection auction is a sealed-bid auction for selling two items. It is also presented as an algorithm. From the algorithm price vectors are generated. The allocation is determined as a function of the agents' answers from the process. Call the two items for sale 1 and 2 and the final prices of interest to be generated from the process are p_1 and p_2 . The idea of obtaining prices by decreasing the length of intervals is maintained. Similarly to before we let p_{1L}^t denote the lower bound of the price for item 1 at iteration t and p_{2L}^t denotes the lower bound for item 2 at iteration t . The corresponding upper bounds for item and 1 and 2 at iteration t are denoted p_{1H}^t and p_{2H}^t .

In order to be able to reach the VCG-outcome it is necessary to elicit more information on preferences than in the single item case. To elicit more information the single item bisection auction will be divided into different sub-processes. Each sub-process is a continuation of the original process and a unique price is generated after in total R iterations, counting from the beginning of the auction. The division into new sub-processes can occur several times and it always creates two new processes. We will call this division of the process a *split* and it is defined as follows.

Definition 3. A *split* in a bisection auction at step t divides the process in two parts. Process 1 consists of agents with a yes answer for item j at price p_j^t , the updated price bounds are $p_{jL}^t = p_j^t$ and $p_{jH}^t = p_{jH}^{t-1}$. The active agents at step t with a no answer for item j are in process 2, the price bounds are $p_{jL}^t = p_{jL}^{t-1}$ and $p_{jH}^t = p_j^t$. Each process continues from iteration $t + 1$.

Agents' in the process with a yes answer at the split are called yes-agents and similarly agents' with a no answer are called no-agents. The auctioneer does not inform the agents about the split, and they cannot infer that a split has occurred from only observing the price pattern in the auction. This is possible since it is a sealed-bid auction and the only information they receive is the announced price. In both processes prices are updated based on the rules of the single item bisection auction, and it is not possible to differentiate among the two processes. Hence, the agents' cannot see that a split has taken place.

For the single item bisection auction it was sufficient to find one price. Here we need two prices, one for item 1 and another for item 2. Building on the process in Algorithm 1 the duo-item bisection auction is constructed. The idea is to use Algorithm 1 for finding prices for item 1 and 2 separately. And by combing Algorithm 1 with splits several prices for each item can be generated, and an sufficient amount of information about agents' preferences can be elicited.

Algorithm 2. *The duo-item bisection auction.*

Start the auction with $p_L^0 = (p_{1L}^0, p_{2L}^0) = (0, 0)$, $p_H^0 = (p_{1H}^0, p_{2H}^0) = (2^R, 2^R)$, $A^0 = N$, and initialize Algorithm 1 with item 1 and keep the price for item 2 fixed at 2^R .

1. Check for each iteration in Algorithm 1 after step 1:
 - (i) If one agent answered yes split the auction and go to 2.
 - (ii) If two agents answered yes split the auction and go to 3.
 - (iii) Otherwise keep on iterating Algorithm 1.
2. Process 1 with the yes-agent continues to step 2 in Algorithm 1, call the price generated $p_1^R(1)$. Process 2 stays at step 1 in Algorithm 1. If one agent report yes another split is made, else keep on iterating and call the price generated $p_1^R(2)$. If

a split occurs both processes continues to step 2 in Algorithm 1. Let $p_1^R(2)$ be the price from the first process and $p_1^R(3)$ from the other.

3. Process 2 with the no-agents continues to step 2 in Algorithm 1, call the price generated $p_1^R(3)$. Process 1 stays at step 1 in Algorithm 1. If one agent report yes another split is made, else keep on iterating call the price generated $p_1^R(1)$. If a split occurs both processes continues to step 2 in Algorithm 1. Let $p_1^R(1)$ be the price from the first process and $p_1^R(2)$ from the other.
4. Start Algorithm 1 with item 2 and set the price for item 1 to 2^R . Repeat step 1 to 3 from above.

Now, using the information from Algorithm 2 an allocation consisting of a price vector and an assignment can be determined. It will be shown later in the paper that the process in Algorithm 2 can be viewed as a method of successively eliciting the information necessary to compute the VCG-outcome. It generates at most three different prices for each item and three associated sets of agents. Each process in Algorithm 2 leads to a distinct price $p_j^R(k)$, where $j = 1, 2$ and $k = 1, 2, 3$. Denote the set of winners by W_j^k . The set of winners for each process consist of agents with a yes answer for the highest associated price in their specific process. For example if the first split is with one agent, then the set of winners equals this agent with the yes answer. Further, let $W_j = W_j^1 \cup W_j^2 \cup W_j^3$ for $j = 1, 2$, and define a function $f : W_1 \times W_2 \mapsto \mathbb{N}$ as $f(i, j) = p_1^R(i) + p_2^R(j)$. Now the allocation can be defined.

The assignment is determined as follows,

$$x_{i'} = 1, x_{j'} = 2, \text{ where } (i', j') \in \arg \max_{(i, j) \in W_1 \times W_2} f(i, j) \text{ and } x_k = 0 \text{ for all other } k \in N. \quad (1)$$

The prices of item 1 and 2 are,

$$\begin{aligned} p_1 &= \max_{(i, j)} f(i, j) - p_2^R(j'), \text{ where } (i, j) \in (W_1 \setminus \{i'\}) \times (W_2 \setminus \{j'\}) \\ p_2 &= \max_{(i, j)} f(i, j) - p_1^R(i'), \text{ where } (i, j) \in (W_1 \setminus \{i'\}) \times (W_2 \setminus \{j'\}). \end{aligned} \quad (2)$$

Below is an example of how the duo-item bisection auction works.

Example 2. Consider a situation with the same agents as in example 1, and add a second item. Call the item from the first example item 1 and the new item 2. The agents α, β, γ

$$V = \begin{pmatrix} v_{\alpha 1} & v_{\alpha 2} \\ v_{\beta 1} & v_{\beta 2} \\ v_{\gamma 1} & v_{\gamma 2} \\ v_{\delta 1} & v_{\delta 2} \end{pmatrix} = \begin{pmatrix} 13 & 4 \\ 9 & 9 \\ 11 & 7 \\ 6 & 5 \end{pmatrix}$$

and δ have private valuations for item 1 and 2. Collect the valuations for item 1 and 2, in a matrix V .

The two first price updates in the duo-item bisection auction are identical to the single item case. In the test after the second price update at step 1 in Algorithm 2 the condition in 1.(i) is satisfied and the first split is made. After the split we move on to step 2 in Algorithm 2. Process 1 with the yes-agent α continues to step 2 in Algorithm 1 with the price bounds updated to $p_L^2 = p^2 = 8$, $p_H^2 = p_H^1 = 16$. Henceforth process 1 follows the rules of Algorithm 1 and a price of 13 is found in round 4, the path is displayed in Table 2. The price $p_1^4(1) = 13$ found equals agent α 's valuation for item 1.

Table 2: Initializing the duo-item bisection auction and the first split for item 1

Round	Price	Agent α	Agent β	Agent γ	Agent δ
1	8	yes	yes	yes	no
2	12	yes	no	no	-
3	14	no	-	-	-
4	13	yes	-	-	-

Process 2 with the no-agents β and γ stays at step 1 in Algorithm 1 and the price bounds are updated to $p_L^2 = p_L^1 = 8$, $p_H^2 = p^2 = 12$. The price in round 3 is updated to 10 and this leads to a new split.

Table 3: The second split for item 1 in the duo-item bisection auction

Round	Price	Agent β	Agent γ
3	10	no	yes

In the process with the yes-agent γ the price bounds are updated to $p_L^3 = p^3 = 10$, $p_H^3 = p_H^2 = 12$ and the price announced is 11. With the price of 11 γ reports yes and we have found her valuation for item 1. The other process with the no-agent β ends up in a

price of 9, which equals her valuation for item 1. Table 4 below shows the final step with agent γ and β leading to the two separate prices.

Table 4: Last rounds for item 1 in the duo-item bisection auction

Round	Price	Agent γ	Round	Price	Agent γ
4	11	yes	4	9	yes

At this point in the duo-item bisection auction we have obtained the three highest valuations for item 1 and can continue with item 2. The procedure in Algorithm 2 is the same for item 2 therefore the various steps are not described as carefully. Table 5 summarizes the path for finding the highest valuation for item 2, it equals the price $p_2^4(1) = 9$.

Table 5: Initializing the duo-item bisection auction for item 2 and the first split

Round	Price	Agent α	Agent β	Agent γ	Agent δ
1	8	no	yes	no	no
2	12	-	no	-	-
3	10	-	no	-	-
4	9	-	yes	-	-

The first split was made already in iteration 1 after the first price announcement for item 2. All agents except β reported no when the price was announced to 8 and they moved on to a new separate process. Table 6 below shows the two first rounds in this process. In the second round of this new process the price is announced to 6 and yet another split occurs.

Table 6: The second split for item 2 in the duo-item bisection auction

Round	Price	Agent α	Agent γ	Agent δ
2	4	yes	yes	yes
3	6	no	yes	no

After the final split the price is set to 5 in the process with the no-agents α and δ , and it is set to 7 with the yes-agent γ . Now the three highest valuations for item 1 and 2 are elicited and an allocation can be determined as described by equation (1) and (2).

Table 7: Last rounds in the duo-item bisection auction for item 2

Round	Price	Agent α	Agent δ	Round	Price	Agent γ
4	5	no	yes	4	7	yes

In the resulting allocation item 1 is given to α and she pays the price of 11, and item 2 is given to β for a price of 7. This corresponds to the VCG outcome.

3.2.1 Modified duo-item bisection auction

Next we will present a modification of the duo-item bisection auction. This modified version will be shown to extract the minimal amount of preferences required to compute the VCG outcome. Algorithm 3 gives us price vectors and sets of potential winners for items 1 and 2. Using these an allocation is determined in the same way as for the standard duo-item bisection auction defined by equation (1) and (2) on page 10. The only difference between Algorithms 2 and 3 is that the latter elicits less information on preferences without giving up the possibility for the VCG outcome.

Algorithm 3. *The modified duo-item bisection auction.*

1. Run Algorithm 2 until item 2 has passed step 1. Generate W_2^1 and the second highest price for item 2.
2. If $|W_1^1 \cup W_2^1| = 1$ continue in Algorithm 2 and generate $p_2^R(1)$. Further if $p_1^R(1) + p_2^R(2) \geq p_1^R(2) + p_2^R(1)$ or $|W_1^2 \cup W_2^2| = 1$, continue in Algorithm 2 and generate the third highest price for item 2.
3. If $|W_1^1 \cup W_2^1| = 2$ consider the two cases.
 - (i) If $|W_2^1| = 2$ continue in Algorithm 2 and elicit $p_2^R(2)$.
 - (ii) If $|W_2^1| = 1$ there are two cases depending on $|W_1^1|$.
 - (a) For $|W_1^1| = 1$, generate $p_2^R(1)$ if $|W_1^2 \cup W_2^2| = 1$ and generate $p_2^R(3)$ if $|W_1^1 \cup W_2^2| = 1$.
 - (b) For $|W_1^1| = 2$, generate $p_2^R(1)$ and generate $p_2^R(3)$ if $|W_1^1 \cup W_2^2| = 2$.
4. If $|W_1^1 \cup W_2^1| \geq 3$ the auction terminates.

Example 3. This example illustrates the modified duo-item bisection auction. The setup is the same as in example 2, we continue with the four agents and their valuations. Nothing changes for item 1 in the modified duo-item bisection auction, the change is for item 2. In the first step of Algorithm 3 the price for item 2 is set to 8 and the answers are collected. The only agent with a yes answer is β , and thereby W_2^1 is generated. Now it remains to find the second highest price $p_2^4(2)$. The auction splits and every agent with a no answer in the first round moves to a new process, which is illustrated in Table 8 below.

Table 8: The modified duo-item bisection auction

Round	Price	Agent α	Agent γ	Agent δ
2	4	yes	yes	yes
3	6	no	yes	no
4	7	-	yes	-

Now, Step 1 in Algorithm 3 is completed with the second highest price generated, and we continue in Algorithm 3. There are two different agents with the highest valuation for item 1 and 2. This makes step 3 the appropriate next step in Algorithm 3. There are three different scenarios at step 3. With $|W_1^1| = |W_2^1| = 1$ we end up in the middle (ii), since both inequalities are violated there is no need to generate more prices and Algorithm 3 ends here. The final part in the auction is identical to example 2, hence agent α is given item 1 to the price of 11 and β buys item 2 for the price of 7. The modified duo-item bisection auction neither elicited the highest nor the third highest price for item 2. In terms of iterations the modified version saved 4 iterations out of 15 in total.

3.3 Multi-item bisection auction

The extension of the duo-item bisection auction to the environment with m items for sale is natural. Instead of eliciting the 3 highest valuations for item 1 and 2, the multi-item bisection auction elicits the $m + 1$ highest valuations, assuming $n > m$. The method is the same as before it builds on splitting up the original auction into separate processes. Similarly to the environment with 2 items for sale the auction begins with item 1 and generates the $m + 1$ highest valuations for item 1. The first split is made when there is a yes answer from $j < m + 1$ agents. The j -th highest valuations for item 1 are elicited in the group with yes-agents and new splits are made if necessary. The remaining valuations

to get the $m + 1$ highest valuations for item 1 are elicited from the group of no-agents from the first split. This procedure continues item per item until the $m + 1$ highest valuations for item m are elicited. The allocation is determined in an analogous manner to the duo-item bisection auction. Based upon the prices' generated from the auction an efficient assignment is found. The price for item j is determined by calculating the difference between the efficient assignment in N and N_{-i} , where agent i is the agent assigned item j .

The main advantages with the multi-item bisection auction vanishes when the number of agents and the number of items is approximately of the same size. Then the multi-item bisection auction basically reveals all information on preferences, and it would have been better to use a direct mechanism from the beginning. The remedy of this would be to create a modification similar to the environment with two items for sale. However, creating such a modification is not as easily done for the general case with m items. The number of possible combinations for efficient assignments grows exponentially with the number of items and it is difficult to find clear cut conditions as with the duo-item auction. The main contribution of the multi-item bisection auction in this context is as an alternative way of solving the standard assignment problem.

4 Main results

In an auction with two items for sale and at least two agents there are three possible efficient allocations. Recall that an agent belonging to the subset $T_{j,k}$ in the partition T_j is denoted i_{jk} , i.e. $i_{jk} \in T_{j,k}$. As remarked earlier the tie-break rule chosen does not matter for our results. The tie-break rule is the following. If two or more efficient allocations exists then the agent with the lowest index is assigned item j . These are the three scenarios, and they will be used extensively in the following proofs.

$$(I) |T_{1,1} \cup T_{2,1}| \geq 2$$

with x given by, $x_{i_{11}} = 1$, $x_{i_{21}} = 2$ and $x_i = 0$ for all $i \in N \setminus \{i_{11}, i_{21}\}$

$$(II) |T_{1,1} \cup T_{2,1}| = 1 \text{ and } v_{i_{11}1} + v_{i_{22}2} \geq v_{i_{12}1} + v_{i_{21}2}$$

with x given by, $x_{i_{11}} = 1$, $x_{i_{22}} = 2$ and $x_i = 0$ for all $i \in N \setminus \{i_{11}, i_{22}\}$

$$(III) |T_{1,1} \cup T_{2,1}| = 1 \text{ and } v_{i_{11}1} + v_{i_{22}2} < v_{i_{12}1} + v_{i_{21}2}$$

with x given by, $x_{i_{12}} = 1$, $x_{i_{21}} = 2$ and $x_i = 0$ for all $i \in N \setminus \{i_{i_{12}}, i_{i_{21}}\}$

As pointed out earlier the single item bisection auction has R iterations. The number of iterations required in the duo-item bisection auction varies, but there is an upper bound. Our first result establish an upper limit on the number of iterations for the duo-item bisection auction.

Proposition 1. *The duo-item bisection auction has an upper limit on the number of iterations of $6(R - 1)$.*

Proof. At most there can be two splits and three prices generated for item 1 and similarly for item 2. Without any split $2R$ iterations is required in the duo-item bisection auction. A split at $t^* < R$ starts two new processes and each process requires $R - t^*$ iterations to generate a price and before the split there was $R - t^*$ iterations. Hence in total a split at t^* adds $R - t^*$ number of iterations. The earliest the first split can arise is in iteration 1 and the second split in the next iteration. Thus, the upper limit for the number of iterations is $2(R + (R - 1) + (R - 2)) = 6(R - 1)$, which is reached if six prices are generated and all four splits happens as early as possible. \square

The VCG outcome is an efficient allocation and the prices are uniquely determined, it is the benchmark for any auction. Loosely speaking the price paid by agent i assigned item j equals the externality agent i imposes to the others by its existence in the economy. Given an efficient assignment x the VCG price of item j depends on how the efficient allocation looks in the economy without the agent who originally was assigned object j . The VCG outcome with two items consists of an assignment and an associated price vector. Define $v(i, j) = v_{i1} + v_{j2}$, using v the VCG outcome can be defined as,

$$x_{i'} = 1, x_{j'} = 2, \text{ where } (i', j') \in \arg \max_{(i, j) \in N \times N} v(i, j) \text{ and } x_k = 0 \text{ for all other } k \in N. \quad (3)$$

The prices for item 1 and 2 are,

$$\begin{aligned} p_1 &= \max_{(i, j)} v(i, j) - v_{j'2}, \text{ where } (i, j) \in (N_{-i'} \times N_{-i'}) \\ p_2 &= \max_{(i, j)} v(i, j) - v_{i'1}, \text{ where } (i, j) \in (N_{-j'} \times N_{-j'}). \end{aligned} \quad (4)$$

Computing the VCG outcome in the generic case with two items requires the three highest valuations for both items. To determine an efficient allocation it is sufficient to know the two highest valuations for item 1 and 2, and for the prices we need the three highest

valuations. Next result establish that the duo-item bisection auction reaches the VCG outcome. In other words it leads to an efficient assignment and the prices in the allocation are equal to the prices given by equation (4) above.

Proposition 2. *The duo-item bisection auction results in the VCG outcome under truthful bidding.*

Proof. Under truthful bidding prices generated in the duo-item bisection auction equals the true valuations. In other words generating prices is the same as eliciting preferences. Algorithm 2 elicits the three highest valuations for item 1 and 2. This is the information on preferences required to compute the VCG outcome. Hence, the function v can be restricted to this domain and still the VCG outcome can be computed by equations (3) and (4) above. Furthermore, the function v equals f on this restricted domain and therefore the allocation found by equation (1) and (2) must be the same as the VCG outcome. \square

The plan for the rest of this section is to prove that the modified duo-item bisection auction elicits the minimal amount of information on preferences, required to find the VCG outcome for any sequential elicitation method and all conceivable valuations V .

Lemma 1. *In any sequential elicitation method reaching the VCG outcome for all conceivable valuations V it is necessary to elicit the three highest valuations for item 1.*

Proof. Suppose there is only one agent with the highest valuation for item 1 and 2, then the highest valuation for item 1 is required to determine an efficient allocation. The next example shows that the second and the third highest valuations are necessary to find the VCG outcome. Consider scenario (I) from page 15 with the efficient assignment $x_{i_{11}} = 1, x_{i_{21}} = 2$ and $|T_{1,1} \cup T_{2,1}| = 2$. Suppose agent i_{21} assigned item 2 also has the second highest valuation for item 1, formally $|T_{1,2} \cup T_{2,1}| = 1$. Similarly agent i_{11} assigned item 1 has the second highest valuation for item 2, formally $|T_{1,1} \cup T_{2,2}| = 1$. Then there are two possible efficient assignments in the economy $N_{-i_{11}}$, either $x_{i_{12}} = 1, x_{i_{23}} = 2$, or $x_{i_{13}} = 1, x_{i_{21}} = 2$. The latter is efficient when $v_{i_{12}1} + v_{i_{23}2} \leq v_{i_{13}1} + v_{i_{21}2}$ and if the inequality is reversed the former is efficient². Hence, we need $v_{i_{12}1}, v_{i_{13}1}, v_{i_{22}2}$ and $v_{i_{21}2}$ to find the VCG price for item 1. Thereby we can conclude that it is necessary to elicit all the three highest valuations for item 1. \square

²If equality both assignments are efficient, and both of them are possible choices.

Lemma 2. *With truthful bidding the modified duo-item bisection auction elicits a necessary amount of information on preferences to reach the VCG outcome for all conceivable valuations V .*

Proof. The proof follows the structure of Algorithm 3. We will show step by step that the valuations' elicited in Algorithm 3 are required to find the VCG outcome. First, to find an efficient assignment the information on who has the highest valuation is required. Next, if we are in scenario (II) or (III), see page 15, the second highest valuation for item 2 is required to find an efficient assignment. For scenario (I) the second highest valuation for item 2 is required when computing the VCG prices. Hence, the information elicited on preferences in step 1 is necessary to find the VCG outcome.

Moving on to step 2 where Algorithm 3 proceeds to when we have scenario (II) or (III). To find the efficient assignment the highest valuation for item 2 is required for both scenarios.

- Let us begin with scenario (II) and see why it is necessary sometimes to elicit the third highest valuation for item 2. The efficient assignment for scenario (II) is to give item 1 to agent i_{11} and item 2 to agent i_{22} . One candidate for the VCG price for item 2 equals $v_{i_{23}2}$, hence the third highest valuation for item 2 should be elicited.
- Looking at scenario (III), where the efficient assignment is to give item 1 to agent i_{12} and item 2 to agent i_{21} . Impose further the condition of $|T_{1,2} \cup T_{2,2}| = 1$. One candidate for the efficient assignment in $N_{-i_{21}}$ is when agent i_{12} keeps item 1 and agent i_{23} is assigned item 2. To evaluate this candidate we need to know the third highest valuation for item 2. Thus, it is necessary to elicit the information on preferences as described in step 2 of Algorithm 3.

Next consider step 3 of Algorithm 3, where we end up if $|T_{1,1} \cup T_{2,1}| = 2$. In the efficient assignment agent i_{11} is given item 1 and agent i_{21} is given item 2. There are two cases in Algorithm 3 at step 3 to cover (i) and (ii).

- First, consider case (i) with two agents having the highest valuation for item 2. Then letting agent i_{11} keep item 1 and giving item 2 to the agent with the third highest valuation is a candidate for an efficient assignment in the economy $N_{-i_{21}}$. Hence, eliciting the third highest valuation is necessary to find the VCG price for item 2.

- The other case (ii) is when one agent has the highest valuation for item 2. To complicate the matter there are two possible subcases (a) and (b), when one agent has the highest valuation for item 2.
 1. In subcase (a) of Algorithm 3 at step 3.(ii), where there is one agent with the highest valuation for item 1. If $|T_{1,2} \cup T_{2,1}| = 1$ the highest valuation for item 2 is elicited. It is required to evaluate the candidate for an efficient assignment in $N_{-i_{11}}$. The candidate consists of giving item 1 to agent i_{13} and item 2 to agent i_{21} . Similarly when $|T_{1,1} \cup T_{2,2}| = 1$ the third highest valuation for item 2 is required to evaluate the efficient assignment in $N_{-i_{21}}$.
 2. In subcase (b) in Algorithm 3 at step 3.(ii) the highest valuation for item 2 is always elicited. Since, one candidate for an efficient assignment in the economy $N_{-i_{11}}$ is to give item 1 to agent i_{12} and item 2 to agent i_{21} . The third highest valuation is elicited only if $|T_{11} \cup T_{22}| = 2$. Because one candidate for an efficient assignment in $N_{-i_{21}}$ is to give item 1 to agent i_{21} and item 2 to agent i_{23} , recall that agent i_{21} also has the highest valuation for item 1.

Now it remains to look at step 4 in Algorithm 3, where we end up if $|T_{1,1} \cup T_{2,1}| \geq 3$. This is straightforward. The VCG price for item 1 respectively 2 equals the second highest valuation for item 1 respectively 2. \square

Lemma 3. *With truthful bidding the modified duo-item bisection auction elicits a sufficient amount of information on preferences to reach the VCG outcome for all conceivable valuations V .*

Proof. The proof goes through each of the three scenarios (I), (II) and (III) on page 15, and for each of them establish that the information on preferences suffices to find the VCG outcome. To begin with we can conclude that in all three scenarios, (I), (II) and (III), the information on preferences suffices to determine an efficient assignment. Lemma 3 boils down to whether the available information on preferences suffices to determine the VCG prices.

Let's first look at scenario (II) and (III). In scenario (II) Algorithm 3 elicits each of the three highest valuations' for item 2, this always suffices to find the VCG outcome. Since, eliciting the three highest valuation for both items is the maximal amount of information required to determine the VCG outcome with two items to allocate. In scenario (III)

both of the two highest valuations' for item 2 are elicited. The third highest valuation for item 2 is not elicited when $|T_{1,2} \cup T_{2,2}| \geq 2$. This causes no problem because agent i_{23} is neither an alternative for item 2 in $N_{-i_{12}}$ nor in $N_{-i_{21}}$.

Now, consider the remaining scenario (I). The case with $|T_{1,1} \cup T_{2,1}| \geq 3$ was covered in the proof of Lemma 2 and there is nothing to be added. Moving on to the other case in scenario (I) with $|T_{1,1} \cup T_{2,1}| = 2$. It is enough to discuss cases when Algorithm 3 does not elicit all three highest valuations for item 2. Therefore it suffices to consider cases in scenario (I) where one agent has the highest valuation for item 2. Taking this together the remaining cases' to cover are in scenario (I) with $|T_{1,1} \cup T_{2,1}| = 2$ and $|T_{2,1}| = 1$

- Begin by looking at the case with $|T_{1,1}| = 2$ and $|T_{2,1}| = 1$. With this collection of preferences we are in step 3.(ii)(a) of Algorithm 3. The highest and the second highest valuation for item 2 are elicited. The third highest valuation is not elicited when $|T_{1,1} \cup T_{2,2}| \geq 3$. This does not cause any problems. The VCG price for item 2 equals $v_{i_{22}}$ and for item 1 there are two candidates for an efficient assignment in $N_{-i_{11}}$. The first candidate is to give item 1 to agent i_{11} and item 2 to agent i_{22} , and in the other item 1 is given to agent i_{12} and item 2 to agent i_{21} . Hence, the third highest valuation is not required.
- Next consider the case with $|T_{1,1}| = |T_{2,1}| = 1$. Here there are four subcases.
 1. When $|T_{1,2} \cup T_{2,1}| \geq 2$ and $|T_{1,1} \cup T_{2,2}| \geq 2$ the VCG price for both items equals the second highest valuation for each item, and we conclude that eliciting the second highest valuation, as is done in Algorithm 3, for item 2 is enough.
 2. Now, suppose one of the conditions changes. Say $|T_{1,2} \cup T_{2,1}| = 1$ and the other condition remains, $|T_{1,1} \cup T_{2,2}| \geq 2$. The VCG price for item 2 is still equal to $v_{i_{22}}$, but the price for item 1 changes. In Algorithm 3 the highest valuation for item 2 is elicited. This is all what is needed to find the VCG price for item 1. Since, agent i_{22} has the highest valuation for item 2 amongst remaining agents in $N_{-i_{11}}$. Hence, no need to elicit the third highest valuation for item 2.
 3. A similar argument can be made when $|T_{1,2} \cup T_{2,1}| \geq 2$ and $|T_{1,1} \cup T_{2,2}| = 1$. The price for item 1 equals $v_{i_{12}}$ and the VCG price for item 2 can be found with the knowledge of the second and the third highest valuation for item 2. Agent i_{21} cannot be assigned any item in $N_{-i_{21}}$.

4. Finally when both $|T_{1,2} \cup T_{2,1}| = 1$ and $|T_{1,1} \cup T_{2,2}| = 1$ all three highest valuations for item 2 are elicited.

Thus, in scenario (I) Algorithm 3 elicits the required amount of information on preferences to find the VCG outcome. \square

Theorem 1. *The modified duo-item bisection auction elicits the minimal amount of information on preferences to reach the VCG outcome for all conceivable valuations V , and for any sequential elicitation method.*

Proof. Lemma 1, 2 and 3 taken together proves Theorem 1. \square

5 Conclusion

We have proposed a multi-item bisection auction, a generalization of the single item auction (Grigorieva et al., 2007). The analysis of the auction has been carried out in the environment with two items for sale. The auction results in the VCG outcome. As part of the analysis in the environment with two items for sale a modified version was presented. We proved that the modified version elicits the minimal amount of preferences required to reach the VCG outcome for any sequential elicitation method. In other words the information acquired is sufficient and necessary for computing the allocation in the VCG outcome. This paper has not discussed the strategic aspects of the multi-item bisection auction. It is a topic for future research.

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