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On symmetry in the formation of stable partnerships

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Abstract

In this note, we examine the connection between the roommate model and the partnership formation model (Talman and Yang, 2011, *Journal of Mathematical Economics* 47, 206-212). Upon noting that both occasionally lack equilibria we look at the *stable partnerships model*, a combination of the former models and interpretable as one with a social planner. We find two sufficient conditions for the existence of stable matchings in the stable partnerships model, where one relates to efficiency, and one (the *symmetry condition*) to fairness. Finally, we provide examples from the fair sharing literature on dividing common values that satisfy the symmetry condition.

Keywords: One-sided matching, partnership formation, symmetry condition, stability
JEL: C62, D02, D60

1. Introduction

In a world of social interaction and competitive business, the formation of partnerships among individuals and likewise among firms plays a significant role. The cornerstone, frequent in practice and attractive in its simplicity, is partnerships in pairs. In this note we deal with the pairwise formation of partnerships with a monetary element added to it: suppose there is a group of economic agents, each and every one having the option to enter a joint venture with another agent; all the individual agents as well as all the possible joint ventures generate values in the form of profits, and the agents want to maximize their own payoffs. Whom, if anyone, an agent chooses to cooperate with crucially comes down to how the profits are split in the various joint ventures - even if you and I make a million dollar profit, I do not want to work with you unless I get a sufficiently large share of it! The questions we address concern how the values of the partnerships are divided among the agents: for example, we examine if there are ways of sharing the values to reach stable and efficient outcomes. These concepts will be defined in detail later on, but it is simple to motivate why they are desirable: an efficient outcome maximizes the sum of values generated in the society and, if stable, does so with agents willingly cooperating, having no incentives to change partners.

Conditions for the existence of stable outcomes have previously been found for a variety of similar models, with the ones most closely related to ours being those of Talman and Yang (2011),

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Eriksson and Karlander (2001), and Rodrigues-Neto (2007). The former two use a different notion of stability than we do and develop their conditions using methods of linear programming and graph theory, respectively. The latter paper varies from this note in that we consider a more general domain of preferences and are more interested in stability and various ways of sharing common values rather than acyclicity of preferences. Rodrigues-Neto (2007, p. 546) also puts forth an argument for why the stable matching is unique in his model, but, as is shown in Proposition 2, this is not the case when considering more general preferences.

The first out of three purposes of this note is to connect the partnership formation model of Talman and Yang (2011) to a specific generalization of the well known roommate problem by Gale and Shapley (1962).[1] The difference between the partnership formation model and roommate model has (at least implicitly) already been described by Sotomayor (2006), but we will connect them in a different way; in doing so a third, alternative model, the *stable partnerships model*, arises as a combination of the two, and it is this one we primarily will focus on.

What will separate the three models is the usage of money; commonly, introducing money (or any divisible good for that matter) into problems featuring indivisible objects adds the possibility of smoothly compensating (or fining) agents to reach an equilibrium. For instance, in an auction for an item the price is adjusted until at most one bidder strictly demands the item, implying an equilibrium state where demand equals supply (see Demange and Gale, 1985; Demange et al., 1986; Shapley and Shubik, 1972). On the other hand, in the case of partnerships and joint ventures, instability may be avoided by altering the way the values are shared among the agents within the partnerships. The key is that, if used wisely, the addition of money can often grant the existence of an equilibrium; nonetheless, a common issue of the partnership formation and roommate models is that equilibria sometimes do not exist, with one possible reason for this being that the agents have too much freedom in deciding how their common values are divided. Accordingly, the idea of this note is that adding restrictions to the way the values are shared may solve the problem.

Roughly, the stable partnerships model can be interpreted as one where a social planner has been put in charge of deciding how the values for all possible joint ventures are to be shared (most importantly, there is a plan also for the joint ventures that do not take place). The second purpose of the note is then to examine whether there always exists a way for the social planner to divide the values such that the outcome is stable and efficient, and this is shown to be possible in Theorem 1. As this to some extent is parallel to *killing two birds with one stone*, the method of reaching such an outcome is a little intricate; the third and final purpose is therefore to find simpler ways of reaching stable outcomes by relaxing the efficiency requirement. Through examples from the fair sharing literature we find a condition (denoted the *symmetry condition*) which, in Theorem 2, is shown to be sufficient for the existence of stable allocations.

To put things into perspective, we can relate the symmetry condition to other conditions such as the pairwise aligned preferences of Pycia (2011) and the "no odd rings" condition of Chung (2000). For the former, the relation is simple as the condition of pairwise aligned preferences is not at all applicable to one-to-one matching (Pycia, 2011, p. 13). Much on the contrary, the latter "no

[1]The generalizations compared to the model of Gale and Shapley (1962) are that the agents are allowed to be single, the number of agents may be odd, and the agents have weak preferences (an agent may be indifferent between distinct agents). This generalization is also considered by, for instance, Chung (2000) and Klaus and Klijn (2010).

odd rings" condition is an extension of ours: once the values are divided into payoffs adhering the symmetry condition, the payoffs correspond to preferences for which the "no odd rings" condition is satisfied. However, the two conditions have very different aims: the purpose of the symmetry condition is to find simple and applicable solutions to the problem of designing stable payoffs; the "no odd rings" condition on the other hand is a characterization of what preferences should be like (or rather, not be like) for the existence of a stable matching - designing payoffs solely based on the "no odd rings" condition is not necessarily an easy task.

The next section contains the formal analysis, starting with the partnership formation model being contrasted with the roommate model, followed by the introduction of the stable partnerships model and the above mentioned results accompanied with short proofs.

2. Definitions and results

Let N be a finite set of agents, with every agent allowed to cooperate with at most one other. We use a matching $\mu : N \to N$, a one-to-one mapping such that $\mu(i) = j$ if and only if $\mu(j) = i$ for all agents i and j, to keep track of the cooperation among the agents. Agents i and j are *matched* if $\mu(i) = j$ and agent i is *single* if $\mu(i) = i$. To every agent and every pair of agents we connect a value, and we denote $v : N \times N \to \mathbb{R}$ the *value function*. In the partnership formation model, the value function is exogenous and used endogenously to choose the matching. The values are then divided into (endogenous) *payoffs* $r^* \in \mathbb{R}^{|N|}$, which are said to be *feasible* if all single agents i get their own value, $r^*_i = v(\{i\})$, and if all different matched agents i and j share their value exactly, $r^*_i + r^*_j = v(\{i, j\})$. A solution to the partnership formation problem, a partnership equilibrium, is a matching and a feasible payoff vector:

Definition 1. *The pair* (μ, r^*) *is a* partnership equilibrium *if* r^* *is feasible,* $r^*_i \geq v(\{i\})$ *for all agents* i*, and* $r^*_i + r^*_j \geq v(\{i, j\})$ *for all different agents* i *and* j*.*

Interesting for our purposes, and we will reflect more on this momentarily, Talman and Yang (2011, p. 208) show that partnership equilibria do not always exist.

Contrary to the partnership formation model, there are no explicit values or payoffs in the generalized roommate problem. Rather, every agent is endowed with a complete and transitive preference relation over N, used to detail who the agent prefers to cooperate with the most, the second most, and so on. We denote that agent i is indifferent between matching with agents j and k as $j \sim_i k$, and that he strictly prefers to be matched with agent j to agent k as $j \succ_i k$. The exogenous preference relations for all agents are collected in $\succsim$.

Definition 2. *An* $|N| \times |N|$ *matrix* r represents the (ordinal) preferences $\succsim$ *if* $r_{i,j} > r_{i,k} \Leftrightarrow j \succ_i k$ *and* $r_{i,j} = r_{i,k} \Leftrightarrow j \sim_i k$ *for all triplets of agents* i*,* j*, and* k*.*

To introduce payoffs and values to the roommate model (making it more comparable to the partnership formation model), we create a payoff matrix r that represents the given preferences $\succsim$ and construct the value function by setting $v(\{i\}) = r_{i,i}$ for all agents i and $v(\{i, j\}) = r_{i,j} + r_{j,i}$ for all different agents i and j (i.e., we make the payoff matrix feasible). A solution to this interpretation of the roommate problem, a stable allocation, is a matching and a feasible payoff matrix such that no agents can block it:

Definition 3. *An allocation* (μ, r) *can be* blocked *by agents i and j (possibly* $i = j$*) if* $r_{i,j} > r_{i,\mu(i)}$ *and* $r_{j,i} > r_{j,\mu(j)}$*. If r is feasible and* (μ, r) *can not be blocked by any agents,* (μ, r) *is* stable.

Note the strict inequalities in the definition of blocking: one agent being better off whilst the other is indifferent is not sufficient. This can be motivated by, for instance, there being a small cost to breaking up (or entering) a partnership. Gale and Shapley (1962) originally only considered strict preferences (making this issue irrelevant), but this notion of stability is used in, for example, Chung (2000), Demange and Gale (1985), Gale and Sotomayor (1985), and Pycia (2011).

Compared to the endogenous payoff vector of the partnership formation model, the exogeneity of the preference relations implies that also the payoff matrix is exogenous. Moreover, the solution concepts, partnership equilibria and stable allocations, are different but clearly connected:

Proposition 1. *If there exists a partnership equilibrium, there exists a stable allocation.*

Proof. Let (μ, r^*) be a partnership equilibrium and r be a feasible payoff matrix such that $r_{i,\mu(i)} = r_i^*$ for all agents i. As (μ, r^*) is a partnership equilibrium, $r_{i,\mu(i)} = r_i^* \geq v(\{i\}) = r_{i,i}$ for all agents i, implying that no agent can block the eventual allocation (μ, r) on his own.

By contradiction, suppose the allocation is not stable. Then there must be different agents i and j such that $r_{i,j} > r_{i,\mu(i)} = r_i^*$ and $r_{j,i} > r_{j,\mu(j)} = r_j^*$. In particular, $r_{i,j} + r_{j,i} = v(\{i, j\}) > r_i^* + r_j^*$, implying that (μ, r^*) is not a partnership equilibrium, which is a contradiction. □

However, again we can find examples for which there exist no stable allocations (e.g., Klaus and Klijn, 2010, p. 650); in fact, Gale and Shapley (1962) introduced the roommate problem merely to illustrate that it *did not* always possess a stable matching (as opposed to the two-sided matching of the marriage problem). This raises a question: what would happen in cases where there are no stable outcomes, for instance in the example of Talman and Yang (2011, p. 208)? Would the agents enter recurring cycles of chaotically dropping out of partnerships and entering new ones? If so, this hints that the agents may need to be kept more in check, and we therefore propose the stable partnerships model where a social planner is introduced for this very purpose; he unequivocally decides how the values are divided, and does so for all possible partnerships. In comparison to the other models, this could be viewed as the payoff matrix being endogenously created, combining the endogenous payoff vector of the partnership formation model and the exogenous payoff matrix of the roommate model.

Next, we examine whether there always exist stable and efficient outcomes, and we then look at a sufficient condition for the existence of stable allocations.

The proofs will revolve around the top partners for the agents, which we formalize as follows:

Definition 4. *Given the payoff matrix r, the* most preferred partners *for agent i in any subset of agents* N' *are collected in the set* $T_i(N', r) = \{j \in N' : r_{i,j} \geq r_{i,k}$ *for all* $k \in N'\}$.

Note that, for any matrices r and r' representing the same preferences and any subset of agents N', $T_i(N', r') = T_i(N', r)$. Furthermore, $T_i(N', r) \neq \emptyset$ if and only if $N' \neq \emptyset$.

Definition 5. *A matching* μ *is* efficient *if*

$$\mu = \arg\max_{\mu'} \Big\{ \sum_{i \in N} r_{i,\mu'(i)} : \mu'(i) = j \Leftrightarrow \mu'(j) = i \text{ for all } i, j \in N \Big\}.$$

There are primarily two noteworthy implications of efficiency: first, if agents i and j are matched by an efficient matching, then $v(\{i, j\}) = r_{i,j} + r_{j,i} \geq r_{i,i} + r_{j,j}$, telling us that $v(\{i, j\})$ can be divided into $r_{i,j} \geq r_{i,i}$ and $r_{j,i} \geq r_{j,j}$. Secondly, if they on the other hand are single, then $r_{i,i} + r_{j,j} \geq r_{i,j} + r_{j,i}$, in turn implying that $r_{i,j} > r_{i,i}$ and $r_{j,i} > r_{j,j}$ can not occur simultaneously.

Theorem 1. *If the matching μ is efficient, there is a payoff matrix r such that (μ, r) is stable.*

Proof. Set $M = \emptyset$.

As long as there are different agents i and j such that $\mu(i) = j$ and $i, j \notin M$, make sure i and j have one another as most preferred partners among $N \setminus M$ by doing the following: divide $v(\{i, j\})$ into $r_{i,j} \geq r_{i,i}$ and $r_{j,i} \geq r_{j,j}$, extend $M = M \cup \{i, j\}$, and let $r_{i,k} \leq r_{i,j}$ and $r_{j,k} \leq r_{j,i}$ for all $k \in N \setminus M$.

When such agents can no longer be found, we are assured that no matched agent can block the eventual allocation. No single agents i and j can block the allocation either, as, by efficiency, $r_{i,j} > r_{i,\mu(i)} = r_{i,i}$ implies that $r_{j,i} < r_{j,\mu(j)} = r_{j,j}$. Remaining parts of r can be set arbitrarily such that it is feasible. As neither single nor matched agents can block the allocation (μ, r), it is stable. □

The above involves quite some tinkering and, in some sense, does not treat all agents equally. Moreover, the payoff matrix r is different depending on the order in which the agents are chosen. Therefore, we want to look for simpler (and, from a fairness point of view, possibly more appealing) ways of reaching stable outcomes, and do this by relaxing the efficiency requirement. Consider, for instance, the following ways of sharing values: in his book on fair division of common values, Moulin (2003) explicitly defines the *even surplus* (denoted *standard* in Ju et al., 2007),

$$r_{i,j} = v(\{i\}) + \frac{v(\{i, j\}) - v(\{i\}) - v(\{j\})}{2},$$

and the *proportional* solution (also mentioned in Banerjee et al., 2001),

$$r_{i,j} = \frac{v(\{i\})}{v(\{i\}) + v(\{j\})} \cdot v(\{i, j\}).$$

Another solution of interest is *equal sharing*, $r_{i,j} = v(\{i, j\})/2$, used by Farrell and Scotchmer (1988). More importantly, these are all included (explicitly shown for the even surplus solution in Corollary 1) in a larger class of solutions satisfying the following condition for any v and N:

Definition 6. *A payoff matrix satisfies the* symmetry condition *if it represent the same preferences as a symmetric*[2] *matrix.*

Lemma 1. *If the payoff matrix r satisfies the symmetry condition, there are agents i and j (possibly $i = j$) in any non-empty subset of agents N' such that $i \in T_j(N', r)$ and $j \in T_i(N', r)$.*

Proof. As r satisfies the symmetry condition, it represents the same preferences as a symmetric matrix r'. Let $T_i \equiv T_i(N', r') = T_i(N', r)$ for all agents i and $N' = \{1, 2, \ldots, n'\}$ without loss of generality. The statement is trivially true if there is an agent i such that $i \in T_i$, and this is henceforth assumed not to be the case.

[2]A matrix r' is symmetric if $r'_{i,j} = r'_{j,i}$

By contradiction, suppose such agents i and j do not exist. Then $i \in T_j \Rightarrow j \notin T_i$. Since $T_i \neq \emptyset$, let $(i+1) \in T_i$ for $1 \leq i \leq (j-1)$ by appropriately rearranging the agents. Then

$$r'_{(j-1),j} > r'_{(j-2),(j-1)} > \cdots > r'_{i,(i+1)} \geq r'_{i,j} \Rightarrow i \notin T_j \text{ for } 1 \leq i \leq j.$$

In particular, $i \notin T_{n'}$ for $1 \leq i \leq n'$ implying that $T_{n'} = \emptyset$, which is a contradiction. □

Lemma 2. *If there are agents i and j (possibly $i = j$) in any non-empty subset of agents N' such that $i \in T_j(N', r)$ and $j \in T_i(N', r)$, there exists a stable allocation.*

Proof. Set $t = 0$ and $N_0 = N$.

As long as $N_t \neq \emptyset$, find agents i and j (possibly $i = j$) such that $i \in T_j(N_t, r)$ and $j \in T_i(N_t, r)$. Set $\mu(i) = j$, $\mu(j) = i$ and $N_{t+1} = N_t \setminus \{i, j\}$. Increase t by one and repeat this step.

By contradiction, suppose (μ, r) is not stable. Then there exists $i \in N_t$ and $j \in N_{t'} \subseteq N_t$ such that $r_{i,j} > r_{i,\mu(i)} \geq r_{i,k}$ for all $k \in N_t$, implying that $j \notin N_t$, which is a contradiction. □

Theorem 2. *If the payoff matrix satisfies the symmetry condition, there exists a stable allocation.*

Proof. Follows immediately by Lemma 1 and Lemma 2. □

Corollary 1. *If the payoff matrix r is created using the even surplus solution, there exists a stable allocation.*

Proof. Define r' such that $r'_{i,j} = 2(r_{i,j} - v(\{i\}))$ for all agents i and j. Then r' is symmetric as $r'_{i,j} = v(\{i, j\}) - v(\{i\}) - v(\{j\}) = r'_{j,i}$, and it represents the same preferences as r since it is constructed using a strictly positive monotonic transformation on r. Thus r satisfies the symmetry condition, and, by Theorem 2, there exists a stable allocation. □

Finally, a result in Rodrigues-Neto (2007) is that the stable allocation is always unique if preferences are acyclic (which they are if the payoff matrix satisfies the symmetry condition). However, this property does not hold under more general preferences:

Proposition 2. *If agents are allowed to have weak preferences, there may be multiple distinct stable allocations even if preferences are acyclic (satisfy the symmetry condition).*

Proof. Let $N = \{1, 2\}$ and $v(\{i, j\}) = r_{i,j} = 0$ for all $(i, j) \in N \times N$. As r is symmetric it must satisfy the symmetry condition, whilst both (μ, r) and (μ', r) using $\mu(1) = 1$, $\mu(2) = 2$ and $\mu'(1) = 2$, $\mu'(2) = 1$ are stable. □

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