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# Budget-Balance, Fairness and Minimal Manipulability\*

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## Abstract

A common real-life problem is to fairly allocate a number of indivisible objects and a fixed amount of money among a group of agents. Fairness requires that each agent weakly prefers his consumption bundle to any other agent's bundle. In this context, fairness is incompatible with budget-balance and non-manipulability (Green and Laffont, 1979). Our approach here is to weaken or abandon non-manipulability. We search for the rules which are minimally manipulable among all fair and budget-balanced rules. First, we show for a given preference profile, all fair and budget-balanced rules are either (all) manipulable or (all) non-manipulable. Hence, measures based on counting profiles where a rule is manipulable or considering a possible inclusion of profiles where rules are manipulable do not distinguish fair and budget-balanced rules. Thus, a “finer” measure is needed. Our new concept compares two rules with respect to their degree of manipulability by counting for each profile the number of agents who can manipulate the rule. Second, we show that maximally linked fair allocation rules are the minimally (individually and coalitionally) manipulable fair and budget-balanced allocation rules according to our new concept. Such rules link any agent to the bundle of a pre-selected agent through a sequence of indifferences.

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# 1 Introduction

We consider the allocation of indivisible objects and a fixed amount of money among a set of agents through a mechanism (Alkan, Demange and Gale, 1991; Svensson, 1983; Tadenuma and Thomson, 1991). The important criterion in this literature is fairness (or envy-freeness) meaning that each agent should like his own consumption bundle (consisting of an object and a monetary compensation) at least as well as that of anyone else.

When analyzing this type of allocation problems, fairness is often coupled with other properties. One such property is non-manipulability which guarantees that no agent can gain by strategic misrepresentation. Another one is budget-balance saying that the sum of monetary compensations should equal the fixed amount of money. A famous result by Green and Laffont (1979) shows that there exists no allocation mechanism that is non-manipulable, fair and budget-balanced. In this paper we will weaken or abandon non-manipulability and offer results that facilitate the comparison of fair and budget-balanced mechanisms according to their level of manipulability when preferences are represented by quasi-linear utility functions.<sup>1</sup>

One way of evaluating the degree of manipulability of a mechanism (e.g. Aleskerov and Kurbanov, 1999; Kelly, 1988, 1993; Maus, Peters and Storcken, 2007a,b) is the idea of counting the number of preference profiles at which a given mechanism is manipulable. A second direction (Pathak and Sönmez, 2012) relies on comparing the sets of preference profiles on which any two mechanisms are manipulable. Previous papers have investigated a number of different problems, including voting rules, matching mechanisms and school choice mechanisms. However, we are not aware of any study with attention to fair and budget-balanced rules.<sup>2</sup>

Our first main result shows for a given preference profile, all fair and budget-balanced rules are either (all) manipulable or (all) non-manipulable. Therefore, measures based on counting profiles where a rule is manipulable and/or considering the inclusion of profiles where a rule is manipulable do not distinguish fair and budget-balanced allocation rules. With respect to those measures, all fair and budget-balanced allocation rules are equally manipulable. The above mentioned measures of minimal manipulability are “coarse” in the sense that preference profiles are categorized as manipulable (for all fair and budget-balanced rules) or non-manipulable (for all fair and budget-balanced rules). For this reason, none of the existing measures are satisfactory when evaluating rules in our context.

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<sup>1</sup>In the early literature (e.g. Moulin, 1980), the primary focus was on restricting the preference domain under which a mechanism is non-manipulable.

<sup>2</sup>Subsequent to this paper Fujinaka and Wakayama (2011) and Andersson, Ehlers and Svensson (2012) adopted a fundamentally different approach by searching for the the fair and budget-balanced allocation rules which minimize the maximal manipulation possibilities (defined in terms of an agent’s utility gain from manipulation) across agents.

In resolving this problem, we introduce a new “finer” measure of minimal manipulability. Because this measure cannot be based solely on the preference domain, a natural approach is to compare two rules via the number of agents who can manipulate the rule at a given preference profile. Then a rule is minimally manipulable (with respect to agents counting) if, for each preference profile, the number of manipulating agents is smaller than or equal to the number of manipulating agents at an arbitrary fair and budget-balanced allocation rule. This guarantees that the minimally manipulable rule is non-manipulable whenever there exists a non-manipulable rule. The main feature of (global) non-manipulability is respected as much as possible in the sense that the ultimate goal of our new notion is to have zero manipulating agents at each preference profile. Our second main result shows that “maximally linked” fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules. Roughly, speaking those rules choose allocations with the maximal number of agents for whom the utility is maximized among all fair and budget-balance allocations.

Showing our main results requires a structural analysis with respect to indifferences at fair and budget-balanced allocations. We show that for any agent  $k$  and any fair and budget-balanced allocation, agent  $k$ 's utility is maximized among all fair and budget-balanced allocations if and only if the allocation is agent  $k$ -linked: any agent can be linked to agent  $k$  through a sequence of agents (an indifference chain) whereby any agent in this sequence is indifferent between his consumption bundle and the bundle received by the next agent in the sequence. Agent  $k$ -linked allocations always exist and all agents are indifferent between all those allocations. The agent  $k$ -linked fair allocation rule chooses for each profile the set of all agent  $k$ -linked allocations. We show that the agent  $k$ -linked fair allocation rules cannot be manipulated by any coalition containing agent  $k$ . As a corollary we obtain that the agent  $k$ -linked fair allocation rule cannot be manipulated by agent  $k$  at any profile.

An indifference component at an allocation is a set of agents such that any two agents can be linked through an indifference chain in this set at this allocation. We show that indifference components are invariant among all fair and budget-balanced allocations, i.e. if  $G$  is an indifference component at a fair and budget-balanced allocation, then  $G$  is an indifference component at *all* fair and budget-balanced allocations. Therefore, if a fair and budget-balanced allocation is agent  $k$ -linked and  $k$  belongs to the indifference component  $G$ , then this allocation is agent  $i$ -linked for all agents  $i$  belonging to  $G$  (and the utility of agent  $i$  is maximized among all fair and budget-balanced allocations). Thus, all agents belonging to  $G$  cannot manipulate the allocation rule choosing those allocations at this profile. We also show that all agents belonging to  $N - G$  can manipulate the allocation rule if  $G$  is an indifference component with maximal cardinality. A maximally linked fair allocation rule chooses for each profile (i) an indifference component  $G$  with maximal cardinality, (ii) some agent  $k$  belonging to  $G$ , and (iii) all agent  $k$ -linked fair allocations. Note that different  $k$ s may be selected for different profiles.

As we mentioned already, our second main result demonstrates that maximally linked fair allocation rules are minimally manipulable with respect to agents counting among all fair and budget-balanced allocation rules. We further show that any fair and budget-

balanced allocation rule, which is not maximally linked, is strongly more manipulable with respect to agents counting than a maximally linked fair allocation rule. We also show that these results are robust with respect to coalitional manipulations. In the same vein as before, when comparing two mechanisms we count the number of coalitions that can manipulate at a given profile. Again, maximally linked fair allocation rules are least coalitionally manipulable among all fair and budget-balanced allocation rules. Finally, when comparing two rules with respect to inclusion of the agents who can manipulate the rule at a profile (à la Pathak and Sönmez, 2012), we show that linked fair allocation rules are minimally manipulable among all fair and budget-balanced allocation rules. Such rules choose for any profile an arbitrary agent  $k$  and then select the agent  $k$ -linked fair allocations for this profile. Here it is possible that the same agent  $k$  is chosen for any profile.

An alternative approach to ours is to abandon budget-balance. A complete characterization of the class of fair and non-manipulable allocation rules has been provided by Andersson and Svensson (2008), Sun and Yang (2003) and Svensson (2009). Any such rule fixes a maximal compensation for each object, and for any profile, a “maximal” fair allocation is chosen without exceeding the fixed compensations for any object. As a result, the allocation rules in this class violate budget-balance. However, in many fair allocation problems, budget-balance is a necessary requirement and non-manipulability must be abandoned. Even though this type of problem has been considered previously, by e.g. Tadenuma and Thomson (1993), Aragoes (1995), Haake, Raith and Su (2000), Klijn (2000), Abdulkadiroğlu, Sönmez and Ünver (2004), Azacis (2008) and Velez (2011), two issues have not been investigated. First, although it is known that each fair and budget-balanced allocation rule is manipulable at some preference profile, a characterization of the preference profiles where successful misrepresentations are possible was missing. Second, there is a large class of fair and budget-balanced allocation rules but it was not known exactly which rules are “minimally” or “least” manipulable. Our paper addresses those two issues.

The paper is organized as follows. In Section 2 we introduce assignment with compensations and fair and budget-balanced allocation rules. In Section 3, our first main result shows that for a given preference profile, all fair and budget-balanced rules are either (all) manipulable or (all) non-manipulable. In Section 4 we discuss different measures of the degree of manipulability of rules. We show that measures which compare different rules via profiles counting or profiles inclusion cannot be used to distinguish among fair and budget-balanced allocation rules. Then we introduce our new criterion of minimal manipulability by counting at each profile the number of agents who can manipulate. In Section 5 we define  $k$ -linked fair allocations. We show that  $k$ -linked fair allocations always exist and that all agents are indifferent between all  $k$ -linked fair allocations. In Section 6 we introduce indifference components and maximally linked fair allocation rules. Our second main result shows that maximally linked fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules. We show that the same result holds if we compare two rules by counting the number of coalitions who can manipulate the rule at the profile. Finally, we show when comparing rules with respect to inclusion of the set of agents who can manipulate, linked fair allocation rules are agents-inclusion-

minimally manipulable among all fair and budget-balanced allocation rules. All technical results and proofs omitted in the main text are relegated to the Appendix.

## 2 Assignment with Compensations

Let  $N = \{1, \dots, n\}$  and  $M = \{1, \dots, m\}$  denote the set of agents and objects, respectively. The number of agents and objects are assumed to coincide, i.e.  $|N| = |M|$ .<sup>3</sup> Each agent  $i \in N$  consumes exactly one object  $j \in M$  together with some amount of money. A consumption bundle is a pair  $(j, x_j) \in M \times \mathbb{R}$  where  $x_j$  is the monetary compensation received when consuming object  $j$ . An allocation  $(a, x)$  is a list of  $|N|$  consumption bundles where  $a : N \rightarrow M$  is a mapping assigning object  $a_i$  to agent  $i \in N$ , and where  $x \in \mathbb{R}^M$  (or  $x : M \rightarrow \mathbb{R}$ ) assigns the amount  $x_j$  of money for the object  $j \in M$ . An allocation  $(a, x)$  is *feasible* if  $a_i \neq a_j$  whenever  $i \neq j$  for  $i, j \in N$ , and  $\sum_{j \in M} x_j \leq 0$ .<sup>4</sup> If  $\sum_{j \in M} x_j = 0$ , then the allocation  $(a, x)$  satisfies *budget-balance*. Let  $\mathcal{A}$  denote the set of feasible and budget-balanced allocations.

Each agent  $i \in N$  has preferences over consumption bundles  $(j, x_j)$  which are represented by continuous utility functions  $u_i : M \times \mathbb{R}^M \rightarrow \mathbb{R}$ . We will write  $u_{ij}(x)$  instead of  $u_i(j, x)$  to denote the utility of agent  $i \in N$  when consuming object  $j \in M$  and receiving compensation  $x_j$  in the distribution vector  $x$ . The utility function  $u_i$  is assumed to be quasi-linear and strictly increasing (or monotonic) in money, i.e.

$$u_{ij}(x) = v_{ij} + x_j \text{ for some } v_{ij} \in \mathbb{R}.$$

A list of utility functions  $u = (u_i)_{i \in N}$  is a preference profile. We also adopt the notational convention of writing  $u = (u_C, u_{-C})$  for  $C \subseteq N$ . The set of preference profiles with utility functions having the above properties is denoted by  $\mathcal{U}$ .

Let  $u \in \mathcal{U}$  and  $(a, x)$  be a feasible allocation. Then  $(a, x)$  is *efficient* if there exists no feasible allocation  $(b, y)$  such that  $u_{ib_i}(y) \geq u_{ia_i}(x)$  for all  $i \in N$  with strict inequality holding for some  $i \in N$ . Obviously, if  $(a, x)$  is efficient, then  $(a, x)$  is budget-balanced.

Throughout we focus on feasible allocations satisfying budget-balance.<sup>5</sup> For convenience, in the following “allocation” stands for “feasible allocation satisfying budget-balance”.

The important concept in this literature is fairness which corresponds to envy-freeness (Foley, 1967). It says that each agent weakly prefers his consumption bundle to any other agent’s bundle.

**Definition 1.** For a given profile  $u \in \mathcal{U}$ , an allocation  $(a, x)$  is *fair* if  $u_{ia_i}(x) \geq u_{ia_j}(x)$  for all  $i, j \in N$ . Let  $F(u)$  denote the set of fair allocations for a given profile  $u \in \mathcal{U}$ .

<sup>3</sup>If  $|N| > |M|$ , then we simply add  $|N| - |M|$  null objects with zero value for all agents.

<sup>4</sup>All our results remain true if the budget constraint is replaced by  $\sum_{j \in M} x_j \leq x_0$  for an arbitrary constant  $x_0 \in \mathbb{R}$ .

<sup>5</sup>When budget-balance is relaxed to  $\sum_{j \in M} x_j \leq 0$ , then general non-manipulability results are possible, see e.g. Andersson and Svensson (2008) or Sun and Yang (2004).

Under fairness, for feasible allocations efficiency is equivalent to budget-balance.<sup>6</sup>

An *allocation rule* is a non-empty correspondence  $\varphi$  choosing for each profile  $u \in \mathcal{U}$  a non-empty set of allocations,  $\varphi(u) \subseteq \mathcal{A}$ , such that  $u_{ib_i}(y) = u_{ia_i}(x)$  for all  $i \in N$  and all  $(a, x), (b, y) \in \varphi(u)$ . Hence, the various allocations in the set  $\varphi(u)$  are utility equivalent. Such a correspondence is called *essentially single-valued*. It is important to note that alternatively we may consider *single-valued* allocation rules choosing for each profile  $u \in \mathcal{U}$  a unique allocation. All our results remain unchanged for single-valued allocation rules.

An allocation rule  $\varphi$  is called *fair* if for any profile  $u \in \mathcal{U}$ ,  $\varphi(u) \subseteq F(u)$ .

### 3 Manipulability and Non-Manipulability

Our first main result will determine the (non-)manipulation possibilities of fair allocation rules.

**Definition 2.** An allocation rule  $\varphi$  is *manipulable* at a profile  $u \in \mathcal{U}$  by an agent  $i \in N$  if there exists a profile  $(\hat{u}_i, u_{-i}) \in \mathcal{U}$  and two allocations  $(a, x) \in \varphi(u)$  and  $(b, y) \in \varphi(\hat{u}_i, u_{-i})$  such that  $u_{ib_i}(y) > u_{ia_i}(x)$ . If the allocation rule  $\varphi$  is not manipulable by any agent at profile  $u \in \mathcal{U}$ , then  $\varphi$  is *non-manipulable* at profile  $u \in \mathcal{U}$ .

Since allocation rules may choose sets of allocations, one may alternatively employ a more conservative notion of manipulability:  $\varphi$  is *strongly manipulable* at a profile  $u \in \mathcal{U}$  by an agent  $i \in N$  if there exists a profile  $(\hat{u}_i, u_{-i}) \in \mathcal{U}$  such that  $u_{ib_i}(y) > u_{ia_i}(x)$  for all  $(a, x) \in \varphi(u)$  and all  $(b, y) \in \varphi(\hat{u}_i, u_{-i})$ . From Svensson (2009, Proposition 3 and its proof) it follows that for any fair allocation rule  $\varphi$  and any profile  $u \in \mathcal{U}$ ,  $\varphi$  is strongly manipulable at profile  $u \in \mathcal{U}$  by  $i \in N$  if and only if  $\varphi$  is manipulable at profile  $u \in \mathcal{U}$  by  $i \in N$ . Hence, we may use the conservative notion of manipulability instead of ours.

It is well-known (Green and Laffont, 1979) that any fair and budget-balanced rule  $\varphi$  is manipulable for some profile  $u \in \mathcal{U}$ . Even though we are primarily interested in manipulation by individuals, it will be interesting to formulate our main results in terms of manipulation by coalitions. We adopt the following version of coalitional manipulability and coalitional non-manipulability.<sup>7</sup> As usual, a coalition is a non-empty subset of  $N$ .

**Definition 3.** An allocation rule  $\varphi$  is (*coalitionally*) *manipulable* at a profile  $u \in \mathcal{U}$  by a coalition  $C \subseteq N$  if there is a profile  $(\hat{u}_C, u_{-C}) \in \mathcal{U}$  and two allocations  $(a, x) \in \varphi(u)$  and  $(b, y) \in \varphi(\hat{u}_C, u_{-C})$  such that  $u_{ib_i}(y) > u_{ia_i}(x)$  for all  $i \in C$ . If the allocation rule  $\varphi$  is not manipulable by any coalition at profile  $u$ , then  $\varphi$  is *coalitionally non-manipulable* at profile  $u \in \mathcal{U}$ .

Our first main result shows that a fair and budget-balanced allocation rule is non-manipulable at a profile if and only if all fair and budget-balanced allocation rules are

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<sup>6</sup>This is due to the fact that any fair allocation must assign the objects efficiently.

<sup>7</sup>Again, in the same vein as above, we may use a more conservative notion of coalitional manipulability where all deviating agents are strictly better off after the deviation for any of the chosen allocations. This would not change any of our results.

non-manipulable at this profile. Furthermore, the same equivalence holds when considering coalitional non-manipulability instead of individual non-manipulability.

**Theorem 1.** Let  $\varphi$  and  $\psi$  be two arbitrary fair and budget-balanced allocation rules. Then  $\varphi$  is (coalitionally) non-manipulable at profile  $u \in \mathcal{U}$  if and only if  $\psi$  is (coalitionally) non-manipulable at profile  $u \in \mathcal{U}$ .

## 4 Minimal Manipulability

Fairness, budget-balance, and (global) non-manipulability are incompatible (Green and Laffont, 1979). Our approach is to weaken or abandon non-manipulability.<sup>8</sup> A natural question is whether there is a “minimally (or least) manipulable” allocation rule among all fair and budget-balanced rules. Several recent contributions<sup>9</sup> use a notion of the degree of manipulability in order to compare the ease of manipulation in allocation mechanisms that are known to be manipulable. The common feature is that these results (except for Theorem 4 in Pathak and Sönmez (2012)) use measures for the degree of manipulability which are based on the preference domain.

To define the various notions of minimal manipulability, given an allocation rule  $\varphi$ , let  $\mathcal{U}^\varphi \subseteq \mathcal{U}$  denote the subset of preference profiles at which  $\varphi$  is manipulable (by some agent). In addition, let  $P^\varphi(u)$  denote the set of agents who can manipulate the allocation rule  $\varphi$  at profile  $u \in \mathcal{U}$ .

In Definitions 4-7, we make weak comparisons of two rules and “more” stands for “weakly more” (like “preferred” stands for “weakly preferred”).

**Definition 4** (Profiles counting). Let  $\varphi$  and  $\psi$  be two allocation rules.

- (a)  $\varphi$  is profiles-counting-more manipulable than  $\psi$  if  $|\mathcal{U}^\varphi| \geq |\mathcal{U}^\psi|$ ; and
- (b)  $\varphi$  and  $\psi$  are profiles-counting-equally manipulable if  $|\mathcal{U}^\varphi| = |\mathcal{U}^\psi|$ .

Note that any two rules can be compared regarding their manipulability with respect to profiles counting. The following partial comparison has been proposed by Pathak and Sönmez (2012).

**Definition 5** (Profiles inclusion). Let  $\varphi$  and  $\psi$  be two allocation rules.

- (a)  $\varphi$  is profiles-inclusion-more manipulable than  $\psi$  if  $\mathcal{U}^\varphi \supseteq \mathcal{U}^\psi$ ; and
- (b)  $\varphi$  and  $\psi$  are profiles-inclusion-equally manipulable if  $\mathcal{U}^\varphi = \mathcal{U}^\psi$ .

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<sup>8</sup>Several papers weaken or abandon budget-balance (Sun and Yang, 2003; Andersson and Svensson, 2008; and Svensson, 2009).

<sup>9</sup>See e.g. Aleskerov and Kurbanov (1999), Kelly (1988, 1993), Maus, Peters and Storcken (2007a,b) or Pathak and Sönmez (2012).



Note that if  $\varphi$  is profiles-inclusion-more manipulable than  $\psi$ , then  $\varphi$  is profiles-counting-more manipulable than  $\psi$ . However, neither of these measures can be used to distinguish fair and budget-balanced allocation rules with respect to their degree of manipulability.

**Proposition 1.** Let  $\varphi$  and  $\psi$  be two fair and budget-balanced allocation rules. Then (i)  $\varphi$  and  $\psi$  are profiles-counting-equally manipulable, and (ii)  $\varphi$  and  $\psi$  are profiles-inclusion-equally manipulable.

*Proof.* By Theorem 1, both  $\mathcal{U}^\varphi = \mathcal{U}^\psi$  and  $|\mathcal{U}^\varphi| = |\mathcal{U}^\psi|$ , which yields the desired conclusion.  $\square$

Since all fair and budget-balanced rules are equally manipulable if the measure is based only on the cardinality and/or set inclusions of subsets in the preference domain, a “finer” notion is needed. Note that an equivalent way of stating (global) non-manipulability (or “strategy-proofness”) is the following. Allocation rule  $\varphi$  is (globally) non-manipulable if

$$|P^\varphi(u)| = 0 \text{ for all } u \in \mathcal{U}. \quad (1)$$

Given the fact that (1) never can be satisfied for fair and budget-balanced rules and the above insights, it is natural to search for rules where  $|P^\varphi(u)|$  is minimized for each profile  $u \in \mathcal{U}$ . This guarantees that the rule is non-manipulable whenever a non-manipulable rule exists for a specific profile, and that the core idea of (global) non-manipulability is respected as much as possible.

**Definition 6** (Agents counting). Let  $\varphi$  and  $\psi$  be two allocation rules. Then  $\varphi$  is agents-counting-more manipulable than  $\psi$  if  $|P^\varphi(u)| \geq |P^\psi(u)|$  for all  $u \in \mathcal{U}$ .

The corresponding notion with respect to inclusion was introduced by Pathak and Sönmez (2012).

**Definition 7** (Agents inclusion). Let  $\varphi$  and  $\psi$  be two allocation rules. Then  $\varphi$  is agents-inclusion-more manipulable than  $\psi$  if  $P^\varphi(u) \supseteq P^\psi(u)$  for all  $u \in \mathcal{U}$ .

While it is clear that these measures are partial comparisons of allocation rules, the following shows the relations among the various measures of the degree of manipulability. For any two allocation rules  $\varphi$  and  $\psi$ , we have:<sup>10</sup>

$$\begin{aligned} & \varphi \text{ is agents-inclusion-more manipulable than } \psi \\ \Rightarrow & \varphi \text{ is agents-counting-more manipulable than } \psi \\ \Rightarrow & \varphi \text{ is profiles-inclusion-more manipulable than } \psi \\ \Rightarrow & \varphi \text{ is profiles-counting-more manipulable than } \psi. \end{aligned}$$

The relations between the different concepts are general and do not depend on our specific model.

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<sup>10</sup>In showing  $\mathcal{U}^\varphi \supseteq \mathcal{U}^\psi$  for the second implication, note that for any  $u \in \mathcal{U}^\psi$  we have  $0 = |P^\psi(u)| \geq |P^\varphi(u)| \geq 0$ . Thus, both  $|P^\psi(u)| = 0$  and  $u \in \mathcal{U}^\varphi$ .

Note that Definitions 4-7 (weakly) compare two rules with respect to their manipulability. Naturally, any of these concepts would strongly compare two rules  $\varphi$  and  $\psi$ , if  $\varphi$  is comparable to  $\psi$  but  $\psi$  is not comparable to  $\varphi$ . In other words, under a strong comparison in Definition 4(a) requires a strict inequality for some profile, in Definition 5(a) a strict inclusion for some profile, in Definition 6 a strict inequality for some profile, and in Definition 7 a strict inclusion for some profile. Actually, as the careful reader may check, Pathak and Sönmez (2012)'s second concept makes (only) a strong comparison in the vein of Definition 7 but requires in addition  $\mathcal{U}^\varphi \supsetneq \mathcal{U}^\psi$ . Of course, again by Theorem 1, in this sense no two fair and budget-balanced rules would be strongly comparable.

## 5 Agent $k$ -linked Allocations

It is well established that the possibility for agents to manipulate a fair allocation rule depends on the structure of the indifference relations at the allocation(s) chosen by the rule.<sup>11</sup> For example, in the single-item Vickrey auction, the number of indifference relations is maximized at the final allocation (i.e., all agents with the second highest bid are indifferent between being assigned the item or not). A systematic description of the indifference structure at fair and budget-balanced allocations is the key to understand under which preference profiles a fair and budget-balanced rule can or cannot be manipulated. This will allow us to define the fair allocation rules which are agents-counting-minimally manipulable and/or agents-inclusion-minimally manipulable among all fair allocation rules. Below we introduce the concepts of indifference chains and agent  $k$ -linked (fair) allocations.

**Definition 8.** Let  $(a, x) \in \mathcal{A}$ .

- (i) For any  $i, j \in N$ , we write  $i \rightarrow_{(a,x)} j$  if  $u_{ia_i}(x) = u_{ia_j}(x)$ .
- (ii) An *indifference chain* at allocation  $(a, x)$  consists of a tuple of distinct agents  $g = (i_0, i_1, \dots, i_k)$  such that  $i_0 \rightarrow_{(a,x)} i_1 \rightarrow_{(a,x)} \dots \rightarrow_{(a,x)} i_k$ .

Note that  $i \rightarrow_{(a,x)} j$  means that agent  $i$  is indifferent between his consumption bundle and agent  $j$ 's consumption bundle, and agent  $i$  is directly linked via indifference to agent  $j$  at allocation  $(a, x)$ . An indifference chain at an allocation is simply a sequence of agents such that any agent in the sequence is indifferent between his bundle and the bundle of the agent following him in the sequence. Indifference chains indirectly link agents via indifference in a sequence of directly linked agents.

The following concept of agent  $k$ -linked allocations will play an important role.

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<sup>11</sup>See for example, Andersson and Svensson (2008), Andersson, Svensson and Yang (2010) or Mishra and Talman (2010) for theoretical results, and Sankaran (1994) or Mishra and Parkes (2010) for efficient procedures to calculate allocations with the maximal number of indifference relations. Similar observations have previously also been made by e.g. Dubey (1982) and Svensson (1991) where the ‘‘tightness’’ of the market is demonstrated to have a significant impact on manipulation possibilities.

**Definition 9.** Let  $(a, x) \in \mathcal{A}$ .

- (i) Agent  $i \in N$  is *linked* to agent  $k \in N$  at allocation  $(a, x)$  if there exists an indifference chain  $(i_0, \dots, i_t)$  at allocation  $(a, x)$  with  $i = i_0$  and  $i_t = k$ .
- (ii) Allocation  $(a, x)$  is *agent  $k$ -linked* if each agent  $i \in N$  is linked to agent  $k \in N$ .

Thus, at an agent  $k$ -linked allocation, each agent is linked to agent  $k \in N$  through some indifference chain. Our next theorem establishes the existence of a  $k$ -linked (fair and budget-balanced) allocation for all  $k \in N$  and all  $u \in \mathcal{U}$ .

**Theorem 2.** For each profile  $u \in \mathcal{U}$  and each  $k \in N$ , there exists an agent  $k$ -linked allocation in  $F(u)$ . Moreover, any allocation that maximizes the utility of agent  $k$  in  $F(u)$  is an agent  $k$ -linked allocation.

Given  $k \in N$  and  $u \in \mathcal{U}$ , let  $\psi^k(u) \subseteq F(u)$  denote the set of all fair and budget-balanced allocations which are agent  $k$ -linked at profile  $u$ . By Theorem 2,  $\psi^k(u)$  is non-empty. Our next result shows that  $\psi^k$  is an allocation rule, i.e., that for any profile all agents are indifferent between all agent  $k$ -linked fair allocations. The allocation rule  $\psi^k$  will be called the *agent  $k$ -linked fair allocation rule*, henceforth.

**Proposition 2.**  $\psi^k$  is an allocation rule.

The corollary below will follow from the proof of Theorem 1.

**Corollary 1.** (i)  $\psi^k$  cannot be manipulated by agent  $k$  at any profile  $u \in \mathcal{U}$ .

- (ii) For any two distinct agents  $i, j \in N$ , there exists no fair and budget-balanced allocation rule  $\varphi$  such that neither  $i$  nor  $j$  can manipulate  $\varphi$  at any profile  $u \in \mathcal{U}$ .

Corollary 1 has the same flavor as the corresponding results in two-sided matching (with men and women): (i) for any agent there exists a stable matching rule which is not manipulable by this agent at any profile; and (ii) there is no stable matching rule which cannot be manipulated by at least one man and at least one woman (Ma, 1995).

## 6 (Maximally) Linked Fair Allocation Rules

We next introduce a more demanding notion of indifference structures, namely indifference components. In each such component any two agents are linked through an indifference chain in this component and there is no superset of this component where any two agents are linked.

**Definition 10.** Let  $(a, x) \in \mathcal{A}$ . An *indifference component* at allocation  $(a, x)$  is a non-empty set  $G \subseteq N$  such that for all  $i, k \in G$  there exists an indifference chain at  $(a, x)$  in  $G$ , say  $g = (i_0, \dots, i_k)$  with  $\{i_0, \dots, i_k\} \subseteq G$ , such that  $i = i_0$  and  $i_k = k$ , and there exists no  $G' \supsetneq G$  satisfying the previous property at allocation  $(a, x)$ .

The next result states an important characteristic of indifference components, namely that if there are two allocations that are fair and budget-balanced at some profile  $u \in \mathcal{U}$  and if there is an indifference component at one of these allocations, then the very same indifference component must be present at the other allocation. In other words, indifference components at fair and budget-balanced allocations only depend on the preference profile  $u \in \mathcal{U}$  because they are invariant with respect to the selected fair and budget-balanced allocation.

**Lemma 1.** Suppose that allocations  $(a, x)$  and  $(b, y)$  are fair and budget-balanced at profile  $u \in \mathcal{U}$ . If  $G$  is an indifference component at allocation  $(a, x)$ , then  $G$  is an indifference component at allocation  $(b, y)$ .

By Lemma 1, the set of indifference components is identical for all fair and budget-balanced allocations in  $F(u)$ . Let

$$\mathcal{G}(u) = \{G \subseteq N : G \text{ is an indifference component at all } (a, x) \in F(u)\},$$

denote the set of all indifference components of fair and budget-balanced allocations at profile  $u \in \mathcal{U}$ . Note that for any  $k \in N$  there exists  $G \in \mathcal{G}(u)$  with  $k \in G$ .

In determining the least manipulable fair and budget-balanced allocation rules, for agent  $k$ -linked fair allocation rules, not only the preference profile, but also the selection of  $k \in N$  may influence the manipulability possibilities. In the search for the agents-counting-minimally manipulable fair and budget-balanced allocation rule, it is important to select the right  $k \in N$  for any given profile  $u \in \mathcal{U}$ . For this reason, the selection of agent  $k$  will be endogenously determined by the profile  $u \in \mathcal{U}$ . The general idea is first to select an indifference component with maximal cardinality, and then some agent  $k$  belonging to this indifference component and finally the set of agent  $k$ -linked fair allocations.

Let

$$\bar{\mathcal{G}}(u) = \{G \in \mathcal{G}(u) : |G| \geq |G'| \text{ for all } G' \in \mathcal{G}(u)\},$$

denote the set of indifference components with maximal cardinality. Let

$$\bar{G}(u) = \cup_{G \in \bar{\mathcal{G}}(u)} G,$$

denote the union of all indifference components with maximal cardinality.

A *selection* is a function  $\kappa : \mathcal{U} \rightarrow N$ . The linked fair allocation rule  $\phi^\kappa$  based on  $\kappa : \mathcal{U} \rightarrow N$  is defined as follows: for all  $u \in \mathcal{U}$ ,  $\phi^\kappa(u) = \psi^{\kappa(u)}(u)$ . In other words, a linked fair allocation rule selects for each  $u$  an agent  $\kappa(u)$  and chooses all  $\kappa(u)$ -linked fair allocations. Note that by Proposition 2,  $\phi^\kappa$  is a well-defined allocation rule because  $\psi^k(u)$  is essentially single-valued for any  $k \in N$  and any  $u \in \mathcal{U}$ . Furthermore, we will say that an allocation rule  $\varphi$  is a *linked* fair allocation rule if there exists a selection  $\kappa$  such that for all  $u \in \mathcal{U}$  we have  $\varphi(u) \subseteq \phi^\kappa(u)$ .

A *maximal selection* is a function  $\kappa : \mathcal{U} \rightarrow N$  such that for all  $u \in \mathcal{U}$  we have  $\kappa(u) \in \bar{G}(u)$ . The maximally linked fair allocation rule  $\phi^\kappa$  is the linked fair allocation rule based on  $\kappa$ . Furthermore, we will say that an allocation rule  $\varphi$  is a *maximally linked* fair allocation

rule if there exists a maximal selection  $\kappa$  such that for all  $u \in \mathcal{U}$  we have  $\varphi(u) \subseteq \phi^\kappa(u)$ . Note that the function  $\kappa$  is a systematic selection from  $\bar{\mathcal{G}}(u)$ . The meaning of “systematic selection” is that there is a well defined rule for selecting  $k$ . This rule can be arbitrary and all our results hold independently of this rule. For example, the rule could be based on a randomized selection from  $\bar{\mathcal{G}}(u)$  or simply the  $k$  with the lowest or highest index in  $\bar{\mathcal{G}}(u)$ .

Our second main result establishes that maximally linked fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules.

**Theorem 3.** Let  $\varphi$  be an arbitrary fair and budget-balanced allocation rule and let  $\phi^\kappa$  be a maximally linked fair allocation rule. Then:

- (i)  $\varphi$  is agents-counting-more manipulable than  $\phi^\kappa$ ; and
- (ii) if  $\phi^\kappa$  is agents-counting-more manipulable than  $\varphi$ , then  $\varphi$  is a maximally linked fair allocation rule.

By (i) in Theorem 3, any fair and budget-balanced allocation rules can be compared to a maximally linked fair allocation rule via agents-counting-manipulability, and by (i) and (ii), any fair and budget-balanced allocation rule, which is not maximally linked fair, is strongly agents-counting-more manipulable (with a strict inequality for some profile in Definition 6) than any maximally linked fair allocation rule. Note that except for degenerate preference profiles where  $N$  is the unique indifference component, the set of fair and budget-balanced allocations is a continuum. Thus, any rule, which chooses for some non-degenerate preference profile an allocation which is not linked, is strongly agents-counting-more manipulable than any maximally linked fair allocation rule. Hence, the comparison is often strict.

In checking the robustness of Theorem 3 we consider the degree of coalitional manipulability. Using the same arguments as above, by Theorem 1 it is in general impossible to define a fair and budget-balanced rule to be less coalitionally manipulable than some other fair and budget-balanced rule if the measure is based only on the cardinality and/or set inclusions of subsets in the preference domain. Let  $Q^\varphi(u)$  denote the coalitions  $C \subseteq N$  that can manipulate the allocation rule  $\varphi$  at profile  $u \in \mathcal{U}$ . We adopt the following notion.

**Definition 11.** Let  $\varphi$  and  $\psi$  be two allocation rules. Then  $\varphi$  is more coalitionally manipulable than  $\psi$  if  $|Q^\varphi(u)| \geq |Q^\psi(u)|$  for all  $u \in \mathcal{U}$ .

The following result states that maximally linked fair allocation rules are minimally coalitionally manipulable among all fair and budget-balanced allocation rules. This result be seen as an extension of Theorem 3 from minimal individual manipulability to minimal coalitional manipulability, i.e., that Theorem 3 is robust with respect to coalitional manipulations.

**Theorem 4.** Let  $\varphi$  be a fair and budget-balanced allocation rule and  $\phi^\kappa$  be a maximally linked fair allocation rule. Then:

- (i)  $\varphi$  is more coalitionally manipulable than  $\phi^\kappa$ ; and

- (ii) if  $\phi^\kappa$  is more coalitionally manipulable than  $\varphi$ , then  $\varphi$  is a maximally linked fair allocation rule.

Finally we will establish that linked fair allocation rules are agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules.<sup>12</sup> We show that any fair and budget-balanced allocation rule is agents-inclusion-more manipulable than some linked fair allocation rule.

**Theorem 5.** Let  $\varphi$  be an arbitrary fair and budget-balanced allocation rule. Then:

- (i) there exists a selection  $\kappa : \mathcal{U} \rightarrow N$  such that  $\varphi$  is agents-inclusion-more manipulable than  $\phi^\kappa$ ; and
- (ii) if  $\phi^\kappa$  is agents-inclusion-more manipulable than  $\varphi$ , then for all  $u \in \mathcal{U}$ ,  $\varphi(u) \subseteq \phi^\kappa(u)$ .

Similar as above for agents-counting-minimal manipulability, Theorem 5 is robust with respect to coalitional manipulability (by considering inclusions of the set of coalitions which can manipulate the rule at a profile).

By (i) and (ii) of Theorem 5, any fair and budget-balanced allocation rule, which is not linked fair, is strongly agents-inclusion-more manipulable (with a strict inclusion for some profile in Definition 7) than some linked fair allocation rule. A direct consequence of (ii) of Theorem 5 is that for any  $k \in N$ , agent  $k$ -linked fair allocation rules are agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules.

**Corollary 2.** Let  $k \in N$  and  $\varphi$  be an arbitrary fair and budget-balanced allocation rule. If  $\psi^k$  is agents-inclusion-more manipulable than  $\varphi$ , then for all  $u \in \mathcal{U}$ ,  $\varphi(u) \subseteq \psi^k(u)$ .

Obviously, by Corollary 2, for distinct  $k, i \in N$ ,  $\psi^k$  and  $\psi^i$  cannot be compared with respect to agent-inclusion-more manipulability.

## APPENDIX.

The following are two well-known properties of fair allocations (see e.g. Svensson, 2009): first, if two allocations are fair at a given profile, then one may interchange both the assignment of objects and the monetary distribution without losing fairness. Obviously, this result holds for fair allocations satisfying budget-balance.

**Lemma 2.** Suppose that allocations  $(a, x)$  and  $(b, y)$  are fair at profile  $u \in \mathcal{U}$ . Then allocations  $(a, y)$  and  $(b, x)$  are also fair at profile  $u \in \mathcal{U}$ .

Second, for fair allocation rules, a unique distribution of money is chosen for any given preference profile.<sup>13</sup>

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<sup>12</sup>The careful reader may note that Theorem 5 is the only new result which is not included in Andersson, Ehlers and Svensson (2010).

<sup>13</sup>To make the presentation self-contained, we include the proof of Lemma 3 (which follows for instance from Lemma 3 in Alkan, Demange and Gale (1991)).

**Lemma 3.** Let  $\varphi$  be a fair allocation rule and  $u \in \mathcal{U}$ . If  $(a, x), (b, y) \in \varphi(u)$ , then  $x = y$ .

*Proof.* Since  $(a, x), (b, y) \in \varphi(u)$ , we have  $u_{ia_i}(x) = u_{ib_i}(y)$  for all  $i \in N$ . By fairness,  $u_{ia_i}(x) \geq u_{ib_i}(x)$ . Thus,  $u_{ib_i}(y) \geq u_{ib_i}(x)$  and  $y_{b_i} \geq x_{b_i}$ . Similarly, we obtain  $x_{b_i} \geq y_{b_i}$ . Hence,  $x = y$ , the desired conclusion.  $\square$

We proceed as follows. First, we show Theorem 2 and Proposition 2 for  $k$ -linked fair allocations. We establish the invariance of indifference components across fair and budget-balanced allocations. In Lemma 4 we show that indifference components are related to “isolated groups” (Definition 12 below) in the following way at fair allocations: if  $N - G$  is an isolated group with maximal cardinality at a fair allocation, then  $G$  is an indifference component.

Second, for any fair and budget-balanced allocation rule and any preference profile, we characterize the set of agents and the set of coalitions who can profitably manipulate the rule at this profile: (i) if a group  $G$  is isolated at a chosen allocation, then any coalition contained in  $G$  can manipulate the rule at this profile; and (ii) if the rule chooses  $k$ -linked fair allocations at this profile, then no coalition containing agent  $k$  can manipulate the rule at this profile. The (non-)manipulability results Theorem 1 and Corollary 1 follow then easily.

Third, we show our minimal manipulability results Theorem 3, Theorem 4, and Theorem 5.

## A Indifferences at Fair and Budget-Balanced Allocations

At an agent  $k$ -linked allocation, each agent is linked to agent  $k \in N$  through some indifference chain. The proof of the existence of a  $k$ -linked (fair and budget-balanced) allocation for all  $k \in N$  and all  $u \in \mathcal{U}$  shows that any utility maximizing allocation for agent  $k \in N$  in  $F(u)$  is agent  $k$ -linked.

**Theorem 2.** For each profile  $u \in \mathcal{U}$  and each  $k \in N$ , there exists an agent  $k$ -linked allocation in  $F(u)$ . Moreover, any allocation that maximizes the utility of agent  $k$  in  $F(u)$  is an agent  $k$ -linked allocation.

*Proof.* Note that an agent  $k \in N$  utility maximizing allocation exists in  $F(u)$  for each profile  $u \in \mathcal{U}$  since  $F(u)$  is compact. Thus, it remains to show that any allocation in  $F(u)$  which maximizes the utility of agent  $k \in N$  is agent  $k$ -linked.

Let  $u \in \mathcal{U}$ ,  $k \in N$  and  $(a, x)$  be an agent  $k$  utility maximizing allocation in  $F(u)$ . By contradiction, suppose that  $(a, x)$  is not  $k$ -linked, i.e., that there is an agent  $l \in N$  which is not linked to agent  $k$ . Let

$$G = \{i \in N : i \text{ is linked to } k \text{ at } (a, x)\} \cup \{k\}.$$

Because  $k \in G$  and  $l \in N-G$ , both  $G$  and  $N-G$  are non-empty. Moreover, by construction,  $u_{ia_i}(x) > u_{ia_j}(x)$  if  $i \in N-G$  and  $j \in G$ . From the Perturbation Lemma in Alkan, Demange and Gale (1991) it then follows that there exists another allocation  $(b, y) \in F(u)$  such that  $y_{a_i} > x_{a_i}$  for all  $i \in G$ .<sup>14</sup> Then by fairness and monotonicity in money, we have

$$u_{ib_i}(y) \geq u_{ia_i}(y) = v_{ia_i} + y_{a_i} > v_{ia_i} + x_{a_i} = u_{ia_i}(x) \text{ for all } i \in G.$$

Because  $k \in G$ , it follows that  $u_{kb_k}(y) > u_{ka_k}(x)$ , which contradicts the fact that  $(a, x)$  maximizes  $k$ 's utility in  $F(u)$ . Hence, any allocation maximizing the utility of agent  $k$  in  $F(u)$  is an agent  $k$ -linked allocation.  $\square$

By Theorem 2,  $\psi^k(u)$  is non-empty. Our next result shows that  $\psi^k$  is an allocation rule, i.e., that for any profile all agents are indifferent between all agent  $k$ -linked fair allocations.<sup>15</sup>

**Proposition 2.**  $\psi^k$  is an allocation rule.

*Proof.* Let  $u \in \mathcal{U}$ . To prove the result we need to establish that if  $(a, x), (b, y) \in \psi^k(u)$ , then  $u_{ia_i}(x) = u_{ib_i}(y)$  for all  $i \in N$ . By Theorem 2, we may suppose without loss of generality that  $(b, y)$  is an allocation that maximizes the utility of agent  $k$  in  $F(u)$ .

We first demonstrate the analogue of Lemma 3 for agent  $k$ -linked fair allocations: if  $(a, x), (b, y) \in \psi^k(u)$ , then  $x = y$ . To see this, note that  $(a, y)$  is also fair by Lemma 2. First, we show that  $(a, y)$  is agent  $k$ -linked if  $(b, y)$  is agent  $k$ -linked. Fairness implies

$$u_{ia_i}(y) = u_{ib_i}(y) \text{ for all } i \in N. \tag{2}$$

Since  $(b, y)$  maximizes the utility of agent  $k$  in  $F(u)$ , (2) implies that  $(a, y)$  also maximizes the utility of agent  $k$  in  $F(u)$ . Thus, by Theorem 2,  $(a, y)$  is agent  $k$ -linked. Hence, without loss of generality we may assume  $a = b$ .

Suppose that the fair allocations  $(a, x)$  and  $(a, y)$  are agent  $k$ -linked but  $x \neq y$ . Then by budget-balance and  $x \neq y$ , there must be two non-empty groups of agents:

$$\begin{aligned} A &= \{i \in N : x_{a_i} > y_{a_i}\}, \\ B &= \{i \in N : x_{a_i} \leq y_{a_i}\}. \end{aligned}$$

Note that for all  $i \in A$  and all  $j \in B$ ,  $u_{ia_i}(x) > u_{ia_i}(y) \geq u_{ia_j}(y) \geq u_{ia_j}(x)$ . Hence, no agent in  $A$  can be linked to any agent in  $B$  at allocation  $(a, x)$ . Because  $(a, x)$  is agent  $k$ -linked, we must have  $k \in A$ . Let  $j \in B$  and  $i \in A$ . By fairness and monotonicity in money,

$$u_{ja_j}(y) \geq u_{ja_j}(x) \geq u_{ja_i}(x) > u_{ja_i}(y).$$

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<sup>14</sup>Because preferences are quasi-linear, this can be simply done by infinitesimally increasing equally the compensations of  $\{a_i : i \in G\}$  and infinitesimally decreasing equally the compensations of  $\{a_i : i \in N-G\}$  (while preserving budget-balance).

<sup>15</sup>Again, to make the presentation self-contained, we include the proof of Proposition 2 (which follows Theorem 6 in Alkan, Demange and Gale (1991)).



Thus, at allocation  $(a, y)$  no agent in  $B$  can be linked to any agent in  $A$ . Hence, by  $k \in A$ , allocation  $(a, y)$  cannot be agent  $k$ -linked which contradicts our assumption.

Let  $(a, x), (b, y) \in \psi^k(u)$  and  $i \in N$ . By the above, we have  $x = y$ . Obviously, if  $a_i = b_i$ , then  $u_{ia_i}(x) = u_{ib_i}(y)$ . If  $a_i \neq b_i$ , then by fairness both  $u_{ia_i}(x) \geq u_{ib_i}(x)$  and  $u_{ia_i}(y) \leq u_{ib_i}(y)$ . Hence, by  $x = y$ , we have  $u_{ia_i}(x) = u_{ib_i}(y)$ , the desired conclusion.  $\square$

Proposition 2 and Theorem 2 imply that the set of agent  $k$ -linked fair allocations and the set of allocations maximizing the utility of agent  $k$  in  $F(u)$  coincide. Hence, all agents are indifferent between all fair allocations which maximize agent  $k$ 's utility in  $F(u)$ .

Below we show that for any profile  $u \in \mathcal{U}$ , the set of indifference components is invariant across all fair and budget-balanced allocations.

**Lemma 1.** Suppose that allocations  $(a, x)$  and  $(b, y)$  are fair and budget-balanced at profile  $u \in \mathcal{U}$ . If  $G$  is an indifference component at allocation  $(a, x)$ , then  $G$  is an indifference component at allocation  $(b, y)$ .

*Proof.* By Lemma 2, we know that  $(a, y)$  is fair. First we show that the indifference component  $G$  is present at  $(a, y)$ .

Because  $G$  is an indifference component at  $(a, x)$ ,  $G$  consists of indifference chains  $g = (i_0, i_1, \dots, i_k)$  such that  $i_k \rightarrow_{(a,x)} i_0$ . Thus, we have  $i_0 \rightarrow_{(a,x)} i_1 \rightarrow_{(a,x)} \dots \rightarrow_{(a,x)} i_k \rightarrow_{(a,x)} i_0$ . We show  $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \dots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$ .

For any  $i \in N$ , let  $\Delta_{a_i} = y_{a_i} - x_{a_i}$ . To obtain a contradiction, suppose that we do not have  $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \dots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$ , say  $u_{i_0 a_{i_0}}(x) = u_{i_0 a_{i_1}}(x)$  but  $u_{i_0 a_{i_0}}(y) > u_{i_0 a_{i_1}}(y)$ . Thus,  $\Delta_{a_{i_0}} > \Delta_{a_{i_1}}$ . Now, fairness is respected among the agents in  $G$  at allocation  $(a, y)$  only if

$$\Delta_{a_{i_j}} \geq \Delta_{a_{i_{j+1}}} \text{ for all } j \in \{0, \dots, k-1\}, \quad (3)$$

$$\Delta_{a_{i_k}} \geq \Delta_{a_{i_0}}. \quad (4)$$

From (3) and  $\Delta_{a_{i_0}} > \Delta_{a_{i_1}}$ , we obtain  $\Delta_{a_{i_0}} > \Delta_{a_{i_k}}$ . Hence, (4) is not satisfied. Thus, allocation  $(a, y)$  cannot be fair, which contradicts our assumption. Hence,  $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \dots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$ . Note that there exists no  $G' \supsetneq G$  such that  $G'$  is an indifference component at  $(a, y)$  because otherwise, using the previous arguments, any two agents in  $G'$  are connected through some indifference chain at  $(a, x)$  in  $G'$  which contradicts the definition of  $G$  being an indifference component at  $(a, x)$ . Thus, the indifference component  $G$  is present at  $(a, y)$ .

Next, we show that  $G$  must be also an indifference component at  $(b, y)$ . Fairness implies

$$u_{ia_i}(y) = u_{ib_i}(y) \text{ for all } i \in N. \quad (5)$$

Let  $j, k \in G$  and suppose that  $j \rightarrow_{(a,y)} k$ . If  $a_k = b_k$ , then by (5),  $j \rightarrow_{(b,y)} k$ . Let  $a_k \neq b_k$  and  $l_1 \in N$  be such that  $a_{l_1} = b_k$ . Obviously, (5) implies  $k \rightarrow_{(a,y)} l_1$ . More generally, let  $l_1, \dots, l_t$  be such that  $a_{l_r} = b_{l_{r-1}}$  with  $r = 2, \dots, t$  and  $a_k = b_{l_t}$ . Note that such a ‘‘cycle’’ exists because  $|N| = |M|$ . Now obviously we have  $k \rightarrow_{(a,y)} l_1, l_r \rightarrow_{(a,y)} l_{r+1}$  for all

$r = 1, \dots, t - 1$ , and  $l_t \rightarrow_{(a,y)} k$ . Since  $k \in G$  and  $G$  is an indifference component at  $(a, y)$ , we must have  $\{l_1, \dots, l_t\} \subseteq G$ .

Now by (5), we have  $u_{jb_j}(y) = u_{ja_j}(y) = u_{ja_k}(y) = u_{jb_{l_t}}(y)$  which implies  $j \rightarrow_{(b,y)} l_t$ . Note that by construction, we also have  $l_1 \rightarrow_{(b,y)} k$  and  $l_r \rightarrow_{(b,y)} l_{r-1}$  for all  $r = 2, \dots, t$ . This means that  $j$  and  $k$  are connected through the indifference chain  $j \rightarrow_{(b,y)} l_t \rightarrow_{(b,y)} l_{t-1} \rightarrow_{(b,y)} \dots \rightarrow_{(b,y)} l_1 \rightarrow_{(b,y)} k$  in  $G$  under  $(b, y)$  (if  $a_k \neq b_k$ ). If  $a_k = b_k$ , then  $j \rightarrow_{(b,y)} k$ . Because this is true for any  $j, k \in G$  such that  $j \rightarrow_{(a,y)} k$ , it also follows that any two agents belonging to  $G$  must be connected through an indifference chain in  $G$  at  $(b, y)$ . Furthermore, there can be no  $G' \supsetneq G$  satisfying this property under  $(b, y)$  because by the same argument  $G'$  would also satisfy this property under  $(a, x)$ , which would contradict the definition of an indifference component.  $\square$

The existence of indifference components is closely related to the presence of isolated groups (or coalitions): a group of agents  $C \subsetneq N$  is isolated if no agent outside this group can be linked to any agent in  $C$ .

**Definition 12.** A group of agents  $C \subsetneq N$  is *isolated* at allocation  $(a, x)$  if  $i \not\rightarrow_{(a,x)} j$  for all  $i \in N - C$  and all  $j \in C$ .

The following relates isolated groups and indifference components.

**Lemma 4.** Let  $\varphi$  be a fair and budget-balanced allocation rule,  $u \in \mathcal{U}$  and  $(a, x) \in \varphi(u)$ . If  $N - G$  is the (possibly empty) isolated group with maximal cardinality at allocation  $(a, x)$ , then  $G$  is an indifference component at allocation  $(a, x)$ .

*Proof.* We first show that all  $i, j \in G$  can be linked via an indifference chain in  $G$ . Suppose not, i.e. there exist  $i, j \in G$  such that  $i$  cannot be linked to  $j$  via some indifference chain in  $G$ . Let

$$H = \{k \in G : k \text{ can be linked to } j \text{ via some indifference chain in } G\}.$$

Since  $i \in G - H$ , we have  $G - H \neq \emptyset$ . Because no agent in  $G - H$  can be linked to any agent in  $H$ , by construction, it follows that  $(N - G) \cup H \subsetneq N$  (by  $i \in G - H$ ), the set  $(N - G) \cup H$  is isolated and  $|(N - G) \cup H| > |N - G|$ , which contradicts the assumption that  $N - G$  is the isolated group with maximal cardinality at allocation  $(a, x) \in \varphi(u)$ .

Now, the proof follows directly because the group  $N - G$  is isolated at allocation  $(a, x)$ , i.e.,  $i \not\rightarrow_{(a,x)} j$  for all  $i \in G$  and all  $j \in N - G$ . Consequently, there is no  $G' \supsetneq G$  such that  $G'$  is an indifference component by Definition 10.  $\square$

## B Manipulability and Non-Manipulability

Below we determine the (non-)manipulation possibilities of fair allocation rules. The first result describes the relation between isolated groups and the possibility to manipulate  $\varphi$  at

a specific profile. We show that any coalition contained in an isolated group can manipulate the fair and budget-balanced allocation rule.<sup>16</sup>

**Lemma 5.** Let  $\varphi$  be a fair and budget-balanced allocation rule,  $u \in \mathcal{U}$  and  $(a, x) \in \varphi(u)$ . If the non-empty group  $G \subsetneq N$  is isolated at allocation  $(a, x)$ , then each coalition  $C \subseteq G$  can manipulate  $\varphi$  at profile  $u \in \mathcal{U}$ .

*Proof.* Let  $(a, x) \in \varphi(u)$ , and suppose that  $G \subsetneq N$  is a non-empty isolated coalition, i.e., that both  $i \not\rightarrow_{(a,x)} j$  and  $u_{ia_i}(x) > u_{ia_j}(x)$  for all  $i \in N - G$  and all  $j \in G$ . Now simultaneously all compensations for objects  $a_i$  ( $i \in G$ ) can be increased by the same amount and all compensations for objects  $a_j$  ( $j \in N - G$ ) can be decreased by the same amount without losing budget-balance and fairness. Hence, there is a number  $\tau > 0$  and  $(a, y) \in F(u)$  such that  $u_{ia_i}(y) > u_{ia_i}(x) + \tau$  for all  $i \in G$  (and  $y_{a_i} > x_{a_i} + \tau$  for all  $i \in G$ ). Fix  $0 < \varepsilon < \tau$  and define for any  $i \in G$  the function  $\hat{u}_i$  as follows: for all  $j \in M$  and all  $x' \in \mathbb{R}^M$ , let

$$\hat{u}_{ij}(x') = (-y_j + \varepsilon_{ij}) + x'_j, \quad (6)$$

where  $\varepsilon_{ij} = 0$  if  $j \neq a_i$  and  $\varepsilon_{ia_i} = \varepsilon > 0$ . Note that  $\hat{v}_{ij} = -y_j + \varepsilon_{ij}$ . Let  $C \subseteq G$  and  $\hat{u}_C = (\hat{u}_i)_{i \in C}$ . By construction of  $\hat{u}_C$ , we have  $(a, y) \in F(\hat{u}_C, u_{-C})$ .<sup>17</sup>

Let  $(b, z) \in \varphi(\hat{u}_C, u_{-C})$ . We first show  $b_i = a_i$  for all  $i \in C$ . Let  $\delta_j = z_j - y_j$  for all  $j \in M$ . Without loss of generality, order  $M$  such that  $\delta_j \geq \delta_{j+1}$  for all  $j = 1, \dots, |M| - 1$ .

If  $z = y$ , then by fairness,  $\hat{u}_{ib_i}(y) = \hat{u}_{ia_i}(y)$  for all  $i \in C$ . Since for all  $i \in C$ ,  $\hat{u}_{ia_i}(y) = \varepsilon$  and  $\hat{u}_{ij}(y) = 0$  for  $j \neq a_i$ , we obtain  $b_i = a_i$  for all  $i \in C$ .

If  $z \neq y$ , then by budget-balance of both  $(b, z)$  and  $(a, y)$ ,  $\delta_1 > 0$  and  $\delta_n < 0$ . Let  $(j_l)_l$  be a subsequence of  $(1, \dots, n)$  such that  $j_l < j_{l+1}$ ,  $\delta_{j_l} > \delta_{j_{l+1}}$  and  $\delta_j = \delta_{j_l}$  if  $j_l \leq j < j_{l+1}$ . Let  $S_l = \{i \in N : j_l \leq a_i < j_{l+1}\}$ . Then for  $i \in S_l$ :

$$\begin{aligned} u_{ia_i}(z) &= u_{ia_i}(y) + \delta_{a_i} \geq u_{ib_i}(y) + \delta_{a_i} > u_{ib_i}(y) + \delta_{b_i} = u_{ib_i}(z) \text{ if } b_i \geq j_{l+1} \text{ and } i \in N - C, \\ \hat{u}_{ia_i}(z) &= z_{a_i} - y_{a_i} + \varepsilon = \delta_{a_i} + \varepsilon > \delta_{b_i} = \hat{u}_{ib_i}(z) \text{ if } b_i \geq j_{l+1} \text{ and } i \in C. \end{aligned}$$

Thus, by fairness, for all  $l$ ,  $i \in S_l$  implies  $j_l \leq b_i < j_{l+1}$ . Moreover, for  $i \in C$ ,  $\hat{u}_{ia_i}(z) = \delta_{a_i} + \varepsilon > \delta_{b_i} = \hat{u}_{ib_i}(z)$  if  $b_i \neq a_i$  and  $b_i \geq j_l$ . Hence, by fairness,  $b_i = a_i$  for all  $i \in C$ .

It remains to prove that  $u_{ib_i}(z) > u_{ia_i}(x)$  for all  $i \in C$ , i.e.,  $\varphi$  is manipulable at  $u$  by coalition  $C$ . From the above, we have  $a_i = b_i$  for all  $i \in C$ . Since  $\varphi$  is fair, we have  $(b, z) \in F(\hat{u}_C, u_{-C})$ . Now we have for all  $i \in C$  with  $b_i \neq 1$ ,

$$\hat{u}_{ib_i}(z) = \hat{u}_{ia_i}(z) = z_{ia_i} - y_{ia_i} + \varepsilon \geq z_{i1} - y_{i1} = \hat{u}_{i1}(z). \quad (7)$$

Because  $\delta_j = z_j - y_j$ , it follows from the above condition that  $\delta_{b_i} \geq \delta_1 - \varepsilon$  for  $i \in C$  with  $b_i \neq 1$ . Note that this inequality holds trivially if  $b_i = 1$  because  $\varepsilon > 0$ . Now this fact, the

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<sup>16</sup>Note that Beviá (2010)'s results do not allow for single-valued allocation rules whereas all our results hold any single-valued allocation rule (and Beviá's Theorem 2.1 does not have any implication for Lemma 5.

<sup>17</sup>Note that for all  $i \in C$ ,  $\hat{u}_{ia_i}(y) = \varepsilon$  and  $\hat{u}_{ij}(y) = 0$  for  $j \neq a_i$ .

definition of  $\delta_j$  and our choice of  $0 < \varepsilon < \tau$ ,  $\delta_1 \geq 0$  and  $a_i = b_i$  for all  $i \in C$ , yield for all  $i \in C$

$$\begin{aligned}
u_{ia_i}(x) &< u_{ia_i}(y) - \tau \\
&= u_{ib_i}(y) - \tau \\
&= v_{ib_i} + z_{b_i} - (z_{b_i} - y_{b_i}) - \tau \\
&= u_{ib_i}(z) - \delta_{b_i} - \tau \\
&\leq u_{ib_i}(z) - \delta_1 - (\tau - \varepsilon), \\
&< u_{ib_i}(z),
\end{aligned}$$

where the first inequality follows from  $u_{ia_i}(y) > u_{ia_i}(x) + \tau$ , the first equality from  $a_i = b_i$  for  $i \in C$ , the second inequality from  $-\delta_{b_i} \leq -(\delta_1 - \varepsilon)$ , and the last inequality from  $\delta_1 \geq 0$  and  $\tau > \varepsilon$ . Hence,  $u_{ia_i}(x) < u_{ib_i}(z)$  for all  $i \in C$ , which is the desired conclusion.  $\square$

The second result shows that the agent  $k$ -linked fair allocation rule cannot be manipulated by any coalition containing agent  $k$ . The intuition is as follows. If agent  $k$  can successfully manipulate the allocation rule, then by fairness agent  $k$  must be assigned a consumption bundle where the monetary compensation increases. Since each agent is linked to agent  $k$ , then each agent must be assigned a consumption bundle where the monetary compensation increases, because if this is not the case then fairness is violated at the new allocation. But then the budget must be exceeded. Hence, agent  $k$  cannot manipulate. The same intuition holds for any fair allocation rule choosing only agent  $k$ -linked fair allocations for some profile.

**Lemma 6.** Let  $\varphi$  be a fair and budget-balanced allocation rule,  $k \in N$  and  $u \in \mathcal{U}$ . If  $\varphi(u) \subseteq \psi^k(u)$ , then no coalition  $C \subseteq N$  containing agent  $k$  can manipulate  $\varphi$  at profile  $u \in \mathcal{U}$ .

*Proof.* Let  $C \subseteq N$  be such that  $k \in C$ . Suppose that  $\varphi$  is manipulable at profile  $u \in \mathcal{U}$  by coalition  $C$ . Then there is a profile  $(\hat{u}_C, u_{-C}) \in \mathcal{U}$  and two allocations  $(a, x) \in \varphi(u)$  and  $(b, y) \in \varphi(\hat{u}_C, u_{-C})$  such that  $u_{ib_i}(y) > u_{ia_i}(x)$  for all  $i \in C$ . Note that  $\varphi(u) \subseteq \psi^k(u)$  and  $(a, x) \in \psi^k(u)$ .

By fairness,  $u_{ia_i}(x) \geq u_{ib_i}(x)$  for all  $i \in C$ . Hence, for all  $i \in C$ ,  $u_{ib_i}(y) > u_{ib_i}(x)$  and  $y_{b_i} > x_{b_i}$ . Because  $(b, y)$  satisfies budget-balance, we must have  $C \subsetneq N$ . We distinguish two cases.

First, suppose  $\{b_i : i \in C\} = \{a_i : i \in C\}$ . Since  $k \in C$  and  $(a, x)$  is an agent  $k$ -linked fair allocation, there exists  $i \in N - C$  and  $j \in C$  such that  $i \rightarrow_{(a,x)} j$ . By  $j \in C$  and  $\{b_i : i \in C\} = \{a_i : i \in C\}$ , we have  $y_{a_j} > x_{a_j}$ . Now  $u_{ia_i}(x) = u_{ia_j}(x)$ , fairness, and monotonicity in money imply

$$u_{ib_i}(y) \geq u_{ia_j}(y) > u_{ia_j}(x) = u_{ia_i}(x) \geq u_{ib_i}(x).$$

Hence,  $y_{b_i} > x_{b_i}$ . Let  $C^1 = C \cup \{i \in N : i \rightarrow_{(a,x)} j \text{ for some } j \in C\}$ . Thus, we have  $y_{b_i} > x_{b_i}$  for all  $i \in C^1$  (and  $C \subsetneq C^1$ ).

Second, suppose  $\{b_i : i \in C\} \neq \{a_i : i \in C\}$ . Let  $i \in N - C$  be such that  $a_i \in \{b_i : i \in C\}$ . Then  $y_{a_i} > x_{a_i}$ , fairness, and monotonicity in money imply

$$u_{ib_i}(y) \geq u_{ia_i}(y) > u_{ia_i}(x) \geq u_{ib_i}(x).$$

Hence,  $y_{b_i} > x_{b_i}$ . Let  $C^1 = C \cup \{i \in N - C : a_i \in \{b_i : i \in C\}\}$ . Thus, we have  $y_{b_i} > x_{b_i}$  for all  $i \in C^1$  (and  $C \subsetneq C^1$ ).

Using the same arguments it follows for any  $l$  that (i) if  $\{b_i : i \in C^l\} = \{a_i : i \in C^l\}$ , then for each  $i \in N$  such that  $i \rightarrow_{(a,x)} j$  for some  $j \in C^l$ , we have  $y_{b_i} > x_{b_i}$ . Let  $C^{l+1} = C^l \cup \{i \in N : i \rightarrow_{(a,x)} j \text{ for some } j \in C^l\}$ ; and (ii) if  $\{b_i : i \in C^l\} \neq \{a_i : i \in C^l\}$ , then for each  $i \in N - C^l$  such that  $a_i \in \{b_i : i \in C^l\}$ , we have  $y_{b_i} > x_{b_i}$ . Let  $C^{l+1} = C^l \cup \{i \in N - C^l : a_i \in \{b_i : i \in C^l\}\}$ .

Because  $N$  is finite and  $(a, x)$  is agent  $k$ -linked, for some  $t$  we obtain  $C^t = N$  and  $y_{b_i} > x_{b_i}$  for all  $i \in C^t$ , which is contradiction to budget-balance of  $(b, y)$ . Hence,  $C$  cannot manipulate  $\varphi$  at profile  $u \in \mathcal{U}$ .  $\square$

The following theorem identifies all preference profiles  $u \in \mathcal{U}$  at which any fair and budget-balanced allocation rule is (coalitionally) non-manipulable.

**Theorem 6.** A fair and budget-balanced allocation rule  $\varphi$  is (coalitionally) non-manipulable at profile  $u \in \mathcal{U}$  if and only if  $N$  is the unique indifference component at profile  $u \in \mathcal{U}$  (i.e.,  $\mathcal{G}(u) = \{N\}$ ).

*Proof.* The “only if” part follows directly from Lemma 5 since there always is an isolated group unless  $N$  is the unique indifference component by Lemma 4. To prove the “if” part, note that if  $N$  is the unique indifference component, any  $(a, x) \in F(u)$  is agent  $i$ -linked for any  $i \in N$  by Lemma 1. Since  $\varphi(u) \subseteq F(u)$ , Lemma 6 implies that no coalition containing  $i \in N$  can manipulate  $\varphi$  at profile  $u \in \mathcal{U}$ . Hence,  $\varphi$  is both non-manipulable at profile  $u \in \mathcal{U}$  and coalitionally non-manipulable at profile  $u \in \mathcal{U}$ .  $\square$

Lemma 1 and Theorem 6 imply our first main result Theorem 1: a fair and budget-balanced allocation rule is non-manipulable at a profile if and only if all fair and budget-balanced allocation rules are non-manipulable at this profile. Furthermore, the same equivalence holds when considering coalitional non-manipulability instead of individual non-manipulability.

**Theorem 1.** Let  $\varphi$  and  $\psi$  be two arbitrary fair and budget-balanced allocation rules. Then  $\varphi$  is (coalitionally) non-manipulable at profile  $u \in \mathcal{U}$  if and only if  $\psi$  is (coalitionally) non-manipulable at profile  $u \in \mathcal{U}$ .

*Proof.* Follows directly from Lemma 1 and Theorem 6.  $\square$

Note that for any  $i \in N$ , there is a unique indifference component  $G \in \mathcal{G}(u)$  such that  $i \in G$  (where  $G = \{i\}$  is possible), i.e., any agent is included in exactly one indifference component at any profile  $u \in \mathcal{U}$ . Given this observation, we determine for any profile the precise number of agents and coalitions who *can* manipulate the agent  $k$ -linked fair

allocation rule. Specifically, we demonstrate that  $\psi^k$  can be manipulated by less than 50% of all coalitions at any profile.

**Theorem 7.** Let  $k \in N$  and  $u \in \mathcal{U}$ .

- (i) If  $k \in S \in \mathcal{G}(u)$ , then  $\psi^k$  is manipulable by exactly  $|N| - |S|$  agents and exactly  $2^{|N|-|S|} - 1$  coalitions at profile  $u \in \mathcal{U}$ .
- (ii)  $\psi^k$  is manipulable by less than 50% of all coalitions at any profile  $u \in \mathcal{U}$ .

*Proof.* To prove (i), note that since  $S$  is an indifference component, for all  $i \in S$  and all  $(a, x) \in \psi^k(u)$ , allocation  $(a, x)$  is agent  $i$ -linked. From Lemma 6 it then follows that no coalition containing agent  $i \in S$  can manipulate  $\psi^k$  at profile  $u \in \mathcal{U}$ . Thus, at most  $2^{|N|-|S|} - 1$  coalitions can manipulate  $\psi^k$  at profile  $u \in \mathcal{U}$ . Lemma 5 guarantees that this bound is tight, i.e., that exactly  $2^{|N|-|S|} - 1$  coalitions can manipulate  $\psi^k$  at profile  $u \in \mathcal{U}$ . Because there are exactly  $|N| - |S|$  non-empty singleton coalitions in the class of coalitions that can gain by manipulation, it follows that exactly  $|N| - |S|$  agents can manipulate  $\psi^k$  at profile  $u \in \mathcal{U}$ .

To prove (ii), note that  $|S| \geq 1$ . Because  $2^{|N|-|S|} \leq 2^{|N|-1}$  for any  $|S| \geq 1$ , it follows from (i) that  $\psi^k$  can be manipulated at profile  $u \in \mathcal{U}$  by at most  $2^{|N|-1} - 1$  coalitions. Since there are  $2^{|N|} - 1$  non-empty coalitions of  $N$  and  $2^{|N|} - 1 = 2(2^{|N|-1} - 1) + 1$ , less than 50% of all coalitions can manipulate  $\psi^k$  at profile  $u \in \mathcal{U}$ .  $\square$

Therefore, if the agent  $k$ -linked fair allocation rule is adopted, then in order to calculate the exact number of manipulating agents and coalitions at a given profile, one only needs to know the number of agents that are included in the indifference component containing agent  $k$ . Because indifference components are invariant with respect to the chosen fair allocation (Lemma 1) it is sufficient to find an arbitrary agent  $k$ -linked fair allocation at the given preference profile to find the exact number of manipulating agents and coalitions. This task can be achieved, for example by using the algorithm in Klijn (2000). Because this algorithm is polynomially bounded, this is not even computationally hard. An algorithm (inspired by Klijn, 2000) for calculating agent  $k$ -linked fair allocations is provided in Andersson, Ehlers and Svensson (2010).

The corollary below follows from the above results.

**Corollary 1.**

- (i)  $\psi^k$  cannot be manipulated by agent  $k$  at any profile  $u \in \mathcal{U}$ .
- (ii) For any two distinct agents  $i, j \in N$ , there exists no fair and budget-balanced allocation rule  $\varphi$  such that neither  $i$  nor  $j$  can manipulate  $\varphi$  at any profile  $u \in \mathcal{U}$ .

Note that Lemma 6 implies that the agent  $k$ -linked fair allocation rule cannot be manipulated by any coalition containing  $k$  at any profile. In particular, the agent  $k$ -linked fair allocation rule is not manipulable by agent  $k$  at any profile  $u \in \mathcal{U}$ , which is the first part of Corollary 1. The second part of Corollary 1 is easy to verify and left to the reader.

**Remark 1.** In a paper subsequent to this, Fujinaka and Wakayama (2011) similar results as ours regarding individual manipulation (possibilities): (a) Proposition 1 and Proposition 2 in Fujinaka and Wakayama (2011) are identical with Theorem 2 and Proposition 2, (b) Theorem 2 of Fujinaka and Wakayama (2011) is equivalent to (restricting attention to individual manipulation) Theorem 6 (using Theorem 2 and Proposition 2), and (c) Corollary 2 of Fujinaka and Wakayama (2011) is equivalent to Corollary 1 (using Lemma 5). There are two important differences between our paper and Fujinaka and Wakayama (2011): on the one hand we allow for multi-valued allocation rules whereas they only consider single-valued allocation rules; on the other hand we consider only quasi-linear utility functions whereas they consider general utility functions satisfying (i) *monotonicity in money*, i.e. for any  $x, y \in \mathbb{R}^M$ , if  $x_j > y_j$ , then  $u_{ij}(x) > u_{ij}(y)$  and (ii) *no infinite desirability in terms of money*, i.e. for any  $j, k \in M$  and any  $x \in \mathbb{R}^M$ , there exists  $y \in \mathbb{R}^M$  such that  $u_{ij}(x) = u_{ik}(y)$ .

## C Minimal Manipulability

First, we show that maximally linked fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules.

**Theorem 3.** Let  $\varphi$  be an arbitrary fair and budget-balanced allocation rule and let  $\phi^\kappa$  be a maximally linked fair allocation rule. Then:

- (i)  $\varphi$  is agents-counting-more manipulable than  $\phi^\kappa$ ; and
- (ii) if  $\phi^\kappa$  is agents-counting-more manipulable than  $\varphi$ , then  $\varphi$  is a maximally linked fair allocation rule.

*Proof.* First, we show (i). Let  $u \in \mathcal{U}$ . Suppose that  $(a, x) \in \varphi(u)$  and  $(b, y) \in \phi^\kappa(u)$ , and let  $N - G$  be a (possibly empty) isolated group with maximal cardinality at allocation  $(a, x) \in \varphi(u)$ . Then  $G$  is an indifference component at allocations  $(a, x)$  and  $(b, y)$  by Lemmas 1 and 4.

Note first that all agents in the isolated coalition  $N - G$  can manipulate  $\varphi$  by Lemma 5. Consequently, at least  $|N - G|$  agents can manipulate  $\varphi$ . Hence, to conclude the proof for (i) we need to show that at most  $|N - G|$  agents can manipulate  $\phi^\kappa$ .

Suppose that  $\kappa(u)$  belongs to the indifference component  $\hat{G} \subseteq \bar{G}(u)$ , and note that  $|\hat{G}| \geq |G|$  by construction of  $\phi^\kappa$ . Since  $\phi^\kappa(u) \subseteq \psi^k(u)$  for all  $k \in \hat{G}$ , it now follows from Lemma 6 that no agent  $k \in \hat{G}$  can manipulate  $\phi^\kappa$  at profile  $u \in \mathcal{U}$ . Thus, at most  $|N - \hat{G}|$  agents can manipulate  $\phi^\kappa$ . The conclusion of (i) then follows directly from the observation that  $|\hat{G}| \geq |G|$  implies  $|N - \hat{G}| \leq |N - G|$ .

For (ii), note that then we have to have  $|N - G| \leq |N - \hat{G}|$  and  $|G| \geq |\hat{G}|$ . Then  $G \subseteq \bar{G}(u)$ . Since  $u \in \mathcal{U}$  was arbitrary, now  $\varphi$  is a maximally linked fair allocation rule.  $\square$

Second, we show that maximally linked fair allocation rules are minimally coalitionally manipulable among all fair and budget-balanced allocation rules.

**Theorem 4.** Let  $\varphi$  be a fair and budget-balanced allocation rule and  $\phi^\kappa$  be a maximally linked fair allocation rule. Then:

- (i)  $\varphi$  is more coalitionally manipulable than  $\phi^\kappa$ ; and
- (ii) if  $\phi^\kappa$  is more coalitionally manipulable than  $\varphi$ , then  $\varphi$  is a maximally linked fair allocation rule.

*Proof.* First, we show (i). Let  $u \in \mathcal{U}$ . Suppose that  $(a, x) \in \varphi(u)$  and  $(b, y) \in \phi^\kappa(u)$ , and let  $N - G$  be the (possibly empty) isolated group with maximal cardinality at allocation  $(a, x) \in \varphi(u)$ . Then  $G$  is an indifference component at allocations  $(a, x)$  and  $(b, y)$  by Lemmas 1 and 4.

Note first that all coalitions in the isolated group  $N - G$  can manipulate  $\varphi$  by Lemma 5. Consequently, at least  $2^{|N-G|} - 1$  coalitions can manipulate  $\varphi$ . Hence, to conclude the proof we need to show that at most  $2^{|N-G|} - 1$  coalitions can manipulate  $\phi^\kappa$ . Suppose now that  $\kappa(u)$  belongs to the indifference component  $\hat{G} \subseteq \bar{G}(u)$ , and note that  $|\hat{G}| \geq |G|$  by construction of  $\phi^\kappa$ . It now follows from Lemma 6 and the construction of  $\phi^\kappa$  that at most  $2^{|\hat{G}|} - 1$  coalitions can manipulate  $\phi^\kappa$ . The conclusion of (i) then follows directly from the observation that  $|\hat{G}| \geq |G|$  implies  $2^{|\hat{G}|} - 1 \leq 2^{|N-G|} - 1$ .

For (ii), note that then we have to have  $2^{|\hat{G}|} - 1 \geq 2^{|N-G|} - 1$  and  $|G| \geq |\hat{G}|$ . Then  $G \subseteq \bar{G}(u)$ . Since  $u \in \mathcal{U}$  was arbitrary, now  $\varphi$  is a maximally linked fair allocation rule.  $\square$

Third, we show that linked fair allocation rules are agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules.

**Theorem 5.** Let  $\varphi$  be an arbitrary fair and budget-balanced allocation rule. Then:

- (i) there exists a selection  $\kappa : \mathcal{U} \rightarrow N$  such that  $\varphi$  is agents-inclusion-more manipulable than  $\phi^\kappa$ ; and
- (ii) if  $\phi^\kappa$  is agents-inclusion-more manipulable than  $\varphi$ , then for all  $u \in \mathcal{U}$ ,  $\varphi(u) \subseteq \phi^\kappa(u)$ .

*Proof.* We construct  $\kappa : \mathcal{U} \rightarrow N$  as follows: for all  $u \in \mathcal{U}$ , if for some  $k \in N$ ,  $\varphi(u) \subseteq \psi^k(u)$ , then we set  $\kappa(u) = k$ , and otherwise  $\kappa(u)$  can be arbitrary.

First, we show (i). Let  $u \in \mathcal{U}$ . If for all  $k \in N$ ,  $\varphi(u) \not\subseteq \psi^k(u)$ , then any agent  $i \in N$  belongs to an isolated group. Now by Lemma 5,  $P^\varphi(u) = N$ . Since  $\phi^\kappa(u) \subseteq \psi^{\kappa(u)}(u)$ , now by Lemma 6,  $P^{\phi^\kappa}(u) \subseteq N - \{\kappa(u)\}$ . Hence,  $P^\varphi(u) \supseteq P^{\phi^\kappa}(u)$ .

If for some  $k \in N$ ,  $\varphi(u) \subseteq \psi^k(u)$ , then by construction of  $\kappa$ , we also have  $\phi^\kappa(u) \subseteq \psi^k(u)$ . But now we have  $P^\varphi(u) \supseteq P^{\phi^\kappa}(u)$ .

Hence, for all  $u \in \mathcal{U}$ ,  $P^\varphi(u) \supseteq P^{\phi^\kappa}(u)$ , and  $\varphi$  is agents-inclusion-more manipulable than  $\phi^\kappa$ , the desired conclusion for (i).

For (ii), note that then we have to have  $P^{\phi^\kappa}(u) \supseteq P^\varphi(u)$ . Then for  $k = \kappa(u)$ , we have  $\phi^\kappa(u) = \psi^k(u)$  and, by  $k \notin P^{\phi^\kappa}(u)$ ,  $\varphi(u) \subseteq \psi^k(u)$ , the desired conclusion.  $\square$



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