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Sets in Excess Demand in Simple Ascending Auctions with Unit-Demand Bidders

T. Andersson · C. Andersson · A.J.J. Talman

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Abstract This paper analyzes the problem of selecting a set of items whose prices are to be updated in the next iteration in so called simple ascending auctions with unit-demand bidders. A family of sets called "sets in excess demand" is introduced, and the main results demonstrate that a simple ascending auction always terminates at the minimum Walrasian equilibrium prices if and only if the selection belongs to this family. The paper also specifies a number of properties of the family of sets in excess demand and relate previously proposed selections to it.

Keywords: Multi-item auctions; unit-demand; excess demand; algorithms.

1 Introduction

Economies with indivisible items and money have received considerable attention in the literature since the pioneering work of Shapley and Shubik (1972). They did not only prove the existence of a Walrasian equilibrium but also that the set of Walrasian price vectors forms a complete lattice. Consequently, there exist unique minimum and maximum Walrasian equilibrium price vectors. The existence result has later been refined and generalized by e.g. Svensson (1983), Demange and Gale (1985) and Alkan et al. (1991). The lattice property has been demonstrated to play a key role when designing strategy-proof mechanisms. For example, Leonard (1983), Demange and Gale (1985) and Andersson and Svensson (2008) demonstrate that by regarding the minimum Walrasian price equilibrium as a direct mechanism for allocating the indivisible items no agent can gain by strategic misrepresentation.

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An obvious field of application for Walrasian pricing mechanisms is auction design. As emphasized by e.g. Ausubel (2004) and Perry and Reny (2005), dynamic auction mechanisms are overwhelmingly more prevalent than their direct counterparts (i.e. sealed-bid auctions) because bidders often fear complete revelation of information. Consequently, a variety of different dynamic auction mechanisms (ascending, descending, bisection etc.) has been designed to handle a number of different prerequisites.

The most elementary type of ascending auction¹ is what we will refer to as a Simple Ascending Auction with unit-demand bidders (SAA, henceforth; see Algorithm 1). An SAA is an ascending auction with bidders that wish to acquire at most one item.² In the special case where only one item is for sale, the basic rules of the auction are rather straightforward. In a multi-item setting, things get more complicated. For example, which termination criteria should be used, and which rule should be applied for selecting the items whose prices are to be updated in the next iteration given that the termination criteria fails to be satisfied? In almost any investigated SAA, so called overdemanded sets of items play a key role in answering these types of questions, see e.g. Demange et al. (1986), Mo et al. (1988), Sankaran (1994) or Sun and Yang (2009). In this context, a set of items is overdemanded, at a given price vector, if the number of bidders demanding only items in the set is greater than the number of items in the set. It is natural that this type of set plays a key role since it is known from a famous theorem by Hall (1935) that a necessary requirement for reaching a Walrasian equilibrium is that all overdemanded sets of items are eliminated. However, it is also well-known that it is in general impossible to reach the minimum Walrasian equilibrium prices by only using information regarding overdemanded sets of items (see e.g. Example 1). Thus, only specific subsets of the family of overdemanded sets can be considered. In for example Demange et al. (1986) and Sun and Yang (2009), the solution is to restrict the attention to the family of minimal overdemanded sets. This family consists of all overdemanded sets with the property that none of its proper subsets is overdemanded. Sankaran (1994), on the other hand, handles the situation by identifying a unique overdemanded set based on the Ford-Fulkerson method (Ford and Fulkerson, 1956). This set is not necessarily minimal overdemanded. A similar approach is adopted in Mo et al. (1988) where a variant of the Ford-Fulkerson method described in Gale (1960) is used (the Gale-procedure, henceforth) to identify a proper overdemanded set. All these selections have one thing in common. Namely, if the selection rule in the SAA is based on any one of them, the auction will converge to the minimum Walrasian equilibrium prices.

Knowing that there exist selections that guarantee that an SAA converges to the minimum Walrasian prices, this paper gives an answer to the following question: Is it possible to characterize the entire class of selections that make an SAA to terminate at the minimum Walrasian equilibrium prices? Apart from the obvious academic interest, this finding is important for a number of reasons. First, for multi-item auctions that do not contain a large number of items (say less than 20), any modern computer can make an exhaustive search through all subsets of the set of items in a matter of seconds to identify sets with specific properties. This means that, for "small" multi-item auctions, it is not necessary that the procedure is polynomial as e.g. the Ford-Fulkerson method and the Gale-procedure. It may for example be more important that convergence occurs on average after a fewer number of iterations. Hence, by characterizing all possible selections, it may be possible to identify e.g. "fast" rules that not necessarily are polynomial. Second, a complete characterization for

¹ This paper focuses on ascending auctions but for each result presented in this paper there are descending auction counterparts.

² Walrasian auction mechanisms in a multi-demand setting have been considered by e.g. Gul and Stacchetti (2000) and Ausubel (2006).

the simplest possible type of ascending auction (i.e., SAA) may give important insights that may be helpful in improving and analyzing more complex multi-item auctions, for example auctions with multi-demand bidders.

To provide a complete characterization, this paper considers a subset of the family of overdemanding sets of items consisting of all "sets in excess demand". Formally, an overdemanding set of items S is in excess demand if the number of items in each subset T of S is strictly smaller than the number of bidders that demand some item in T and in addition only demand items in S . Sets in excess demand have a number of attractive properties. First, if a set is minimal overdemanding, at a given price vector, then it is also in excess demand. Hence, the family of minimal overdemanding sets is a subset of the family of sets in excess demand (Theorem 2). Second, if two sets are in excess demand, at a given price vector, so is their union (Theorem 1). Thus, by taking the union of all sets in excess demand, at given prices, it is possible to construct a unique set in excess demand with a largest cardinality. We demonstrate that the selections proposed by Mo et al. (1988) and Sankaran (1994) in fact are given by this unique largest set (Theorem 3).

Because the selections in Mo et al. (1988), Sankaran (1994) and Demange et al. (1986) are based on sets of items that belong to specific subsets of the family of sets in excess demand, this paper establishes a common framework and a link between these papers. In addition, the paper specifies an ascending auction mechanism, called the Excess Demand Ascending Auction (EDAA), where in every iteration the prices for items belonging to one of the sets in excess demand are updated. It is demonstrated that an SAA always converges to the minimum Walrasian equilibrium prices if and only if it is an EDAA.

This section ends by describing the similarities between this paper and the unpublished manuscript by Mo et al. (1988).³ The point of departure in Mo et al. (1988) is the claim in Demange et al. (1986) that it is possible to compute minimal overdemanding sets by using the Gale-procedure, and the main result establishes that the output of the extended Gale-procedure is in fact not a minimal over-demanding set but instead a so-called "purely overdemanding set" with maximal cardinality. Because (i) their definition of a purely overdemanding set is equivalent to our definition of excess demand, and (ii) the Gale-procedure is a special case of the Ford–Fulkerson method, their Proposition 2 is identical to our Theorem 3. In addition, the observation in their Proposition 1 that the union of two purely overdemanding sets results in a purely overdemanding set is identical to our Theorem 1. Finally, Mo et al. (1988) define a modified version of the auction algorithm in Demange et al. (1986) which uses the idea that price increments should be based on the pure overdemanding set with maximal cardinality. This algorithm is a special case of our EDAA. We also remark that none of our characterization results (Theorems 5 and 6) are included or discussed in Mo et al. (1988), i.e., their focus is on the understanding and the output of the Gale-procedure and its relation to the minimum Walrasian equilibrium prices whereas our main results completely characterize the class of simple ascending auctions that always terminate at the minimum Walrasian equilibrium prices.

The paper is organized as follows. Section 2 introduces the economic model. Sets in excess demand are defined and analyzed in Section 3. Section 4 provides the characterization results. Some of the proofs are delegated to the Appendix.

³ The paper by Mo et al. (1988) neither has been published nor is available on the Internet. For this reason, we regard our article as an independent work. We are extremely grateful to Al Roth for providing us with a copy of the paper of Mo et al. (1988) in April, 2011, after we had completed an earlier version of this paper.

2 The Model

The finite set of bidders is denoted by B and the finite set of items is denoted by I . Each item $i \in I$ has a price p_i and a reservation price r_i . For simplicity and without loss of generality the reservation price of each item is set equal to zero, i.e. $r = 0$. The value of item $i \in I$ to bidder $b \in B$ is given by the number v_{bi} . These values are assumed to be integers since in reality no bidder can specify a monetary value more closely than to the nearest dollar or cent. For each bidder there is a so-called null-item, denoted by 0, and having value, i.e., $v_{b0} = 0$ for all $b \in B$. It is also assumed that a null-item has always price zero, denoted by $p_0 = 0$. Prices of the items are gathered in a vector p . A price vector p is said to be feasible if $p_0 = 0$ and $p_i \geq 0$ for all $i \in I$. For notational simplicity we let $I^* = I \cup \{0\}$ and $I^+(p) = \{i \in I^* : p_i > 0\}$. The demand correspondence for bidder $b \in B$ at price vector p is defined by:

$$D_b(p) = \{i \in I^* : v_{bi} - p_i \geq v_{bj} - p_j \text{ for all } j \in I^*\}.$$

A feasible price vector p is said to be a Walrasian equilibrium price vector if there is an assignment $x : B \mapsto I^*$ such that $x_b \in D_b(p)$ for all $b \in B$ and if $b' \neq b$ and $x_b = x_{b'}$ then $x_b = 0$, i.e. each bidder is assigned an item from his demand set and if two bidders are assigned the same item then both bidders are assigned the null-item. The pair (p, x) is a Walrasian equilibrium if p is a Walrasian equilibrium price vector and $x_b \neq i$ for all $b \in B$ implies $p_i = 0$, i.e. when an item is not assigned to some bidder, then its price equals the zero reservation price. As demonstrated by Shapley and Shubik (1972) the set of competitive price vectors is non-empty and forms a complete lattice. Thus, the existence of a unique minimum Walrasian equilibrium price vector p^{\min} is guaranteed.

This section ends by defining the simplest possible ascending auction format for unit-demand bidders and multiple items. This type of auction is called a Simple Ascending Auction (SAA, henceforth).

Algorithm 1 (Simple Ascending Auction, SAA) Initialize the price vector to the reservation prices, $p^0 := r$. For each iteration $t = 0, 1, 2, \dots$:

1. Collect the demand sets $D_b(p^t)$ of every bidder $b \in B$.
2. If there is no overdemanded set of items at p^t (Definition 1), the algorithm is terminated.
3. Choose an overdemanded set of items $S^t \subseteq I$.
4. Compute the updated price vector p^{t+1} whose elements are given by:

$$p_i^{t+1} = \begin{cases} p_i^t + 1 & \text{if } i \in S^t, \\ p_i^t & \text{otherwise.} \end{cases}$$

5. Set $t := t + 1$ and start a new iteration from Step 1.

3 Sets in Excess Demand

The focus in this section is the set of items S^t to be chosen in Step 3 of Algorithm 1 (SAA). More explicitly, we define a class of sets, called "sets in excess demand", that belongs to the family of overdemanded sets, at given prices, and analyze it in detail. The following section then demonstrates that an SAA always converges to the minimum equilibrium prices if and only if S^t belongs to this class in all steps of the algorithm.

Let now p be a given feasible price vector, i.e., $p_0 = 0$ and $p_i \geq 0$ for all $i \in I$. The bidders that *only* demand items in the set $S \subseteq I$ at prices p , and the bidders that demand *some* item in the set $S \subseteq I$ at prices p are collected in the sets

$$\begin{aligned} O(S, p) &= \{b \in B : D_b(p) \subseteq S\}, \\ U(S, p) &= \{b \in B : D_b(p) \cap S \neq \emptyset\}, \end{aligned}$$

respectively. By definition it holds that $O(S, p) \subseteq U(S, p)$. Consequently, $U(S, p) \cap O(S, p) = O(S, p)$.

Definition 1 A set of items S is overdemanded at prices p if $S \subseteq I$ and $|O(S, p)| > |S|$.

Definition 2 A set of items S is weakly underdemanded at prices p if $S \subseteq I^+(p)$ and $|U(S, p)| \leq |S|$.⁴

Note that Definition 2 requires that S is a subset of $I^+(p)$. Thus, no set S where the price of some item equals the reservation price (zero) can be weakly underdemanded. Consequently, at the reservation prices, no set of items is weakly underdemanded.

Definition 3 A set of items S is in excess demand⁵ at prices p if $S \subseteq I$ and:

$$|U(T, p) \cap O(S, p)| > |T| \text{ for each non-empty } T \subseteq S. \quad (1)$$

A set of items S is therefore in excess demand if it is overdemanded and the number of items in each proper subset T is strictly smaller than the number of bidders that demand some item in T and in addition only demand items in S . The family of sets in excess demand at prices p is given by:

$$ED(p) = \{S \subseteq I : |U(T, p) \cap O(S, p)| > |T| \text{ for each non-empty } T \subseteq S\}.$$

The following result shows that the union of two sets in excess demand is also a set in excess demand.

Theorem 1 If $S \in ED(p)$ and $T \in ED(p)$, then $S \cup T \in ED(p)$.

Proof By Definition 3, we need to prove that the following condition is satisfied:

$$|U(K, p) \cap O(S \cup T, p)| > |K| \text{ for each non-empty } K \subseteq S \cup T. \quad (2)$$

Note first that if $b \in O(S, p)$ or $b \in O(T, p)$ then $b \in O(S \cup T, p)$. This together with the observation that there may exist a bidder b with $b \notin O(S, p) \cup O(T, p)$ but $b \in O(S \cup T, p)$ gives: $(U(L, p) \cap O(R, p)) \subseteq (U(L, p) \cap O(S \cup T, p))$ for each $L \subseteq R$ and $R \in \{S, T\}$. Consequently:

$$|U(L, p) \cap O(S \cup T, p)| \geq |U(L, p) \cap O(R, p)| > |L| \text{ for each } L \subseteq R \text{ and } R \in \{S, T\}.$$

The last inequality follows from Definition 3 since $S \in ED(p)$ and $T \in ED(p)$ by assumption. This implies that condition (2) holds if $K \subseteq S$ or $K \subseteq T$.

⁴ This definition is from Mishra and Talman (2010, Definition 3) and it differs slightly from the definition of underdemand in Sotomayor (2002) in the sense that Sotomayor (2002) assumes that there is a dummy bidder that can be allocated more than one item and in addition demands each item at the reservation prices.

⁵ In Mo et al. (1988) such a set of items is called a purely overdemanded set.

It remains to prove that condition (2) holds when $K \subseteq S \cup T$ and both $K \not\subseteq S$ and $K \not\subseteq T$. Let $A = K \cap S$ and $C = K \setminus S$, then both A and C are nonempty, $A \cap C = \emptyset$, and $K = A \cup C$. Since $A \subseteq S$ and $C \subseteq T \setminus S$ it holds that:

$$(U(A, p) \cap O(S, p)) \cap (U(C, p) \cap O(T, p)) = \emptyset.$$

Moreover, because $S \in ED(p)$ and $T \in ED(p)$ by assumption, it follows from Definition 3 that both $|U(A, p) \cap O(S, p)| > |A|$ and $|U(C, p) \cap O(T, p)| > |C|$. These facts, together with the observation that there may exist a bidder b with $b \notin O(S, p) \cup O(T, p)$ but $b \in O(S \cup T, p)$, give:

$$\begin{aligned} |U(K, p) \cap O(S \cup T, p)| &= |U(A \cup C, p) \cap O(S \cup T, p)| \\ &\geq |U(A \cup C, p) \cap (O(S, p) \cup O(T, p))| \\ &\geq |(U(A, p) \cap O(S, p)) \cup (U(C, p) \cap O(T, p))| \\ &= |U(A, p) \cap O(S, p)| + |U(C, p) \cap O(T, p)| \\ &> |A| + |C| \\ &= |A \cup C| \\ &= |K|, \end{aligned}$$

which concludes the proof.

Corollary 1 If $ED(p) \neq \emptyset$, then there is a unique set $S^* \in ED(p)$ with maximum cardinality.

The remaining part of this section demonstrates that (i) the notion of excess demand is a weaker notion than minimal overdemand introduced by Demange et al. (1986), and (ii) the selection of an overdemanded set based on the Ford-Fulkerson (or its special case: the Gale-procedure) method corresponds to the unique set in excess demand with maximum cardinality.

3.1 Minimal Overdemand

An overdemanded set of items is minimal overdemanded if none of its proper subsets is overdemanded. Let p be a given feasible price vector.

Definition 4 A set of items S is minimal overdemanded at prices p if S is overdemanded at prices p and:

$$|T| \geq |O(T, p)| \text{ for all } T \subset S.$$

The family of minimal overdemanded sets of items at prices p is given by:

$$MOD(p) = \{S \subseteq I : |O(S, p)| > |S| \text{ and } |T| \geq |O(T, p)| \text{ for all } T \subset S\}.$$

The following result establishes that the notion of sets in excess demand is a weaker notion than minimal overdemand. This result can be traced back to van der Laan and Yang (2008, Lemma 3.2) where it is demonstrated that any minimal overdemanded set satisfies the property of excess demand (both when bidders are financially constrained and when they are not). Note, however, that they do not analyze sets in excess demand, their lemma is only introduced to facilitate some of their proofs.

Theorem 2 If $S \in MOD(p)$, then $S \in ED(p)$.

Proof Suppose that $S \in MOD(p)$ and $S \notin ED(p)$. $S \in MOD(p)$ implies that $|O(S, p)| > |S|$ and $|O(T, p)| \leq |T|$ for all $T \subset S$. $S \notin ED(p)$ implies that there exists a non-empty $K \subset S$ such that:

$$|U(K, p) \cap O(S, p)| \leq |K|.$$

Moreover, K is a proper subset of S , since $|O(S, p)| > |S|$ and $U(S, p) \cap O(S, p) = O(S, p)$. Let $S' = S \setminus K$ and note that S' is a non-empty and proper subset of S . It then follows that $O(S', p) = O(S, p) \setminus (U(K, p) \cap O(S, p))$. Thus:

$$\begin{aligned} |O(S', p)| &= |O(S, p) \setminus (U(K, p) \cap O(S, p))| \\ &\geq |O(S, p)| - |U(K, p) \cap O(S, p)| \\ &> |S| - |K| \\ &= |S'|. \end{aligned}$$

But then S' is a proper overdemanded subset of S at prices p , which contradicts that $S \in MOD(p)$.

Note that the converse of Theorem 2 is not true, i.e., a set in excess demand need not be a minimal overdemanded set (see e.g. Example 2 in Section 4).

3.2 The Ford-Fulkerson Selection

The Ford and Fulkerson (1956) method is a classical method for network flow problems, that can be used to find a feasible assignment of maximum cardinality. Let p be again a given feasible price vector. By feasible assignment we mean a set $X(p) \subseteq \{(b, i) : b \in B, i \in I^*\}$ of bidder-item pairs at prices p such that $i \in D_b(p)$ for all $(b, i) \in X(p)$ and such that any $b \in B$ and $i \in I$ is part of at most one pair from $X(p)$. No assumption is made on the cardinality of $X(p)$, but if $|X(p)| = |B|$ each bidder can be assigned an item from his demand set and p is therefore a Walrasian equilibrium price vector.

Starting from $X(p) = \emptyset$, the Ford and Fulkerson (1956) method iteratively updates $X(p)$ based on augmenting paths. An augmenting path with respect to $X(p)$ is a sequence of the form $\mathcal{P} = (b_0, i_0, \dots, b_n, i_n)$ such that $(b_j, i_{j-1}) \in X(p)$ for $j = 1, 2, \dots, n$, $i_j \in D_{b_j}(p)$ for $j = 0, 1, \dots, n$, and $i_j \neq 0$ for $j \neq n$. This characterization corresponds to a bipartite matching problem modified to account for the null-item.

Algorithm 2 (Ford-Fulkerson) Initialize the feasible assignment to the empty set, $X(p) := \emptyset$. For each iteration:

1. Find an augmenting path $\mathcal{P} = (b_0, i_0, \dots, b_n, i_n)$ with respect to $X(p)$, e.g. using Algorithm 3. If no augmenting path exists, then $X(p)$ is a maximal feasible assignment and terminate the algorithm.
2. Augment the assignment along \mathcal{P} :

$$X(p) := (X(p) \setminus \{(b_j, i_{j-1}) : j \in \{1, 2, \dots, n\}\}) \cup \{(b_j, i_j) : j \in \{0, 1, \dots, n\}\}.$$

3. Start a new iteration from Step 1.

The augmentation in Step 2 of Algorithm 2 increases the cardinality of $X(p)$ by one. Thus $|X(p)|$ is strictly increasing as long as there is an augmenting path.

Algorithm 3 (Breadth-first Search for an Augmenting Path) Given a feasible assignment $X(p)$, let the initial set of bidders be defined by:

$$B_0 = \{b \in B : (b, i) \notin X(p) \text{ for all } i \in I^*\}, \quad (3)$$

and label any $b \in B_0$ with s . All other bidders and items are initially unlabeled. Introduce the iteration counter n . For each iteration $n = 0, 1, \dots$:

1. Define the set of items:

$$I_n = \{i \in I^* \setminus \cup_{j=0}^{n-1} I_j : (b, i) \notin X(p) \text{ and } i \in D_b(p) \text{ for some } b \in B_n\}. \quad (4)$$

Label all $i \in I_n$ with a respective b .

2. If $0 \in I_n$ or there is an $i \in I_n$ such that $(b, i) \notin X(p)$ for all $b \in B$, a shortest augmenting path has been found and terminate the algorithm.
3. If $I_n = \emptyset$, no augmenting path exists and terminate the algorithm.
4. Define the set of bidders:

$$B_{n+1} = \{b \in B \setminus \cup_{j=0}^n B_j : (b, i) \in X(p) \text{ for some } i \in I_n\}. \quad (5)$$

Label all $b \in B_{n+1}$ with a respective i .

5. Set $n := n + 1$ and start a new iteration from Step 1.

In case Algorithm 3 terminates with $I_n = \emptyset$, no augmenting path exists and the assignment $X(p)$ is feasible and has maximum cardinality. Otherwise a shortest augmenting path $(b_0, i_0, \dots, b_n, i_n)$, with $b_j \in B_j$ and $i_j \in I_j$ for $j = 0, 1, \dots, n$, with respect to $X(p)$ can be found by backtracking the assigned labels starting from 0 if $0 \in I_n$, or from any $i \in I_n$ such that $(b, i) \notin X(p)$ for all $b \in B$, until a bidder with label s is encountered. The breadth-first search is guaranteed to find a shortest augmenting path if one exists. Several possible labels may exist in Steps 1 and 4 of Algorithm 3, and there may be more than one shortest augmenting path. The Ford-Fulkerson method converges regardless of the path and labels chosen.

Theorem 3 Let $S = \cup_{j=0}^n I_j$ be the set of labeled items upon termination of Algorithm 3 in the last Ford-Fulkerson iteration at prices p , then S is equal to the unique set S^* in excess demand having maximum cardinality at prices p .

Proof See the Appendix.

4 The Excess Demand Ascending Auction

Since we are particularly interested in simple ascending auctions where the selection in Step 3 is made from the family of sets in excess demand at prices p^t , we next define an ascending auction called the Excess Demand Ascending Auction (EDAA, henceforth).

Algorithm 4 (Excess Demand Ascending Auction, EDAA) The Excess Demand Ascending Auction is a Simple Ascending Auction where $S^t \in ED(p^t)$ in Step 3 of Algorithm 1.

We observe from Theorem 2 that the algorithm described in Demange et al. (1986) is a special case of Algorithm 4, since price increments in this algorithm are based on minimal overdemanded sets. We also note that the modifications of the algorithm in Demange et al. (1986) based on the extended Gale-procedure proposed by Mo et al. (1988) and the Ford-Fulkerson method proposed by Sankaran (1994) are special cases of Algorithm 4. This

is because the selections in Mo et al. (1988) and Sankaran (1994) is the set described in Theorem 3.⁶

It is well-known (see e.g. Demange et al., 1986) that Algorithm 1 can identify the minimum Walrasian equilibrium price vector p^{\min} for proper selections in Step 3. To define the whole class of selections that guarantee that Algorithm 1 always terminates at p^{\min} , weakly underdemanded sets of items play an important role. The family of weakly underdemanded sets of items at prices p is denoted by $WUD(p) = \{S \subseteq I^+(p) : |U(S, p)| \leq |S|\}$, and the reason for its importance is the following theorem from Mishra and Talman (2010).

Theorem 4 A price vector p equals p^{\min} if and only if $OD(p) = \emptyset$ and $WUD(p) = \emptyset$.

The main insight from the above theorem is that if the aim is to identify an SAA that converges to the minimum Walrasian equilibrium price vector, one needs to specify rules for the price increments which guarantee that the family of weakly underdemanded sets remains empty. The next result demonstrates that by defining price increments based on sets in excess demand, the family of weakly underdemanded sets will always stay empty.

Lemma 1 Let p be a feasible vector of prices. Suppose that $S \in ED(p)$ and $WUD(p) = \emptyset$, and let:

$$\bar{p}_i = \begin{cases} p_i + 1 & \text{if } i \in S, \\ p_i & \text{otherwise.} \end{cases} \quad (6)$$

Then $WUD(\bar{p}) = \emptyset$.

Proof We need to show that for an arbitrary $T \subseteq I^+(\bar{p})$ it holds that:

$$|U(T, \bar{p})| > |T|. \quad (7)$$

Three different cases are considered.

Case (i) $T \subseteq S$. We first make three observations. First, $(U(T, \bar{p}) \cap O(S, p)) \subseteq U(T, \bar{p})$ because $O(S, p) \subseteq B$. Second, $D_b(p) \subseteq D_b(\bar{p})$ for all $b \in O(S, p)$ by construction of \bar{p} . Consequently, if $b \in U(T, p) \cap O(S, p)$, then $b \in U(T, \bar{p}) \cap O(S, p)$, implying that $|U(T, \bar{p}) \cap O(S, p)| \geq |U(T, p) \cap O(S, p)|$. Third, $|U(T, p) \cap O(S, p)| > |T|$ since $S \in ED(p)$. From these three observations, we conclude:

$$|U(T, \bar{p})| \geq |U(T, \bar{p}) \cap O(S, p)| \geq |U(T, p) \cap O(S, p)| > |T|,$$

which demonstrates that condition (7) is satisfied when $T \subseteq S$.

Case (ii) $T \subseteq I^+(p) \setminus S$. Since $WUD(p) = \emptyset$ and $T \subseteq I^+(p)$ it follows that $|U(T, p)| > |T|$. Moreover, by construction of \bar{p} , an item $i \in T$ belongs to $D_b(\bar{p})$ if it belongs to $D_b(p)$, and therefore $U(T, p) \subseteq U(T, \bar{p})$. Hence:

$$|U(T, \bar{p})| \geq |U(T, p)| > |T|,$$

i.e., condition (7) is satisfied when $T \subseteq I^+(p) \setminus S$.

Case (iii) $T = A \cup C$ where $\emptyset \neq A \subseteq S$ and $\emptyset \neq C \subseteq I^+(p) \setminus S$. Clearly, $A \cap C = \emptyset$. By construction of \bar{p} it holds that if $i \in C$ belongs to $D_b(p)$ then it also belongs to $D_b(\bar{p})$. Thus, $U(C, p) \subseteq U(C, \bar{p})$, and as a consequence:

$$|U(T, \bar{p})| = |U(A \cup C, \bar{p})| = |U(A, \bar{p}) \cup U(C, \bar{p})| \geq |U(A, \bar{p}) \cup U(C, p)|. \quad (8)$$

⁶ The algorithms in Mo et al. (1988) and Sankaran (1994) are terminated if the assignment $X(p)$ of maximal cardinality (identified by Algorithm 2) has the property that $|X(p)| = |B|$. This termination criterium may of course also be used in Algorithms 1 and 4.

Because, by construction of \bar{p} , $\bar{p}_i > p_i$ for $i \in A$ and $\bar{p}_i = p_i$ for $i \in C$, it follows that $U(A, \bar{p}) \cap U(C, p) = \emptyset$, and therefore:

$$|U(A, \bar{p}) \cup U(C, p)| = |U(A, \bar{p})| + |U(C, p)|. \quad (9)$$

Since $A \subseteq S$ it follows from Case (i) that $|U(A, \bar{p})| > |A|$. Moreover, because $WUD(p) = \emptyset$ and $C \subseteq I^+(p)$ it follows that $|U(C, p)| > |C|$. These observations together with (8) and (9) yield:

$$|U(T, \bar{p})| > |A| + |C| = |A \cup C| = |T|,$$

which concludes the proof.

A direct consequence of the above lemma is the following convergence result.

Theorem 5 The Excess Demand Ascending Auction (EDAA) always converges to the unique minimum Walrasian price equilibrium p^{\min} in a finite number of iterations.

Proof The EDAA starts with reservation price vector r . By definition it holds that $WUD(r) = \emptyset$. From Lemma 1 and the price increases prescribed by Step 4 of Algorithm 4 it follows that $WUD(p^t) = \emptyset$ at any iteration t . Moreover, at any iteration t it holds that $ED(p^t) \neq \emptyset$ as long as there exists at least one overdemanded set of items at p^t . Now the proof follows directly from Theorem 4 and the observation that as soon as the price of an item becomes sufficiently large, the item cannot belong to any overdemanded set.

As demonstrated in Theorem 5, EDAA always converges to the minimum Walrasian equilibrium prices p^{\min} . A related problem is of course to investigate if this is the only SAA that *always* terminates at p^{\min} , i.e., is EDAA a complete characterization of the class of SAA that identifies the minimum Walrasian equilibrium prices. The next result demonstrates that this indeed is the case. Before providing this result, we first demonstrate, using a simple example, that unless the family of weakly underdemanded sets stays empty throughout the whole ascending price adjustment procedure, the process need not converge to the minimum Walrasian equilibrium prices. In other words, the selection in Step 3 of Algorithm 1 must guarantee that $WUD(p^t)$ stays empty in each iteration otherwise it will be impossible to reach the minimum Walrasian equilibrium prices for all possible valuation profiles.

Example 1 Let $B = \{1, 2, 3\}$ and $I = \{1, 2\}$. The values v_{bi} , $b \in B$, $i \in I$, are given by the matrix:

$$\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ v_{31} & v_{32} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \quad (10)$$

At the reservation prices $r = (0, 0)$, the bidders' demand sets are $D_b(r) = \{2\}$ for all $b \in B$. Hence, $ED(r) = MOD(r) = \{\{2\}\}$ and $OD(r) = \{\{2\}, \{1, 2\}\}$. Suppose now that the price increments in Step 3 of Algorithm 1 are based on the overdemanded set $\{1, 2\} \notin ED(r)$. Then $p^1 = (0, 1, 1)$ and, as a consequence, $WUD(p^1) = \{1\}$. Because $p^{\min} = (0, 1, 0)$, the minimum Walrasian equilibrium prices can not be reached in such an ascending process. \square

Theorem 6 A Simple Ascending Auction (SAA) always converges to the minimum Walrasian Equilibrium Prices if and only if it is an Excess Demand Ascending Auction (EDAA).

Proof The "if" part follows from Theorem 5. To prove the "only if" part, recall from Example 1 that the selection of the set $S^t \subseteq I$ in Step 3 at iteration t of Algorithm 1 must guarantee that $WUD(p^{t+1}) = \emptyset$ otherwise the process need not terminate at p^{\min} . Hence, it remains to show that if $S^t \notin ED(p^t)$ then $WUD(p^{t+1}) \neq \emptyset$.

If $S^t \notin ED(p^t)$, then $|U(T, p^t) \cap O(S^t, p^t)| \leq |T|$ for some $T \subseteq S^t$ by Definition 3. We start by making two key observations. First, if $b \notin U(T, p^t) \cap O(S^t, p^t)$, then $D_b(p^{t+1}) \cap T = \emptyset$. To see this, suppose that $b \notin U(T, p^t) \cap O(S^t, p^t)$ meaning that there is some item $i \in D_b(p^t)$ that do not belong to S^t . But then the conclusion follows, since $p_i^{t+1} = p_i^t$ whereas $p_j^{t+1} = p_j^t + 1$ for all $j \in T$. Second, if $b \in U(T, p^t) \cap O(S^t, p^t)$, then $D_b(p^{t+1}) \cap T \neq \emptyset$. This follows from the construction (6), $T \subseteq S^t$, and the definitions of $U(\cdot)$ and $O(\cdot)$.

From the above two observations, we conclude that $D_b(p^{t+1}) \cap T \neq \emptyset$ if and only if $b \in U(T, p^t) \cap O(S^t, p^t)$. But then $U(T, p^{t+1}) = U(T, p^t) \cap O(S^t, p^t)$. Using this fact and $|U(T, p^t) \cap O(S^t, p^t)| \leq |T|$ give:

$$|U(T, p^{t+1})| = |U(T, p^t) \cap O(S^t, p^t)| \leq |T|,$$

which demonstrates that $T \in WUD(p^{t+1})$. Hence, $WUD(p^{t+1}) \neq \emptyset$.

We end this paper by providing an example that illustrates all possible paths from the reservation prices to the minimum Walrasian equilibrium prices that can be obtained by SAA. By Theorem 6, all these paths are provided by EDAA.

Example 2 Let $B = \{1, 2, 3, 4, 5\}$ and $I = \{1, 2, 3\}$. The values v_{bi} , $b \in B$, $i \in I$, are given by the matrix:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \\ v_{41} & v_{42} & v_{43} \\ v_{51} & v_{52} & v_{53} \end{bmatrix} = \begin{bmatrix} 24 & 8 & 32 \\ 0 & 12 & 66 \\ 99 & 66 & 53 \\ 85 & 30 & 18 \\ 45 & 74 & 94 \end{bmatrix}. \quad (11)$$

At the reservation prices $r = (0, 0, 0)$ the bidders' initial demand sets are $D_1(r) = D_2(r) = D_5(r) = \{3\}$ and $D_3(r) = D_4(r) = \{1\}$. At the reservation prices r we have:

$$\begin{aligned} ED(r) &= \{\{1\}, \{3\}, \{1, 3\}\}, \\ MOD(r) &= \{\{1\}, \{3\}\}. \end{aligned}$$

In Figure 1, all possible paths from the seller's reservation prices $r = (0, 0, 0)$ to the minimum Walrasian equilibrium prices $p^{\min} = (79, 46, 66)$ obtainable by Algorithm 4 are represented in the form of a graph.⁷ Each vertex corresponds to a price vector reachable by Algorithm 4 and each arc is associated with a specific set in excess demand used to update prices. The solid arcs and the shaded vertices are not reachable using the algorithm in Demange et al. (1986). The unique path through the graph identified by the algorithms in Mo et al. (1988) and Sankaran (1994) is the path marked by bold (solid or dashed) arcs to the far right in the figure. Notice that there are several paths which are shorter than the latter path. \square

Appendix: Proof of Theorem 3

Lemma 2 Suppose that Algorithm 3 terminates with $I_n = \emptyset$. Let $S = \cup_{j=0}^n I_j$, then $O(S, p) = \cup_{j=0}^n B_j$, and for each $i \in S$ there is some $b \in O(S, p)$ such that $(b, i) \in X(p)$.

⁷ In the graph a set S^t is fixed until some bidder b with $D_b(p^t) \subseteq S^t$ becomes indifferent to some item $i \notin D_b(p^t)$. In Figure 1, this can be seen e.g. for the set $S^0 = \{3\}$ and the price vector $p^0 = (0, 0, 0)$ where $D_1(p^0) = \{3\}$ but $D_1(\hat{p}) = \{1, 3\}$ for $\hat{p} = (0, 0, 8)$. Such an approach has previously been considered in related problems by e.g. Abdulkadiroğlu et al. (2004) and Andersson and Andersson (2011).

Proof To show that $O(S, p) = \cup_{j=0}^n B_j$ we prove that $b \in \cup_{j=0}^n B_j$ if and only if $D_b(p) \subseteq S$. We do this by considering three mutually exclusive cases.

Case (i) $(b, i) \in X(p)$ for some $i \notin S$. Then $D_b(p) \not\subseteq S$, so it must be shown that b is not an element of any B_j . The condition in (5) that $(b', i') \in X(p)$ for some $i' \in I_j$ is always false for $b' = b$ since $i \notin S$. It follows that $b \notin \cup_{j=0}^n B_j$.

Case (ii) $(b, i) \in X(p)$ for some $i \in S$. Let $i \in I_m$ for $0 \leq m < n$, where the inequality follows from the assumption $I_n = \emptyset$. Equation (5) gives:

$$B_{m+1} = \{b' \in B \setminus \cup_{j=0}^m B_j : (b', i') \in X(p) \text{ for some } i' \in I_m\},$$

i.e. $b \in B_{m+1}$. To show that $D_b(p) \subseteq S$ we need only to compute the set of items I_{m+1} in the next iteration from (4):

$$\begin{aligned} I_{m+1} &= \{i' \in I^* \setminus \cup_{j=0}^m I_j : (b', i') \notin X(p) \text{ and } i' \in D_{b'}(p) \text{ for some } b' \in B_{m+1}\} \\ &\supseteq \{i' \in I^* \setminus \cup_{j=0}^m I_j : i' \in D_b(p) \setminus \{i\}\} \\ &= D_b(p) \setminus \cup_{j=0}^m I_j, \end{aligned}$$

where $i \in I_m$ has been used in the last equality.

Case (iii) $(b, i) \notin X(p)$ for all $i \in I^*$. From (3) we have $b \in B_0$. Insertion in (4) yields $D_b(p) \subseteq I_0$, which concludes the proof of the first statement.

To prove the second statement we observe that $0 \notin S$, and for each $i \in S$ there is some $b \in B$ such that $(b, i) \in X(p)$. Otherwise the algorithm would terminate in Step 2, which violates the assumption $I_n = \emptyset$. From Case (ii) above it follows that $b \in O(S, p)$.

Lemma 3 Suppose that Algorithm 3 terminates with $I_n = \emptyset$, and let $S = \cup_{j=0}^n I_j$. If $b \notin O(S, p)$, then $(b, i) \in X(p)$ for some $i \notin S$.

Proof The result follows from Lemma 2, and Cases (i) and (iii) from its proof.

Proof of Theorem 3. To prove the theorem, we demonstrate that if Algorithm 3 terminates with $I_n = \emptyset$, then $S = \cup_{j=0}^n I_j$ is the set in excess demand with maximum cardinality at prices p .

By assumption Algorithm 3 is not terminated in Step 2, so it follows that $0 \notin S$, and thus $S \subseteq I$. Lemma 2 states that all items in S are assigned to bidders in $O(S, p)$. It follows directly that $|U(T, p) \cap O(S, p)| \geq |T|$ for all $T \subseteq S$. To prove that $S \in ED(p)$, suppose $|U(T, p) \cap O(S, p)| = |T|$ for some $T \subseteq S$.

Let $K = \{b \in O(S, p) : (b, i) \in X(p) \text{ for some } i \in T\}$ denote the set of bidders in $O(S, p)$ that are assigned an item from T . If $|U(T, p) \cap O(S, p)| = |T|$ then $D_b(p) \cap T = \emptyset$ holds for all $b \in O(S, p) \setminus K$. Using Lemma 2 we have from (4) that $I_n \cap T \neq \emptyset$ only if $B_n \cap K \neq \emptyset$, and from (5) that $B_{n+1} \cap K \neq \emptyset$ only if $I_n \cap T \neq \emptyset$. Since the initial set of bidders satisfies $B_0 \cap K = \emptyset$ we obtain the contradictions $T \not\subseteq S$ and $K \not\subseteq O(S, p)$. Thus $|U(T, p) \cap O(S, p)| > |T|$ for all $T \subseteq S$, which completes the proof that $S \in ED(p)$.

Because S is a set in excess demand we know from Corollary 1 that there exists a unique $S^* \in ED(p)$ of maximum cardinality, and from Theorem 1 it follows that $S \subseteq S^*$. To prove that $S = S^*$, suppose $S \subset S^*$.

Denote the set of bidders that are assigned an item in $S^* \setminus S$ by $L = \{b \in B \setminus O(S, p) : (b, i) \in X(p) \text{ for some } i \in S^* \setminus S\}$, and the set of assigned items in $S^* \setminus S$ by $T = \{i \in S^* \setminus S : (b, i) \in X(p) \text{ for some } b \in B \setminus O(S, p)\}$. Without loss of generality we assume that $T \neq \emptyset$. Otherwise $O(S^*, p) = O(S, p)$ by Lemma 3, and $U(S^* \setminus S, p) \cap O(S^*, p) = \emptyset$ after applying Lemma 2, thus implying that S^* is not in excess demand.

Note that $D_b(p) \not\subseteq S^*$ may hold for some $b \in L$, and that $U(T, p) \cap O(S^*, p) = \emptyset$ from Lemma 2. We obtain:

$$\begin{aligned} |U(T, p) \cap O(S^*, p)| &= |U(T, p) \cap (O(S^*, p) \setminus O(S, p))| \\ &\leq |U(T, p) \cap L| = |T|. \end{aligned}$$

Hence, if $S \subset S^*$, then S cannot be in excess demand, thereby proving that S is the set in excess demand with maximum cardinality. \square

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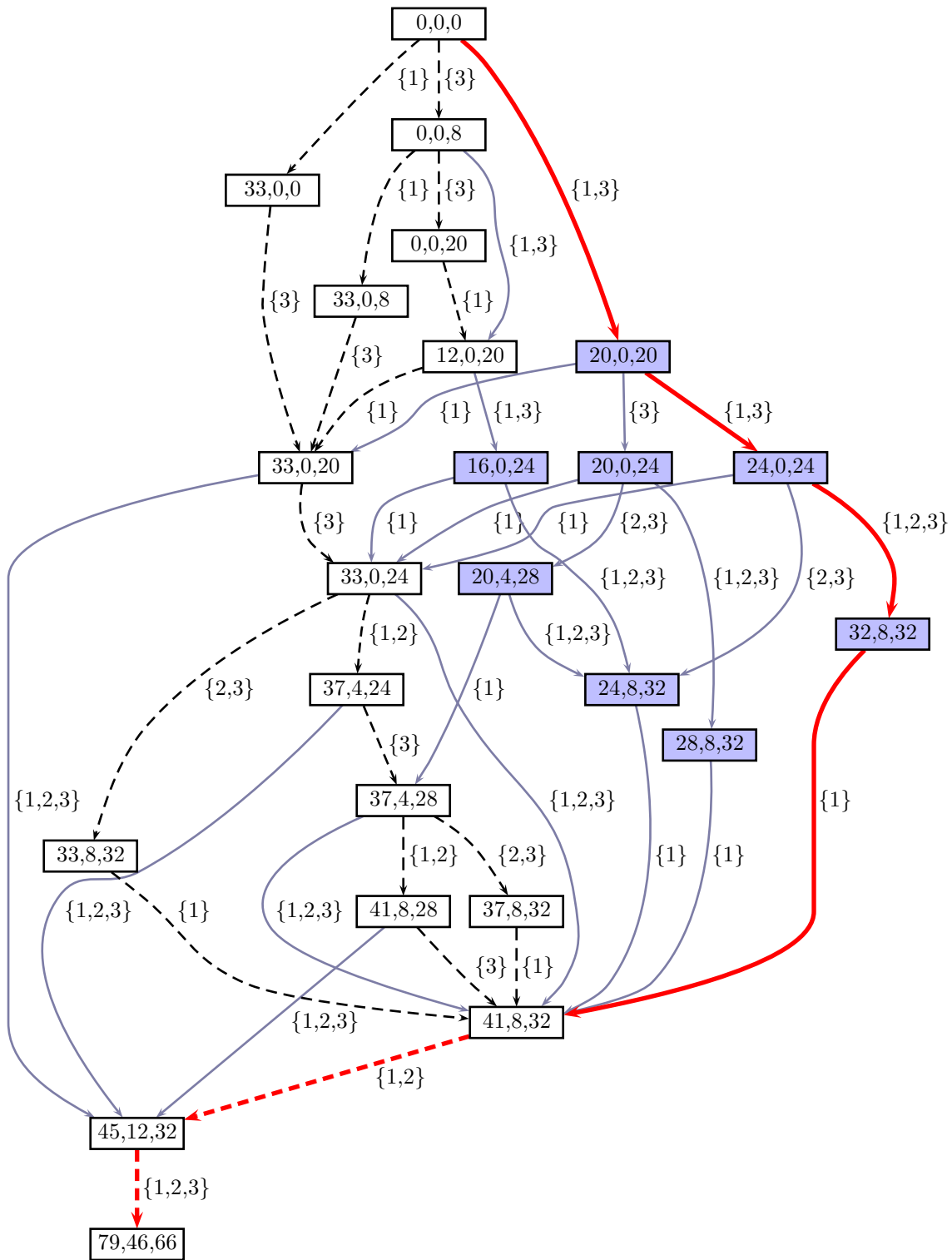


Fig. 1 Graph corresponding to all price paths of the EDAA in Example 2. Vertices are labeled with current prices and arcs are labeled with the sets in excess demand used to update prices.