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Strategy-proofness and Fixed-price Allocation of Indivisible Goods - a characterization proof

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Abstract

In this paper we considered the classical Shapley-Scarf (1974) "house allocation model", where in addition there is a perfectly divisible good (money). The problem is to characterize all strategy-proof, nonbossy and individually rational allocation mechanisms. The finding is that only a fixed-price allocation mechanism is consistent with these presumptions. Miyagawa (2001) first proved this result. Here we give an alternative and comparatively short proof of the characterization result.

Keywords: Strategy-proof, Indivisible object, Housing market.
JEL Classification Numbers: C71, C78, D71, D78.

1 Introduction

In two recent studies, Miyagawa (2001) and Svensson and Larsson (2002), the class of strategy-proof, nonbossy, individually rational and surjective allocation mechanisms for the classical Shapley-Scarf (1974) "house allocation problem", where a divisible good called money was added, was examined. The outcomes of the mechanisms were fixed-price allocations, and hence, the existence of a divisible good did not alter the findings in the original house allocation model substantially; in that case the class was the class of core mechanisms (Ma (1994), Svensson (1999)).

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However, the proof of the characterization was substantially more demanding in the case with a divisible good than without. The basic proof was given in Miyagawa (2001), while Svensson and Larsson (2002) used an additional assumption called weak decentralization. The objective of the present study is to give an alternative to Miyagawa's proof. This alternative takes as a point of departure the model without money. Given this, the arguments become rather simple for the model with money and the proof comparatively short.

The paper is organized in the following way. In Section 2 the formal model is shortly introduced, while Section 3 contains some results from previous studies used in the proof of the theorem in this paper. The proof is also given in Section 3.

2 The model

Here notation and assumptions made in the basic model are shortly presented - for a more detailed presentation, see e.g. Svensson and Larsson (2002). We also gather some known results which will be used in the proof of the theorem.

2.1 Basic concepts

There is a finite number n of *individuals,* $N = \{1, 2, ..., n\}$, and the same number of *objects* (called houses), $A = \{a_1, a_2, ..., a_n\}$. We simply let $A = N$. Sometimes we will use a "no object" denoted a_0 or 0. There is also a fixed quantity $e_0 \in \mathcal{R}_+$ of a divisible good (called money) initially distributed among the individuals. The *endowments* of individual $i \in N$ are (a_i, e_i), with $e_0 = \Sigma_i e_i$. The individuals have *quasi-linear preferences* over consumption bundles $(a, x) \in (A \cup \{0\}) \times \mathcal{R}$ represented by utility functions $u_i(a) + x$ for $i \in N$, with $u_i(a) > u_i(0)$ for all $a \in A$ and all $i \in N$. Hence the individuals consume at most one house and an amount of money. The set of all such utility functions u_i defined in $A \cup \{0\}$ is denoted U. *Preference profiles,* or for short *profiles*, are elements in $\mathcal{U} = U^n$. A profile $u = (u_1, u_2, \dots u_n)$ can also be denoted (u_i, u_{-i}) for $i \in N$, or (u_S, u_{-S}) for $S \subset N$, where u_S contains utility functions u_i with $i \in S$ and u_{-S} contains the remaining utility functions.

An *allocation* is a mapping $\alpha = (\varphi, \tau)$ from N to $A \times \mathcal{R}$, such that φ is injective and (resource balance) $\Sigma_i \tau_i = e_0$. φ is the allocation of the objects (a permutation of N), while τ is called the *income distribution.* Incomplete allocations of objects is called *assignments*, i.e. $\varphi : N \to A \cup \{0\}$ is an assignment if $\varphi(i) = \varphi(j)$ implies that $i = j$ or $\varphi(i) = 0$. Note that an

assignment φ is an allocation of the objects if $\varphi(i) \neq 0$ for all $i \in N$. An allocation of the objects is also called a complete assignment. Let $\mathcal{A}$ denote the set of assignments.

For two assignments $\varphi, \delta \in \mathcal{A}$ such that $\varphi(i) = 0$ if $\delta(i) \neq 0$, the *sum* $\varphi+\delta$ is defined as the assignment $(\varphi+\delta)(i) = \varphi(i)$ if $\varphi(i) \neq 0$ and $(\varphi+\delta)(i) = \delta(i)$ if $\delta(i) \neq 0$.

An assignment φ is a *cycle* in $G \subset N$ if $\#G \geq 2$ and there is a bijection $\pi : N \to N$ such that

- $G = \{i_0, i_1, \ldots, i_n\}$ with $i_0 \in N$ and $i_{j+1} = \pi(i_j)$ for all $j < n$,
- $\varphi(i_j) = a_{i_{j+1}}$ for all $j < n$, and $\varphi(i) = 0$ for $i \notin G$.

Obviously each complete assignment φ can be decomposed into a number of disjunct cycles φ_j, $j = 1, 2, \ldots, l-1$, and a residual φ_l with $\varphi_l(i) = a_i$ or 0, such that $\varphi = \Sigma_{j=1}^{l} \varphi_j$. Each φ_j is a reallocation of the objects within a group G_j and $N = \cup_j G_j$, $G_j \cap G_k = \varnothing$ if $j \neq k$.

Of particular interest are the *fixed-price allocations*.

Definition *For a given profile $u \in \mathcal{U}$, an allocation $\alpha = (\varphi, \tau)$ is a fixed-price allocation if there is a price vector $p \in \mathcal{R}^n$ and a decomposition in cycles $\varphi = \Sigma_{j=1}^{l} \varphi_j$ such that for an i with $\varphi_j(i) \neq 0$,*

- $\tau_i = e_i + p_i - p_{\varphi(i)}$,
- $u_i(\varphi(i)) - p_{\varphi(i)} = \max\{u_i(a) - p_a;\ a = \varphi_s(k) \text{ for some } k \text{ with } s \geq j\}$.

■

This type of rationing method is the well-known *Gale's top trading cycle principle* (TTCP). If e.g. $p = 0$ the cycle φ_1 contains the top ranked objects in A of the individuals in the cycle, while the cycle φ_2 contains the top ranked objects among the remaining objects $(A - \{\varphi_1(i)\}_i)$ of the individuals in that cycle. Similarly for the other cycles. On the other hand, if p is chosen so that *all* individuals receive their top ranked object in A, then $\alpha = (\varphi, \tau)$ is a Walrasian allocation.

2.2 Allocation mechanisms

An allocation mechanism, or for short a *mechanism,*[1] is a mapping $f = (g, t)$ from the set of profiles $\mathcal{U}$ to the set of allocations, where $g(u)$ is the allocation

[1] An alternative name of "allocation mechanism" in this context is *social choice function* or *allocation rule.*

of the objects (called the *assignment function*) and $t_i(u)$ (called the *transfer function*) denotes the quantity of money individual $i \in N$ receives by f.

A mechanism $f = (g, t)$ is *manipulable* if there is a profile $u \in \mathcal{U}$ such that for some individual $i \in N$,

$$u_i(g_i(v_i, u_{-i})) + t_i(v_i, u_{-i}) > u_i(g_i(u)) + t_i(u)$$

for some preferences $v_i \in U$. If the mechanism is not manipulable, it is *strategy-proof (SP)*.

A mechanism f is *nonbossy*[2] *(NB)* if for all preferences $v_i \in U$ and profiles $u \in \mathcal{U}$, $f(v_i, u_{-i}) = f(u)$ when $f_i(v_i, u_{-i}) = f_i(u)$. (A strategy-proof and nonbossy mechanism will be abbreviated a SPNB mechanism.)

A mechanism f is *individually rational* (*IR*) if for all $i \in N$ and for all $u \in \mathcal{U}$,

$$u_i(g_i(u)) + t_i(u) \geq u_i(a_i) + e_i.$$

We will also consider assignment functions g that are *surjective*, i.e. functions such that for each allocation φ of the objects there is a profile $u \in \mathcal{U}$ such that $g(u) = \varphi$.

3 The theorem

The theorem of this paper can now be formulated as:

Theorem *If $f = (g, t)$ is a strategy-proof, nonbossy and individually rational mechanism with g surjective, then $f(u)$ is a fixed-price allocation for almost all $u \in \mathcal{U}$.*

Remark. The mechanism is determined for almost all profiles. This means that we consider only those profiles where there is a strict ranking of the various commodity bundles. Due to Proposition 1 below, the range of the mechanism contains only a finite number of income distributions so the outcome of the mechanism is determined for almost all profiles.

There is one simple idea, which is used throughout the proof of the theorem. We show that if not all outcomes of the mechanism are characterized by the same trading prices, it is very easy to obtain inconsistency with strategy-proofness by letting all but one individual have "ordinal" preferences. In that case the outcome of the mechanism is determined by well-known rules from the case without money, rules that are gathered in the next section.

[2]The concept of nonbossiness is due to Satterthwaite and Sonnenschein (1981).

3.1 Some well-known results

The following results will be used in the proof of the theorem and are well-known from other studies. They are stated without a proof.

Proposition 1 *Let $f = (g, t)$ be a SPNB mechanism. There is then is a (distribution-) function T mapping complete assignments on income distributions such that for all $u \in \mathcal{U}$, $T(g(u)) = t(u)$.*

The proposition entails that the distribution of the divisible good is completely determined by the allocation of the objects. Of course, this also entails that the outcome of a SPNB mechanism in general cannot by Pareto efficient. For a proof of this fundamental proposition, see Ohseto (1999) in the case with one indivisible good or in the general case, Schummer (2000), or Miyagawa (2001), or Svensson and Larsson (2002). The theorem of this paper shows that the range of T is restricted in a particular way by, in addition to SPNB, also assume IR.

Proposition 2 *Let $f = (g, t)$ be a SPNB and IR mechanism with g surjective and t constant. The assignment function $g(u)$ is then given by the TTCP for all $u \in \mathcal{U}$ representing strict preferences among the objects.*

The condition $t(u)$ constant for all u means of course that the model is the original Shapley-Scarf house allocation model with no money to allocate. The outcome $g(u)$ is also the core allocation given the preference profile u. This result was first proved in Ma (1994). Ma's proof did not presume nonbossiness and surjectivity but instead Pareto efficiency. However, a nonbossiness assumption simplifies the proof somewhat, see Svensson (1999).

The content of the following lemma is also well-known.

Ordinal and cardinal monotonicity (M):

Lemma 1 *Let $f = (g, t)$ be a SPNB mechanism and $u \in \mathcal{U}$. Then, for any $i \in N$, $f(v_i, u_{-i}) = f(u)$ if $v_i \in U$ and $v_i(g_i(u)) - v_i(a) > u_i(g_i(u)) - u_i(a)$ for all $a \in A - \{g_i(u)\}$. If $u_i, v_i \in U^*$ and $g_i(u) = a$, then $f(v_i, u_{-i}) = f(u)$ if $v_i(b) > v_i(a)$ for all $b \in A$ such that $u_i(b) > u_i(a)$.*

Here $U^* \subset U$ denotes a set of "ordinal" preferences, i.e. preferences where the ranking of the commodity bundles is uniquely derermined by the ranking of the objects. For an exact definition U^*, see below.

For a proof of the first (cardinal) part of the lemma, see e.g. Svensson and Larsson (2002) (see also Roberts (1984) in a somewhat different context). Proofs for the (ordinal) second part of the lemma can be found in many places in the literature, e.g. in Svensson (1999).

3.2 Proof of the theorem

In the proof of the theorem some arguments are used frequently and they are collected in a number of lemmas below. In all lemmas 2 - 7 we consider a SPNB and IR mechanism $f = (g, t)$ with g surjective, a presumption that is no further repeated in the formulations of the lemmas.

First let us define a subset $\mathcal{U}^* \subset \mathcal{U}$ of "ordinal" preference profiles. Since the available quantity of money is bounded and because of Proposition 1, the income distribution will not influence the individual rankings of the various allocations in a SPNB mechanism if the utility distances between the objects are sufficiently large. Therefore, for a SPNB mechanism f, let $m \in \mathcal{R}_+$ be a number such that $m > 2 \max |T_i(\varphi)|$, $i \in N$, $\varphi \in \mathcal{A}$, where φ is a complete assignment and T the distribution function corresponding to f. Moreover let,

$$U^* = \{u \in U;\ |u(a) - u(b)| > m \text{ for all } a, b \in A,\ a \neq b\}$$

and $\mathcal{U}^* = U^{*^n}$. Of course, the number m and the sets U^* and $\mathcal{U}^*$ depend on f. To simplify notation we will write $a \succ b$ if $u(a) > u(b)$ and $a \succcurlyeq b$ if $a \succ b$ or $a = b$, for preferences $u \in U^*$.

Also let $M_i(a) = \{u_i \in U^*;\ a \succcurlyeq a_i \succcurlyeq b \text{ for all } b \neq a, a_i\}$, i.e. the set of "ordinal" preferences of $i \in N$ such that the object $a \in A$ is top ranked by i (i.e. is a *m*aximal element), while i's endowment a_i is ranked first or second best.

In the proof of the theorem two allocations will be used to "calibrate" the model; is the allocation φ^0 of the endowments, i.e. $\varphi^0(i) = a_i$ for all i, and it is a cycle φ^* defined by $\varphi^*(i) = a_{i+1}$ if $i < n$ and $\varphi^*(n) = a_1$. We will first (Lemma 5 - 7 below) consider mechanisms where $T(\varphi^0) = T(\varphi^*) = 0$. The objective is to prove that an implication of this is that $T(\varphi) = 0$ for all complete assignments φ. The complete proof will then easily follow.

The following notation will also be usefull**:** Call a cycle *dense* if for some $p, q \in N$, $\pi(q) = p$ while $\pi(i) = i + 1$ or $\pi(i) = 0$ for $i \neq q$. A cycle is called a *transposition* if only two individuals have received a non-zero object. Note that for a dense cycle the corresponding group of individuals is $G = \{i \in N;\ p \leq i \leq q\}$ if $p < q$ and $G = \{i \in N;\ i \geq p \text{ and } i \leq q\}$ if $p > q$.

Top trading cycle procedure for $\mathcal{U}^*$ (TTCP):

Lemma 2 *If $u \in \mathcal{U}^*$ then $g(u)$ is given by the TTCP.*

Proof. Consider the restricted mechanism $f^* = (g^*, t^*) : \mathcal{U}^* \to \mathcal{A}$, where $g^*(u) = g(u)$ and $t_i^*(u) = e_i$ for $u \in \mathcal{U}^*$. First, f^* is SP since $u_i(g_i(v_i, u_{-i})) +$

$e_i > u_i(g_i(u))+e_i$ implies that $u_i(g_i(v_i, u_{-i}))+t_i(v_i, u_{-i}) > u_i(g_i(u))+t_i(u)$ for $u \in \mathcal{U}^*$ and f is SP. Second, f^* is NB since f is NB and t^* constant. Third, the mechanism f^* is IR because f is IR and $u_i(g_i(u))+t_i(u) \geq u_i(a_i)+e_i$ implies that $u_i(g_i(u))+e_i \geq u_i(a_i)+e_i$ for $u \in \mathcal{U}^*$. Finally, that g^* is surjective follows from monotonicity and the fact that g is surjective. Now by Proposition 2, $g^*(u)$ is given by the TTCP. ■

Independence (I):

Lemma 3 *Let $\varphi \in \mathcal{A}$ be a complete assignment and let $G \subset N$ be such that φ_G is a cycle in G. If $u \in \mathcal{U}$ and for all $i \in G$, $u_i \in M_i(\varphi(i))$, then $g_G(u_G, v_{-G}) = \varphi_G$ for all profiles $v \in \mathcal{U}$.*

Proof. Let $v \in \mathcal{U}$ and suppose that $g(u_G, v_{-G}) = \delta$. Also let $w_i \in M_i(\delta(i))$ for all $a \neq \delta(i)$, for all $i \in N - G$. Then for sufficiently large $w_i(\delta(i))$, $g(u_G, w_{-G}) = \delta$ by monotonicity. But $(u_G, w_{-G}) \in \mathcal{U}^*$ so by Lemma 2, $\delta_G = \varphi_G$. ■

Pareto dominance (PD):

Lemma 4 *Let $\varphi, \delta \in \mathcal{A}$ and $k \in N$, and let $u_k \in U$, $u_i \in U^*$ for all $i \neq k$ be such that $\varphi(i) \succcurlyeq_i \delta(i)$ for all $i \neq k$ while $u_k(\varphi(k))+T_k(\varphi) > u_k(\delta(k))+T_k(\delta)$. Then $g(u) \neq \delta$.*

Proof. Let $g(u) = \delta$ and $v_i \in M_i(\varphi(i))$ for all $i \in N$. Then by (ordinal) monotonicity (Lemma 1), $g(u_k, v_{-k}) = \delta$. Moreover, $g(v) = \varphi$ by TTCP (Lemma 2). This is not consistent with SP since $u_k(\varphi(k))+T_k(\varphi) > u_k(\delta(k))+T_k(\delta)$. Hence, $g(u) \neq \delta$ must be the case. ■

Lemma 5 *Let $\varphi \in \mathcal{A}$ be complete assignment with $\varphi = \varphi_1 + \varphi_2$ being the corresponding partition in cycles. If φ_1 is a dense cycle and $T(\varphi^0) = T(\varphi^*) = 0$, then $T(\varphi) = 0$.*

Proof. $\varphi = \varphi_1+\varphi_2$ means that φ_1 is a cycle while $\varphi_2(i) = a_i$ or 0. To simplify notation we can, with no loss of generality, assume that the set of individuals involved in the cycle is $G = \{i \in N;\ 1 \leq i \leq q\}$, and hence $\varphi_1(i) = \varphi^*(i)$ for $i < q$. Moreover, for $q = n$, $\varphi = \varphi^*$ and then $T(\varphi) = 0$, so consider $q < n$. Assume that $T(\varphi) = \tau \neq 0$. Then by the resource balance condition and IR, $\tau_i < 0$ for some $i \in G$.

Now separate three cases; (i) there is a number $r < q$ with $\tau_r < 0$, (ii) $\tau_q < 0$ and $\tau_s > 0$ for some $s < q$, and (iii) $\tau_q < 0$ and $\tau_s > 0$ for some $s > q$. Also let $u \in \mathcal{U}^*$ satisfy $u_i \in M_i(\varphi^*(i))$ for all $i \in N$.

(i) In this case, let $v_r \in U$ and $v_q \in U^*$ satisfy

$v_r(\varphi(r)) > v_r(a_r) > v_r(\varphi(r)) + \tau_r > v_r(a) + m$ for all $a \neq a_r, \varphi(r)$,

$v_q : a_1 \succ a_{q+1} \succ a_q \succ a$ for all $a \neq a_1, a_q, a_{q+1}$.

Then $g(v_q, u_{-q}) = \varphi$ by TTCP (Lemma 2) and $g(v_r, v_q, u_{-r,q}) = \varphi^*$ by IR and Pareto dominance (Lemma 4). But this is a contradiction to strategy-proofness since $0 = T_r(\varphi^*) > T_r(\varphi)$ and $\varphi(r) = \varphi^*(r)$.

(ii) Now let $v_s \in U$ and $v_q \in U^*$ satisfy

$v_s(\varphi(s)) + \tau_s > v_s(a_s) > v_s(\varphi(s)) > v_s(a) + m$ for all $a \neq a_s, \varphi(s)$,

$v_q : a_{q+1} \succ a_1 \succ a_q \succ a$ for all $a \neq a_1, a_q, a_{q+1}$.

Then $g(v_q, u_q) = \varphi^*$ by TTCP and $g(v_q, v_s, u_{-q,s}) = \varphi$ by IR and Pareto dominance. But this is a contradiction to strategy-proofness.

(iii) Finally, let $v_q \in U^*$ be as in (ii) and $v_s \in U^*$ satisfy

$v_s(a_{s+1}) + \tau_s > v_s(a_s) > v_s(a_{s+1}) > v_s(a) + m$ for all $a \neq a_s, a_{s+1}$.

Then $g_s(v_q, v_s, u_{-q,s}) = a_s$ or a_{s+1} by Pareto dominance and IR. If $g_s(v_q, v_s, u_{-q,s}) = a_s$ then $g(v_q, v_s, u_{-q,s}) = \varphi$ by Pareto dominance. But $g(v_s, u_{-s}) = \varphi^*$ and hence $i = q$ can manipulate. If $g_s(v_q, v_s, u_{-q,s}) = a_{s+1}$ then $g(v_q, v_s, u_{-q,s}) = \varphi^*$ by Pareto dominance. But $g(v_q, u_{-q}) = \varphi$ and hence $i = s$ can manipulate.

In conclusion, $T(\varphi) \neq 0$ leads to a contradiction in any case and hence $T(\varphi) = 0$ must prevail. ■

Lemma 6 *Let $\varphi \in \mathcal{A}$ be complete assignment with $\varphi = \varphi_1 + \varphi_2$ being the corresponding partition in cycles. If φ_1 is a transposition and $T(\varphi^0) = T(\varphi^*) = 0$, then $T(\varphi) = 0$.*

Proof. Let $\varphi(p) = a_q$, $\varphi(q) = a_p$ and $\varphi(i) = a_i$ for the remaining i:s. Assume that $p < q$. If $p = q - 1$, the proof follows from Lemma 5. Now consider $p < q - 1$ and assume that $T(\varphi) = \tau \neq 0$. Then $\tau_p < 0$ or $\tau_q < 0$ by IR and resource balance. If $\tau_q < 0$ let $u \in \mathcal{U}^*$ be a profile that satisfies: $u_i \in M_i(a_{i+1})$ for $p < i < q$, $u_q \in M_i(a_p)$, $u_i \in M_i(a_i)$ for $i < p$ and for $i > q$ while for $i = p$, $a_q \succ_p a_{p+1} \succ_p a_p \succ_p a_j$ for all $j \neq q, p, p + 1$. Moreover, let $v_q \in U$ satisfy

$v_q(a_p) > v_q(a_q) > v_q(a_p) + \tau_q > v_q(a_j) + m$ for all $j \neq p, q$.

Denote by $\delta \in \mathcal{A}$ the assignment: $\delta(i) = a_i$ if $i < p$ or $i > q$, $\delta(i) = a_{i+1}$ if $p \leq i < q$ and $\delta(q) = a_p$.

Then $g(u) = \varphi$ by TTCP and $g(v_q, u_{-q}) = \delta$ by Pareto dominance and Lemma 6. But this is a contradiction to strategy-proofness. This means that $T(\varphi) = 0$ must prevail also in this case.

In the case $\tau_p < 0$ we can argue in the same way as above; for $p < q - 1$, let δ' be a dense cycle in the group $\{q, q+1, \ldots n, 1, \ldots p\} \subset N$. ■

Lemma 7 *Let $\varphi \in \mathcal{A}$ be a complete assignment. Then $T(\varphi) = 0$ if $T(\varphi^*) = T(\varphi^0) = 0$.*

Proof. The lemma will be proved by induction over the number of exchanges in a complete assignment φ. We will assume that $T(\varphi) = 0$ if at most k individuals in the assignment φ have changed their objects and prove that then also $T(\varphi) = 0$ if $k+1$ individuals have changed objects in an assignment φ. If $k = 2$ then $T(\varphi) = 0$ by Lemma 6, and then the lemma follows by induction. So our *induction assumption* is

$$T(\varphi) = 0 \text{ for all complete } \varphi \in \mathcal{A} \text{ such that } \#\{i \in N;\ \varphi(i) \neq a_i\} \leq k.$$

Let φ be a complete assignment such that $\#\{i \in N;\ \varphi(i) \neq a_i\} = k + 1$. There is a decomposition $\varphi = \Sigma_{j=1}^{l} \varphi_j$ in cycles φ_j, $j < l$ and $\varphi_l(i) = a_i$ or 0. To prove the lemma, assume that $T(\varphi) = \tau \neq 0$. Then by resource balance and IR, $\tau_i < 0$ for some $i \in G_j = \{i;\ \varphi_j(i) \neq 0\}$, $j < l$. We consider two cases separately; (i) $\#G_j > 2$ for some $j < l$, (ii) $\#G_j \leq 2$ for all j:s.

(i) To simplify notation and with no loss of generality, we can assume that $\#G_1 > 2$, that φ_1 is a dense cycle in $G_1 = \{1, 2, \ldots, q\}$, and that $\tau_1 < 0$. Now let $u \in \mathcal{U}^*$ satisfy $u_i \in M_i(\varphi(i))$ for all $i \neq 2$, while $\varphi(2) \succcurlyeq_2 a_1 \succcurlyeq_2 a_2 \succcurlyeq_2 a$ for all $a \in A - \{a_1,\ \varphi(2)\}$. Also let $v_1 \in U$ satisfy

$$v_1(\varphi(1)) > v_1(a_1) > v_1(\varphi(1)) + \tau_1 > v_1(a) + m \text{ for all } a \in A - \{a_1,\ \varphi(1)\}.$$

Denote by $\delta \in \mathcal{A}$ the assignment: $\delta(1) = \varphi(1) = a_2$, $\delta(2) = a_1$, $\delta(i) = a_i$ for $2 < i \leq q$, and $\delta(i) = \varphi(i)$ for $i > q$.

Then $g(u) = \varphi$ by TTCP and $g(v_1, u_{-1}) = \delta$ by IR and Pareto dominance. This is a contradiction to strategy-proofness, and hence, $T(\varphi) = 0$ must prevail in this case.

(ii) Now $\#G_j = 2$ for all $j < l$. If $l = 2$ then $T(\varphi) = 0$ by Lemma 6, so consider $l > 2$. Then, with no loss of generality, we can assume that $\varphi = \varphi_1 + \varphi_2 + \ldots$, $G_1 = \{1, 2\}$, $G_2 = \{3, 4\}$ and $T_1(\varphi) = \tau_1 < 0$.

Let $u \in \mathcal{U}^*$ satisfy $u_1 \in M_1(a_2), u_4 \in M_4(a_3)$ and for $i = 2, 3$, $a_1 \succcurlyeq_2 a_3 \succcurlyeq_2 a_2 \succcurlyeq_2 a$ and $a_1 \succcurlyeq_3 a_4 \succcurlyeq_3 a_3 \succcurlyeq_3 a$ for all remaining $a \in A$, while $u_i \in M_i(\varphi(i))$ for $i > 4$. Moreover, let $v_1 \in U$ satisfy

$v_1(a_2) > v_1(a_1) > v_1(a_2) + \tau_1 > v_1(a) + m$ for all $a \in A - \{a_1, a_2\}$.

Denote by $\delta \in \mathcal{A}$ the assignment: $\delta(1) = a_2$, $\delta(2) = a_3$, $\delta(3) = a_1$, $\delta(4) = a_4$ and $\delta(i) = \varphi(i)$ for $i > 4$.

Then $g(u) = \varphi$ by TTCP and $g(v_1, u_{-1}) = \delta$ by IR and Pareto dominance. This is a contradiction to strategy-proofness, and hence, $T(\varphi) = 0$ must prevail also in this case. This completes the proof. ■

Proof of the theorem. Let $p, q \in \mathcal{R}^n$ and, given the set A of objects and the mechanism f, define a set $A' = \{(a_j, p_j);\ a_j \in A\}$. Also let $f' = (g', t')$ be a derived mechanism, where $g'_i(u) = (a_j, p_j)$ if $g_i(u) = a_j$ and $t'_i(u) = \tau_i + p_j + q_i$, where $\tau_i = t_i(u)$. The utility of the bundle $[(a_j, p_j),\ x]$, $x \in \mathcal{R}$, is $u_i(a_j) - p_j + x$. This entails that the utility of the outcome of f'_i for $i \in N$ is:

$$u_i(g_i(u)) - p_j + \tau_i + p_j + q_i = u_i(g_i(u)) + \tau_i + q_i,$$

i.e. equal to the utility of the outcome of f_i minus the constant q_i.

Now it immediately follows that f' is SPNB with g' surjective. Moreover, let (a'_i, e'_i) with $a'_i = (a_i, p_i)$ and $e'_i = e_i + p_i + q_i$ be the endowments of i in the mechanism f'. Then IR of f implies IR of f':

$$u_i(g_i(u)) + \tau_i + q_i \geq u_i(a_i) + e_i + q_i = (u_i(a_i) - p_i) + e'_i.$$

Finally, let p and q be chosen according to

$$p_1 = 0 \text{ and } p_{i+1} = p_i + e_i - \tau^*_i \text{ where } T(\varphi^*) = \tau^*, \text{ and } q_i = -e_i - p_i.$$

Then $T'(\varphi^0) = 0$ $(e'_i = 0)$ and $T'(\varphi^*) = 0$ because

$$T'_i(\varphi^*) = T_i(\varphi^*) + p_{i+1} + q_i = \tau^*_i + p_{i+1} + q_i = p_i + e_i + q_i = 0.$$

The presumptions for Lemma 7 are now satisfied for f' and then $T'(\varphi) = 0$ for all complete assignments φ. But then, $0 = T'_i(\varphi) = \tau_i + p_{\varphi(i)} - (e_i + p_i)$ and hence, $\tau_i = e_i + p_i - p_{\varphi(i)}$. Moreover, by Proposition 2, $g'(u)$ is given by the TTCP for all $u \in \mathcal{U}$ such that $u_i(a_j) - p_j \neq u_i(a_k) - p_k$ if $j \neq k$. This means that there is a decomposition $\varphi = \Sigma_{j \leq l} \varphi_j$ such that for an i with $a_j = \varphi_k(i) \neq 0$,

$$u_i(a_j) - p_j = \max\{u_i(a_s) - p_s;\ a_s = \varphi_r(i') \text{ for some } i' \in N \text{ and some } r \geq k\}.$$

Hence, the allocation (φ, τ) is a fixed-price allocation. This completes the proof. ■

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