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# General Equilibrium Pricing of Trading Strategy Risk

by

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## Abstract

When forward contracts are involved in dynamic portfolio strategies, incurred profits or losses that accrue at each instant are locked-in in the forward position up to the contract maturities. The discounted value of these gains or losses at each date  $t$  is part of investors' wealth. This discounting thus gives rise to an interest rate risk. Therefore, investors are bound to have a constrained bond position and, being unable to diversify away the corresponding systematic risk, they must be compensated for it. We derive the general equilibrium of a dynamic financial market in which the investors' opportunity set includes non-redundant forward contracts. We show that Breeden's (1979) intertemporal consumption-based CAPM equation for forward contracts contains an extra term relative to that for cash assets. We name this term a strategy risk premium. It compensates investors for the (systematic) risk that stems from their very portfolio strategies when the latter involve non-redundant forward contracts. We also show that Merton's (1973) multi-beta intertemporal CAPM must be amended for forward contracts to exhibit adjusted risk premia for the market portfolio and all relevant state variables, as opposed to the usual (non-adjusted) risk premia for cash assets. While the traditional intertemporal CAPMs shows that only the systematic risks related to asset return fluctuations are priced, we finally show that systematic trading strategy risk is also priced. Finally, none of our results depends on the usual cash-and-carry relationship, which in general will not hold.

**JEL Classification:** D52, E52, G11, G12.

## I. INTRODUCTION

The economic significance of forward and forward-related instruments is not disputable. Forward contracts, interest rate and currency swaps and exotic (structured) financial vehicles using forwards have known so huge a development they have dwarfed primitive cash markets both in terms of liquidity and volume of transactions. In particular, most market makers on interest and exchange rate products traded over-the-counter and most major corporations worldwide use all kinds of forwards for hedging purposes. Many such instruments are written on tradable financial assets or storable commodities, which implies that they are fundamentally redundant instruments whose prices are trivial to compute, using the cash-and-carry relationship. However, the proportion of forward contracts that cannot be considered as redundant, even as a first approximation, is increasing. Some of these are or will shortly be written on non-tradable economic variables [see Shiller (1993), Sumner (1995) and Athanasoulis, Shiller and van Wincoop (1999)]. A representative example is the forward contracts on a Consumer Price Index that could be launched in the near future by various Central Banks [see Cowen (1997), Dowd (1994) and Sumner (1997)]. Comparable forward contracts could be expanded to other macro-economic aggregates, such as the Gross National Product and monetary aggregates [Sumner (1995)]. Other famous examples are weather or, more generally, nature-linked derivatives. Another category includes forward contracts written on non-storable commodities (such as electricity), which have recently attracted much attention. Ongoing projects include non-redundant forwards written on computer memory storage capacity, on emission credits and on bank credits. These propositions largely (but not exclusively) focus on forward contracts. They tend to neglect futures and options, probably because the underlying non-traded assets or economic aggregates are not sufficiently volatile to warrant the development of such markets, and because arbitrage activity seems to be an important motivation for participants in futures markets and strict arbitrage is impossible for non-redundant futures.<sup>1</sup>

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<sup>1</sup> See Horrigan (1987), Dowd (1995), Sumner (1995) and Athanasoulis et al. (1999). In the US, for instance, a market for futures contracts written on the CPI was organized but failed, probably because the CPI was not volatile enough.

The pricing of non-redundant forwards, in spite of the economic significance of the latter, has been neglected in the literature and still leaves something to be desired. Cox, Ingersoll and Ross (1981), hereafter CIR, derived several important results regarding the relationship between the prices of a forward contract and of its corresponding futures contract when such contracts are redundant, for instance because the financial market is already complete before their introduction. They also showed that Merton's (1973) intertemporal capital asset pricing model (hereafter ICAPM) and Breeden's (1979) consumption-based CAPM (hereafter CCAPM) do not hold for forwards although they hold for futures. Yet, they did not provide the solutions for the forward price equations. One could be tempted to object that knowing the cash-and-carry relationship and the CAPM equation(s) for the underlying cash asset and the relevant discount bond, it is a trivial matter to derive the corresponding CAPM-like equation(s) for the forward. Yet, this objection would miss the point. First, the cash-and-carry relationship in general does not hold for forward contracts, whether they are redundant or not. Second, this procedure assumes away the issue of how the introduction of forwards in an incomplete economy affects the equilibrium prices of existing cash assets. Third, and most importantly, one cannot provide a convincing economic interpretation of the result since the precise mechanism that makes the traditional CAPM(s) invalid for forwards is not exhibited.

Richard and Sundaresan (1981) went a step further and derived a general equilibrium pricing of non-redundant forward (and futures) contracts. They showed in particular that the simple cash-and-carry pricing equation that characterizes redundant forward contracts still holds for non-redundant ones in special cases. However, their pioneering work has three limitations. The first is that they did not provide either a CAPM-like equation for the latter contracts, even in the special case where they are redundant. Consequently, they could not propose a convincing answer to the question regarding forwards (redundant or not) raised by CIR. The second limitation is that the cash-and-carry relationship does not hold, even for simple (with linear pay-off) forward contracts, when the underlying asset pays *stochastic* dividend(s) and/or provides a *stochastic* convenience yield (which is the case on most commodities). To derive their results, the authors had to assume these stochastic features away. The last limitation lies in that they considered only the case of forward contracts with linear pay-offs for which, in their infinite horizon setting, the cash-and-carry prices are easily derived as the equilibrium forward prices.

A solution to the issue of pricing forward contracts, in particular non-redundant ones, thus is interesting per se since it fills a gap in financial asset price modeling. Yet there is another important reason why such a result is useful and awaited for. The pricing of redundant derivative assets by arbitrage is well understood and mastered by both academics and practitioners. However, a general or partial equilibrium model is needed to value correctly any non-redundant derivative asset as well as any primitive asset. The payoff(s) of the asset is (are) discounted by a risk-adjusted expected return. This risk adjustment is made for the systematic risk associated with the relevant asset only, which is correct for cash instruments. However, forward contracts will be shown to give rise to another systematic risk associated with the investment and/or hedging strategy. Therefore, an additional risk premium will have to be included in the discount rate.

In this paper, we propose a more general model in which the cash-and-carry equation does not necessarily hold for non-redundant forwards, yet leads to CAPM-like pricing equations for the latter in equilibrium. It is important to note at the outset that we adopt a *broad definition* of what we call a forward contract: it is a contingent claim to a pay-off that is *any function, not necessarily linear*, of some units of a predetermined good or asset at a given maturity. No cash payment is ever involved before the contract expires. Along the way, we provide a convincing explanation as to why the traditional CAPMs do not hold for these contracts, and will show that this is also the case for the less interesting redundant ones.

We derive both a CCAPM and a multi-beta ICAPM for these claims and offer an intuitive economic interpretation of the results. The difference between these models and the two traditional CAPMs that hold for cash assets (and futures contracts alike) is explicitly related to an additional risk that stems from the very portfolio strategy involving forward contracts. Suppose for ease of exposition that the investor trades on only one such contract. When the latter is part of the optimal dynamic strategy, the profit or loss that occurs is locked-in in the forward position up to the contract maturity. Thus it is the *discounted value* of the accumulated gain or loss at each instant that is part of the investor's wealth. This necessary discounting thus gives rise to an interest rate risk. Therefore, when using forward contracts in a dynamic trading strategy, the investor is bound to have an implicit (almost surely non-zero) constrained discount bond position. Hence, she must bear two risks on the forward

component of her wealth. The first is traditional and associated with the random fluctuations of the forward price, as for any risky cash asset. The second is novel and due to interest rate moves that randomly affect the value of her forward position, for a *given* forward price. Since she is unable to diversify away the corresponding systematic risk *ex ante*, she must be given an incentive to bear it. Consequently, the expected “return” on a forward (non cash) asset will include at equilibrium an extra premium compensating for a risk different from that of the forward asset itself. This is the reason why the traditional CAPMs for cash assets cannot hold for forward contracts.

This result gives rise to the interesting issue of what this risk premium becomes if interest rate risk can be perfectly hedged. The (somewhat subtle) answer is that the premium will not be affected. When the profit or loss on the forward position taken at date  $t$  is known at date  $t^+$ , the associated risk will be a pure interest rate risk. Since presumably the investor would like optimally to hedge it, the possibility to hedge perfectly or not does in fact matter. However, at date  $t$  when the investor trades new contracts, he does not know yet what will be at date  $t^+$  the value of his position to be hedged against interest rate risk since this value depends on the fluctuation of the forward price between  $t$  and  $t^+$ , which is random. Therefore, at time  $t$  the investor knows that what he will get at time  $t^+$  is not the (random) change in the forward price, for which he obviously demands a risk premium, as for any risky cash asset, but the (twice random) present value of that change that depends on the level of interest rates at date  $t^+$ . It is this extra risk that commands an additional premium. In summary, even if interest rate risk in the economy can be perfectly hedged, it does not help the investor since he does not know *ex ante* the position to hedge. This implies in particular that the possibility to trade the discount bond whose maturity coincides with that of the forward contract is irrelevant for the additional premium to be present.

Our findings help clarify the intuition that, when an investor trades forward contracts, she bears an additional, interest rate related, risk. More importantly yet, we show a not so intuitive result we believe is fundamental. It concerns the way the presence of this forward trading strategy risk affects the equilibrium expected returns on the other (cash) assets traded in the economy. It turns out that only the expected “return” on the forward is affected, the required returns on the other assets remaining unchanged.

Our main results can be summarized as follows. In addition to the consumption-related risk premium, the CCAPM for forward contracts is shown to contain a term different from the traditional one relevant for cash assets, namely the riskless rate of interest. We name this novel term a *strategy risk premium*. It compensates the investor for the (systematic) risk that stems from his very *portfolio strategy* when the latter involves non-redundant forward contracts. Moreover, this result extends to redundant contracts and is valid regardless of whether the cash-and-carry relationship holds true. We also show that the multi-beta ICAPM for forward contracts contains *adjusted* risk premia for the market portfolio and each state variable, not the usual risk premia relevant for cash assets. Again, the adjustment is due to the systematic interest rate risk stemming from the trading strategy itself. While the traditional ICAPM shows that only the systematic risks associated with the random fluctuations of a cash asset itself are priced, the ICAPM that holds for forward contracts indicates that the systematic trading *strategy risk* is also priced. In a celebrated special case due to CIR (1985), we show that the magnitude of the strategy risk premium may be as large as that of the traditional risk premium.

The remainder of the paper is organized as follows. In Section II, we present the economic framework and describe an investor's wealth dynamics in an incomplete financial market. Section III provides the investor's optimal demands for risky assets including forwards. Section IV derives the CCAPM and the multi-beta ICAPM that apply to forward contracts and compares them with the usual ones valid for cash assets and futures. The last section offers some concluding remarks. For convenience, all proofs are gathered in a mathematical appendix.

## II. ECONOMIC FRAMEWORK

We provide first a description of the economy and then an explicit expression for the gain/loss process generated by continuous trading on forward contracts. Finally, we derive the investors' wealth dynamics.

### *2.1 Technologies and financial assets*

Following CIR (1985), we assume an economy in which there is a single physical good, the numéraire, which may be allocated to consumption or investment. We



assume that there are  $N$  technological investment opportunities in the economy, all producing the same good. When an amount  $\eta_i(t)$  of this consumption good is invested in technology  $i$ , ( $i = 1, \dots, N$ ), the change in its value over time is governed by the following stochastic differential equation (SDE):

$$d\eta_i(t) = \eta_i(t)\mu_{\eta_i}(t, Y(t))dt + \eta_i(t)\sigma_{\eta_i}(t, Y(t))'dZ(t) \quad (1)$$

where  $Z(t)$  is an  $(N + K) \times 1$  dimensional Wiener process in  $\mathbb{R}^{N+K}$ ,  $Y(t)$  is a  $K \times 1$  dimensional vector of state variables,  $\mu_{\eta_i}(t, Y(t))$  is a bounded valued function of  $t$  and  $Y$ , and  $\sigma_{\eta_i}(t, Y(t))$  is a bounded  $(K + N) \times 1$  vector valued function of  $t$  and  $Y$ . The Wiener process is defined on a usual complete probability space  $(\mathbf{W}, \mathbf{F}, \mathbf{P})$  where  $\mathbf{P}$  is the historical probability measure. Like in CIR (1985), equation (1) specifies the growth of an initial investment when the output of the process is continually reinvested in that same process.

The movement of the  $K$  state variables is determined by the following system of SDEs:

$$dY(t) = \mu_Y(t, Y(t))dt + \Sigma_Y(t, Y(t))dZ(t) \quad (2)$$

where  $\mu_Y(t, Y(t))$  is a bounded  $K \times 1$  vector valued function of  $t$  and  $Y$  and  $\Sigma_Y(t, Y(t))$  is a bounded  $K \times (N + K)$  matrix valued function of  $t$  and  $Y$ .

When convenient, we will write the dynamics of the  $N$  technological processes in the following form:

$$d\eta(t) = I_\eta \mu_\eta(t, Y(t))dt + I_\eta \Sigma_\eta(t, Y(t))dZ(t) \quad (3)$$

where  $I_\eta$  is an  $N \times N$  diagonal matrix valued function of  $\eta(t)$  whose  $i^{\text{th}}$  diagonal element is  $\eta_i(t)$ ,  $\mu_\eta(t, Y(t))$  is an  $N \times 1$  dimensional vector whose  $i^{\text{th}}$  component is  $\mu_{\eta_i}(t, Y(t))$  and finally  $\Sigma_\eta(t, Y(t))$  is a  $(K \times N) + K$  matrix valued function whose  $i^{\text{th}}$  element is  $\sigma_{\eta_i}(t, Y(t))$ .

Investors have also access to an instantaneously riskless asset (money market account) yielding an interest rate  $r(t)$  in the numeraire good at which they can lend or borrow. It is in zero net supply. The diffusion process followed by  $r(t)$  is endogenous and will be made explicit in the special case examined at the end of Section III.

In addition to the  $N$  technological investment opportunities and the riskless asset, non-redundant forward contracts, which are contingent claims to a specified amount of the physical good at a given maturity, are also available for trade in this economy. These financial claims are in zero net supply. We consider forward contracts only because, unlike cash primary assets or most options, no cash payments are involved until maturity. Since there is a single good in this economy, we assume without real loss of generality that there exists one forward contract only that, together with the  $N$  technologies and the riskless asset, forms the basis of our financial market.<sup>2</sup> It is a claim to a payoff that is any deterministic function, not necessarily linear, of some units of the physical good, with maturity  $T$ .

Hence investors have access to  $(N+2)$  trading instruments. Note that none of our results depends on whether the financial market is complete or not, provided the forward contract we are interested in is not redundant.

The price of the forward contract solves the following SDE:

$$dF(t) = F(t)\mu_F dt + F(t)\sigma_F' dZ(t) \quad (4)$$

where the notation  $F(t)$  is short for  $F(t, T)$ ;  $\mu_F$  and  $\sigma_F$ , respectively, are assumed a bounded valued function of  $t$  and  $Y$ , and a bounded  $(N + K) \times 1$  vector valued function of  $t$  and  $Y$ . The explicit dependence of the drift and diffusion parameters on  $t$  and  $Y(t)$  has been omitted for simplicity. Part of our problem is to determine these functions endogenously in market equilibrium. We stress that  $F(t, T)$  is the *price* of the forward, not its *market value*. In particular, its initial value is zero while its price is  $F(0, T)$ .

The variance-covariance matrix  $\Sigma_\eta \Sigma_\eta'$  is assumed to be positive definite. The variance-covariance matrix of the percent changes in all risky asset prices

(technologies and forward contract), i.e.  $\Sigma \Sigma'$  where  $\Sigma \equiv \begin{bmatrix} \Sigma_\eta \\ \sigma_F \end{bmatrix}$ , is also assumed to be

positive definite.

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<sup>2</sup> Richard and Sundaresan's (1981) economy contains  $N$  consumption goods, thus  $N$  forward contracts written on them. This easy generalization would not add any economic insight to our results and, in particular, our pricing equations would remain identical.

In our CIR-like economy, we assume for simplicity that no bond is actually traded. We stress that this assumption is used only to make easier the interpretation of the investors' optimal strategies and is completely innocuous. Relaxing it would not affect any result herein derived. Indeed, the price of any risky cash asset will be seen to obey the usual CAPMs. Bonds, as is well known, are no exception. The only difference, apart from an investor's optimal strategy slightly more complicated, would be that, were the number of such bonds large enough to ensure the market's completeness, some optimal hedge to be defined later on would be perfect as opposed to imperfect.

Now, the riskless asset process  $r(t)$  determines an implicit yield curve whose characteristics are endogenous. In particular, let  $P(t, T)$ , or  $P(t)$  for short, be the price at time  $t$  of an implicit pure discount bond whose maturity  $T$  coincides with that of the forward contract. Its dynamics then obeys the following SDE (for  $t$  such that  $0 \leq t < T$ ):

$$dP(t) = P(t)\mu_p(t, Y(t))dt + P(t)\sigma_p(t, Y(t))' dZ(t) \quad (5)$$

Note that since its price is a function of the term structure that is endogenous, the parameters of the SDE (5) are, like those of SDE (4), determined endogenously.

Finally, trading in technologies and all financial assets takes place continuously in a frictionless market<sup>3</sup> and only at equilibrium prices.

## 2.2 Profit and loss from forward trading

Let us turn now to the value of an investor's position resulting from his trading in the forward contract. Let  $\Theta$  be the number of forward contracts *held* (as opposed to *traded*) at time  $t$  ( $t < T$ ). The profit/loss account for the forward contracts is equal to:

$$X(t) = P(t) \int_0^t \Theta(u) dF(u) \quad (6)$$

The RHS of (6) is the value at date  $t$  of the profit or loss generated by the forward position. Since this cumulative algebraic gain is cashed-in or -out at the contract maturity date  $T$  only, it must be discounted using  $P(t)$ .

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<sup>3</sup> We take exception of the fact that the technologies cannot be shorted.

### 2.3 Wealth dynamics

Let  $\mathbf{a}$  be the vector of proportions of wealth invested in the technologies,  $\gamma$  the proportion of wealth invested in the riskless asset and  $C$  the instantaneous consumption rate. Given equation (6), an investor's wealth dynamics writes:

$$\begin{aligned} dW = & \left[ W\alpha' \mu_\eta + \left( P \int_0^t \Theta dF \right) \mu_P + P\Theta F \mu_F + P\Theta F \sigma_F' \sigma_P + W\gamma r - C \right] dt \\ & + \left[ W\alpha' \Sigma_\eta + \left( P \int_0^t \Theta dF \right) \sigma_P' + P\Theta F \sigma_F' \right] dZ \end{aligned} \quad (7)$$

This dynamics, using the definition of wealth  $W \equiv W\gamma + W\alpha' \mathbf{1}_N + P \int_0^t \Theta dF$ , can be rewritten as follows:

$$\begin{aligned} dW = & \left[ W\alpha' \mu_\eta + (W - W\alpha' \mathbf{1}_N - W\gamma) \mu_P + P\Theta F \mu_F \right] dt \\ & + \left[ P\Theta F \sigma_F' \sigma_P + W\gamma r - C \right] dt \\ & + \left[ W\alpha' \Sigma_\eta + (W - W\alpha' \mathbf{1}_N - W\gamma) \sigma_P' + P\Theta F \sigma_F' \right] dZ \end{aligned} \quad (8)$$

or

$$\begin{aligned} dW = & \left[ W\alpha' (\mu_\eta - \mu_P \mathbf{1}_N) + W\mu_P + P\Theta F (\mu_F + \sigma_F' \sigma_P) \right] dt \\ & + \left[ W\gamma (r - \mu_P) - C \right] dt \\ & + \left[ W\alpha' (\Sigma_\eta - \mathbf{1}_N \sigma_P') + W(1 - \gamma) \sigma_P' + P\Theta F \sigma_F' \right] dZ \end{aligned} \quad (9)$$

Finally, to simplify notations, we denote by  $\theta$  the ratio of the present value of the forward position to the investor's wealth, i.e.,  $\theta \equiv \frac{1}{W} P\Theta F$ . Therefore, the wealth dynamics is finally given by:

$$\begin{aligned} dW = & \left[ W\alpha' (\mu_\eta - \mu_P \mathbf{1}_N) + W\mu_P + W\theta (\mu_F + \sigma_F' \sigma_P) \right] dt \\ & + \left[ W\gamma (r - \mu_P) - C \right] dt \\ & + \left[ W\alpha' (\Sigma_\eta - \mathbf{1}_N \sigma_P') + W(1 - \gamma) \sigma_P' + W\theta \sigma_F' \right] dZ \end{aligned} \quad (10)$$

We are now fully equipped to compute optimal demands and then equilibrium quantities.

### III. OPTIMAL DEMANDS FOR RISKY ASSETS

Once the dynamics of the investor's wealth has been derived, the traditional stochastic dynamic programming approach à la Merton (1971) is used to find the optimal demand for risky assets. The modern martingale approach cannot be applied in its usual form to a set of assets that includes (non cash) forward contracts and its application subject to a number of modifications would not add any economic insight.

An investor in this economy maximizes the expected utility of his intertemporal consumption stream subject to his budget constraint. Thus, the optimal consumption and portfolio rules must solve:

$$\text{Max } E \left[ \int_t^\tau U(s, C(s)) ds \right] \quad (11)$$

subject to equation (10) and to positive  $\alpha_i$ 's and positive consumption  $P$  -a.s.

where  $U$  is an increasing and strictly concave Von Neuman-Morgenstern utility function respecting the usual Inada conditions,  $E$  is the expectation operator conditional on the current endowment and the state of the economy. For simplicity, we do not introduce a bequest motive at terminal date  $\tau$ . Moreover, in maximizing (11), the investor is assumed to limit his attention to admissible controls only. Recall from the previous section that the state of the economy is fully characterized by the vector  $Y(t)$  of state variables and wealth  $W(t)$ . Finally, we assume without loss of generality that the investor's horizon  $\tau$  is larger than or equal to the forward contract expiration date  $T$ .

Before deriving equilibrium results, we analyze in some depth the optimal strategy followed by investors in our economy. This strategy is described in the following proposition.

**Proposition 1.** *The investor's optimal demands for risky assets are equal to:*

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \end{pmatrix} = (\Sigma \Sigma')^{-1} \begin{pmatrix} \mu_\eta - \mathbf{1}_N r \\ \mu_F + \sigma_F' \sigma_P \end{pmatrix} \begin{pmatrix} -\frac{J_W}{W J_{WW}} \end{pmatrix} + (\Sigma \Sigma')^{-1} \Sigma \Sigma_Y' \begin{pmatrix} -\frac{J_{WY}}{W J_{WW}} \end{pmatrix} - (\Sigma \Sigma')^{-1} \Sigma \sigma_P \frac{X}{W} \quad (12)$$

where  $\Sigma\Sigma'$ , with  $\Sigma \equiv \begin{bmatrix} \Sigma_\eta \\ \sigma_F \end{bmatrix}$ , is the instantaneous variance-covariance matrix of the returns on the technologies and the percent changes in the forward contract price, and  $X$  is the present value of the profit or loss generated by trading on the forward.

We stress first that in spite of the presence of  $X$  in the last term of the RHS of equation (12), this solution is indeed in closed form. This is because  $X(t)$  depends on the holdings  $\hat{\alpha}(t)$  and  $\hat{\theta}(t)$  of time  $t$ , and is therefore known at date  $t$  when the new holdings are chosen. For instance, at date  $t = 0$ ,  $\hat{\alpha}(0)$  and  $\hat{\theta}(0)$  are selected with  $X(0)$  equal to zero.

Recall that in addition to the traded assets (the forward contract, the riskless asset and the technologies), the agent has a constrained discount bond position stemming from her profit or loss generated by forward trading. Since the latter is not part of the investor's optimization program, it does not appear in (12).

The demand for risky assets thus contains three components. The first term on the RHS of (12) is the usual mean-variance speculative component while the last two are hedging components. The first hedging component is the traditional Merton-Breeden hedge. It is an information-based and dynamic term whose purpose is to hedge wealth against unfavorable shifts of the  $K$  economic state variables that drive the opportunity set. Note that this component is, as usual, preference dependent, and that  $J_{WY}$  ( $\equiv \partial J_W / \partial Y$ ) is the cross-partial derivative that represents the effect of the state variable  $Y$  on the marginal value of wealth. In the case of logarithmic (Bernoulli) utility, the investor has a myopic behavior and this hedge would disappear since the cross partial derivative  $J_{WY}$  would vanish.

The second hedging component results specifically from trading in forward contracts and would not of course be present in absence of the latter. It involves the fraction of wealth  $X/W$  corresponding to the forward position value times a covariance/variance ratio. Therefore, as intuition suggests, it depends on the fraction of wealth that has been generated so far on the forward position. This term however, unlike the preceding one, does not qualify as a Merton-Breeden component for two reasons. First, it is not a hedge against future random levels of state variables. Second, it does not depend on the utility term ( $-J_{WY}/J_{WW}$ ), although it is not preference free

since both optimal  $X$  and  $W$  do depend on the investor's utility. Rather, this hedging term is due to the forward position not being marked-to-market, hence bearing an additional (interest rate) risk on the cumulative algebraic gain that has accrued so far. Thus, the source of this additional risk is to be found in the *forward trading strategy* itself<sup>4</sup>. Because they anticipate that one period ahead the current value of their forward position will have changed, investors will optimally hedge against the interest rate risk brought about by their very strategy. Thus, this hedging component is not due to the presence of an exogenous source of non-diversifiable risk but results from the *endogenous* risk brought about by the particular nature of (non cash) forward contracts.

Through the third component of their optimal strategy, investors can achieve a perfect hedge of the additional interest rate risk or not depending on whether the financial market is complete or incomplete. But this does not of course affect equation (12). However, that part of wealth that is so hedged against interest rate fluctuations will earn the riskfree rate  $r(t)$  if perfect hedging is possible, and will earn something else (a random return) if it is not.

An important consequence of (12) is that, since the last term is not a Merton/Breeden hedging term, this third component will *not* lead to a risk premium for any risky asset in equilibrium. This result implies in particular that the technologies, although risky assets, will not command any risk premium in addition to the one delivered by standard pricing models, as will be shown below. Therefore, the introduction in the economy of non-redundant forwards does not *formally* affect cash asset returns, although it certainly alters all asset equilibrium prices.

The difference between forward contracts and standard cash assets (here, technologies) comes from the speculative part in (12). While for risky cash assets the usual excess return  $(\mu_\eta - \mathbf{1}_N r)$  is involved, for forward contracts the novel excess return  $(\mu_F - (-\sigma_F' \sigma_P))$  appears. Since there is no actual investment in a forward

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<sup>4</sup> Lioui and Poncet (2000 (a) and (b), and 2001) have derived a similar result in various frameworks to solve different but related issues. In particular, they have pointed out the crucial theoretical difference in this regard between futures and forward contracts. Empirically, this difference may be large or not, depending on the specific parameters of the diffusion processes.

contract (no cost-of-carry), the riskless rate of return  $r$  is absent from the forward excess return. However, since the forward is nonetheless a risky asset, its instantaneous expected return  $\mu_F$  is corrected by a term that comes directly from the interest rate risk stemming from the profit-and-loss account. This expected excess return  $\left(\mu_F - \left(-\sigma_F' \sigma_P\right)\right)$  will be further elaborated on in the following sub-section.

#### IV. INTERTEMPORAL GENERAL EQUILIBRIUM

In an economy where the investment opportunity set driven by an arbitrary number of stochastic state variables, Merton (1973) derived an ICAPM that exhibits two (or more) betas, one vis-à-vis the market portfolio and one (or more) vis-à-vis the state variable(s). In another seminal paper, Breeden (1979) showed that this multi-beta ICAPM reduces to a single beta one when the beta is measured with respect to aggregate consumption (CCAPM). This is because the ultimate concern of all investors is real consumption, and that the latter (stochastic) variable encompasses all the sources of risk affecting the economy. However, CIR (1981) showed later that neither Merton's nor Breeden's models hold for non-redundant forward contracts but derived no alternative equations. We will provide versions of the CCAPM and the multi-beta ICAPM that hold for such forward contracts.

To obtain equilibrium relationships, we now assume, like CIR (1985) and many others, that a representative investor exists. Thus, aggregate wealth and individual wealth are equal.<sup>5</sup> In addition, the following two market-clearing conditions must hold in equilibrium:

- i) the total amount invested in the technologies equals total wealth (i.e.,  $\hat{\alpha}' \mathbf{1}_N = 1$ ),
- and
- ii) the net positions in the forward market and the riskless asset are zero (i.e.,  $\hat{\gamma} = 0$  and  $\hat{\theta} = 0$ ).

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<sup>5</sup> Alternatively, and equivalently, we can assume as in Richard and Sundaresan (1981) that all investors are identical. Then each individual wealth is a fixed and known fraction of aggregate wealth. See their footnote 3 on page 351, for a brief discussion of this assumption.



#### 4.1 Consumption-based CAPM

The representative investor is obviously concerned by the equilibrium expected return on the technologies and on the forward contract. He is also concerned by the expected return on the implicit cash bond that is part of his wealth. Thus our second proposition contrasts the CCAPM for the implicit cash bond of maturity  $T^6$  and the forward contract.

**Proposition 2.** *The consumption-based CAPM for the implicit pure discount bond  $P(t)$  is expressed as:*

$$\mu_P = r - \frac{\hat{C}U_{\hat{C}\hat{C}}}{U_{\hat{C}}} \sigma_P' \sigma_{\hat{C}} \quad (13)$$

where  $\sigma_{\hat{C}}$  is the instantaneous volatility of the relative changes in aggregate consumption, and the consumption-based CAPM for the forward contract  $F(t)$  writes:

$$\mu_F = -\sigma_F' \sigma_P - \frac{\hat{C}U_{\hat{C}\hat{C}}}{U_{\hat{C}}} \sigma_F' \sigma_{\hat{C}} \quad (14)$$

As expected, Breeden's traditional CCAPM holds for the cash bond (equation (13)). Investors are fairly compensated by the market for the systematic risk they bear from holding (in positive or negative amounts) the implicit discount bond generated by their forward position. In addition, this CCAPM of course holds for the technologies themselves, as is apparent from the optimal demand for  $\hat{\alpha}$  [equation (12)]. This is because these are in effect cash assets.

However, the CCAPM for the forward contract is different (equation (14)). As noted before, investors must be compensated for the two risks they bear. The risk associated with the random fluctuations of the forward price itself gives rise to the usual

premium  $-\frac{\hat{C}U_{\hat{C}\hat{C}}}{U_{\hat{C}}} \sigma_F' \sigma_{\hat{C}}$ , and the risk associated with the forward trading strategy

generates a premium equal to minus the instantaneous covariance between the relative changes in the forward contract and the relative changes in the discount bond whose

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<sup>6</sup> Recall that there is no explicit discount bond in our CIR-like economy.

maturity is equal to that of the forward contract  $(-\sigma_F' \sigma_P)$ . It is interesting to remark that the sign of this covariance term is indeterminate.<sup>7</sup>

We call this novel term  $(-\sigma_F' \sigma_P)$  a *strategy risk premium*. Thus, the “return”<sup>8</sup> on a forward contract compensates also for the systematic risk that results from the covariance between the contract and its associated discount bond. The economic intuition behind this result is as follows. When trading on forward contracts, the investor generally has an implicit discount bond position and, being unable to diversify the corresponding systematic risk, she must be compensated for it. An important feature of this additional premium is that, in contrast to the premium related to the consumption risk of the forward price, it is preference free. This not so intuitive result can be explained as follows. First, at equilibrium the accumulated profit or loss  $X$  is obviously zero for the set of all investors (or the representative individual). And  $X/W$  was noted to be the only preference-dependent element of the second hedging term of equation (12). Second, under the assumption of homogeneous beliefs, the covariance  $(-\sigma_F' \sigma_P)$  is common to all investors (or is the belief of the representative agent). This term thus plays to some extent the role of a riskless interest rate. In effect, rewriting equations (14) under an “excess return” form yields:

$$\mu_F - (-\sigma_F' \sigma_P) = -\frac{\hat{C}U_{\hat{C}\hat{C}}}{U_{\hat{C}}} \sigma_F' \sigma_{\hat{C}}$$

Thus, the RHS of the rewritten equation has exactly the same structure as the RHS of equation (13). This RHS does of course depend on the investor’s utility function.

An interesting remark concerns the relationship between our new CCAPM (14) and the classic cash-and-carry relationship. One can easily show that if we *assume* that the cash-and-carry relationship holds, then equation (14) follows essentially from Itô’s

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<sup>7</sup> To be rigorous, let us recall that the interest rate  $r$  is a real rate in this moneyless economy and could actually be negative in equilibrium. Therefore, both  $r$  and  $(-\sigma_F' \sigma_P)$  have indeterminate signs.

<sup>8</sup> The word “return” is written within quotes since, as noted previously, no initial payment is involved with a forward contract. The correct phrase is “percent change in the forward price”. We use both expressions equivalently.

lemma.<sup>9</sup> Therefore, result (14) holds for both redundant and non-redundant forwards and does not depend on whether the cash-and-carry equation holds or not. However, our derivation not being grounded on this assumption is more general. In our setting, the extra term emerges from Itô's lemma applied to the present value of the profit and loss account. In addition, we can provide a clear economic interpretation of the result in terms of trading strategy risk, interpretation that is impossible from the cash-and-carry relation.

#### 4.2 Multi-beta ICAPM

The one-to-one relationship that is known to exist in general equilibrium between the CCAPM and the multi-beta ICAPM seems on a priori grounds to make the derivation of the latter useless. Yet the volatilities of financial assets appearing in equations (13) and (14) are endogenously determined, so that it is useful to characterize them as a function of the volatilities of the state variables and aggregate wealth. This leads to the multi-beta ICAPM that (i) is interesting in its own right, (ii) is easier to submit to empirical testing<sup>10</sup>, and (iii) has obvious and well known implications as to the current hotly debated issue of financial asset predictability. Our third proposition thus provides the ICAPM for the implicit discount bond and the forward contract.

**Proposition 3.** *The general equilibrium expected return on the discount bond  $P(t)$  is given by:*

$$\mu_P = r + \frac{W}{P} P_W \lambda_W + \sum_{j=1}^K \frac{W}{P} P_{Y_j} \lambda_{Y_j} \quad (15)$$

*and the general equilibrium expected percent change in the forward contract is equal to:*

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<sup>9</sup> See the derivation in the Appendix following the proof of Proposition 2.

<sup>10</sup> Testing the CCAPM is well known to be very difficult due to severe practical and empirical problems. For instance, one has to measure the “representative individual’s” utility and its relevant derivatives, consumption data suffer from too low a frequency, their accuracy depends on the quality of the samples that are used, and expenses rather than consumption itself are in fact measured.

$$\mu_F = \frac{W}{F} F_W (\lambda_W - \sigma_P' \sigma_W) + \sum_{j=1}^K \frac{F_{Y_j}}{F} (W \lambda_{Y_j} - \sigma_P' \sigma_{Y_j}) \quad (16)$$

where  $\sigma_W$  is the instantaneous volatility of aggregate wealth,  $P_x$  and  $F_x$  denote partial derivatives of the two asset prices, and where:

$$\lambda_W = \left( -\frac{W J_{WW}}{J_W} \right) \sigma_W' \sigma_W + \sum_{j=1}^K \left( -\frac{J_{WY_j}}{J_W} \right) \sigma_W' \sigma_{Y_j} \quad (17)$$

$$\lambda_{Y_j} = \left( -\frac{W J_{WW}}{J_W} \right) \sigma_W' \sigma_{Y_j} + \sum_{k=1}^K \left( -\frac{J_{WY_k}}{J_W} \right) \sigma_{Y_j}' \sigma_{Y_k} \quad (18)$$

Since in equilibrium aggregate wealth  $W$  is the value of the market portfolio, (15) and (16) are ICAPM-like equations. While (15) is standard, (16) is not: In addition to the fact that  $r$  does not appear, as previously explained, the various risk premia affecting the forward expected “return” must be *adjusted* for the systematic interest rate risk linked to the presence of the implicit discount bond. This theoretical result also bears on empirical testing. When a CAPM-like model is applied to forward contracts, the same linear risk-return relationship that is tested for cash assets is assumed to hold, namely:

$$\text{Percent change in the forward price} = a \text{ constant} + \sum_{j=1}^K \beta_j Y_j + a \text{ noise.}$$

Since the  $\sigma_P' \sigma_{Y_j}$  are actually not constant, the above regression model is misspecified, and its empirically estimated parameters will be biased.

Incidentally, one may wonder what the multi-beta ICAPM looks like for redundant *cash* (as opposed to forward) *contingent claims* that are replicable using, in particular, the forward contract. The answer is given in the following proposition.

**Proposition 4.** *The general equilibrium expected return on an attainable contingent claim  $S^i$  is given by:*

$$\mu_{S^i} - r = \frac{W}{S^i} S_W^i \lambda_W + \sum_{j=1}^K \frac{W}{S^i} S_{Y_j}^i \lambda_{Y_j} \quad (19)$$

The multi-beta ICAPM holds in its usual form, without adjustment for the risk premia. This was expected since the financial claim is a cash asset. However, the replicating strategy is different from what is known. Since forward contracts are used, hedging the interest rate risk will in effect be involved. It is worth noting that this ICAPM holds whether the latter risk can be hedged perfectly (in special cases) or not (in the general case).

Now, the following proposition shows that the partial differential equation (PDE) for the forward contract is different from the one governing cash assets. We will then apply it to a celebrated special case in order to obtain a closed form solution.

**Proposition 5.** *The price of the forward contract satisfies the following PDE:*

$$\begin{aligned}
0 = & \frac{1}{2} W^2 \sigma_W' \sigma_W F_{WW} + \sum_{j=1}^K \sigma_W' \sigma_{Y_j} F_{WY_j} + \frac{1}{2} \sum_{j=1}^K \sum_{k=1}^K \sigma_{Y_k}' \sigma_{Y_j} F_{Y_j Y_k} \\
& + \left( \hat{\alpha}' \mu_\eta W - W \lambda_W + W \sigma_W' \sigma_P - \hat{C} \right) F_W + \sum_{j=1}^K \left( \mu_{Y_j} - W \lambda_{Y_j} + \sigma_P' \sigma_{Y_k} \right) F_{Y_j} + F_t
\end{aligned} \tag{20}$$

*subject to the relevant boundary condition for the forward price as dictated by the terms of the contract.*

Our valuation PDE turns out to have a unique characteristic not present, to the best of our knowledge, in the literature on contingent claims pricing. This is due to the presence of several terms involving the discount bond volatility  $\sigma_P$ . This implies that to solve for this PDE and thus find the price of the forward contract, we must solve a system of two simultaneous equations. Namely, one must solve the (usual) PDE for the discount bond together with PDE (20). Fortunately, this can be done in two steps. We obtain first the price of the discount bond, and then derive by substitution the price of the forward contract. We provide below a specific example that gives the solution without formally resorting to actually solve the two PDEs.

#### 4.3 A celebrated special case

The investors' optimal strategy having been derived and interpreted in full generality, we can now focus on a special case where the cash-and-carry relationship will be shown to hold for a simple (linear) forward contract to deliver an amount  $\eta(\tau)$  of the

consumption good. This example is chosen as it represents the benchmark case. We stress that it is not redundant with the case studied by Richard and Sundaresan (1981) since the latter postulate that the representative agent's horizon is infinite. We consider a specialized version of our economy due to Cox et al. (1985). There is a single state variable that obeys the square root process:

$$dY = \varepsilon_Y (Y + \kappa_Y) dt + \sigma_Y \sqrt{Y} dz \quad (21)$$

and a single technology whose dynamics is given by:

$$\frac{d\eta}{\eta} = \mu_\eta Y dt + \sigma_\eta \sqrt{Y} dz \quad (22)$$

where  $\varepsilon_Y$ ,  $\kappa_Y$ ,  $\sigma_Y$ ,  $\mu_\eta$ , and  $\sigma_\eta$  are constants.

One should notice that, although this economy is affected by one source of uncertainty only, the market is still incomplete because of the short sale constraint imposed on trading on the technology. In other words, even a forward contract written on the technology is not redundant.

Finally, the representative investor is assumed to have a logarithmic utility:

$$U(t, C(t)) = e^{-\rho t} \text{Ln} C(t) \quad (23)$$

In such a framework, our first results easily obtain, gathered in the following proposition.

**Proposition 6.** *The equilibrium aggregate wealth is equal to:*

$$W(t) = \frac{1 - e^{-\rho(\tau-t)}}{1 - e^{-\rho\tau}} W(0) e^{-\rho t} \eta(t), \quad (24)$$

*the equilibrium aggregate consumption is proportional to wealth:*

$$\hat{C}(t) = \frac{\rho}{1 - e^{-\rho(\tau-t)}} W(t) = \frac{\rho}{1 - e^{-\rho\tau}} W(0) e^{-\rho t} \eta(t), \quad (25)$$

*and the equilibrium spot rate is given by:*

$$r(t) = (\mu_\eta - \sigma_\eta^2) Y(t) \quad (26)$$

We are now fully equipped to price a forward contract. Consider for instance a simple (with linear pay-off) forward contract of maturity  $\tau$  to deliver an amount  $\eta(\tau)$  of the consumption good. Its price is shown in the Appendix to be equal to:

$$F(t) = \frac{\eta(t)}{P(t, \tau)} \quad (27)$$

The familiar cash-and-carry relationship is recovered in this special case. However, we stress that it is grounded on a general equilibrium model, not on mere absence of arbitrage opportunity. The latter is indeed ineffective since the forward contract is not redundant. This important and elegant result was first obtained by Richard and Sundaresan (1981). However, they derived it in a somewhat simpler setting where the representative investor has an infinite horizon while the forward contract has finite maturity, which may introduce confusion and lead to the question as to why the forward contract is not longer-lived.

Now, it is well known that, in the CIR (1985) model, the price of a discount bond with maturity  $\tau$  writes:

$$P(t, \tau) = A(t, \tau) e^{B(t, \tau)r(t)} \quad (28)$$

where  $A(t, \tau)$  and  $B(t, \tau)$  are deterministic functions.

Applying Ito's lemma, we have:

$$\begin{aligned} \sigma_P &= B(t, \tau) (\mu_\eta - \sigma_\eta^2) \sigma_Y \sqrt{Y} \\ \sigma_{\hat{C}} &= \sigma_\eta \sqrt{Y} \\ \sigma_F &= (\sigma_\eta - B(t, \tau) (\mu_\eta - \sigma_\eta^2) \sigma_Y) \sqrt{Y} \end{aligned} \quad (29)$$

Using (14) and substituting for (29), it follows that:

$$\begin{aligned} \mu_F &= -\sigma_F \sigma_P + \sigma_F \sigma_{\hat{C}} \\ &= -B(t, \tau) (\mu_\eta - \sigma_\eta^2) \sigma_Y \sigma_\eta Y + B(t, \tau) (\mu_\eta - \sigma_\eta^2) \sigma_Y B(t, \tau) (\mu_\eta - \sigma_\eta^2) \sigma_Y^2 Y \\ &\quad - B(t, \tau) (\mu_\eta - \sigma_\eta^2) \sigma_Y \sigma_\eta Y + \sigma_\eta^2 Y \end{aligned} \quad (30)$$

The presence of the first risk premium in the RHS of the first part of equation (30) should not be downplayed under the false pretext that the (second) usual risk premium associated with consumption risk is present. It may very well be that the

compensation for the strategy risk is of the same magnitude, or even larger, than the compensation for the consumption risk. The pricing of non-redundant forward contracts is actually different from that of cash assets. Again, it is important to note that the usual cash-and-carry derivation, while applicable to this special case, would not by itself lead to the decomposition of the drift  $\mu_F$  into the two components present in the first equality (30) and its interpretation. A general equilibrium model is required.

## V. CONCLUDING REMARKS

We have developed a simple general equilibrium model of a one-good economy where the financial market is perfect but possibly incomplete. Our findings should help to highlight a specific feature of the financial market equilibrium, namely that the expected return on a forward (non cash) asset includes a premium for a risk other than the risk of the asset price. This result has been derived without the introduction of any exogenous (background) risk. Expanding our results to markets with frictions could be of course a natural extension of this work. The approach followed here could equally help analyzing other issues in financial economics such as corporate hedging and financial market optimal design.



## MATHEMATICAL APPENDIX

*Proof of Proposition 1.* Let  $J(t, W(t), Y(t))$  be the value (indirect utility) function at date  $t \in [0, T]$  and let  $L(t)J$  be the differential generator of  $J$ . We assume that  $J$  exists and is an increasing and strictly concave function of  $W$ . This assumption encompasses several of the assumptions made by CIR (1985). Let  $\psi \equiv LJ + U$ . Then the necessary and sufficient conditions for optimality are:

$$\psi_{\hat{c}} = U_{\hat{c}} - J_W \leq 0 \quad (a)$$

$$\hat{C}\psi_{\hat{c}} = 0 \quad (b)$$

$$\begin{aligned} \psi_{\hat{\alpha}} = & W(\mu_\eta - \mu_P \mathbf{1}_N)J_W + W(\Sigma_\eta - \mathbf{1}_N \sigma_P)' \Sigma_Y' J_{WY} + \\ & \left[ \begin{aligned} & (\Sigma_\eta - \mathbf{1}_N \sigma_P)' (\Sigma_\eta - \mathbf{1}_N \sigma_P)' \hat{\alpha} + (1 - \hat{\gamma}) (\Sigma_\eta - \mathbf{1}_N \sigma_P)' \sigma_P \\ & + (\Sigma_\eta - \mathbf{1}_N \sigma_P)' \sigma_F \hat{\theta} \end{aligned} \right] W^2 J_{WW} \leq 0 \end{aligned} \quad (c)$$

$$\hat{\alpha}' \psi_{\hat{\alpha}} = 0 \quad (d)$$

$$\begin{aligned} \psi_{\hat{\theta}} = & W(\mu_F + \sigma_F' \sigma_P)J_W + W\sigma_F' \Sigma_Y' J_{WY} \\ & + \left[ \sigma_F' (\Sigma_\eta - \mathbf{1}_N \sigma_P)' \hat{\alpha} + (1 - \hat{\gamma}) \sigma_F' \sigma_P + \sigma_F' \sigma_F \hat{\theta} \right] W^2 J_{WW} = 0 \end{aligned} \quad (e)$$

$$\begin{aligned} \psi_{\hat{\gamma}} = & W(r - \mu_P)J_W - W\sigma_P' \Sigma_Y' J_{WY} \\ & - \left[ \hat{\alpha}' (\Sigma_\eta - \mathbf{1}_N \sigma_P)' \sigma_P + (1 - \hat{\gamma}) \sigma_P' \sigma_P + \hat{\theta} \sigma_F' \sigma_P \right] W^2 J_{WW} = 0 \end{aligned} \quad (f)$$

with usual notations for partial derivatives of the value function with respect to each of its arguments. The hat  $\hat{\cdot}$  above a variable denotes an optimal value. Equation (a) is the standard envelope condition while equations (b) and (d) guarantee that the optimal value for the consumption process and the proportions of wealth invested in the technologies are non-negative.

Using (c), (e) and (f), we have:

$$\begin{aligned}
& (\mu_\eta - \mu_P \mathbf{1}_N) \mathbf{J}_W + (\Sigma_\eta - \mathbf{1}_N \sigma_P') \Sigma_Y' \mathbf{J}_{WY} + \\
& \left[ \begin{aligned} & (\Sigma_\eta - \mathbf{1}_N \sigma_P') (\Sigma_\eta - \mathbf{1}_N \sigma_P')' \hat{\alpha} + (1 - \hat{\gamma}) (\Sigma_\eta - \mathbf{1}_N \sigma_P') \sigma_P \\ & + (\Sigma_\eta - \mathbf{1}_N \sigma_P') \sigma_F \hat{\theta} \end{aligned} \right] \mathbf{W} \mathbf{J}_{WW} = 0 \tag{c'}
\end{aligned}$$

$$\begin{aligned}
& (\mu_F + \sigma_F' \sigma_P) \mathbf{J}_W + \Sigma_F \Sigma_Y' \mathbf{J}_{WY} \\
& + \left[ \sigma_F' (\Sigma_\eta - \mathbf{1}_N \sigma_P')' \hat{\alpha} + (1 - \hat{\gamma}) \sigma_F' \sigma_P + \sigma_F' \sigma_F \hat{\theta} \right] \mathbf{W} \mathbf{J}_{WW} = 0 \tag{e'}
\end{aligned}$$

$$\begin{aligned}
& (r - \mu_P) \mathbf{J}_W - \sigma_P' \Sigma_Y' \mathbf{J}_{WY} \\
& - \left[ \hat{\alpha}' (\Sigma_\eta - \mathbf{1}_N \sigma_P') \sigma_P + (1 - \hat{\gamma}) \sigma_P' \sigma_P + \hat{\theta} \sigma_F' \sigma_P \right] \mathbf{W} \mathbf{J}_{WW} = 0 \tag{f'}
\end{aligned}$$

(c') hold for any vector  $\hat{\alpha}$  satisfying the non-negativity constraint. Solving for (c') and (e') and using (f') to eliminate  $\hat{\gamma}$  yields the desired result.

Proof of Proposition 2. Using the market clearing conditions and (e) yields:

$$\mu_F = -\sigma_F' \sigma_P - \sigma_F' \left( \frac{\mathbf{W} \mathbf{J}_{WW}}{\mathbf{J}_W} \hat{\alpha}' \Sigma_\eta + \frac{\mathbf{J}_{WY}}{\mathbf{J}_W} \Sigma_Y' \right) \tag{g}$$

Using the market clearing conditions, the aggregate wealth dynamics writes:

$$dW = \left[ \mathbf{W} \hat{\alpha}' \mu_\eta - \hat{C} \right] dt + \left[ \mathbf{W} \hat{\alpha}' \Sigma_\eta \right] dZ \tag{h}$$

In our Markovian setting, the control variables may be written in a feedback form.

Therefore, the optimal consumption process  $\hat{C}(t, \cdot)$  writes  $\hat{C}(t, W, Y)$ . Applying Ito's lemma then yields:

$$\frac{d\hat{C}}{\hat{C}} = \mu_{\hat{C}} dt + \sigma_{\hat{C}}' dZ \tag{i}$$

where:

$$\sigma_{\hat{C}} = \frac{\mathbf{W} \hat{C}_W \Sigma_\eta' \hat{\alpha} + \Sigma_Y' \hat{C}_Y}{\hat{C}} \tag{j}$$

From the envelope condition (a), we have:

$$U_{\hat{c}} = J_w, \hat{C}_w U_{\hat{c}\hat{c}} = J_{ww} \text{ and } \hat{C}_Y U_{\hat{c}\hat{c}} = J_{wY} \quad (k)$$

and therefore:

$$\hat{C}_w \frac{U_{\hat{c}\hat{c}}}{U_{\hat{c}}} = \frac{J_{ww}}{J_w} \text{ and } \hat{C}_Y \frac{U_{\hat{c}\hat{c}}}{U_{\hat{c}}} = \frac{J_{wY}}{J_w} \quad (l)$$

Using this, we have:

$$\frac{WJ_{ww}}{J_w} \hat{\alpha}' \Sigma_\eta + \frac{J_{wY}}{J_w} \Sigma_Y = \frac{U_{\hat{c}\hat{c}}}{U_{\hat{c}}} \left( W \hat{C}_w \hat{\alpha}' \Sigma_\eta + \hat{C}_Y \Sigma_Y \right) \quad (m)$$

Using this and (g) yields the second part of Proposition 2.

The first part of the proposition follows from the same approach applied to (f) instead of (e).

Derivation of (14) when the cash-and-carry relation holds. Consider a forward contract of maturity T written on a technology. Assuming the price of this contract satisfies the cash-and-carry relation, we have:

$$F(t, T) = \frac{\eta(t)}{P(t, T)} \quad (n)$$

Using:

$$\begin{aligned} \frac{dF}{F} &= \mu_F dt + \sigma_F' dZ \\ \frac{d\eta}{\eta} &= \mu_\eta dt + \sigma_\eta' dZ \\ \frac{dP}{P} &= \mu_P dt + \sigma_P' dZ \end{aligned} \quad (o)$$

and since the CCAPM holds for cash assets, we have:

$$\begin{aligned} \mu_P - r &= - \frac{\hat{C}U_{\hat{c}\hat{c}}}{U_{\hat{c}}} \sigma_P' \sigma_{\hat{c}} \\ \mu_S - r &= - \frac{\hat{C}U_{\hat{c}\hat{c}}}{U_{\hat{c}}} \sigma_S' \sigma_{\hat{c}} \end{aligned} \quad (p)$$

Applying Itô's lemma to (n), one gets:

$$\frac{dF}{F} = (\mu_\eta - \mu_P - \sigma_\eta' \sigma_P + \sigma_P' \sigma_P) dt + (\sigma_\eta - \sigma_P)' dZ \quad (q)$$

and therefore:

$$\begin{aligned} \mu_F &= \mu_\eta - \mu_P - \sigma_\eta' \sigma_P + \sigma_P' \sigma_P \\ \sigma_F &= \sigma_\eta - \sigma_P \end{aligned} \quad (r)$$

Using the preceding results, one gets:

$$\begin{aligned} \mu_F &= \mu_\eta - \mu_P - \sigma_\eta' \sigma_P + \sigma_P' \sigma_P \\ &= r - \frac{\hat{C}U_{\hat{C}\hat{C}}}{U_{\hat{C}}} \sigma_\eta' \sigma_{\hat{C}} - r + \frac{\hat{C}U_{\hat{C}\hat{C}}}{U_{\hat{C}}} \sigma_P' \sigma_{\hat{C}} - (\sigma_\eta - \sigma_P)' \sigma_P \\ &= -\frac{\hat{C}U_{\hat{C}\hat{C}}}{U_{\hat{C}}} (\sigma_\eta - \sigma_P)' \sigma_{\hat{C}} - (\sigma_\eta - \sigma_P)' \sigma_P \\ &= -\frac{\hat{C}U_{\hat{C}\hat{C}}}{U_{\hat{C}}} \sigma_F' \sigma_{\hat{C}} - \sigma_F' \sigma_P \end{aligned} \quad (s)$$

which is equation (14).

*Proof of Proposition 3.* The price at each time  $t$  of the discount bond  $P(t, \cdot)$  can also be written in a feedback form as  $P(t, Y(t), W(t))$ . Applying Ito's lemma yields the following expression for  $\sigma_P$ :

$$\sigma_P = \frac{1}{P} (\Sigma_\eta' \hat{\alpha} W P_W + \Sigma_Y' P_Y) \quad (t)$$

where  $P_Y$  is the  $K \times 1$  vector of partial derivatives of the discount bond price with respect to the state variables. Substituting for  $\sigma_P$  given by (t) into (13) and rearranging terms by using definitions (17) and (18) yields the desired result (15).

The same approach is followed to prove (16), using (14) instead of (13).

*Proof of Proposition 4.* To prove this result, we replicate the contingent claim to be priced by constructing a portfolio  $\bar{F}$  comprising the optimally invested wealth, the forward contract and the riskless asset. Throughout the proof, to ease the notation, we will denote the value of the riskless asset at each date  $t$  by  $F^0$  and by  $F^1$  the value of

the optimally invested wealth. The number of units of each one of the assets held by the investor will be written as  $\theta_i$ . The value of the contingent claim at each time  $t$  is equal to the value of the replicating strategy. Therefore, we have:

$$\bar{F} = \theta_0 F^0 + P \int_0^t \theta dF + \theta_1 F^1 \quad (u)$$

Since it is a self-financing trading strategy, its dynamics is given by:

$$d\bar{F} = \theta_0 dF^0 + dP \int_0^t \theta dF + P\theta dF + \theta d \langle F; P \rangle + \theta_1 dF^1 \quad (v)$$

where  $d \langle F; P \rangle$  stands for the instantaneous covariance between the discount bond price changes and the forward price changes.

From (15), the instantaneous expected rate of return on the optimally invested wealth is  $\lambda_w + r$  and its volatility, from (h), is  $\hat{\alpha}' \Sigma_\eta$ . Using these and equations (5) and (7), (v) can be rewritten:

$$d\bar{F} = \left[ \begin{array}{l} \theta_0 F^0 r + (\bar{F} - \theta_0 F^0 - \theta_1 F^1) \mu_P + P\theta F \mu_F + \theta_1 F^1 r \\ + \lambda_w \theta_1 + P\theta F \sigma_F' \sigma_P \end{array} \right] dt \quad (w)$$

$$+ \left[ (\bar{F} - \theta_0 F^0 - \theta_1 F^1) \sigma_P' + P\theta F \sigma_F' + \theta_1 F^1 (\hat{\alpha}' \Sigma_\eta) \right] dZ$$

Substituting for (15) and (16), one gets the expected return on the contingent claim:

$$\begin{aligned} \mu_{\bar{F}} \bar{F} &= \theta_0 F^0 r + \theta_1 F^1 r + \theta_1 \lambda_w \\ &+ (\bar{F} - \theta_0 F^0 - \theta_1 F^1) \left( r + \frac{1}{P} [\lambda_w \quad \lambda_{Y_1} \quad \dots \quad \lambda_{Y_k}] [P_w \quad P_{Y_1} \quad \dots \quad P_{Y_k}]' \right) \quad (x) \\ &+ P [\lambda_w \quad \lambda_{Y_1} \quad \dots \quad \lambda_{Y_k}] [F_w \quad F_{Y_1} \quad \dots \quad F_{Y_k}]' \end{aligned}$$

Rearranging terms, one gets:

$$\mu_{\bar{F}} - r = \frac{W}{\bar{F}} [\lambda_w \quad \lambda_{Y_1} \quad \dots \quad \lambda_{Y_k}] \begin{bmatrix} F_w & P_w & 1 \\ F_{Y_1} & P_{Y_1} & 0 \\ \cdot & \cdot & \cdot \\ F_{Y_k} & P_{Y_k} & 0 \end{bmatrix} \left[ \begin{array}{l} P\theta \\ \frac{1}{P} (\bar{F} - \theta_0 F^0 - \theta_1 F^1) \\ \theta_1 \end{array} \right] \quad (y)$$

Equation (t) relative to the volatility  $\sigma_P$  of the discount bond applies as well to that of the forward contract. Thus:

$$\sigma_F = \frac{1}{F} \left( \Sigma_\eta' \hat{\alpha} W F_W + \Sigma_Y' F_Y \right) \quad (z)$$

From (w), the instantaneous volatility of the contingent claim price is equal to:

$$\left( \bar{F} - \theta_0 F^0 - \theta_1 F^1 \right) \sigma_P + P \theta F \sigma_F + \theta_1 F^1 \left( \Sigma_\eta' \hat{\alpha} \right) \quad (aa)$$

This can be written as:

$$\begin{aligned} & \left( \bar{F} - \theta_0 F^0 - \theta_1 F^1 \right) \sigma_P + P \theta F \sigma_F + \theta_1 F^1 \left( \Sigma_\eta' \hat{\alpha} \right) \\ &= \begin{bmatrix} \sigma_P & \sigma_F & \Sigma_\eta' \hat{\alpha} \end{bmatrix} \begin{bmatrix} \bar{F} - \theta_0 F^0 - \theta_1 F^1 \\ P \theta F \\ \theta_1 F^1 \end{bmatrix} \end{aligned} \quad (ab)$$

Using (t) and (z), we obtain:

$$\begin{aligned} & \left( \bar{F} - \theta_0 F^0 - \theta_1 F^1 \right) \sigma_P + P \theta F \sigma_F + \theta_1 F^1 \left( \Sigma_\eta' \hat{\alpha} \right) \\ &= \begin{bmatrix} W \Sigma_\eta' \hat{\alpha} & \Sigma_Y' \end{bmatrix} \begin{bmatrix} F_W & P_W & 1 \\ F_{Y_1} & P_{Y_1} & 0 \\ \cdot & \cdot & \cdot \\ F_{Y_K} & P_{Y_K} & 0 \end{bmatrix} \begin{bmatrix} P \theta \\ \frac{1}{P} \left( \bar{F} - \theta_0 F^0 - \theta_1 F^1 \right) \\ \theta_1 \end{bmatrix} \end{aligned} \quad (ac)$$

Applying Ito's lemma to  $\bar{F} = \bar{F}(t, W, Y)$  and identifying terms, we have:

$$\begin{bmatrix} \bar{F}_W & \bar{F}_{Y_1} & \cdot & \cdot & \bar{F}_{Y_K} \end{bmatrix}' = \begin{bmatrix} F_W & P_W & 1 \\ F_{Y_1} & P_{Y_1} & 0 \\ \cdot & \cdot & \cdot \\ F_{Y_K} & P_{Y_K} & 0 \end{bmatrix} \begin{bmatrix} P \theta \\ \frac{1}{P} \left( \bar{F} - \theta_0 F^0 - \theta_1 F^1 \right) \\ \theta_1 \end{bmatrix} \quad (ad)$$

Using (y) and (ad) and substituting  $S^i$  for  $\bar{F}$  yields the desired result, equation (19).

Proof of Proposition 5. Applying Ito's lemma to  $F(t, Y, W)$  we get:

$$\begin{aligned} \mu_F F &= \frac{1}{2} W^2 \sigma_{WW} F_{WW} + \sum_{j=1}^K \sigma_{WY_j} F_{WY_j} + \frac{1}{2} \sum_{j=1}^K \sum_{k=1}^K \sigma_{Y_j Y_k} F_{Y_j Y_k} \\ &+ \left( \hat{\alpha}' \mu_\eta W - \hat{C} \right) F_W + \sum_{j=1}^K \mu_{Y_j} F_{Y_j} + F_t \end{aligned} \quad (ae)$$

Using (ae) and (16) yields the desired result, equation (20).

Proof of Proposition 6. In the special case of a logarithmic utility function, the standard solution for the investor's value function (see Ingersoll (1987) page 274) is:

$$J(t, W) = e^{-\rho t} \frac{1 - e^{-\rho(\tau-t)}}{\rho} \text{Ln}W(t) \quad (\text{af})$$

Using the envelope condition, we have:

$$\hat{C}(t) = \frac{\rho}{1 - e^{-\rho(\tau-t)}} W(t)$$

which is equation (25).

Now, using (25), we have in equilibrium:

$$dW(t) = W(t) \left( \mu_\eta - \frac{\rho}{1 - e^{-\rho(\tau-t)}} \right) dt + W(t) \sigma_\eta dZ(t) \quad (\text{ag})$$

which yields the desired result (24).

Using (c) and (f) and the fact that all the wealth is invested in the technology in equilibrium, we have:

$$\begin{aligned} (\mu_\eta - \mu_P) J_W + \sigma_\eta (\sigma_\eta - \sigma_P) W J_{WW} &= 0 \\ (r - \mu_P) J_W - \sigma_P \sigma_\eta W J_{WW} &= 0 \end{aligned} \quad (\text{ah})$$

Solving this system yields:

$$r = \mu_\eta - \sigma_\eta^2 \quad (\text{ai})$$

Identifying from (22) yields the desired result (26).

Proof of equation (27). We can use here the modern pricing kernel methodology, since investors are myopic, implying that random changes in the investment opportunity set do not matter to them. The price of the forward contract is such that:

$$0 = E_t [\Lambda(\tau)(\eta(\tau) - F(\tau))] \quad (\text{aj})$$

where  $\Lambda(\tau)$  is the pricing kernel defined by  $\Lambda(t) = U_{\hat{C}}(t, \hat{C})$ .

Thus, (aj) can be written as:

$$F(t) E_t [\Lambda(\tau)] = E_t [\Lambda(\tau) \eta(\tau)] \quad (\text{ak})$$

Consider a discount bond maturing at time  $\tau$ . Its price at time  $t$  is such that:

$$P(t, \tau)\Lambda(t) = E_t[\Lambda(\tau)1] \quad (\text{al})$$

Therefore, we have:

$$F(t)P(t, \tau)\Lambda(t) = E_t[\Lambda(\tau)\eta(\tau)] \quad (\text{am})$$

By definition, and using (23), (24) and (25), we have:

$$\Lambda(t) = U_{\hat{C}}(t, \hat{C}) = e^{-\rho t} \frac{1}{\hat{C}} = \frac{1}{\frac{\rho}{1 - e^{-\rho\tau}} W(0)\eta(t)} \quad (\text{an})$$

Using (am), and (an) for  $t = t$  and  $t = \tau$ , and simplifying the integrals involved yields the desired result (27).



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