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## Article

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## Article

# Transformations of Telegraph Processes and Their Financial Applications

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**Abstract:** In this paper, we consider non-linear transformations of classical telegraph process. The main results consist of deriving a general partial differential Equation (PDE) for the probability density (pdf) of the transformed telegraph process, and then presenting the limiting PDE under Kac's conditions, which may be interpreted as the equation for a diffusion process on a circle. This general case includes, for example, classical cases, such as limiting diffusion and geometric Brownian motion under some specifications of non-linear transformations (i.e., linear, exponential, etc.). We also give three applications of non-linear transformed telegraph process in finance: (1) application of classical telegraph process in the case of balance, (2) application of classical telegraph process in the case of dis-balance, and (3) application of asymmetric telegraph process. For these three cases, we present European call and put option prices. The novelty of the paper consists of new results for non-linear transformed classical telegraph process, new models for stock prices based on transformed telegraph process, and new applications of these models to option pricing.

**Keywords:** classical telegraph equation; transformations of telegraph equation; asymmetric telegraph equation; Black–Scholes formula; European call and put options

**MSC:** Primary 60K35; Secondary 60K99; 60K15



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## 1. Introduction

In 1951, Goldstein and Kac (see [Goldstein \(1951\)](#); [Kac \(1974\)](#) and also [Kac \(1950\)](#)) proposed an interesting random motion model for the movement of a particle on the line (or one dimension). The particle was moving at a constant velocity  $v$  in any of the two directions and traveling a random distance drawn from an exponential probability distribution with parameter  $\lambda$ . Therefore, this is a random motion driven by a Poisson process with intensity  $\lambda$ . After one movement, the particle changes its direction of motion in the opposite direction under the same stochastic conditions. This particle motion can be modeled as a random motion governed by a switching Poisson process with alternating directions and having exponentially distributed holding times.

In an independent manner, Goldstein and Kac solved this problem and they found that the solution satisfies the one-dimensional telegraph equation, which has a similar mathematical form as the Heaviside telegraph equation appearing in deterministic problems of wave propagation in electrical transmission lines, namely:

$$\frac{\partial^2 f(t, x)}{\partial t^2} + 2\lambda \frac{\partial f(t, x)}{\partial t} = v^2 \frac{\partial^2 f(t, x)}{\partial x^2}.$$

In the case of the random motion probabilistic model this equation is also called the Goldstein—Kac telegraph equation or classical telegraph equation.

This seminal work have been extended in many publications worldwide by introducing variations of this basic idea. For instance, we should mention applications in financial market theory of the one-dimensional jump telegraph process, which is a generalization of the telegraph process, [Ratanov \(2007, 2010\)](#), [López and Ratanov \(2012, 2014\)](#), and [Ratanov and Melnikov \(2008\)](#). Explicit formula for the distribution of the integrated telegraph process (or Kac's process) first appeared in [Janssen and Siebert \(1981\)](#). Its proof was presented in [Steutel \(1985\)](#) (see also [Orsingher \(1990\)](#) for more details). Some connections between telegraph equation and heat equation may be found in [Janssen \(1990\)](#) along with asymptotic properties of integrated telegraph process. Distributions of the integrated telegraph process were obtained in [Di Masi et al. \(1994\)](#) in both symmetric (intensities of transitions are same) and asymmetric cases (intensities of transitions are different). In the hydrodynamic limit, this process approximates the diffusion process on the line. Some probabilistic analysis of the telegrapher's process with drift by means of relativistic transformations were considered in [Beghin et al. \(2001\)](#). Variety of transformations of telegraph process and its association with many areas were studied by Orsingher; see [Orsingher \(1985\)](#); [Orsingher and Beghin \(2009\)](#); [Orsingher and De Gregorio \(2007\)](#); [Orsingher and Ratanov \(2002\)](#); [Orsingher and Somella \(2004\)](#); [Orsingher \(1990\)](#). They include hyperbolic equations, fractional diffusion equations, random flights, planar and cyclic random motions, among others. Some recent works consider a telegraph equation with time-dependent coefficients [Angelani and Garra \(2019\)](#), Markov-modulated Lévy processes with two different regimes of restarting [Ratanov \(2020\)](#), some generalizations of the classical Black–Scholes models in finance [Stoynov \(2019\)](#), jump-telegraph process with exponentially distributed interarrival times [Di Crescenzo and Meoli \(2018\)](#), and a model to describe the vertical motions in the Campi Flegrei volcanic region consisting of a Brownian motion process driven by a generalized telegraph process [Travaglino et al. \(2018\)](#).

The application of the telegraph process for option prices was studied in [Ratanov \(2007\)](#); [Ratanov and Melnikov \(2008\)](#); [Ratanov \(2010\)](#), [Kolesnik and Ratanov \(2013\)](#). Some applications of classical telegraph process in finance were considered in [Pogorui et al. \(2021b\)](#).

We note that the classical telegraph process is the simplest case of one-dimensional random evolutions (REs). A good introduction to RE may be found in [Pinsky \(1991\)](#). Random evolutions driven by the hyper-parabolic operators were considered in [Kolesnik and Pinsky \(2011\)](#). Many ideas, methods, classifications, applications, and examples of REs are presented in the handbook [Swishchuk \(1997\)](#). Many applications of REs in finance and insurance are considered in [Swishchuk \(2000\)](#).

In this paper, we consider transformations of classical telegraph process. We also give three applications of transform telegraph process in finance: (1) application of classical telegraph process ( $h(x) = x$ ) in the case of balance; (2) application of classical telegraph process ( $h(x) = x$ ) in the case of dis-balance; and (3) application of asymmetric telegraph process ( $h(x)$  has a special form presented below) in finance. The novelty of the paper consists of new results for transformed classical telegraph process, new models for stock prices, and new applications of these models to option pricing. Function  $h(x)$  is used to generalize classical telegraph equation to obtain, e.g., asymmetric telegraph process, diffusion process on a circle, etc.

The main idea of application of telegraph process in finance is the following one: Instead of the geometric Brownian motion (GBM) we propose the following model for the price  $S_t$  of a stock at time  $t$  :

$$S_t^e = S_0 \exp(x(t)),$$

where  $x(t) = \int_0^t v(s)ds$ ,  $v(t)$  is a continuous-time Markov chain with state space  $(v_1, v_2)$ , and with  $\lambda_i$  being the rates of the exponential waiting times,  $i = 1, 2$ .

To satisfy Kac's conditions, we consider the scaled model for stock price:

$$S_t^\epsilon = S_0 \exp\left(\epsilon^{-1} \int_0^t v\left(\frac{s}{\epsilon^2}\right) ds\right),$$

and then taking a limit when  $\epsilon \rightarrow 0$ . In this manner, process  $S_t^\epsilon$  converges weakly to GBM  $\hat{S}_t = S_0 e^{\mu t + \sigma W(t)}$ , with specific constants  $\mu$  and  $\sigma$ , and  $\{W(t)\}$  is the Wiener process. We use the latter model to calculate European call and put options pricing.

For example, the modeling of cash flow in high-frequency and algorithmic trading can be fitted by this model. If transactions happen in a short time (i.e., every millisecond), then because of the scale  $1/\epsilon^2$ ,  $v(t/\epsilon^2)$  is switched quickly between two states  $\{v_1, v_2\}$  according to Markov chain  $\zeta(t/\epsilon^2)$ . Hence,  $S_t^\epsilon$  is changing quickly as well. If we use a longer time interval  $t/\epsilon^2$ , instead of time  $t$  in milliseconds, we can fit this model for different purposes such as market making, liquidation, acquisition, etc., purposes. Thus, we can apply our asymptotic results considered above. Our modeling approach is an alternative to the well-known Black–Scholes model based on geometric Brownian motion. It is well-known that Brownian motion (Wiener process) has some mathematical difficulties that make it difficult to fit to real data. For instance, it has trajectories continuous everywhere but differentiable nowhere, it is fractal with Hausdorff dimension equals 1.5, it has zero length of free path, and infinite velocity at any point of time.

The paper is organized as follows: Section 2 considers transformations of classical telegraph process. Section 3 presents financial applications of transformed telegraph process. Section 4 concludes the paper.

## 2. Transformations of Classical Telegraph Process

Suppose that  $x(t)$  is the telegraph process, such that

$$\frac{d}{dt}x(t) = (-1)^{\zeta(t)}v, \quad (1)$$

where  $v > 0$ ,  $\zeta(t)$  is an alternating Markov process with phase space  $X = \{0, 1\}$  and infinitesimal generator matrix (or intensity matrix)

$$Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}.$$

Consider a differentiable function  $h : \mathbf{R} \rightarrow \mathbf{R}$ , such that there exists the inverse function  $h^{-1}$ . Let us introduce the following process,

$$z(t) = h(x(t)). \quad (2)$$

Then,

$$\frac{d}{dt}z(t) = \frac{d}{dt}h(x(t)) = h'(x(t))(-1)^{\zeta(t)}v = h'\left(h^{-1}(z(t))\right)(-1)^{\zeta(t)}v.$$

The bivariate process  $\{z(t), \zeta(t)\}$  is a Markov process [Korolyuk and Swishchuk \(1995\)](#), [Swishchuk \(1997, 2000\)](#), and its infinitesimal generator  $L$  is given by

$$L\varphi(x, n) = h'\left(h^{-1}(x)\right)(-1)^n v \frac{\partial}{\partial x} \varphi(x, n) + Q\varphi(x, n),$$

where  $Q\varphi(x, n) = q_{n0}\varphi(x, 0) + q_{n1}\varphi(x, 1)$ ,  $n = 1, 2$ .

Or in more details

$$L\varphi(x, 0) = v h'\left(h^{-1}(x)\right) \frac{\partial}{\partial x} \varphi(x, 0) - \lambda \varphi(x, 0) + \lambda \varphi(x, 1)$$

$$L\varphi(x, 1) = -v h'\left(h^{-1}(x)\right) \frac{\partial}{\partial x} \varphi(x, 1) - \lambda \varphi(x, 1) + \lambda \varphi(x, 0).$$

Denote by  $f(x, n, t)$ ,  $x \in \mathbf{R}$ ,  $t \geq 0$  the pdf of the process  $\{z(t), \xi(t)\}$  in the case when  $\xi(t) = n$ ,  $n \in \{0, 1\}$ . Then,

$$\frac{\partial}{\partial t} f(x, n, t) = Lf(x, n, t)$$

or in more details

$$\frac{\partial}{\partial t} f(x, 0, t) = vh'(h^{-1}(x)) \frac{\partial}{\partial x} f(x, 0, t) - \lambda f(x, 0, t) + \lambda f(x, 1, t)$$

$$\frac{\partial}{\partial t} f(x, 1, t) = -vh'(h^{-1}(x)) \frac{\partial}{\partial x} f(x, 1, t) - \lambda f(x, 1, t) + \lambda f(x, 0, t).$$

Equivalently, these equations can be expressed in matrix form as follows:

$$\begin{pmatrix} \frac{\partial}{\partial t} - vh'(h^{-1}(x)) \frac{\partial}{\partial x} + \lambda & -\lambda \\ -\lambda & \frac{\partial}{\partial t} + vh'(h^{-1}(x)) \frac{\partial}{\partial x} + \lambda \end{pmatrix} \begin{pmatrix} f(x, 0, t) \\ f(x, 1, t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let us define the following notation,

$$f(x, t) := f(x, 0, t) + f(x, 1, t).$$

It is easily seen that  $f(x, t)dx$  is the pdf of  $z(t)$  at  $x$ , i.e.,  $f(x, t)dx = P(z(t) \in (x, x + dx))$ . Then,  $f(x, t)$  satisfies the following equation,

$$\det \begin{pmatrix} \frac{\partial}{\partial t} - vh'(h^{-1}(x)) \frac{\partial}{\partial x} + \lambda & -\lambda \\ -\lambda & \frac{\partial}{\partial t} + vh'(h^{-1}(x)) \frac{\partial}{\partial x} + \lambda \end{pmatrix} f(x, t) = 0,$$

or equivalently,

$$\left( \frac{\partial^2}{\partial t^2} - v^2 \left( h'(h^{-1}(x)) \right)^2 \frac{\partial^2}{\partial x^2} + 2\lambda \frac{\partial}{\partial t} \right) f(x, t) = 0. \quad (3)$$

It is well-known under Kac's conditions [Kac \(1950\)](#) that the telegraph process  $x(t)$  converges weakly to Wiener process  $W(t)$  and, hence,  $h(x(t))$  converges weakly to  $h(W(t))$ . Passing in the last equation to the Kac's limit [Kac \(1950\)](#), i.e., when  $\lambda \rightarrow +\infty$  and  $v \rightarrow +\infty$  in such a way that  $v^2/\lambda \rightarrow \sigma^2 > 0$ , we obtain

$$\frac{\partial}{\partial t} f_{h(w)}(x, t) = \frac{\sigma^2}{2} (h'(h^{-1}(x)))^2 \frac{\partial^2}{\partial x^2} f_{h(w)}(x, t), \quad (4)$$

where  $f_{h(w)}(x, t)$  is the pdf of the process  $h(W(t))$ .

On the other hand, the pdf  $f(x, t)$  of the telegraph process with the initial density distribution  $g(x)$  and equally probable velocities  $v$  and  $-v$  satisfies the telegraph equation

$$\left( \frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2} + 2\lambda \frac{\partial}{\partial t} \right) f(x, t) = 0, \quad t \geq 0, x \in \mathbf{R}, \quad (5)$$

and initial conditions

$$f(x, 0) = g(x), \quad \frac{\partial}{\partial t} f(x, t) \Big|_{t=0} = 0.$$

It is well-known that the unique solution of Cauchy problem (5) is given by the following formula

$$f(x, t) = \frac{e^{-\lambda t}}{2} (g(vt - x) + g(vt + x)) + \frac{e^{-\lambda t}}{2v} \int_{-vt}^{vt} g(y + x) p(y, t) dy,$$

where  $p(y, t) = \left( \lambda I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - y^2} \right) + \frac{\partial}{\partial t} I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - y^2} \right) \right)$ .

In the particular case, when the telegraph process starts from  $x = 0$ , with equally probable velocities  $v$  and  $-v$ , its pdf  $f(x, t)$  satisfies the telegraph equation

$$\left( \frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2} + 2\lambda \frac{\partial}{\partial t} \right) f(x, t) = 0 \quad (6)$$

with initial conditions  $f(x, 0) = \delta(x)$ ,  $\frac{\partial}{\partial t} f(x, t) \Big|_{t=0} = 0$ , and is given by

$$f(x, t) = \frac{e^{-\lambda t}}{2} (\delta(vt - x) + \delta(vt + x)) + \frac{e^{-\lambda t}}{2v} \left( \lambda I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - x^2} \right) + \frac{\partial}{\partial t} I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - x^2} \right) \right), \quad (7)$$

where  $(vt)^2 \geq x^2$ .

It is also well-known that if a random variable  $X$  has the pdf  $f_X(x)$  and  $Y = h(X)$  and there exists the inverse function  $h^{-1}$ , then the pdf  $f_Y(y)$  of  $Y$  is as follows:

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{dh^{-1}}{dx}(y) \right|.$$

Therefore, the solution  $f_h(x, t)$  of Equation (2) with initial conditions  $f_h(x, 0) = g(h^{-1}(x))$ ,  $\frac{\partial}{\partial t} f_h(x, t) \Big|_{t=0} = 0$  is given by the following formula:

$$f_h(x, t) = \frac{e^{-\lambda t}}{2} \left| \frac{dh^{-1}}{dx}(x) \right| \times \left\{ \left( g(vt - h^{-1}(x)) + g(vt + h^{-1}(x)) \right) + \frac{1}{v} \int_{-vt}^{vt} g(y + h^{-1}(x)) \left( \lambda I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - y^2} \right) + \frac{\partial}{\partial t} I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - y^2} \right) \right) dy \right\},$$

where  $(vt)^2 \geq (h^{-1}(x))^2$ .

In particular, the solution  $f_h(x, t)$  of Equation (2) with initial conditions  $f_h(x, 0) = \delta(h^{-1}(x))$ ,  $\frac{\partial}{\partial t} f_h(x, t) \Big|_{t=0} = 0$  is given by the following formula:

$$f_h(x, t) = \frac{e^{-\lambda t}}{2} \left| \frac{dh^{-1}}{dx}(x) \right| \times \left\{ \left( \delta(vt - h^{-1}(x)) + \delta(vt + h^{-1}(x)) \right) + \frac{1}{v} \left( \lambda I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - (h^{-1}(x))^2} \right) + \frac{\partial}{\partial t} I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - (h^{-1}(x))^2} \right) \right) \right\},$$

where  $(vt)^2 \geq (h^{-1}(x))^2$ .

Now, let us consider a case where  $h$  is a differentiable mapping  $h : \mathbf{R} \rightarrow \mathbf{R}$  or  $(\mathbf{C})$  and the inverse function does not necessarily exist for it.

In particular, in the case where  $h(x) = e^{ix}$ , we have

$$\frac{d}{dt} z(t) = \frac{d}{dt} e^{ix(t)} = ie^{ix(t)} \frac{d}{dt} x(t) = iz(t)(-1)^{\xi(t)} v.$$

Hence,

$$\frac{d}{dt} \cos x(t) = v(-1)^{\xi(t)+1} \sin x(t) \frac{d}{dt} \sin x(t) = v(-1)^{\xi(t)} \cos x(t).$$

By assuming  $\alpha(t) = \cos x(t)$  and  $\beta(t) = \sin x(t)$ , we have

$$\frac{d}{dt}\alpha(t) = v(-1)^{\xi(t)+1}\beta(t)$$

$$\frac{d}{dt}\beta(t) = v(-1)^{\xi(t)}\alpha(t).$$

The process  $\{(\alpha(t), \beta(t)), \xi(t)\}$  is Markov and its infinitesimal generator  $L$  is given by the formula [Korolyuk and Swishchuk \(1995\)](#)

$$\begin{aligned} L\varphi(\alpha, \beta, \xi(t)) \\ = v(-1)^{\xi(t)+1}\beta\frac{\partial}{\partial\alpha}\varphi(\alpha, \beta, \xi(t)) + v(-1)^{\xi(t)}\alpha\frac{\partial}{\partial\beta}\varphi(\alpha, \beta, \xi(t)) + Q\varphi(\alpha, \beta, \xi(t)). \end{aligned}$$

Denoting by  $f(\alpha, \beta, n, t)$ ,  $x \in \mathbf{R}$ ,  $t \geq 0$  the pdf of the process  $\{(\alpha(t), \beta(t)), \xi(t)\}$  in the special case when  $\xi(t) = n$ ,  $n \in \{0, 1\}$ , we have

$$\frac{\partial}{\partial t}f(\alpha, \beta, n, t) = Lf(\alpha, \beta, n, t)$$

or in more details

$$\begin{aligned} \frac{\partial}{\partial t}f(\alpha, \beta, 0, t) = \\ -v\beta\frac{\partial}{\partial\alpha}f(\alpha, \beta, 0, t) + v\alpha\frac{\partial}{\partial\beta}f(\alpha, \beta, 0, t) - \lambda f(\alpha, \beta, 0, t) + \lambda f(\alpha, \beta, 1, t)\frac{\partial}{\partial t}f(\alpha, \beta, 1, t) \\ = v\beta\frac{\partial}{\partial\alpha}f(\alpha, \beta, 1, t) - v\alpha\frac{\partial}{\partial\beta}f(\alpha, \beta, 2, t) - \lambda f(\alpha, \beta, 1, t) + \lambda f(\alpha, \beta, 0, t). \end{aligned} \quad (8)$$

Passing to the polar coordinate system  $\alpha = r \cos \varphi$ ,  $\beta = r \sin \varphi$ , we have

$$\frac{\partial}{\partial\alpha}f = \frac{\partial}{\partial r}f \cos \varphi - \frac{\partial}{\partial\varphi}f \frac{\sin \varphi}{r} \frac{\partial}{\partial\beta}f = \frac{\partial}{\partial r}f \sin \varphi - \frac{\partial}{\partial\varphi}f \frac{\cos \varphi}{r}. \quad (9)$$

Taking into account that  $f$  does not depend on  $r$  and substituting (9) into (8), we have

$$\frac{\partial}{\partial t}f(\varphi, 0, t) = v \sin^2 \varphi \frac{\partial}{\partial\varphi}f(\varphi, 0, t) - v \cos^2 \varphi \frac{\partial}{\partial\varphi}f(\varphi, 0, t) + \lambda f(\varphi, 1, t)$$

$$\frac{\partial}{\partial t}f(\varphi, 1, t) = -v \sin^2 \varphi \frac{\partial}{\partial\varphi}f(\varphi, 1, t) + v \cos^2 \varphi \frac{\partial}{\partial\varphi}f(\varphi, 0, t) - \lambda f(\varphi, 1, t) + \lambda f(\varphi, 0, t).$$

Denote by

$$f(\varphi, t) := f(\varphi, 0, t) + f(\varphi, 1, t).$$

In much the same manner as we obtained (3), we have

$$\det \begin{pmatrix} a_{11} & -\lambda \\ -\lambda & a_{22} \end{pmatrix} f(\varphi, t) = 0,$$

where

$$\begin{aligned} a_{11} &= \frac{\partial}{\partial t} - v \sin^2 \varphi \frac{\partial}{\partial\varphi} + v \cos^2 \varphi \frac{\partial}{\partial\varphi} + \lambda \\ a_{22} &= \frac{\partial}{\partial t} + v \sin^2 \varphi \frac{\partial}{\partial\varphi} - v \cos^2 \varphi \frac{\partial}{\partial\varphi} + \lambda. \end{aligned}$$

It is easily verified that

$$\det \begin{pmatrix} a_{11} & -\lambda \\ -\lambda & a_{22} \end{pmatrix} = \frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} - \cos^2 2\varphi \frac{\partial^2}{\partial\varphi^2} + \frac{\sin 4\varphi}{2} \frac{\partial}{\partial\varphi}.$$



Therefore, the pdf  $f(\varphi, t)$  satisfies the following equation:

$$\left( \frac{\partial^2}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} - v^2 \cos^2 2\varphi \frac{\partial^2}{\partial \varphi^2} + v^2 \frac{\sin 4\varphi}{2} \frac{\partial}{\partial \varphi} \right) f(\varphi, t) = 0, \quad (10)$$

with initial conditions  $f(\varphi, 0) = \delta(\varphi)$ ,  $\frac{\partial}{\partial \varphi} f(\varphi, t) \Big|_{\varphi=0} = 0$ .

Passing to the Kac's limit in (10) as  $\lambda \rightarrow \infty$  and  $v \rightarrow \infty$ , such that  $\frac{v^2}{\lambda} \rightarrow \sigma^2 > 0$  we obtain

$$\frac{\partial}{\partial t} f(\varphi, t) = \frac{\sigma^2}{2} \left( \cos^2 2\varphi \frac{\partial^2}{\partial \varphi^2} f(\varphi, t) - \frac{\sin 4\varphi}{2} \frac{\partial}{\partial \varphi} f(\varphi, t) \right).$$

By analogy with a diffusion process on a line, this equation can be interpreted as the equation for a diffusion process on a circle [De Gregorio and Iafrate \(2020\)](#).

**Remark 1.** Let us consider the case where  $h(x) = e^x$ , i.e.,  $z(t) = e^{x(t)}$ . Then we have  $h'(h^{-1}(x)) = x$ . That is,

$$\left( \frac{\partial^2}{\partial t^2} - v^2 x^2 \frac{\partial^2}{\partial x^2} + 2\lambda \frac{\partial}{\partial t} \right) f(x, t) = 0, \quad t \geq 0, x \in \mathbf{R}. \quad (11)$$

The solution to the last equation with the initial conditions  $f_h(x, 0) = \delta(\ln(x)) = \delta(x - 1)$ , and  $\frac{\partial}{\partial t} f_h(x, t) \Big|_{t=0} = 0$ , is given by the following formula:

$$f(x, t) = \frac{e^{-\lambda t}}{2|x|} (\delta(vt - \ln(x)) + \delta(vt + \ln(x))) + \frac{1}{v} \left( \lambda I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - (\ln x)^2} \right) + \frac{\partial}{\partial t} I_0 \left( \frac{\lambda}{v} \sqrt{(vt)^2 - (\ln x)^2} \right) \right), \quad (12)$$

where  $(vt)^2 \geq (\ln x)^2$ .

**Remark 2.** We note that if  $\lambda \rightarrow \infty$  and  $v \rightarrow \infty$ , such that  $\frac{v^2}{\lambda} \rightarrow \sigma^2 > 0$ , i.e., under the Kac's condition [Kac \(1974\)](#), the process  $e^{x(t)}$  weakly converges to the geometric Brownian motion  $e^{W(t)}$  that is used for many authors for modeling of a stock price in the Black–Scholes formula.

### 3. Financial Applications of Transformed Telegraph Process

In this section, we give three applications of transformations of telegraph process in finance: (1) application of classical telegraph process ( $h(x) = x$ ) in the case of balance; (2) application of classical telegraph process ( $h(x) = x$ ) in the case of dis-balance; and (3) application of asymmetric telegraph process ( $h(x)$  has a special form presented below) in finance.

We note that the asymmetric telegraph process is a telegraph process where the particle is allowed to move forward or backward with two different velocities,  $v_1, v_2$ , [Beghin et al. \(2001\)](#), [De Gregorio and Iafrate \(2020\)](#), and [López and Ratanov \(2014\)](#). Furthermore, two different velocity switching rates  $\lambda_1$  and  $\lambda_2$ , are allowed in this process. Thus, the underlying telegraph signal can be modeled as a continuous-time Markov chain  $\{X(t), t \geq 0\}$  with state space  $(v_1, -v_2)$ , where  $\lambda_i$  is the rate of the exponential sojourn time when the telegraph signal is in state  $(-1)^{i+1}v_i$ ,  $i = 1, 2$ , and  $X(0)$  is uniformly distributed on  $(v_1, -v_2)$ . Then, the asymmetric telegraph process is defined as  $x(t) = \int_0^t X(s)ds$ .

It is straightforward to recover the classical telegraph process as a particular case when  $v_1 = v_2 = v$ , and  $\lambda_1 = \lambda_2 = \lambda$ .

Thus, in our case, the balance condition for classical telegraph process means  $(v_1 + v_2)/2 = 0$ , i.e.,  $v_2 = -v_1$ , and this is the case because  $v_1 = v_2 = v$ ; thus,  $v + (-v) = 0$ . Dis-balance condition for classical telegraph process means that  $(v_1 + v_2)/2 \neq 0$ , i.e.,  $v_2 \neq -v_1$ . Thus, we have different velocities  $v_1, v_2$ , that satisfy the dis-balance condition.



We note that in the case of asymmetric telegraph process the transformed telegraph process may be presented by the following function  $h(x)$  :

$$h(x(t)) = \int_0^t \left[ \frac{v_1 - v_2}{2} + \frac{v_1 + v_2}{2} (\mathbf{1}_{X(0)=v_1} - \mathbf{1}_{X(0)=-v_2}) (-1)^{N(s)} \right] ds,$$

where  $N(t)$  is a counting process which has intensities  $\lambda_i$  and  $\lambda_{i+(-1)^{i+1}}$  for switching times  $S_{2i-1}$  and  $S_{2i}$ , respectively,  $i = 1, 2$ .

### 3.1. Application of Classical Telegraph Process in Finance: Balance Case

It is possible to derive an analog model of the Black–Scholes formula after an application of the asymptotic results presented in [Pogorui and Rodríguez-Dagnino \(2008, 2009\)](#); [Pogorui et al. \(2021a\)](#).

The classical symmetric telegraph process is defined as follows: a particle is allowed to move forward or backward with velocities  $v, -v$ , in an alternate manner, and the process has a velocity switching rate  $\lambda$ . Hence, the underlying telegraph signal can be modeled as a continuous-time Markov chain  $\{X(t), t \geq 0\}$  with state space  $(v, -v)$ , where  $\lambda$  is the rate of the exponential sojourn time in the interval when the telegraph signal is in state  $(-1)^{i+1}v$ . Therefore, the classical symmetric telegraph process is defined as  $x(t) = \int_0^t X(s) ds$ .

The probability law of the asymmetric telegraph process has an absolutely continuous component  $f$  that satisfies the following hyperbolic Equation (see [Beghin et al. \(2001\)](#)):

$$\frac{\partial^2 f(t, x)}{\partial t^2} = v^2 \frac{\partial^2 f(t, x)}{\partial x^2} - 2\lambda \frac{\partial f(t, x)}{\partial t}.$$

By taking the limits for  $\lambda, v \rightarrow +\infty$ , such that

$$\frac{v}{\sqrt{\lambda}} \rightarrow \sigma,$$

one obtains the governing equation of a Brownian motion (Kac-type condition). it is also possible to show that the classical symmetric telegraph process converges in distribution to a Brownian motion, i.e.,

$$x(t) \rightarrow_d \sigma W(t),$$

where  $W(t)$  is a Wiener process (standard Brownian motion) and

$$\sigma := \frac{v}{\sqrt{\lambda}}.$$

Let us consider the following model for a stock price:

$$S_t = S_0 e^{x(t)}, \quad (13)$$

where  $x(t)$  is a classical symmetric telegraph process. Under above-mentioned Kac's conditions we can state that

$$S_t = S_0 e^{x(t)} \rightarrow_d \hat{S}_t := S_0 e^{\sigma W(t)}.$$

After applying Itô's formula ([Shreve \(2004\)](#)), we found  $\hat{S}_t$  satisfies the following stochastic differential Equation (SDE):

$$d\hat{S}_t = b\hat{S}_t dt + \sigma\hat{S}_t dW(t), \quad (14)$$

where

$$b := \frac{\sigma^2}{2}.$$

Now, let us define the following process:

$$Z(t) := \exp \left\{ \left( \frac{b-r}{\sigma} \right) W_t - \frac{1}{2} \left( \frac{b-r}{\sigma} \right)^2 t \right\},$$

where  $r > 0$  is the interest rate, and  $W_t = W(t)$ .

Then, it is not difficult to see that  $\mathbf{E}_P[Z(t)] = 1$ , hence according to Novikov's result [Novikov \(1980\)](#), this process is a positive martingale. Now, let us define the probability measure  $Q$  (recall  $\mathbf{E}_P[Z(t)] = 1$ ) on a complete probability space  $(\Omega, \mathcal{F})$ :

$$Q(A) := \mathbf{E}_P[Z(T)\mathbf{1}_A],$$

where  $\mathbf{1}_A$  is the indicator operator of the set  $A \in \mathcal{F}$ , and the process  $Z(t)$  is defined above.

In a similar manner, we can define the following process:

$$\hat{W}_t := W_t - \left( \frac{b-r}{\sigma} \right) t,$$

where  $b$  is defined above.

We can find that the stochastic process  $\hat{W}_t$  is a standard Wiener process after applying Girsanov's theorem [Shreve \(2004\)](#), under the probability measure  $Q$ . After this fact, measure  $Q$  is called a risk-neutral or martingale measure. Then, our stock price  $\hat{S}_t$  in (14), under the risk-neutral measure  $Q$ , satisfies the following SDE:

$$d\hat{S}_t = r\hat{S}_t dt + \sigma\hat{S}_t d\hat{W}_t, \quad \sigma = \frac{v}{\sqrt{\lambda}}. \quad (15)$$

Therefore, we can write the equivalent Black-Scholes formula for European call option price  $C(t)$  for our model in (15):

$$C(t) = S_0 N(d_1) - Ke^{-r(T-t)} N(d_2), \quad (16)$$

where

$$d_1 := \frac{\ln(S_0/K) + (r + v^2/2\lambda)(T-t)}{(v/\sqrt{\lambda})\sqrt{T-t}}, \quad (17)$$

$$d_2 := d_1 - (v/\sqrt{\lambda})\sqrt{T-t},$$

and

$$N(x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$$

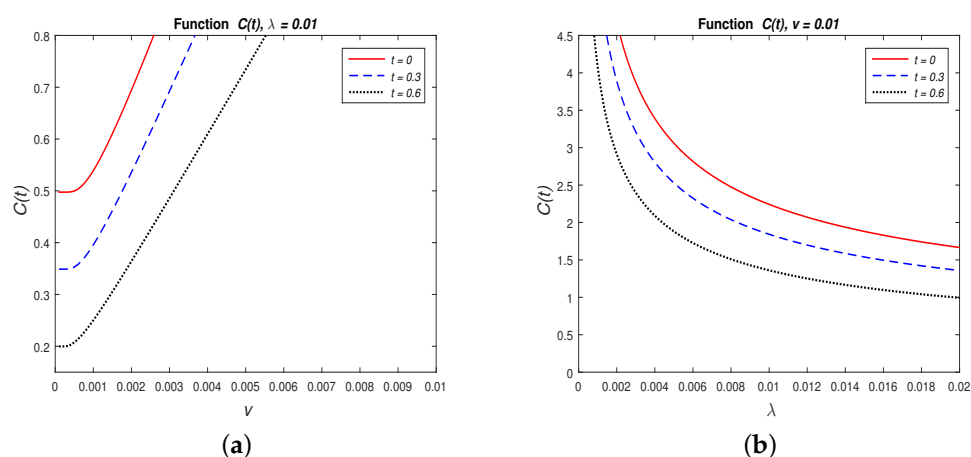
is the cdf (cumulative distribution function) of a standard normal random variable with zero mean and unit variance,  $K$  is a strike price, and  $T$  is the maturity.

**Example 1 (European call option for limiting telegraph process in balance case).** *Let us suppose the following numerical values:  $S_0 = \$50$ ,  $K = \$50$ ,  $r = 0.01$ ,  $v = 0.01$ ,  $\lambda = 0.01$ ,  $T = 1$ . Then, applying Formulas (16) and (17), we can obtain the following European call option price at time  $t = 0$ :*

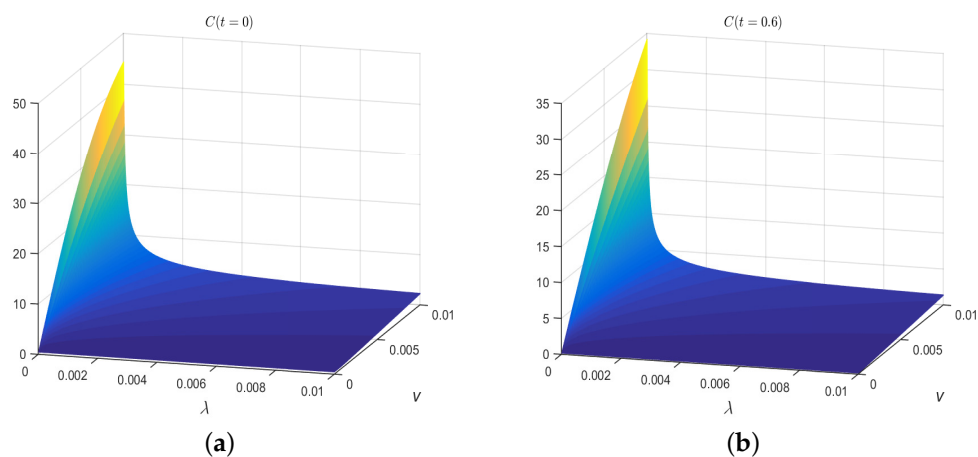
$$C(0) = 50 \times 0.559617 - 50 \times e^{-0.01 \times 1} \times 0.519938 = \$2.2426$$

or  $C(0) = 224.26$  cents.

Below we show the numerical values of time evolution of  $C(t)$  dependent on  $\lambda$  (upon fixing  $v$ ), on  $v$  (upon fixing  $\lambda$ ), see [Figure 1](#), and of  $C(t)$  as a function of  $v$  and  $\lambda$  after fixing  $t$ , see [Figure 2](#).

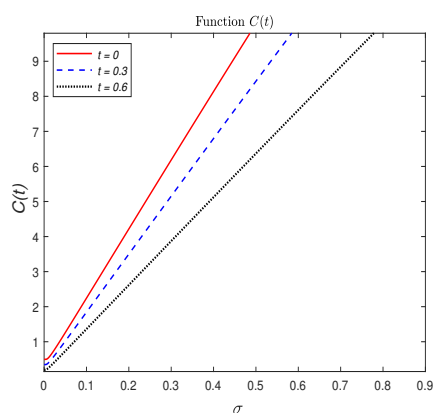


**Figure 1.** Time evolution of European call option price  $C(t)$  as a function of  $v$  (a) and  $\lambda$  (b).



**Figure 2.** Time evolution of European call option price  $C(t)$  as a function of  $v$  and  $\lambda$  for  $t = 0$  (a) and  $t = 0.6$  (b).

**Remark 3.** In Figure 3 we show the cost behavior according to Equation (16) as a function of  $\sigma$ . The Black–Scholes limit case occurs by letting  $\sigma = v/\sqrt{\lambda} = 0.01/\sqrt{0.01} = 0.1$ .



**Figure 3.** Time evolution of European call option price  $C(t)$  as a function of  $\sigma$ .

Now, we can say that on longer time interval the BS formula works better but on shorter time interval our formula produces a better performance. The same for volatility: If volatility is bigger than 0.1, then  $C(t)$  is bigger.

### 3.2. Application of Classical Telegraph Process in Finance: Dis-Balance Case

Now, we study the one-dimensional transport process in the case of dis-balance. We consider first the scaled telegraph process and its limiting case (Section 3.2.1). Then, we applied the limiting process to option pricing; the stock price in this case is modeled as a geometric limiting telegraph process (Section 3.2.2).

#### 3.2.1. Asymptotic Results for Scaled Telegraph Process

Consider a Markov process  $\{\xi(t), t \geq 0\}$  with two states  $\{0, 1\}$  and the generator matrix

$$Q = \lambda \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let us introduce the following random evolution or transport process

$$x(t) = x + \int_0^t v(s) ds, \quad (18)$$

where

$$v(t) = \begin{cases} v_0, & \text{when } \xi(t) = 0; \\ v_1, & \text{when } \xi(t) = 1. \end{cases}$$

The generator  $A$  of the bi-variate process

$$\{\zeta(t) = (x(t), \xi(t)), t \geq 0\}$$

is as follows

$$\begin{aligned} A\varphi(x, 0) &= v_0 \frac{\partial}{\partial x} \varphi(x, 0) + \lambda \varphi(x, 1) - \lambda \varphi(x, 0) \\ A\varphi(x, 1) &= v_1 \frac{\partial}{\partial x} \varphi(x, 1) + \lambda \varphi(x, 0) - \lambda \varphi(x, 1), \end{aligned}$$

where  $\varphi \in D(A)$  is the domain of  $A$ , and  $x \in \mathbf{R}$ .

The generator  $A$  can be interpreted in the following equivalent manner: Denote by  $\mathbf{Z} = \mathbf{R} \times \{0, 1\}$  and

$$T_t \varphi(x, i) = \int_{\mathbf{R}} \varphi(z) P\{\zeta(t) \in dz \mid \zeta(0) = (x, i)\}, \quad i \in \{0, 1\}.$$

Then,

$$A\varphi(x, i) = \lim_{\Delta t \rightarrow 0^+} \frac{T_{\Delta t} \varphi(x, i) - \varphi(x, i)}{\Delta t}.$$

Considering  $u_i(x, t) = T_t \varphi(x, i)$ , we have

$$\frac{\partial}{\partial t} u_i(x, t) = A T_t \varphi(x, i) = A u_i(x, t).$$

Let us introduce the scaled telegraph process

$$x_\varepsilon(t) = x_0 + \frac{1}{\varepsilon} \int_0^t v\left(\frac{s}{\varepsilon^2}\right) ds, \quad (19)$$

with velocities  $v_i/\varepsilon, i \in \{0, 1\}$ . It is easily seen that  $v\left(\frac{t}{\varepsilon^2}\right)$  is Markovian and its generator is of the form  $\frac{1}{\varepsilon^2} Q$ .

Thus, we have the system of Kolmogorov differential equations:

$$\frac{\partial}{\partial t} u_0^\varepsilon(x, t) = \frac{v_0}{\varepsilon} \frac{\partial}{\partial x} u_0^\varepsilon(x, t) + \frac{\lambda}{\varepsilon^2} u_1^\varepsilon(x, t) - \frac{\lambda}{\varepsilon^2} u_0^\varepsilon(x, t)$$

$$\frac{\partial}{\partial t} u_1^\varepsilon(x, t) = \frac{v_1}{\varepsilon} \frac{\partial}{\partial x} u_1^\varepsilon(x, t) + \frac{\lambda}{\varepsilon^2} u_0^\varepsilon(x, t) - \frac{\lambda}{\varepsilon^2} u_1^\varepsilon(x, t), \quad (20)$$

or in matrix form

$$\frac{\partial}{\partial t} \mathbf{u}^\varepsilon(x, t) = \frac{1}{\varepsilon} V \nabla \mathbf{u}^\varepsilon(x, t) + \frac{1}{\varepsilon^2} Q \mathbf{u}^\varepsilon(x, t) \quad (21)$$

where

$$\mathbf{u}^\varepsilon(x, t) = \begin{pmatrix} u_0^\varepsilon(x, t) \\ u_1^\varepsilon(x, t) \end{pmatrix}, \quad V = \begin{pmatrix} v_0 & 0 \\ 0 & v_1 \end{pmatrix},$$

and

$$V \nabla = \begin{pmatrix} v_0 \frac{\partial}{\partial x} & 0 \\ 0 & v_1 \frac{\partial}{\partial x} \end{pmatrix}, \quad Q = \lambda \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now, consider the potential matrix of  $\zeta(t)$ , [Korolyuk and Swishchuk \(1995\)](#), [Swishchuk \(1997, 2000\)](#):

$$R_0 = \int_0^\infty (\Pi - e^{Qt}) dt,$$

where  $e^{Qt} = \{p_{ij}(t); i, j \in \{0, 1\}\}$  are transition probabilities, and

$$\Pi = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

is the projector matrix on  $N(Q) = \ker(Q)$ . It is easily verified that  $R_0 = \Pi - (Q + \Pi)^{-1}$ .

The balance condition implies  $\frac{v_0 + v_1}{2} = 0$ , i.e.,  $v_1 = -v_0$ . It is easily verified that the balance condition can be also written as  $\Pi V \Pi = 0$  (see [Korolyuk and Swishchuk \(1995\)](#), [Swishchuk \(1997, 2000\)](#)).

We are interested in the following dis-balance case:  $v_0 = v + \Delta_1$  and  $v_1 = -v - \Delta_2$ , where  $\Delta_i = \varepsilon a_i$ ,  $i = 1, 2$ ,  $\varepsilon > 0$ . It is easy to see that the generator of  $\zeta_\varepsilon(t) = (x_\varepsilon(t), \zeta(\frac{t}{\varepsilon^2}))$  has the following form:

$$A_\varepsilon = \frac{1}{\varepsilon} V \nabla + A \nabla + \frac{1}{\varepsilon^2} Q,$$

where

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Denote as  $u_i^\varepsilon(x, t) = T_t^\varepsilon \varphi(x, i) = \int_{\mathbb{Z}} \varphi(z) P\{\zeta_\varepsilon(t) \in dz / \zeta_\varepsilon(0) = (x, i)\}$ .

Then, much in the same way we obtain the following matrix equation

$$\frac{\partial}{\partial t} \mathbf{u}^\varepsilon(x, t) = \frac{1}{\varepsilon} V \nabla \mathbf{u}^\varepsilon(x, t) + A \nabla \mathbf{u}^\varepsilon(x, t) + \frac{1}{\varepsilon^2} Q \mathbf{u}^\varepsilon(x, t). \quad (22)$$

By applying asymptotic expansion, [Korolyuk and Swishchuk \(1995\)](#), we have:

$$\mathbf{u}^\varepsilon(x, t) = \mathbf{u}^{(0)}(x, t) + \sum_{n=1}^{\infty} \varepsilon^n \left( \mathbf{u}^{(n)}(x, t) + \mathbf{v}^{(n)}\left(x, t/\varepsilon^2\right) \right), \quad (23)$$

where  $\mathbf{u}^{(n)}(x, t)$ ,  $n = 0, 1, 2, \dots$  are the regular terms of the expansion whereas  $\mathbf{v}^{(n)}(x, t/\varepsilon^2)$ ,  $n = 1, 2, \dots$  are the singular ones.

Then, by substituting (23) into (22), we obtain

$$\begin{aligned} Q \mathbf{u}^{(0)}(x, t) &= 0, & Q \mathbf{u}^{(1)}(x, t) + V \nabla \mathbf{u}^{(0)}(x, t) &= 0, \\ Q \mathbf{u}^{(2)}(x, t) + V \nabla \mathbf{u}^{(1)}(x, t) + A \nabla \mathbf{u}^{(0)}(x, t) - \frac{\partial}{\partial t} \mathbf{u}^{(0)}(x, t) &= 0, \\ Q \mathbf{u}^{(k+2)}(x, t) + V \nabla \mathbf{u}^{(k+1)}(x, t) + A \nabla \mathbf{u}^{(k)}(x, t) - \frac{\partial}{\partial t} \mathbf{u}^{(k)}(x, t) &= 0, \end{aligned} \quad (24)$$

for  $k \geq 0$ .

Thus,  $\mathbf{u}^{(0)}(x, t) \in N(Q)$ , i.e.,  $\Pi \mathbf{u}^{(0)}(x, t) = \mathbf{u}^{(0)}(x, t)$ .

From (24) it follows that

$$\mathbf{u}^{(1)}(x, t) = R_0 V \nabla \mathbf{u}^{(0)}(x, t) + c_1(t), \quad (25)$$

where  $c_1(t) \in N(Q)$ .

Much in the same way, we have

$$\begin{aligned} Q \mathbf{u}^{(2)}(x, t) &= \frac{\partial}{\partial t} \mathbf{u}^{(0)}(x, t) - V \nabla \mathbf{u}^{(1)}(x, t) \\ &= \frac{\partial}{\partial t} \mathbf{u}^{(0)}(x, t) - V \nabla R_0 V \nabla \mathbf{u}^{(0)}(x, t) - V \nabla c_1(t) - A \nabla \mathbf{u}^{(0)}(x, t). \end{aligned} \quad (26)$$

From the properties of  $\Pi$  it follows that

$$\begin{aligned} \Pi Q \mathbf{u}^{(2)}(x, t) &= 0 \\ &= \frac{\partial}{\partial t} \mathbf{u}^{(0)}(x, t) - \Pi V \nabla R_0 V \nabla \Pi \mathbf{u}^{(0)}(x, t) - \Pi A \nabla \Pi \mathbf{u}^{(0)}(x, t). \end{aligned} \quad (27)$$

Hence, the first term  $\mathbf{u}^{(0)}(x, t)$  of the Expansion (23) satisfies the diffusion Equation (27).

The matrix Equation (22) can be written as follows:

$$\begin{aligned} &\left( \begin{array}{cc} \frac{\partial}{\partial t} - \frac{v}{\varepsilon} \frac{\partial}{\partial x} - a_1 \frac{\partial}{\partial x} + \frac{\lambda}{\varepsilon^2} & \frac{\lambda}{\varepsilon^2} \\ \frac{\lambda}{\varepsilon^2} & \frac{\partial}{\partial t} + \frac{v}{\varepsilon} \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial x} + \frac{\lambda}{\varepsilon^2} \end{array} \right) \mathbf{u}^{(\varepsilon)}(x, t) \\ &= \Psi \mathbf{u}^{(\varepsilon)}(x, t) = 0. \end{aligned}$$

It is easily seen that the function  $f_\varepsilon(x, t) = u_0^\varepsilon(x, t) + u_1^\varepsilon(x, t)$  is the solution of the following equation:

$$\det(\Psi) f_\varepsilon(x, t) = 0. \quad (28)$$

Let us write (28) in more detail as follows:

$$\begin{aligned} &\left[ \left( 2\lambda \frac{\partial}{\partial t} - v^2 \frac{\partial^2}{\partial x^2} + \lambda(a_2 - a_1) \frac{\partial}{\partial x} \right) \right. \\ &\left. + \varepsilon^2 \left( \frac{\partial^2}{\partial t^2} + (a_2 - a_1) \frac{\partial^2}{\partial x \partial t} - \frac{v(a_2 + a_1)}{\varepsilon} \frac{\partial^2}{\partial x^2} - a_1 a_2 \frac{\partial^2}{\partial x^2} \right) \right] \\ &\times f_\varepsilon(x, t) = 0. \end{aligned} \quad (29)$$

Let us define the notation  $u^{(0)}(x, t) := (u_0(x, t), u_1(x, t))$  and  $f_0(x, t) := u_0(x, t) + u_1(x, t)$ . Since  $\mathbf{u}^\varepsilon(x, t) > \mathbf{u}^{(0)}(x, t)$  as  $\varepsilon > 0$  [Korolyuk and Swishchuk \(1995\)](#), we have  $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x, t) = f_0(x, t)$ .

From (29) it follows that

$$\left( \frac{\partial}{\partial t} - \frac{v^2}{2\lambda} \frac{\partial^2}{\partial x^2} + \frac{a_2 - a_1}{2} \frac{\partial}{\partial x} \right) f_0(x, t) = 0. \quad (30)$$

Hence, if  $a_2 \neq a_1$ , then  $f_0(x, t)$  satisfies the diffusion Equation (30) with drift  $\frac{a_1 - a_2}{2}$  and diffusion  $\frac{v^2}{\lambda}$ .

**Remark 4.** It follows from (30) that scaled telegraph process  $x_\varepsilon(t)$  in (19) weakly converges to a diffusion process with drift coefficient  $\frac{a_1 - a_2}{2}$  and diffusion coefficient  $\frac{v^2}{\lambda}$ .

### 3.2.2. Application in Finance: Black-Scholes Formula for Geometric Limiting Telegraph Process

The well-known geometric Brownian motion (GBM) is used to model a price  $S_t$  of a stock (Shreve (2004)) at time  $t$ , such that

$$S_t = S_0 \exp(\mu t + \sigma W_t), \quad (31)$$

where  $\mu$  and  $\sigma$  are the drift and volatility of the stock, respectively, and  $\{W_t\}$  is the Wiener process.

After substituting  $\mu = \frac{a_1 - a_2}{2}$  and  $\sigma = \frac{v}{\sqrt{\lambda}}$  in (31) the diffusion equation for the process  $\{\mu t + \sigma W_t\}$  can be obtained. Therefore, the Black-Scholes formula is obtained by considering an exponential Brownian motion for the share price  $S_t$ .

As a consequence, we propose the following formula for the price  $S_t^\epsilon$  of a stock at time  $t$  (see (19)):

$$S_t^\epsilon = S_0 \exp\left(\epsilon^{-1} \int_0^t v\left(\frac{s}{\epsilon^2}\right) ds\right),$$

where  $v(s)$  is defined in (19) above. This formula represents an alternative to the formulation based on GBM.

This new formulation can be used to model cash flow in high-frequency and algorithmic trading. In many financial applications transactions happen in short periods of time (every millisecond), thus the stochastic process  $v(t/\epsilon^2)$  is switched quickly between two states  $\{v_0, v_1\}$  according to an underlying Markov chain  $\xi(t/\epsilon^2)$ , because of the scale  $1/\epsilon^2$ , and it means that  $S_t^\epsilon$  is changing quickly as well. In some cases, we can assume longer periods of time  $t$  (instead of time in milliseconds), thus  $t/\epsilon^2$  can be used and it might be needed in applications, such as liquidation, acquisition, market making, etc. Therefore, we can apply our asymptotic results considered above for a better model fitting in these cases. For instance, below we show how to obtain an option pricing formula that is analogue to the Black-Scholes formula, for our model of a stock price. Of course, our modeling approach and results may be applied to other problems in mathematical finance, such as portfolio optimization, optimal control, etc.

Thus, by applying the results from the previous subsection we can state the following weak convergence

$$S_t^\epsilon \rightarrow_{\epsilon \rightarrow 0} \hat{S}_t,$$

in Skorokhod topology, where

$$\hat{S}_t = S_0 \exp\left\{\frac{a_1 - a_2}{2}t + \frac{v}{\sqrt{\lambda}}W_t\right\}.$$

Now, applying Itô's formula (Shreve (2004)), then  $\hat{S}_t$  satisfies the stochastic differential equation:

$$d\hat{S}_t = \hat{\mu}\hat{S}_t dt + \frac{v}{\sqrt{\lambda}}\hat{S}_t dW_t,$$

where

$$\hat{\mu} := \frac{a_1 - a_2}{2} + \frac{v^2}{2\lambda}.$$

Let us define the following stochastic process:

$$Z(t) := \exp\left\{\left(\frac{\hat{\mu} - r}{v/\sqrt{\lambda}}\right)W_t - \frac{1}{2}\left(\frac{\hat{\mu} - r}{v/\sqrt{\lambda}}\right)^2 t\right\},$$

where  $r > 0$  is the interest rate.

Hence,  $\mathbb{E}_P[Z(t)] = 1$ , and using Novikov's result (Novikov (1980)), we conclude that this process is a positive martingale.



Let us define the new probability measure  $Q$  (recall  $\mathbf{E}_P[Z(t)] = 1$ ) on a complete probability space  $(\Omega, \mathcal{F})$  :

$$Q(A) := \mathbf{E}_P[Z(T)\mathbf{1}_A],$$

where  $\mathbf{1}_A$  is an indicator operator of the set  $A \in \mathcal{F}$ , and  $Z(t)$  is the stochastic process defined above.

We also define the stochastic process:

$$\hat{W}_t := W_t - \left( \frac{\hat{\mu} - r}{v/\sqrt{\lambda}} \right) t,$$

where  $\hat{\mu}$  is defined above.

After applying Girsanov's theorem (Shreve (2004)), under the probability measure  $Q$ , the stochastic process we conclude that  $\hat{W}_t$  is a standard Wiener process. We call measure  $Q$  a risk-neutral or martingale measure. Then, under the risk-neutral measure  $Q$  our stock price  $\hat{S}_t$  satisfies the following SDE:

$$d\hat{S}_t = r\hat{S}_t dt + \frac{v}{\sqrt{\lambda}} \hat{S}_t d\hat{W}_t.$$

Therefore, we can write the Black–Scholes formula for European put option price  $P(t)$  for our model:

$$P(t) = Ke^{-r(T-t)}N(-d_2) - S_0N(-d_1) \quad (32)$$

where

$$d_1 := \frac{\ln(S_0/K) + (r + v^2/2\lambda)(T-t)}{(v/\sqrt{\lambda})\sqrt{T-t}}, \quad (33)$$

$$d_2 := d_1 - (v/2\sqrt{\lambda})\sqrt{T-t},$$

and  $N(x)$  is the cdf of a standard normal random variable with zero mean and unit variance,  $K$  is a strike price, and  $T$  is the maturity.

We note that

$$-d_1 := \frac{\ln(K/S_0) - (r + v^2/2\lambda)(T-t)}{(v/\sqrt{\lambda})\sqrt{T-t}},$$

and

$$-d_2 = (v/2\sqrt{\lambda})\sqrt{T-t} - d_1.$$

We also note that European call option price is:

$$C(t) = S_0N(d_1) - Ke^{-r(T-t)}N(d_2),$$

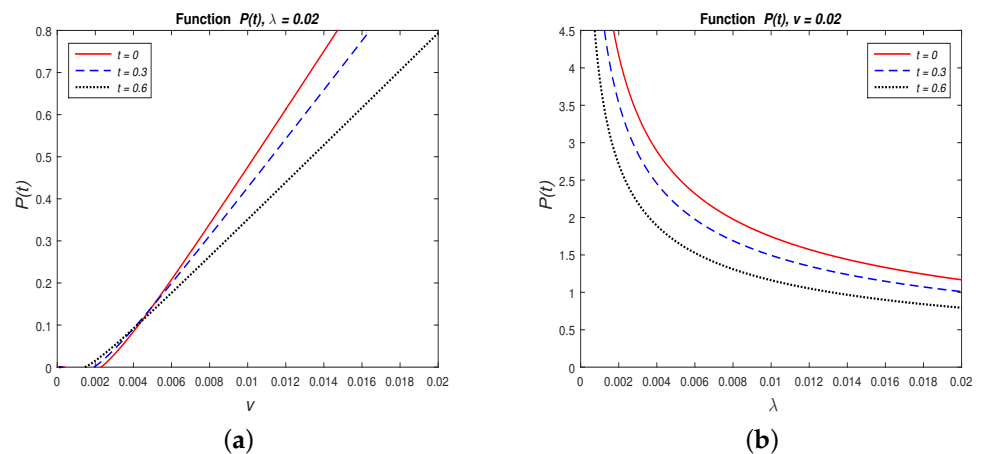
where  $d_1, d_2$  are defined in (33).

**Example 2 (European Put Option for Limiting Telegraph Process).** Suppose the numerical values  $S_0 = \$50, K = \$50, r = 0.01, v = 0.02, \lambda = 0.02, T = 1$ . Therefore, applying Formulas (32) and (33), we obtain the following European put option price at time  $t = 0$  :

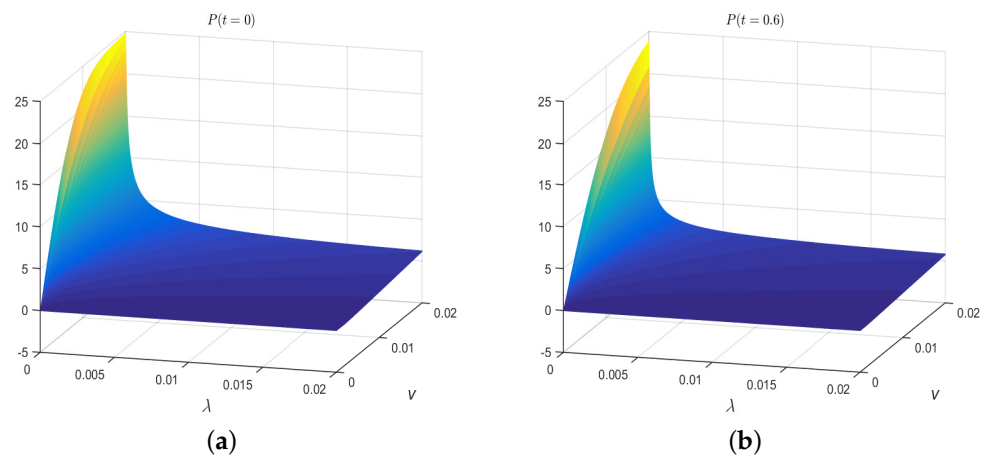
$$P(0) = 50 \times e^{-0.01 \times 1} \times 0.471814 - 50 \times 0.44376854 = \$1.167542$$

or  $P(0) = 116.7542$  cents.

Below we present some graphs of the time evolution of  $P(t)$  as a function of  $\lambda$  (upon fixing  $v$ ), and as a function of  $v$  (upon fixing  $\lambda$ ); see Figure 4, and on  $v$  and  $\lambda$  after fixing  $t$ ; see Figure 5.

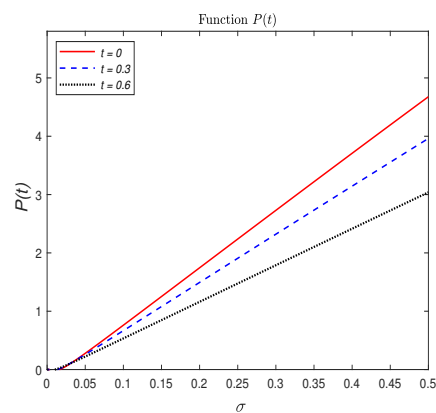


**Figure 4.** Time evolution of European put option price  $P(t)$  and dependent on  $v$  (a) and  $\lambda$  (b).



**Figure 5.** Time evolution of European put option price  $P(t)$  and dependent on  $v$  and  $\lambda$  for  $t = 0$  (a) and  $t = 0.6$  (b).

**Remark 5.** In Figure 6 we show the cost behavior according to Equation (32) as a function of  $\sigma$ . The Black–Scholes limit case occurs by letting  $\sigma = 0.14$ .



**Figure 6.** Time evolution of European put option price  $P(t)$  as a function of  $\sigma$ .

Now, we can say that on longer time interval the BS formula works better but on shorter time interval our formula produces a better performance. The same for volatility: If volatility is bigger than 0.14, then  $P(t)$  is bigger.

### 3.3. Asymmetric Telegraph Process and Its Financial Application

The asymmetric telegraph process is a telegraph process where the particle is allowed to move forward or backward with two different velocities,  $v_1, v_2$ ; and it has been studied in [Beghin et al. \(2001\)](#) and [López and Ratanov \(2014\)](#). Furthermore, this stochastic process can have two different velocity switching rates  $\lambda_1$  and  $\lambda_2$ . Hence, the underlying telegraph stochastic signal is modeled as a continuous-time Markov process  $(X(t), t \geq 0)$  with state space  $(v_1, -v_2)$ , where  $\lambda_i$  is the rate of the exponential sojourn time when the telegraph signal is in state  $(-1)^{i+1}v_i$ ,  $i = 1, 2$ , and  $X(0)$  is uniformly distributed on  $(v_1, -v_2)$ . Hence, the asymmetric telegraph process is defined as  $x(t) = \int_0^t X(s)ds$ . It is clear that the classical telegraph process can be recovered as a particular case when  $v_1 = v_2 = c$ ,  $\lambda_1 = \lambda_2 = \lambda$ . The probability law of the asymmetric telegraph process has an absolutely continuous component  $f$  and it satisfies the following hyperbolic Equation (see [Beghin et al. \(2001\)](#)):

$$\begin{aligned} \frac{\partial^2 f(t, x)}{\partial t^2} &= (v_1 v_2) \frac{\partial^2 f(t, x)}{\partial x^2} + (v_2 - v_1) \frac{\partial^2 f(t, x)}{\partial t \partial x} - (\lambda_1 + \lambda_2) \frac{\partial f(t, x)}{\partial t} \\ &+ \frac{1}{2} [(v_2 - v_1)(\lambda_1 + \lambda_2) - (\lambda_2 - \lambda_1)(v_1 + v_2)] \frac{\partial f(t, x)}{\partial t}. \end{aligned}$$

Under Kac-type conditions we can take the limit in the first equation, and we can obtain the governing equation of a Brownian motion with drift. Hence, after taking the limits for  $\lambda_i, v_i \rightarrow +\infty$  in such a manner that

$$\frac{\lambda_1}{\lambda_2} \rightarrow v^2 > 0, \quad \frac{v_i}{\sqrt{\lambda_i}} \rightarrow \sigma_i, \quad i = 1, 2, \quad \frac{\lambda_2 v_1 - \lambda_1 v_2}{\lambda_1 + \lambda_2} \rightarrow \delta,$$

it is not difficult to show that the marginal distributions of the asymmetric telegraph process converges to a drifted Brownian motion

$$x(t) \rightarrow_d \delta t + \sigma W(t),$$

where  $W(t)$  is a standard Brownian motion and

$$\sigma := \frac{\sigma_1 \sigma_2}{\sqrt{(\sigma_1^2 + \sigma_2^2)/2}}.$$

**Remark 6.** We note that for the symmetric case,  $v_1 = v_2 = c$ ,  $\lambda_1 = \lambda_2 = \lambda$ , the symmetric telegraph process  $x(t)$  under Kac's conditions converges to standard Wiener process with volatility  $\sigma = c/\sqrt{\lambda}$ :

$$x(t) \rightarrow_d (c/\sqrt{\lambda})W(t).$$

Let us consider the following model for a stock price:

$$S_t = S_0 e^{x(t)},$$

where  $x(t)$  is a telegraph process. Under above-mentioned Kac's conditions we can state that

$$S_t = S_0 e^{x(t)} \rightarrow_d \hat{S}_t := S_0 e^{\delta t + \sigma W(t)}.$$

Applying Itô's formula ([Shreve \(2004\)](#)), the stochastic process  $\hat{S}_t$  satisfies the following SDE:

$$d\hat{S}_t = a\hat{S}_t dt + \sigma\hat{S}_t dW_t,$$

where

$$a := \delta + \frac{\sigma^2}{2}.$$

Let us define the following stochastic process:

$$Z(t) := \exp \left\{ \left( \frac{a-r}{\sigma} \right) W_t - \frac{1}{2} \left( \frac{a-r}{\sigma} \right)^2 t \right\},$$

where  $r > 0$  is the interest rate.

Hence,  $E_P[Z(t)] = 1$ , and applying Novikov's result (Novikov (1980)), we conclude that this process is a positive martingale. Now, let us define the new probability measure  $Q$  (recall  $E_P[Z(t)] = 1$ ) on a complete probability space  $(\Omega, \mathcal{F})$  :

$$Q(A) := E_P[Z(T)\mathbf{1}_A],$$

where  $\mathbf{1}_A$  is an indicator operator of the set  $A \in \mathcal{F}$ , and  $Z(t)$  is the stochastic process defined above.

We can also define the following process:

$$\hat{W}_t := W_t - \left( \frac{a-r}{\sigma} \right) t,$$

where  $a$  is defined above.

By applying Girsanov's theorem (Shreve (2004)), under the probability measure  $Q$ , then we conclude that the stochastic process  $\hat{W}_t$  is a standard Wiener process. We call measure  $Q$  a risk-neutral or martingale measure. Thus, under the risk-neutral measure  $Q$  our stock price  $\hat{S}_t$  satisfies the following SDE:

$$d\hat{S}_t = r\hat{S}_t dt + \sigma\hat{S}_t d\hat{W}_t.$$

Therefore, we can write the alternative form of Black–Scholes formula for European call option price  $C(t)$  for our model:

$$C(t) = S_0 N(d_1) - K e^{-r(T-t)} N(d_2), \quad (34)$$

where

$$d_1 := \frac{\ln(S_0/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad (35)$$

$$d_2 := d_1 - \sigma\sqrt{T-t},$$

$$\sigma := \frac{\sigma_1\sigma_2}{\sqrt{(\sigma_1^2 + \sigma_2^2)/2}}, \quad \sigma_i := \frac{v_i}{\sqrt{\lambda_i}}, \quad (36)$$

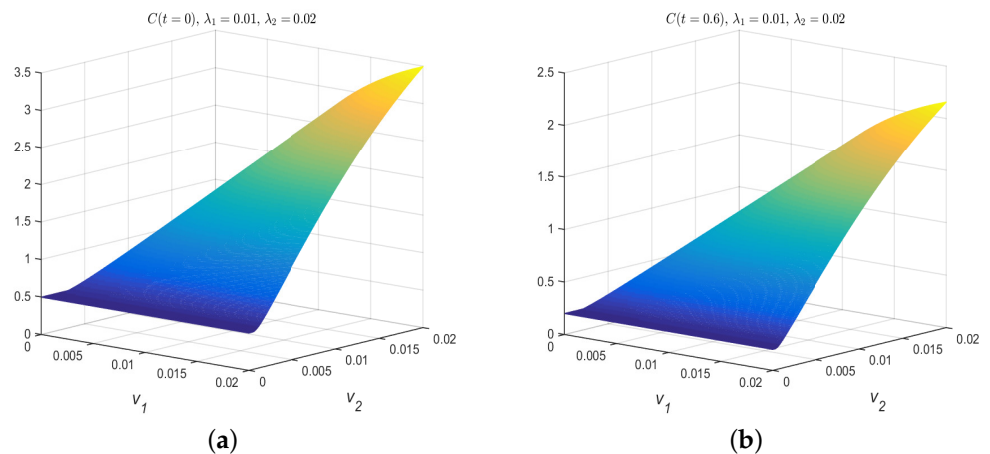
and  $N(x)$  is the cdf of a zero mean normal random variable with unit variance,  $K$  is a strike price, and  $T$  is the maturity.

**Example 3 (European Call Option for Asymmetric Limiting telegraph Process).** Suppose the numerical values  $S_0 = \$50, K = \$50, r = 0.01, v_1 = 0.01, v_2 = 0.02, \lambda_1 = 0.01, \lambda_2 = 0.02, T = 1$ . Therefore, after applying formulas (34)–(36), we have  $\sigma_1 = 0.1, \sigma_2 = 0.141421356$ , thus  $\sigma = 0.1155$ , and that European call option price at time  $t = 0$  is:

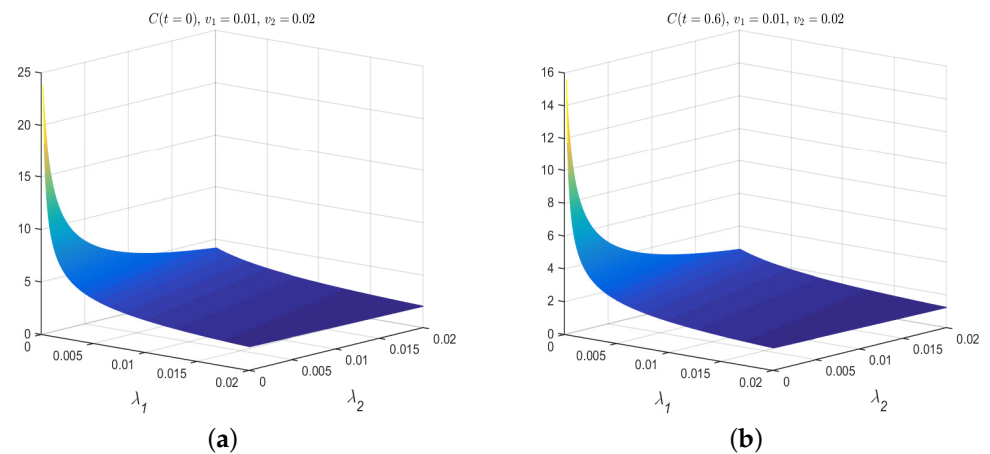
$$C(0) = 50 \times 0.557383 - 50 \times e^{-0.01 \times 1} \times 0.51151487 = \$2.547891$$

or  $C(0) = 254.7891$  cents.

Now, below we present some graphs of the time evolution of  $C(t)$  dependent on  $\lambda_1, \lambda_2$  (upon fixing  $v_i$ ), on  $v_1, v_2$  (upon fixing  $\lambda_i$ ), see Figure 7, and on  $v_1$  and  $\lambda_1$ , see Figure 8.

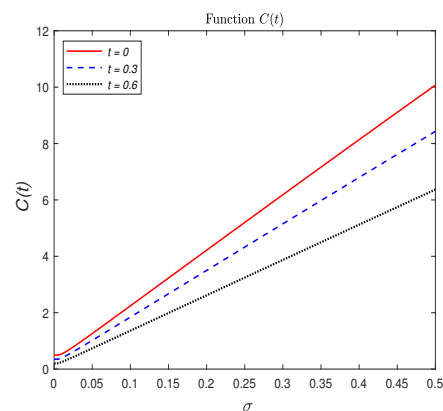


**Figure 7.** Time evolution of European call option price  $C(t)$  by fix  $\lambda$ ,  $t = 0$  (a) and  $t = 0.6$  (b) in the asymmetric case.



**Figure 8.** Time evolution of European call option price  $C(t)$  for fix  $v$ ,  $t = 0$  (a) and  $t = 0.6$  (b) in the asymmetric case.

**Remark 7.** In Figure 9 we show the cost behavior according to Equation (34) as a function of  $\sigma$ . The Black–Scholes limit case occurs by letting  $\sigma = 0.115079291$  (according to (36)).



**Figure 9.** Time evolution of European call option price  $C(t)$  as a function of  $\sigma$ .

Now, we can say that on longer time interval the BS formula works better but on shorter time interval our formula produces a better performance. The same for volatility: If volatility is bigger than 0.115079291, then  $C(t)$  is bigger.

#### 4. Conclusions and Future Work

In this paper, we considered transformations of classical telegraph process. We also presented three applications of transform telegraph process in finance: (1) application of classical telegraph process in the case of balance, (2) application of classical telegraph process in the case of dis-balance, and (3) application of asymmetric telegraph process in finance. For these three cases, we presented European call and put option prices. The novelty of the paper consists of new results for transformed classical telegraph process, new models for stock prices and new applications of these models to option pricing.

As for the future work we could consider other problems in mathematical finance, such as portfolio optimization, optimal control, etc. Furthermore, we will calibrate  $\lambda$ ,  $v$  and  $\sigma$  according to high-frequency and algorithmic trading (HFT) real data to see a better fit of our model. We will also try to apply our models of a stock price to optimization problems in HFT, such as optimal liquidation, acquisition, and market making. We will perform a comparative analysis of different models, including ours, in finance based on real data, as well.

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