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Article

# De Finetti's Control Problem with Parisian Ruin for Spectrally Negative Lévy Processes

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**Abstract:** We consider de Finetti's stochastic control problem when the (controlled) process is allowed to spend time under the critical level. More precisely, we consider a generalized version of this control problem in a spectrally negative Lévy model with exponential Parisian ruin. We show that, under mild assumptions on the Lévy measure, an optimal strategy is formed by a barrier strategy and that this optimal barrier level is always less than the optimal barrier level when classical ruin is implemented. In addition, we give necessary and sufficient conditions for the barrier strategy at level zero to be optimal.

**Keywords:** stochastic control; spectrally negative Lévy processes; optimal dividends; Parisian ruin; log-convexity; barrier strategies

## 1. Introduction and Main Result

In the 1950s, Bruno de Finetti (1957) formulated the following stochastic control problem: find the dividend strategy maximizing the expected present value of the dividend payments associated with an insurance surplus process. Presently, this control problem is known as de Finetti's optimal dividends problem. Another active field of research in insurance mathematics is the analysis of Parisian implementation delays in the recognition of default (see e.g., (Landriault et al. 2011; Loeffen et al. 2013)) and/or in the design of dividend strategies (see e.g., (Albrecher et al. 2011; Dassios and Wu 2009)). In what follows, we formulate and solve an extension of de Finetti's control problem with Parisian ruin.

### 1.1. Problem Formulation

On a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ , let  $X = \{X_t, t \geq 0\}$  be a spectrally negative Lévy process with Laplace exponent  $\theta \mapsto \psi(\theta)$  and with  $q$ -scale functions  $\{W^{(q)}, q \geq 0\}$  given by

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = (\psi(\theta) - q)^{-1},$$

for all  $\theta > \Phi(q) = \sup \{\lambda \geq 0: \psi(\lambda) = q\}$ . Recall that

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty (e^{-\theta z} - 1 + \theta z \mathbf{1}_{(0,1]}(z)) \nu(dz),$$

where  $\gamma \in \mathbb{R}$  and  $\sigma \geq 0$ , and where  $\nu$  is a  $\sigma$ -finite measure on  $(0, \infty)$ , called the Lévy measure of  $X$ , satisfying

$$\int_0^\infty (1 \wedge x^2) \nu(dx) < \infty.$$

For more details on spectrally negative Lévy processes and scale functions, see e.g., (Kuznetsov et al. 2012; Kyprianou 2014).

In what follows, we will use the following notation: the law of  $X$  when starting from  $X_0 = x$  is denoted by  $\mathbb{P}_x$  and the corresponding expectation by  $\mathbb{E}_x$ . We write  $\mathbb{P}$  and  $\mathbb{E}$  when  $x = 0$ .

Let the spectrally negative Lévy process  $X$  be the underlying surplus process. A dividend strategy  $\pi$  is represented by a non-decreasing, left-continuous and adapted stochastic process  $L^\pi = \{L_t^\pi, t \geq 0\}$ , where  $L_t^\pi$  represents the cumulative amount of dividends paid up to time  $t$  under this strategy, and such that  $L_0^\pi = 0$ . For a given strategy  $\pi$ , the corresponding controlled surplus process  $U^\pi = \{U_t^\pi, t \geq 0\}$  is defined by  $U_t^\pi = X_t - L_t^\pi$ . The stochastic control problem considered in this paper involves the time of Parisian ruin (with rate  $p > 0$ ) for  $U^\pi$  defined by

$$\sigma_p^\pi = \inf \left\{ t > 0 : t - g_t^\pi > e_p^{g_t^\pi} \text{ and } U_t^\pi < 0 \right\},$$

where  $g_t^\pi = \sup \{0 \leq s \leq t : U_s^\pi \geq 0\}$ , with  $e_p^{g_t^\pi}$  an independent random variable, following the exponential distribution with mean  $1/p$ , associated with the corresponding excursion below 0 (see (Baurdoux et al. 2016) for more details). Please note that, without loss of generality, we have chosen 0 to be the critical level.

**Remark 1.** Recall that  $X$  and  $L^\pi$  are adapted to the filtration. Set  $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$ , i.e., the smallest  $\sigma$ -algebra containing  $\mathcal{F}_t$ , for all  $t \geq 0$ . It is implicitly assumed that  $\mathcal{F}_\infty$  is strictly less than  $\mathcal{F}$  and that all exponential clocks are independent of  $\mathcal{F}_\infty$ .

A strategy  $\pi$  is said to be admissible if a dividend payment is not larger than the current surplus level, i.e.,  $L_{t+}^\pi - L_t^\pi \leq U_t^\pi$ , for all  $t < \sigma_p^\pi$ , and if no dividends are paid when the controlled surplus is negative, i.e.,  $t \mapsto L_t^\pi \mathbf{1}_{(-\infty, 0)}(U_t^\pi) \equiv 0$ . The set of admissible dividend strategies will be denoted by  $\Pi_p$ . These two conditions are motivated by the following interpretation: if  $U^\pi$  enters the interval  $(-\infty, 0)$ , then a period of *financial distress* begins. Consequently, dividend payments should not cause an excursion under the critical level nor should they be made during those critical periods.

Fix a discounting rate  $q \geq 0$ . The value function associated with an admissible dividend strategy  $\pi \in \Pi_p$  is defined by

$$v_\pi(x) = \mathbb{E}_x \left[ \int_0^{\sigma_p^\pi} e^{-qt} dL_t^\pi \right], \quad x \in \mathbb{R}.$$

The goal is to find the optimal value function  $v_*$  defined by

$$v_*(x) = \sup_{\pi \in \Pi_p} v_\pi(x)$$

and an optimal strategy  $\pi_* \in \Pi_p$  such that

$$v_{\pi_*}(x) = v_*(x),$$

for all  $x \in \mathbb{R}$ . Because of the Parisian nature of the time of ruin considered in this control problem, we have to deal with possibly negative starting capital.

### 1.2. Main Result and Organization of the Paper

Let us introduce the family of horizontal barrier strategies, also called reflection strategies. For  $b \in \mathbb{R}$ , the (horizontal) barrier strategy at level  $b$  is the strategy denoted by  $\pi^b$  and with cumulative amount of dividends paid until time  $t$  given by  $L_t^b = \left( \sup_{0 < s \leq t} X_s - b \right)_+$ , for  $t > 0$ . If  $X_0 = x > b$ ,

then  $L_{0+}^b = x - b$ . Please note that, if  $b \geq 0$ , then  $\pi_b \in \Pi_p$ . The corresponding value function is thus given by

$$v_b(x) = \mathbb{E}_x \left[ \int_0^{\sigma_p^b} e^{-qt} dL_t^b \right],$$

for all  $x \in \mathbb{R}$ , where  $\sigma_p^b$  is the time of Parisian ruin (with rate  $p > 0$ ) for the controlled process  $U_t^b = X_t - L_t^b$ .

Before stating the main result of this paper, recall that the tail of the Lévy measure is the function  $x \mapsto \nu(x, \infty)$ , where  $x \in (0, \infty)$ , and that a function  $f: (0, \infty) \rightarrow (0, \infty)$  is log-convex if the function  $\log(f)$  is convex on  $(0, \infty)$ .

**Theorem 1.** Fix  $q \geq 0$  and  $p > 0$ . If the tail of the Lévy measure is log-convex, then an optimal strategy for the control problem is formed by a barrier strategy.

The original version of de Finetti’s optimal dividends problem, i.e., when the time of ruin is the first passage time below the critical level (intuitively, when  $p \rightarrow \infty$ ), has been extensively studied. In a spectrally negative Lévy model, following the work of Avram et al. (2007), an important breakthrough was made by Loeffen (2008); in the latter paper, a sufficient condition, on the Lévy measure  $\nu$ , is given for a barrier strategy to be optimal. This condition was further relaxed by Loeffen and Renaud (2010); in this other paper, it is shown that if the tail of the Lévy measure is log-convex then a barrier strategy is optimal for de Finetti’s optimal dividends problem with an affine penalty function at ruin (if we set  $S = K = 0$  in that paper, we recover the classical problem). To the best of our knowledge, this still stands as the mildest condition for the optimality of a barrier strategy in a spectrally negative Lévy model. Finally, note that Czarna and Palmowski (2014) have considered de Finetti’s control problem with deterministic Parisian delays.

The rest of the paper is organized as follows. First, we provide an alternative interpretation of the value function and we fill the gap between the models with classical ruin and no ruin. Then, we compute the value function of an arbitrary horizontal barrier strategy and find the optimal barrier level  $b_p^*$  (see the definition in (9)). Finally, we derive the appropriate verification lemma for this control problem and prove that, under our assumption on the Lévy measure, the barrier strategy at level  $b_p^*$  is optimal.

## 2. More on the Value Function

Please note that for  $\pi \in \Pi_p$  and  $x < 0$ , using the strong Markov property and the spectral negativity of  $X$ , we can easily verify that

$$v_\pi(x) = \mathbb{E}_x \left[ e^{-q\tau_0^+} \mathbf{1}_{\{\tau_0^+ < \mathbf{e}_p\}} \right] v_\pi(0) = e^{\Phi(p+q)x} v_\pi(0), \tag{1}$$

where  $\tau_0^+ = \inf \{t > 0: X_t > 0\}$  and where  $\mathbf{e}_p$  is an independent exponentially distributed random variable with mean  $1/p$ , thanks to the well-known fluctuation identity (see e.g., (Kyprianou 2014))

$$\mathbb{E}_x \left[ e^{-r\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \infty\}} \right] = e^{-\Phi(r)(b-x)}, \quad x \leq b, \tag{2}$$

where  $\tau_b^+ = \inf \{t > 0: X_t > b\}$ .

Interestingly, we can show that (see the proof of Lemma 1 below), for any  $\pi \in \Pi_p$ , we have

$$v_\pi(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} dL_s^\pi \right]. \tag{3}$$

Using this last identity, we can argue that using Parisian ruin with rate  $p$  fills the gap between the model with classical ruin (no delay,  $p \rightarrow \infty$ ) and the model with no ruin (infinite delays,  $p \rightarrow 0$ ). Indeed, using (3), we see directly that

$$v_\pi(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} dL_s^\pi \right] \xrightarrow{p \rightarrow 0} \mathbb{E}_x \left[ \int_0^\infty e^{-qs} dL_s^\pi \right].$$

On the other hand, as

$$\int_0^\infty e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} dL_s^\pi = \int_0^\infty e^{-qs} \mathbf{1}_{\{s \leq \sigma_\infty^\pi\}} dL_s^\pi + \int_0^\infty e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} \mathbf{1}_{\{s > \sigma_\infty^\pi\}} dL_s^\pi,$$

where  $\sigma_\infty^\pi := \inf \{t > 0: U_t^\pi < 0\}$ , we obtain

$$v_\pi(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} dL_s^\pi \right] \xrightarrow{p \rightarrow \infty} \mathbb{E}_x \left[ \int_0^{\sigma_\infty^\pi} e^{-qs} dL_s^\pi \right].$$

**Remark 2.** Note also that the expression of the value function given in (3) tells us that the current control problem amounts to a control problem with no ruin and in which the dividend payments are penalized by the occupation time of the surplus process. Indeed, from this point of view, the discount factor increases with the time spent below zero by the surplus process.

### 3. Horizontal Barrier Strategies

Before computing the value function of an arbitrary barrier strategy at level  $b$ , we have to define another family of scale functions, also called second  $q$ -scale functions of  $X$ .

#### 3.1. Second Family of Scale Functions

The so-called second scale functions are defined by: for each  $q, \theta \geq 0$  and for  $x \in \mathbb{R}$ , let

$$Z_q(x, \theta) = e^{\theta x} \left( 1 - (\psi(\theta) - q) \int_0^x e^{-\theta y} W^{(q)}(y) dy \right). \tag{4}$$

Please note that for  $x \leq 0$  or for  $\theta = \Phi(q)$ , we have  $Z_q(x, \theta) = e^{\theta x}$ . The second scale functions have appeared in the literature in various forms; see e.g., (Albrecher et al. 2016; Avram et al. 2015; Ivanovs and Palmowski 2012).

In what follows,  $Z'_q(x, \theta)$  will represent the derivative with respect to the first argument. Consequently, for  $x > 0$ , we have  $Z'_q(x, \theta) = \theta Z_q(x, \theta) - (\psi(\theta) - q)W^{(q)}(x)$  and, for  $x < 0$ , we have  $Z'_q(x, \theta) = \theta e^{\theta x}$ .

In this paper, we will encounter the function  $Z_q$  when  $\theta = \Phi(p + q)$ , that is the function

$$Z_q(x, \Phi(p + q)) = e^{\Phi(p+q)x} \left( 1 - p \int_0^x e^{-\Phi(p+q)y} W^{(q)}(y) dy \right),$$

from which we deduce that, for  $x > 0$ ,

$$Z'_q(x, \Phi(p + q)) = \Phi(p + q)Z_q(x, \Phi(p + q)) - pW^{(q)}(x). \tag{5}$$

Consequently, set  $Z'_q(0, \Phi(p + q)) = \Phi(p + q) - pW^{(q)}(0)$ . Since we assume that  $p > 0$ , we have that  $\Phi(p + q) > \Phi(q)$  and we can write

$$Z_q(x, \Phi(p + q)) = p \int_0^\infty e^{-\Phi(p+q)y} W^{(q)}(x + y) dy, \quad x \in \mathbb{R}. \tag{6}$$

Then, for  $x > 0$ , we have

$$Z'_q(x, \Phi(p + q)) = p \int_0^\infty e^{-\Phi(p+q)y} W^{(q)'}(x + y) dy, \tag{7}$$

which is well defined since  $W^{(q)}$  is differentiable almost everywhere (see e.g., Lemma 2.3 in (Kuznetsov et al. 2012)). Clearly,  $x \mapsto Z_q(x, \Phi(p + q))$  is a non-decreasing continuous function. In fact, it will be proved in Appendix B that if the tail of the Lévy measure is log-convex, then  $Z'_q(\cdot, \Phi(p + q))$  is a log-convex function on  $(0, \infty)$ .

### 3.2. Value Function of a Barrier Strategy

Here is the value of an arbitrary admissible barrier strategy:

**Proposition 1.** For  $q, b \geq 0$ , the value function associated with  $\pi_b$  is given by

$$v_b(x) = \begin{cases} \frac{Z_q(x, \Phi(p+q))}{Z'_q(b, \Phi(p+q))} & \text{for } x \in (-\infty, b], \\ x - b + v_b(b) & \text{for } x \in (b, \infty). \end{cases} \tag{8}$$

**Proof.** This result has appeared before in the literature. See for example Equation (15) in (Albrecher and Ivanovs 2014) or Equation (46) in (Avram and Zhou 2016). Nevertheless, we provide an alternative proof in Appendix A.  $\square$

### 3.3. Optimal Barrier Level

As defined in (Loeffen 2008; Loeffen and Renaud 2010), the optimal barrier level in de Finetti’s classical control problem is given by

$$b_\infty^* = \sup \left\{ b \geq 0: W^{(q)'}(b) \leq W^{(q)'}(x), \text{ for all } x \geq 0 \right\}.$$

Similarly, let us define the candidate for the optimal barrier level for the current version of this control problem by

$$b_p^* = \sup \left\{ b \geq 0: Z'_q(b, \Phi(p + q)) \leq Z'_q(x, \Phi(p + q)), \text{ for all } x \geq 0 \right\}. \tag{9}$$

**Proposition 2.** Fix  $q \geq 0$  and  $p > 0$ . Suppose the tail of the Lévy measure is log-convex. Then,  $0 \leq b_p^* \leq b_\infty^*$  and  $b_p^* > 0$  if and only if

$$\Phi(p + q) - pW^{(q)}(0) < \frac{p}{\Phi(p + q)} W^{(q)'}(0+). \tag{10}$$

Equivalently,  $b_p^* > 0$  if and only if one of the following three cases hold:

- (a)  $\sigma > 0$  and  $(\Phi(p + q))^2 / p < 2/\sigma^2$ ;
- (b)  $\sigma = 0$  and  $v(0, \infty) = \infty$ ;
- (c)  $\sigma = 0$ ,  $v(0, \infty) < \infty$  and

$$\frac{c\Phi(p + q)}{p} \left( \Phi(p + q) - \frac{p}{c} \right) < \frac{q + v(0, \infty)}{c},$$

where  $c = \gamma + \int_0^1 xv(dx)$ .

**Proof.** See the proof in Appendix B.  $\square$

First of all, note from Proposition 2 that the optimal barrier level  $b_p^*$ , when Parisian ruin with rate  $p$  is implemented, is always lower than the optimal barrier level  $b_\infty^*$  when classical ruin is used.

In cases (a) and (c), the value of  $b_p^*$  can be either positive or zero, depending on the parameters of the model. It is clear from the condition in (10) that, when  $q > 0$ , if the Parisian rate  $p$  is small enough (large delays), then  $b_p^* = 0$ ; in words, if Parisian delays are infinite (no ruin), then it is better to start paying out dividends right away. However, when  $q = 0$  (no discounting), if Parisian delays are infinite (no ruin), then  $b_p^* > 0$  if and only if  $\mathbb{E}[X_1] > 0$ .

Also, in case (a), if  $X_t = ct + \sigma B_t$  is a Brownian motion with drift, then

$$\Phi(p + q) = \frac{1}{\sigma^2} \left( \sqrt{c^2 + 2\sigma^2(p + q)} - c \right)$$

and we can verify that  $b_p^* = 0$  as soon as the Brownian coefficient  $\sigma$  is large enough.

**Remark 3.** In Section 4 of (Avram and Minca 2017), economic principles for evaluating the efficiency of a surplus process are discussed. One of them is that the optimal barrier level be equal to zero.

Interestingly, the condition in (c) can be re-written as follows:

$$\frac{c\Phi(p + q)}{p} \mathbb{E} \left[ \int_0^{\sigma_\infty^0} e^{-qt} dL_t^0 \right] < \mathbb{E} \left[ \int_0^{\sigma_p^0} e^{-qt} dL_t^0 \right] = v_0(0).$$

Indeed, when  $\sigma = 0$  and  $v(0, \infty) < \infty$ , it is known (see Equation (3.14) in (Avram et al. 2007)) that

$$\mathbb{E} \left[ \int_0^{\sigma_\infty^0} e^{-qt} dL_t^0 \right] = \frac{c}{q + v(0, \infty)}$$

and, from Proposition 1, we have

$$\mathbb{E} \left[ \int_0^{\sigma_p^0} e^{-qt} dL_t^0 \right] = \frac{1}{\Phi(p + q) - pW^{(q)}(0+)}.$$

#### 4. Verification Lemma and Proof of the Main Result

Define the operator  $\Gamma$  associated with  $X$  by

$$\Gamma v(x) = \gamma v'(x) + \frac{\sigma^2}{2} v''(x) + \int_0^\infty \left( v(x - z) - v(x) + v'(x)z \mathbf{1}_{(0,1]}(z) \right) \nu(dz), \tag{11}$$

where  $v$  is a function defined on  $\mathbb{R}$  such that  $\Gamma v(x)$  is well defined. We say that a function  $v$  is sufficiently smooth if it is continuously differentiable on  $(0, \infty)$  when  $X$  is of bounded variation and twice continuously differentiable on  $(0, \infty)$  when  $X$  is of unbounded variation.

Next is the verification lemma of our stochastic control problem. As the controlled process is now allowed to spend time below the critical level, it is different from the classical verification lemma (see (Loeffen 2008)).

**Lemma 1.** Let  $\Gamma$  be the operator defined in (11). Suppose that  $\hat{\pi} \in \Pi_p$  is such that  $v_{\hat{\pi}}$  is sufficiently smooth and that, for all  $x \in \mathbb{R}$ ,

$$\left( \Gamma - q - p \mathbf{1}_{(-\infty, 0)} \right) v_{\hat{\pi}}(x) \leq 0$$

and, for all  $x > 0$ ,  $v'_{\hat{\pi}}(x) \geq 1$ . In this case,  $\hat{\pi}$  is an optimal strategy for the control problem.

**Proof.** Set  $w := v_{\hat{\pi}}$  and let  $\pi \in \Pi_p$  be an arbitrary admissible strategy. As  $w$  is sufficiently smooth, applying an appropriate change-of-variable/version of Ito’s formula to the joint process  $(t, \int_0^t \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr, U_t^\pi)$  yields

$$\begin{aligned} & e^{-qt-p \int_0^t \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} w(U_t^\pi) - w(U_0^\pi) \\ &= \int_0^t e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} \left[ (\Gamma - q)w(U_s^\pi) - p \mathbf{1}_{(-\infty,0)}(U_s^\pi) w(U_s^\pi) \right] ds \\ & - \int_0^t e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} w'(U_{s-}^\pi) dL_s^\pi + M_t^\pi \\ & + \sum_{0 < s \leq t} e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} \left[ w(U_{s-}^\pi - \Delta L_s^\pi) - w(U_{s-}^\pi) + w'(U_{s-}^\pi) \Delta L_s^\pi \right], \end{aligned} \tag{12}$$

where  $M^\pi = \{M_t^\pi, t \geq 0\}$  is a (local) martingale.

Consider an independent (of  $\mathcal{F}_\infty$ ) Poisson process with intensity measure  $p dt$  and jump times  $\{T_i^p, i \geq 1\}$ . Therefore, we can write

$$e^{-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} = \mathbb{P}_x \left( T_i^p \notin \{r \in (0, s] : U_r^\pi < 0\}, \text{ for all } i \geq 1 | \mathcal{F}_\infty \right) = \mathbb{E}_x \left[ \mathbf{1}_{\{\sigma_p^\pi > s\}} | \mathcal{F}_\infty \right]$$

and consequently

$$\mathbb{E}_x \left[ \int_0^t e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} dL_s^\pi \right] = \mathbb{E}_x \left[ \int_0^t e^{-qs} \mathbb{E}_x \left[ \mathbf{1}_{\{\sigma_p^\pi > s\}} | \mathcal{F}_\infty \right] dL_s^\pi \right] = \mathbb{E}_x \left[ \int_0^{\sigma_p^\pi \wedge t} e^{-qs} dL_s^\pi \right],$$

where we used the definition of a Riemann-Stieltjes integral and the monotone convergence theorem for conditional expectations.

Now, as for all  $x \in \mathbb{R}$ ,

$$(\Gamma - q - p \mathbf{1}_{(-\infty,0)}) w(x) \leq 0$$

and, for all  $x > 0$ ,  $w'(x) \geq 1$ , using standard arguments (see e.g., (Loeffen 2008)) and our definition of an admissible strategy, e.g., that  $L^\pi$  is identically zero when  $U^\pi$  is below zero, we get

$$w(x) \geq \mathbb{E}_x \left[ \int_0^\infty e^{-qs-p \int_0^s \mathbf{1}_{(-\infty,0)}(U_r^\pi) dr} dL_s^\pi \right] = \mathbb{E}_x \left[ \int_0^{\sigma_p^\pi} e^{-qs} dL_s^\pi \right] = v_\pi(x).$$

This concludes the proof.  $\square$

The rest of this section is devoted to proving Theorem 1, i.e., proving that an optimal strategy for the control problem is formed by the barrier strategy at level  $b^* := b_p^*$ .

By the definition of  $b^*$  given in (9), for  $0 \leq x \leq b^*$ , we have

$$v'_{b^*}(x) = \frac{Z'_q(x, \Phi(p+q))}{Z'_q(b^*, \Phi(p+q))} \geq 1.$$

By the definition of  $v_{b^*}$ , for  $x > b^*$ , we have  $v'_{b^*}(x) = 1$ . This means  $v'_{b^*}(x) \geq 1$ , for all  $x \geq 0$ .

Please note that for any  $x \in \mathbb{R}$ , we have

$$\begin{aligned} (\Gamma - q - p) e^{\Phi(p+q)x} &= e^{\Phi(p+q)x} \left( \gamma \Phi(p+q) + \frac{\sigma^2}{2} \Phi^2(p+q) \right) \\ & + e^{\Phi(p+q)x} \left[ \int_0^\infty \left( e^{-\Phi(p+q)z} - 1 + \Phi(p+q)z \mathbf{1}_{(0,1]}(z) \right) \nu(dz) - (q+p) \right] \\ &= e^{\Phi(p+q)x} [\psi(\Phi(p+q)) - (q+p)] = 0. \end{aligned} \tag{13}$$

Consequently, for  $x < 0$ , we have

$$(\Gamma - q - p) Z_q(x, \Phi(p+q)) = 0$$



and, for  $x \geq 0$ , using (6), we have

$$(\Gamma - q) Z_q(x, \Phi(p + q)) = p \int_0^\infty e^{\Phi(p+q)y} (\Gamma - q) W^{(q)}(x + y) dy = 0,$$

since  $(\Gamma - q) W^{(q)}(x) = 0$  for all  $x > 0$  (see e.g., (Biffis and Kyprianou 2010)). Please note that under our assumption,  $W^{(q)}$  is sufficiently smooth. Indeed, by Theorem 1.2 in (Loeffen and Renaud 2010), if the tail of the Lévy measure is log-convex, then  $W^{(q)'}$  is log-convex. Therefore,  $W^{(q)''}(x)$  exists and is continuous for almost all  $x \in (0, \infty)$ ; see e.g., (Roberts and Varberg 1973).

As a consequence, and since  $v_{b^*}$  is smooth in  $x = b^*$ , we have

$$\left(\Gamma - q - p\mathbf{1}_{(-\infty, 0)}\right) v_{b^*}(x) = 0, \quad \text{for } x \leq b^*.$$

All that is now left to verify is that  $(\Gamma - q) v_{b^*}(x) \leq 0$ , for all  $x > b^*$ . It can be done following the same steps as in the proof of Theorem 2 in (Loeffen 2008), thanks to the fact that, under our assumption on the Lévy measure, the function  $Z'_q(\cdot, \Phi(p + q))$  is sufficiently smooth (see the details in Appendix B). The details are left to the reader.

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### Appendix A. Proof of Proposition 1

To prove this result, we can adapt the methodology used in the proof of Proposition 1 of (Renaud and Zhou 2007) (for the case  $k = 1$  in that paper). Let us define  $\kappa^p$  as the time of Parisian ruin with rate  $p$  for  $X$  or, said differently, the time of Parisian ruin when the pay-no-dividend strategy, i.e., the strategy  $\pi$  with  $L_t^\pi \equiv 0$ , is implemented. More precisely, define

$$\kappa^p = \inf \left\{ t > 0 : t - g_t > e_p^{g_t} \text{ and } X_t < 0 \right\},$$

where  $g_t = \sup \{0 \leq s \leq t : X_s \geq 0\}$ . Let us also define, for  $a \in \mathbb{R}$ , the stopping time

$$\tau_a^+ = \inf \{t > 0 : X_t > a\}.$$

It is known that (see e.g., Equation (16) in (Lkabous and Renaud 2019)), for  $x \leq a$ ,

$$\mathbb{E}_x \left[ e^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < \kappa^p\}} \right] = \frac{Z_q(x, \Phi(p + q))}{Z_q(a, \Phi(p + q))}. \tag{A1}$$

As in Renaud and Zhou (2007), we can show that

$$\begin{aligned} & \left(v_b(b) + \frac{1}{n}\right) \mathbb{E}_{b-1/n} \left[ e^{-q\tau_b^+} \mathbf{1}_{\{\tau_b^+ < \kappa^p\}} \right] \\ & \leq v_b(b) \leq \left(v_b(b) + \frac{1}{n}\right) \mathbb{E}_b \left[ e^{-q\tau_{b+1/n}^+} \mathbf{1}_{\{\tau_{b+1/n}^+ < \kappa^p\}} \right] + o(1/n). \end{aligned} \tag{A2}$$

The result for  $x = b$  follows by taking a limit and then the result for  $0 \leq x \leq b$  follows by using again the identity in (A1). Finally, if  $x < 0$ , then using (1) we have

$$v_b(x) = e^{\Phi(p+q)x} \frac{Z_q(0, \Phi(p+q))}{Z'_q(b, \Phi(p+q))} = \frac{Z_q(x, \Phi(p+q))}{Z'_q(b, \Phi(p+q))}.$$

### Appendix B. Proof of Proposition 2

Recall from (7) that, for  $x \in (0, \infty)$ , we have

$$Z'_q(x, \Phi(p+q)) = p \int_0^\infty e^{-\Phi(p+q)y} W^{(q)'}(x+y) dy. \tag{A3}$$

By Theorem 1.2 in (Loeffen and Renaud 2010), if the tail of the Lévy measure is log-convex, then  $W^{(q)'}$  is log-convex. Using the properties of log-convex functions, as presented in (Roberts and Varberg 1973), we can deduce that  $x \mapsto pe^{-\Phi(p+q)y}W^{(q)'}(x+y)$  is log-convex on  $(0, \infty)$ , for any fixed  $y \in (0, \infty)$ . Then, as Riemann integrals are limits of partial sums, we have that  $x \mapsto Z'_q(x, \Phi(p+q))$  is also a log-convex function on  $(0, \infty)$ . In particular,  $Z'_q(\cdot, \Phi(p+q))$  is convex on  $(0, \infty)$ , so we can write, for some fixed  $c > 0$ ,

$$Z'_q(x, \Phi(p+q)) = Z'_q(c, \Phi(p+q)) + \int_c^x Z_q''^-(y, \Phi(p+q)) dy,$$

where  $Z_q''^-(\cdot, \Phi(p+q))$  is the left-hand derivative of  $Z'_q(\cdot, \Phi(p+q))$ . Since  $Z_q''^-(\cdot, \Phi(p+q))$  is increasing and  $\lim_{x \rightarrow \infty} Z'_q(x, \Phi(p+q)) = \infty$ , we have that the function  $Z'_q(\cdot, \Phi(p+q))$  is ultimately strictly increasing. This proves that  $b_p^*$  is well-defined.

It is known that  $W^{(q)'}$  is strictly increasing on  $(b_\infty^*, \infty)$ ; see (Loeffen and Renaud 2010). Then, using together the representations of  $Z'_q(x, \Phi(p+q))$  given in (5) and (7), we obtain

$$\begin{aligned} Z_q''(x, \Phi(p+q)) &= \Phi(p+q)p \int_0^\infty e^{-\Phi(p+q)y} W^{(q)'}(x+y) dy - pW^{(q)'}(x) \\ &> pW^{(q)'}(x) \int_0^\infty \Phi(p+q)e^{-\Phi(p+q)y} dy - pW^{(q)'}(x) = 0, \end{aligned} \tag{A4}$$

for all  $x > b_\infty^*$ . In other words,  $x \mapsto Z'_q(x, \Phi(p+q))$  is strictly increasing on  $(b_\infty^*, \infty)$ . Consequently,  $b_p^* \leq b_\infty^*$ .

The rest of the proof is similar to Lemma 3 in (Kyprianou et al. 2012), where a function closely related to one of the representations of  $Z'_q(x, \Phi(p+q))$  appears. For simplicity, set  $g(x) = Z'_q(x, \Phi(p+q))$ . Using (5), we can write, for  $x > 0$ ,

$$g'(x) = \Phi(p+q) \left( g(x) - \frac{p}{\Phi(p+q)} W^{(q)'}(x) \right).$$

It follows that  $g'(x) > 0$  (resp.  $g'(x) < 0$ ) if and only if  $g(x) > \frac{p}{\Phi(p+q)} W^{(q)'}(x)$  (resp.  $g(x) < \frac{p}{\Phi(p+q)} W^{(q)'}(x)$ ). This means  $g(b) > \frac{p}{\Phi(p+q)} W^{(q)'}(b)$  for  $b < b_p^*$  and  $g(b) < \frac{p}{\Phi(p+q)} W^{(q)'}(b)$  for  $b > b_p^*$ . If  $b_p^* > 0$  then  $g(b_p^*) = (p/\Phi(p+q))W^{(q)'}(b_p^*)$ .

We deduce that  $b_p^* > 0$  if and only if  $g(0+) < (p/\Phi(p+q))W^{(q)'}(0+)$ , where  $g(0+) = \Phi(p+q) - pW^{(q)}(0)$ . Written differently, we have  $b_p^* > 0$  if and only if

$$\Phi(p+q) - pW^{(q)}(0) < \frac{p}{\Phi(p+q)} W^{(q)'}(0+).$$

If  $\sigma > 0$ , then  $W^{(q)}(0) = 0$  and  $W^{(q)'}(0+) = 2/\sigma^2$ , which implies that  $b_p^* > 0$  if and only if

$$\frac{(\Phi(p+q))^2}{p} < \frac{2}{\sigma^2}.$$

If  $\sigma = 0$  and  $\nu(0, \infty) = \infty$ , then  $W^{(q)'}(0+) = \infty$ , which implies that  $b_p^* > 0$ . Finally, if  $\sigma = 0$  and  $\nu(0, \infty) < \infty$ , then  $W^{(q)}(0) = 1/c$ , where  $c > 0$  is the drift, and  $W^{(q)'}(0+) = (q + \nu(0, \infty))/c^2$ , which implies that  $b_p^* > 0$  if and only if

$$\Phi(p+q) - \frac{p}{c} < \frac{p}{\Phi(p+q)} \frac{q + \nu(0, \infty)}{c^2}.$$

## References

- Albrecher, Hansjörg, and Jevgenijs Ivanovs. 2014. Power identities for Lévy risk models under taxation and capital injections. *Stochastic Systems* 4: 157–72. [CrossRef]
- Albrecher, Hansjörg, Eric C. K. Cheung, and Stefan Thonhauser. 2011. Randomized observation periods for the compound Poisson risk model dividends. *ASTIN Bulletin* 41: 645–72.
- Albrecher, Hansjörg, Jevgenijs Ivanovs, and Xiaowen Zhou. 2016. Exit identities for Lévy processes observed at Poisson arrival times. *Bernoulli* 22: 1364–82. [CrossRef]
- Avram, Florin, and Andreea Minca. 2017. On the central management of risk networks. *Advances in Applied Probability* 49: 221–37. [CrossRef]
- Avram, Florin, and Xiaowen Zhou. 2016. On fluctuation theory for spectrally negative Lévy processes with Parisian reflection below, and applications. *Theory of Probability and Mathematical Statistics* 95: 17–40. [CrossRef]
- Avram, Florin, Zbigniew Palmowski, and Martijn R. Pistorius. 2007. On the optimal dividend problem for a spectrally negative Lévy process. *The Annals of Applied Probability* 17: 156–80. [CrossRef]
- Avram, Florin, Zbigniew Palmowski, and Martijn R. Pistorius. 2015. On Gerber-Shiu functions and optimal dividend distribution for a Lévy risk process in the presence of a penalty function. *The Annals of Applied Probability* 25: 1868–935. [CrossRef]
- Bardoux, Erik J., Juan Carlos Pardo, José Luis Pérez, and Jean-François Renaud. 2016. Gerber-Shiu distribution at Parisian ruin for Lévy insurance risk processes. *Journal of Applied Probability* 53: 572–84. [CrossRef]
- Biffis, Enrico, and Andreas E. Kyprianou. 2010. A note on scale functions and the time value of ruin for Lévy insurance risk processes. *Insurance: Mathematics and Economics* 46: 85–91. [CrossRef]
- Czarna, Irmína, and Zbigniew Palmowski. 2014. Dividend problem with Parisian delay for a spectrally negative Lévy risk process. *Journal of Optimization Theory and Applications* 161: 239–56. [CrossRef]
- Dassios, Angelos, and Shanle Wu. 2009. On barrier strategy dividends with Parisian implementation delay for classical surplus processes. *Insurance: Mathematics and Economics* 45: 195–202. [CrossRef]
- De Finetti, Bruno. 1957. Su un' impostazione alternativa della teoria collettiva del rischio. *Transactions of the XVth International Congress of Actuaries* 2: 433–43.
- Ivanovs, Jevgenijs, and Zbigniew Palmowski. 2012. Occupation densities in solving exit problems for Markov additive processes and their reflections. *Stochastic Processes and Their Applications* 122: 3342–60. [CrossRef]
- Kuznetsov, Alexey, Andreas E. Kyprianou, and Victor Rivero. 2012. *The Theory of Scale Functions for Spectrally Negative Lévy Processes*. Lévy Matters. Berlin/Heidelberg: Springer.
- Kyprianou, Andreas E. 2014. *Fluctuations of Lévy Processes with Applications—Introductory Lectures*, 2nd ed. Heidelberg: Springer.
- Kyprianou, Andreas E., Ronnie Loeffen, and José-Luis Pérez. 2012. Optimal control with absolutely continuous strategies for spectrally negative Lévy processes. *Journal of Applied Probability* 49: 150–66. [CrossRef]
- Landriault, David, Jean-François Renaud, and Xiaowen Zhou. 2011. Occupation times of spectrally negative Lévy processes with applications. *Stochastic Processes and Their Applications* 121: 2629–41. [CrossRef]
- Lkabous, Mohamed Amine, and Jean-François Renaud. 2019. A unified approach to ruin probabilities with delays for spectrally negative Lévy processes. *Scandinavian Actuarial Journal*. [CrossRef]

- Loeffen, Ronnie L. 2008. On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes. *The Annals of Applied Probability* 18: 1669–80. [[CrossRef](#)]
- Loeffen, Ronnie L., and Jean-François Renaud. 2010. De Finetti's optimal dividends problem with an affine penalty function at ruin. *Insurance: Mathematics and Economics* 46: 98–108. [[CrossRef](#)]
- Loeffen, Ronnie, Irmina Czarna, and Zbigniew Palmowski. 2013. Parisian ruin probability for spectrally negative Lévy processes. *Bernoulli* 19: 599–609. [[CrossRef](#)]
- Renaud, Jean-François, and Xiaowen Zhou. 2007. Distribution of the present value of dividend payments in a Lévy risk model. *Journal of Applied Probability* 44: 420–27. [[CrossRef](#)]
- Roberts, A. W., and D. E. Varberg. 1973. Convex functions. In *Pure and Applied Mathematics*. New York and London: Academic Press, vol. 57.



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